Karapinar et al. *Advances in Difference Equations* (2020) 2020:539
https://doi.org/10.1186/s13662-020-02992-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Study on Pata E-contractions

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Abstract

In this paper, we introduce the notion of an α–$\tilde{\zeta}$–$\mathcal{E}$–Pata contraction that combines well-known concepts, such as the Pata contraction, the *E*-contraction and the simulation function. Existence and uniqueness of a fixed point of such mappings are investigated in the setting of a complete metric space. An example is stated to indicate the validity of the observed result. At the end, we give an application on the solution of nonlinear fractional differential equations.

MSC: Primary 46T99; 54E50; secondary 54H10

Keywords: Fixed point; Pata type contraction; *E*-contraction; Fractional integral equation; Orbital admissible mapping

1 Introduction

In 2015, Khojasteh et al. [1] initiated the concept of simulation functions.

Definition 1.1 ([1]) A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function if the following conditions hold:

(ζ_1) $\zeta(\varkappa, y) < y - \varkappa$ for all $\varkappa, y > 0$;

(ζ_2) if $\{\varkappa_n\}$, $\{y_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} \varkappa_n = \lim_{n\to\infty} y_n > 0$, then

$$\limsup_{n\to\infty} \zeta(\varkappa_n, y_n) < 0. \tag{1.1}$$

We denote by $\mathcal{Z}$ the family of all above simulation functions.

Let $(\mathcal{X}, d)$ be a metric space and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a function. A mapping $\hbar : \mathcal{X} \to \mathcal{X}$ is called α-orbital admissible if the following condition holds:

$$\alpha(\nu, \hbar\nu) \geq 1 \quad \text{implies} \quad \alpha\big(\hbar\nu, \hbar^2\nu\big) \geq 1, \tag{1.2}$$

for all $\nu \in \mathcal{X}$. Moreover, an α-orbital admissible mapping is called triangular α-orbital admissible if for all $\nu, \omega \in \mathcal{X}$, we have

$$\alpha(\nu, \omega) \geq 1 \quad \text{and} \quad \alpha(\omega, \hbar\omega) \geq 1 \quad \Longrightarrow \quad \alpha(\nu, \hbar\omega) \geq 1. \tag{1.3}$$

Definition 1.2 A set $\mathcal{X}$ is said to be regular with respect to a given function $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ if for each sequence $\{\nu_n\}$ in $\mathcal{X}$ such that $\alpha(\nu_n, \nu_{n+1}) \geq 1$ for all n and $\nu_n \to \nu \in \mathcal{X}$ as $n \to \infty$, then $\alpha(\nu_n, \nu) \geq 1$ for all n.

The notion of α-admissible $\mathcal{Z}$-contractions with respect to a given simulation function was merged and used by Karapinar in [2]. Using this new type of contractive mappings, he investigated the existence and uniqueness of a fixed point in standard metric spaces.

Definition 1.3 ([2]) Let T be a self-mapping defined on a metric space $(\mathcal{X}, d)$. If there exist a function $\zeta \in \mathcal{Z}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that

$$\zeta\big(\alpha(\nu,\omega)d(T\nu, T\omega), d(\nu,\omega)\big) \geq 0 \quad \text{for all } \nu, \omega \in \mathcal{X}, \tag{1.4}$$

then we say that T is an α-admissible $\mathcal{Z}$-contraction with respect to ζ.

Theorem 1.4 ([2]) *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be an α-admissible $\mathcal{Z}$-contraction with respect to ζ. Suppose that*:

(a) *T is triangular α-orbital admissible*;
(b) *there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$*;
(c) *T is continuous.*

Then there is $\nu_* \in \mathcal{X}$ such that $T\nu_* = \nu_*$.

Remark 1.5 The continuity condition in Theorem 1.4 can be replaced by the "regularity" condition, which is considered in Definition 1.2.

We will consider the following set of functions:

$$\mathcal{Z} = \big\{\psi : [0,1] \to [0,\infty) \mid \psi \text{ is continuous at zero with } \psi(0) = 0\big\}$$

and we denote

$$\|\nu\| = d(\nu, \nu_0), \quad \text{for an arbitrary but fixed } \nu_0 \in \mathcal{X}.$$

Several interesting extensions and generalizations of the Banach contraction principle [3] appeared in the literature. For instance, see [4–10]. Among these generalizations, we cite the paper of Pata [11]. Since then, much work appeared in the same direction; see [12–15].

Theorem 1.6 ([11]) *Let $(\mathcal{X}, d)$ be a complete metric space and let $\Lambda \geq 0$, $\lambda \geq 1$, $\beta \in [0,\lambda]$ be fixed constants. The mapping $\hbar : \mathcal{X} \to \mathcal{X}$ has a fixed point in $\mathcal{X}$ if the inequality*

$$d(\hbar\nu, \hbar\omega) \leq (1-\varepsilon)d(\nu,\omega) + \Lambda(\varepsilon)^\lambda \psi(\varepsilon)\big[1 + \|\nu\| + \|\omega\|\big]^\beta, \tag{1.5}$$

is satisfied for every $\varepsilon \in [0,1]$ and $\psi \in \mathcal{Z}$.

Definition 1.7 Let $(\mathcal{X}, d)$ be a metric space. We say that $\hbar : \mathcal{X} \to \mathcal{X}$ is a Pata type Zamfirescu mapping if for all $\nu, \omega \in \mathcal{X}$, $\psi \in \mathcal{Z}$ and for every $\varepsilon \in [0,1]$, $\hbar$, it satisfies the following inequality:

$$d(\hbar\nu, \hbar\omega) \leq (1-\varepsilon)\mathcal{M}(\nu,\omega) + \Lambda(\varepsilon)^\lambda \psi(\varepsilon)\big[1 + \|\nu\| + \|\omega\| + \|\hbar\nu\| + \|\hbar\omega\|\big]^\beta, \tag{1.6}$$

where

$$\mathcal{M}(\nu,\omega) = \max\left\{ d(\nu,\omega), \frac{d(\nu,\hbar\nu)+d(\omega,\hbar\omega)}{2}, \frac{d(\nu,\hbar\omega)+d(\omega,\hbar\nu)}{2} \right\}$$

and $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0,\lambda]$ are constants.

Theorem 1.8 ([16]) *Let $(\mathcal{X}, d)$ be a complete metric space and let $\hbar : \mathcal{X} \to \mathcal{X}$ be a Pata type Zamfirescu mapping. Then $\hbar$ has a unique fixed point in $\mathcal{X}$.*

We state the following useful known lemma.

Lemma 1.9 *Let $(\mathcal{X}, d)$ be a complete metric space and $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. If the sequence $\{u_n\}$ is not Cauchy, then there exist $e > 0$ and subsequences $\{u_{n_l}\}$ and $\{u_{m_l}\}$ of $\{u_n\}$ such that*

$$\lim_{n\to\infty} d(u_{n_l+1}, u_{m_l+1}) = e \tag{1.7}$$

and

$$\lim_{n\to\infty} d(u_{n_l}, u_{m_l}) = \lim_{n\to\infty} d(u_{n_l+1}, u_{m_l}) = \lim_{n\to\infty} d(u_{n_l}, u_{m_l+1}) = e. \tag{1.8}$$

In this paper, we combine the concepts of simulation functions and α-admissibility to give a generalized Pata type fixed point result. At the end, we present an application on fractional calculus.

2 Main results

We denote by $\tilde{\mathcal{Z}}$ the set of all functions $\tilde{\zeta} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following condition:

($\tilde{\zeta}_1$) $\tilde{\zeta}(\chi, y) \leq y - \chi$ for all $\chi, y > 0$.

Definition 2.1 Let $(\mathcal{X}, d)$ be a metric space and $\phi \in \Phi$. Let $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0,\lambda]$ be fixed constants. A triangular α-orbital admissible mapping $\hbar : \mathcal{X} \to \mathcal{X}$ is called an α-$\tilde{\zeta}$-$\mathcal{E}$- Pata contraction if there exists a function $\tilde{\zeta} \in \tilde{\mathcal{Z}}$ such that, for every $\varepsilon \in [0,1]$, the following condition is satisfied:

$$\tilde{\zeta}\big(\alpha(\nu,\omega) d(\hbar\nu, \hbar\omega), (1-\varepsilon)\mathcal{E}(\nu,\omega) + \mathcal{S}(\nu,\omega)\big) \geq 0 \tag{2.1}$$

for all $\nu, \omega \in \mathcal{X}$, where

$$\mathcal{E}(\nu,\omega) = \max\left\{ \begin{array}{l} d(\nu,\omega) + |d(\nu,\hbar\nu) - d(\omega,\hbar\omega)|, \\ \frac{d(\nu,\hbar\nu)+d(\omega,\hbar\omega)+|d(\nu,\hbar\nu)-d(\omega,\hbar\omega)|}{2}, \\ \frac{d(\nu,\hbar\omega)+d(\omega,\hbar\nu)+|d(\nu,\hbar\nu)-d(\omega,\hbar\omega)|}{2} \end{array} \right\} \tag{2.2}$$

and

$$\mathcal{S}(\nu,\omega) = \Lambda \varepsilon^{\lambda} \psi(\varepsilon)\big[1 + \|\nu\| + \|\omega\| + \|\hbar\nu\| + \|\hbar\omega\|\big]^{\beta}. \tag{2.3}$$

Remark 2.2 It is clear that any Pata type Zamfirescu mapping is also an α-$\tilde{\zeta}$-$\mathcal{E}$- Pata mapping. Indeed, letting $\alpha(\nu,\omega)=1$ and $\tilde{\zeta}(\chi,y)=y-\chi$, the inequality (2.1) becomes

$$\begin{aligned} d(\hbar\nu,\hbar\omega) &\leq (1-\varepsilon)\mathcal{E}(\nu,\omega)+\mathcal{S}(\nu,\omega) \\ &= (1-\varepsilon)\mathcal{E}(\nu,\omega)+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1+\|\nu\|+\|\omega\|+\|\hbar\nu\|+\|\hbar\omega\|\big]^{\beta}. \end{aligned}$$

Moreover, note that $\mathcal{M}(\nu,\omega)\leq\mathcal{E}(\nu,\omega)$ for all $\nu,\omega\in\mathcal{X}$.

Theorem 2.3 *Every α–$\tilde{\zeta}$–$\mathcal{E}$–Pata contraction $\hbar$ on a complete metric space $(\mathcal{X},d)$ possesses a fixed point if*

(*i*) *there exists $u_0\in\mathcal{X}$ such that $\alpha(u_0,\hbar u_0)\geq 1$;*
(*ii*) *$\hbar$ is triangular α-orbital admissible;*
(*iii*) *either $\hbar$ is continuous, or the set $\mathcal{X}$ is regular.*

If in addition we assume that the following condition is satisfied:

(*iv*) *$\alpha(z^*,\nu^*)\geq 1$ for all $z^*,\nu^*\in Fix_{\mathcal{X}}(\hbar)$,*

then such a fixed point of $\hbar$ is unique.

Proof Let $u_0\in\mathcal{X}$ be a point such that $\alpha(u_0,\hbar u_0)\geq 1$. On account of the assumption that $\hbar$ is a triangular α-orbital admissible mapping, we derive that

$$\alpha(u_0,\hbar u_0)\geq 1 \quad\Rightarrow\quad \alpha\big(\hbar u_0,\hbar^2u_0\big)\geq 1,$$

and iteratively we find

$$\alpha\big(\hbar^n u_0,\hbar^{n+1}u_0\big)\geq 1 \quad \text{for every } n\in\mathbb{N}. \tag{2.4}$$

Moreover, by (2.4) together with (1.3), we have

$$\alpha(u_0,\hbar u_0)\geq 1 \quad\text{and}\quad \alpha\big(\hbar u_0,\hbar^2u_0\big)\geq 1 \quad\Rightarrow\quad \alpha\big(u_0,\hbar^2u_0\big)\geq 1.$$

Again, iteratively, one writes

$$\alpha\big(u_0,\hbar^n u_0\big)\geq 1 \quad \text{for every } n\in\mathbb{N}. \tag{2.5}$$

Starting from this point $u_0\in\mathcal{X}$, we build an iterative sequence $\{u_n\}$ where $u_n=\hbar u_{n-1}=\hbar^n u_0$ for $n=1,2,3,\dots$. We can presume that any two consequent terms of this sequence are distinct. Indeed, if, on the contrary, there exists $i_0\in\mathbb{N}$ such that

$$u_{i_0}=u_{i_0+1}=\hbar u_{i_0},$$

then u_{i_0} is a fixed point. To avoid this, we will assume in the following that for all $n\in\mathbb{N}$

$$u_n\neq u_{n+1} \quad\Leftrightarrow\quad d(\hbar u_{n-1},\hbar u_n)=d(u_n,u_{n+1})>0.$$

We mention that (2.4) can be rewritten as

$$\alpha(u_n,u_{n+1})\geq 1, \tag{2.6}$$

respectively,

$$\alpha(u_0, u_n) \geq 1, \tag{2.7}$$

for any $n \in \mathbb{N}$. In the sequel, we will denote $d(\nu, u_0) = \|\nu\|$ for all $\nu \in X$.

Since $\hbar$ is an α-$\tilde{\zeta}$-$\mathcal{E}$-Pata contraction, we have

$$\tilde{\zeta}\big(\alpha(u_{n-1}, u_n) d(\hbar\nu_{n-1}, \hbar u_n), (1-\varepsilon)\mathcal{E}(\nu_{n-1}, u_n) + \mathcal{S}(\nu_{n-1}, u_n)\big) \geq 0.$$

Thus, taking into account $(\tilde{\zeta}_1)$, together with (2.6) we get

$$\begin{aligned} d(u_n, u_{n+1}) &= d(\hbar u_{n-1}, \hbar u_n) \\ &\leq \alpha(u_{n-1}, u_n) d(\hbar u_{n-1}, \hbar u_n) \\ &\leq (1-\varepsilon)\mathcal{E}(u_{n-1}, u_n) + \mathcal{S}(u_{n-1}, u_n), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{E}(u_{n-1}, u_n) &= \max\left\{ \begin{array}{c} d(u_{n-1}, u_n) + |d(u_{n-1}, \hbar u_{n-1}) - d(u_n, \hbar u_n)| \\ \frac{d(u_{n-1}, \hbar u_{n-1}) + d(u_n, \hbar u_n) + |d(u_{n-1}, \hbar u_{n-1}) - d(u_n, \hbar u_n)|}{2} \\ \frac{d(u_{n-1}, \hbar u_n) + d(u_n, \hbar u_{n-1}) + |d(u_{n-1}, \hbar u_{n-1}) - d(u_n, \hbar u_n)|}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(u_{n-1}, u_n) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|, \\ \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|}{2} \\ \frac{d(u_{n-1}, u_{n+1}) + d(u_n, u_n) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|}{2} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(u_{n-1}, u_n) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|, \\ \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|}{2} \\ \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(u_{n-1}, u_n) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|, \\ \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|}{2} \end{array} \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}(u_{n-1}, u_n) &= \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n-1}\| + \|u_n\| + \|\hbar u_{n-1}\| + \|\hbar u_n\|\big]^{\beta} \\ &= \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n-1}\| + \|u_n\| + \|u_n\| + \|u_{n+1}\|\big]^{\beta} \\ &= \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n-1}\| + 2\|u_n\| + \|u_{n+1}\|\big]^{\beta}. \end{aligned}$$

Denoting by $\gamma_n = d(u_{n-1}, u_n)$, we have

$$\mathcal{E}(u_{n-1}, u_n) \leq \max\left\{\gamma_n + |\gamma_n - \gamma_{n+1}|, \frac{\gamma_n + \gamma_{n+1} + |\gamma_n - \gamma_{n+1}|}{2}\right\}.$$

Thus, (2.8) becomes

$$\begin{aligned} \gamma_{n+1} \leq{}& (1-\varepsilon)\max\left\{\gamma_n + |\gamma_n - \gamma_{n+1}|, \frac{\gamma_n + \gamma_{n+1} + |\gamma_n - \gamma_{n+1}|}{2}\right\} \\ &+ \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n-1}\| + 2\|u_n\| + \|u_{n+1}\|\big]^{\beta}. \end{aligned} \tag{2.9}$$

We claim that the sequence $\{\gamma_n\}$ is non-increasing. Indeed, if we suppose the contrary that, for some p, $\gamma_p < \gamma_{p+1}$, and so $\max\{\gamma_p, \gamma_{p+1}\} = \gamma_{p+1}$, then we have $|\gamma_p - \gamma_{p+1}| = \gamma_{p+1} - \gamma_p$.

$$\mathcal{E}(u_{n-1}, u_n) \le \gamma_{n+1}. \tag{2.10}$$

Consequently, from (2.9), we get, for such an integer p,

$$\gamma_{p+1} \le (1-\varepsilon)\gamma_{p+1} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{p-1}\| + 2\|u_p\| + \|u_{p+1}\|\big]^{\beta}. \tag{2.11}$$

The above inequality is true for all $\varepsilon \in [0,1]$. In particular, for $\varepsilon = 0$, we get $\gamma_{p+1} \le \gamma_{p+1}$, which clearly is a contradiction. In this case, we find that the sequence $\{\gamma_n\}$ is non-increasing. So we can find a non-negative real number γ such that

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = \lim_{n\to\infty} \gamma_n = \gamma.$$

We claim that $\gamma = 0$. In order to prove this, we have to show that the sequence $\{\kappa_n\}$ is bounded, where $\kappa_n = \|u_n\| = d(u_n, u_0)$. Since the sequence $\{d(u_n, u_{n+1})\}$ is non-increasing, we have

$$d(u_n, u_{n+1}) = \gamma_n \le \kappa_1 = d(u_1, u_0).$$

By the triangle inequality, we get

$$\begin{aligned}\kappa_n &= d(u_n, u_0) \le d(u_n, u_{n+1}) + d(u_{n+1}, u_1) + d(u_1, u_0)\\ &= d(u_n, u_{n+1}) + d(\hbar u_n, \hbar u_0) + \kappa_1 \le d(\hbar u_n, \hbar u_0) + 2\kappa_1.\end{aligned} \tag{2.12}$$

On account of (2.5), regarding that $\hbar$ is an α-$\tilde{\zeta}$-Pata-$\mathcal{E}$ contraction, we have

$$\begin{aligned}0 &\le \tilde{\zeta}\big(\alpha(u_0, u_n)d(\hbar u_0, \hbar u_n), (1-\varepsilon)\mathcal{E}(u_0, u_n) + \mathcal{S}(u_0, u_n)\big)\\ &\le (1-\varepsilon)\mathcal{E}(u_0, u_n) + \mathcal{S}(u_0, u_n) - \alpha(u_0, u_n)d(\hbar u_0, \hbar u_n).\end{aligned}$$

Taking into account (2.7), this is equivalent to

$$\begin{aligned}d(\hbar u_n, \hbar u_0) = d(\hbar u_0, \hbar u_n) &\le \alpha(u_0, u_n)d(\hbar u_0, \hbar u_n)\\ &\le (1-\varepsilon)\mathcal{E}(u_0, u_n) + \mathcal{S}(u_0, u_n)\\ &= (1-\varepsilon)\max\left\{\begin{array}{l} d(u_n, u_0) + |d(u_n, \hbar u_n) - d(u_0, \hbar u_0)|,\\ \frac{d(u_n,\hbar u_n)+d(u_0,\hbar u_0)+|d(u_n,\hbar u_n)-d(u_0,\hbar u_0)|}{2}\\ \frac{d(u_n,\hbar u_0)+d(u_0,\hbar u_n)+|d(u_n,\hbar u_n)-d(u_0,\hbar u_0)|}{2}\end{array}\right\}\\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_n\| + \|u_0\| + \|\hbar u_n\| + \|\hbar u_0\|\big]^{\beta}\\ &= (1-\varepsilon)\max\left\{\begin{array}{l} d(u_n, u_0) + |d(u_n, u_{n+1}) - d(u_0, u_1)|,\\ \frac{d(u_n,u_{n+1})+d(u_0,u_1)+|d(u_n,u_{n+1})-d(u_0,u_1)|}{2}\\ \frac{d(u_n,u_1)+d(u_0,u_{n+1})+|d(u_n,u_{n+1})-d(u_0,u_1)|}{2}\end{array}\right\}\\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_n\| + \|u_0\| + \|u_{n+1}\| + \|u_1\|\big]^{\beta}\end{aligned}$$

$$\begin{aligned}
&\leq (1-\varepsilon)\max\left\{\begin{array}{l}\kappa_n + |\gamma_n - \kappa_1|, \\ \frac{\gamma_n+\kappa_1+|\gamma_n-\kappa_1|}{2} \\ \frac{\kappa_n+\kappa_1+\kappa_n+\gamma_n+|\gamma_n-\kappa_1|}{2}\end{array}\right\} \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\kappa_n+\gamma_n+\kappa_n+\kappa_1]^{\beta} \\
&\leq (1-\varepsilon)\max\{\kappa_n+\kappa_1-\gamma_n, \kappa_1, \kappa_1+\kappa_n\} \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+2\kappa_n+2\kappa_1]^{\beta}.
\end{aligned}$$

Using (2.12) and the above inequality, we get

$$\begin{aligned}
\kappa_n &\leq d(\hbar u_n, \hbar u_0) + 2\kappa_1 \\
&\leq (1-\varepsilon)\max\{\kappa_n+\kappa_1-\gamma_n, \kappa_1, \kappa_1+\kappa_n\} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+2\kappa_n+2\kappa_1]^{\beta} + 2\kappa_1 \\
&\leq (1-\varepsilon)(\kappa_n+\kappa_1) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+2\kappa_n+2\kappa_1]^{\beta} + 2\kappa_1.
\end{aligned}$$

Moreover, since $\beta \leq \lambda$, we have

$$\begin{aligned}
\varepsilon\kappa_n &\leq (3-\varepsilon)\kappa_1 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+2\kappa_n+2\kappa_1]^{\beta} \\
&\leq (3-\varepsilon)\kappa_1 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+2\kappa_n+2\gamma_1]^{\lambda} \\
&= (3-\varepsilon)\kappa_1 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)(1+2\kappa_n)^{\lambda}\left[1+\frac{2\kappa_1}{1+2\kappa_n}\right]^{\lambda} \\
&\leq 3\kappa_1 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)2^{\lambda}\kappa_n^{\lambda}\left(1+\frac{1}{2\kappa_n}\right)^{\lambda}(1+2\kappa_1)^{\lambda}.
\end{aligned}$$

Now, supposing that the sequence $\{\kappa_n\}$ is not bounded, there exists a subsequence $\{\kappa_{n_l}\}$ of $\{\kappa_n\}$ such that $\kappa_{n_l} \to \infty$ as $l \to \infty$. In this case, letting $\varepsilon = \varepsilon_l = \frac{1+3\kappa_1}{\kappa_{n_l}} (\in [0,1])$, the above inequality yields

$$\begin{aligned}
1 &\leq \Lambda 2^{\lambda}\left[\varepsilon^{\lambda}\kappa_n^{\lambda}\right](1+2\kappa_1)^{\lambda}\left(1+\frac{1}{2\kappa_{n_l}}\right)^{\lambda}\psi(\varepsilon_l) \\
&\leq \Lambda 2^{\lambda}(1+3\kappa_1)^{\lambda}(1+2\kappa_1)^{\lambda}\left(1+\frac{1}{2\kappa_{n_l}}\right)^{\lambda}\psi(\varepsilon_l) \\
&\leq \Lambda 2^{\lambda}(1+3\kappa_1)^{2\lambda}\left(1+\frac{1}{2\kappa_{n_l}}\right)^{\lambda}\psi(\varepsilon_l) \to 0 \quad \text{as } l \to \infty.
\end{aligned}$$

This is a contradiction. Thus, we conclude that our presumption is false and then the sequence $\{\kappa_n\}$ is bounded. Furthermore, there exists $\mathcal{K} > 0$ such that $\kappa_n \leq \mathcal{K}$ for all $n \in \mathbb{N}$.

Let us go back now and prove that $\gamma = 0$ (where $\gamma = \lim_{n\to\infty}\gamma_n$). In view of (2.10) and the fact that the sequence $\{\gamma_n\}$ is non-increasing, one writes

$$\mathcal{E}(u_{n-1}, u_n) \leq 2\gamma_n - \gamma_{n+1}.$$

Recall that

$$\mathcal{S}(u_{n-1}, u_n) \leq \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\left[1+\|u_{n-1}\|+2\|u_n\|+\|u_{n+1}\|\right]^{\beta} \leq \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+4\mathcal{K}]^{\beta}.$$

Taking into account that $\hbar$ is an α-$\tilde{\zeta}$-E_* contraction, keeping in mind (2.6) and using $(\tilde{\zeta}_1)$, we have

$$\begin{aligned}0 &\leq \tilde{\zeta}\bigl(\alpha(u_{n-1},u_n)d(\hbar u_{n-1},\hbar u_n),(1-\varepsilon)\mathcal{E}(u_{n-1},u_n)+\mathcal{S}(u_{n-1},u_n)\bigr)\\ &\leq (1-\varepsilon)(1-\varepsilon)\mathcal{E}(u_{n-1},u_n)+\mathcal{Z}(u_{n-1},u_n)-\alpha(u_{n-1},u_n)d(\hbar u_{n-1},\hbar u_n).\end{aligned}$$

We have

$$\begin{aligned}\gamma_n = d(u_n,u_{n+1}) &\leq \alpha(u_{n-1},u_n)d(\hbar u_{n-1},\hbar u_n)\\ &\leq (1-\varepsilon)\mathcal{E}(u_{n-1},u_n)+\mathcal{S}(u_{n-1},u_n)\\ &\leq (1-\varepsilon)(2\gamma_n-\gamma_{n+1})+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+4\mathcal{K}]^{\beta}).\end{aligned} \tag{2.13}$$

Letting $n\to\infty$ in the previous inequality, we obtain

$$\gamma \leq (1-\varepsilon)\gamma+\Lambda\varepsilon^{\lambda}(1+4\mathcal{K})^{\beta}\psi(\varepsilon),$$

which is equivalent to

$$\gamma \leq \Lambda\varepsilon^{\lambda-1}(1+4\mathcal{K})^{\beta}\psi(\varepsilon).$$

When $\varepsilon\to 0$, we get $\gamma\leq 0$. Therefore,

$$\gamma = \lim_{n\to\infty} d(u_n,u_{n+1}) = 0. \tag{2.14}$$

As a next step, we claim that $\{u_n\}$ is a Cauchy sequence. On the contrary, assuming that the sequence is not Cauchy, it follows from Lemma 1.9 that there exist $e>0$ and subsequences $\{u_{n_l}\}$ and $\{u_{m_l}\}$ such that (1.7) and (1.8) hold. Replacing $\nu=u_{n_l}$ and $\omega=u_{m_l}$ in (2.1), we have

$$\begin{aligned}0 &\leq \tilde{\zeta}\bigl(\alpha(u_{n_l},u_{m_l})d(\hbar u_{n_l},\hbar u_{m_l}),(1-\varepsilon)\mathcal{E}+\mathcal{S}(u_{n_l},u_{m_l})\bigr)\\ &\leq (1-\varepsilon)\mathcal{E}(u_{n_l},u_{m_l})+\mathcal{S}(u_{n_l},u_{m_l})-\alpha(u_{n_l},u_{m_l})d(\hbar u_{n_l},\hbar u_{m_l}),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\mathcal{E}(u_{n_l},u_{m_l}) &= \max\left\{\begin{array}{c} d(u_{n_l},u_{m_l})+|d(u_{n_l},\hbar u_{n_l})-d(u_{m_l},\hbar u_{m_l})|,\\ \frac{d(u_{n_l},\hbar u_{n_l})+d(u_{m_l},\hbar u_{m_l})+|u_{n_l},\hbar u_{n_l})-d(u_{m_l},\hbar u_{m_l})|}{2}\\ \frac{d(u_{n_l},\hbar u_{m_l})+d(u_{m_l},\hbar u_{n_l})+|u_{n_l},\hbar u_{n_l})-d(u_{m_l},\hbar u_{m_l})|}{2}\end{array}\right\}\\ &= \max\left\{\begin{array}{c} d(u_{n_l},u_{m_l})+|d(u_{n_l},u_{n_l+1})-d(u_{m_l},u_{m_l+1})|,\\ \frac{d(u_{n_l},u_{n_l+1})+d(u_{m_l},u_{m_l+1})+|u_{n_l},u_{n_l+1})-d(u_{m_l},u_{m_l+1})|}{2}\\ \frac{d(u_{n_l},u_{m_l+1})+d(u_{m_l},u_{n_l+1})+|u_{n_l},u_{n_l+1})-d(u_{m_l},u_{m_l+1})|}{2}\end{array}\right\}.\end{aligned}$$

The triangular α-orbital admissibility of $\hbar$ shows that $\alpha(u_{n_l},u_{m_l})\geq 1$. Thus,

$$d(u_{n_l+1},u_{m_l+1}) \leq (1-\varepsilon)\mathcal{E}(u_{n_l},u_{m_l})+\mathcal{S}(u_{n_l},u_{m_l}). \tag{2.16}$$

Letting $l \to \infty$ and taking into account (2.14) and Lemma 1.9, we have

$$\lim_{l\to\infty} \mathcal{E}(u_{n_l}, u_{m_l}) = e. \tag{2.17}$$

At the same time, one writes

$$\begin{aligned}
\mathcal{S}(u_{n_l}, u_{m_l}) &= \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n_l}\| + \|u_{m_l}\| + \|\hbar u_{n_l}\| + \|\hbar u_{m_l}\|\big]^{\beta} \\
&= \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\big[1 + \|u_{n_l}\| + \|u_{m_l}\| + \|u_{n_l+1}\| + \|u_{m_l+1}\|\big]^{\beta} \\
&\leq \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\mathcal{K}]^{\beta}.
\end{aligned}$$

Denoting by $a_l = d(u_{n_l+1}, u_{m_l+1})$ and $b_l = (1-\varepsilon)\mathcal{E}(u_{n_l}, u_{m_l}) + \mathcal{S}(u_{n_l}, u_{m_l})$, by Lemma 1.9, it follows that

$$a_l \to e \quad \text{and} \quad \limsup_{l\to\infty} b_l \leq (1-\varepsilon)e + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\mathcal{K}]^{\beta}.$$

Thus, passing to the limit as $l \to \infty$ in (2.16), we get

$$e = \limsup_{l\to\infty} a_l \leq \limsup_{l\to\infty} b_l \leq \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\mathcal{K}]^{\beta}.$$

Furthermore,

$$e \leq (1-\varepsilon)e + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\mathcal{K}]^{\beta},$$

i.e.,

$$e \leq \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)[1 + 4\mathcal{K}]^{\beta}.$$

That is, $e = 0$. Therefore, $\{u_n\}$ is a Cauchy sequence in the complete metric space. For this reason, there exists $v^* \in \mathcal{X}$ such that $u_n \to v^*$, as $n \to \infty$.

Furthermore, in the case that $\hbar$ is a continuous mapping, we get $\hbar v^* = v^*$, that is, v^* is a fixed point of $\hbar$.

Now, suppose that $\mathcal{X}$ is regular. From (2.1), one writes

$$\tilde{\zeta}\big(\alpha(u_n, v^*)d(\hbar u_n, \hbar v^*), (1-\varepsilon)\mathcal{E}(u_n, v^*) + \mathcal{S}(u_n, v^*)\big). \tag{2.18}$$

Using the regularity of $\mathcal{X}$ and $(\tilde{\zeta}_1)$, we get

$$d(\hbar u_n, \hbar v^*) \leq \alpha(u_n, v^*)d(\hbar u_n, \hbar v^*) \leq (1-\varepsilon)\mathcal{E}(u_n, v^*) + \mathcal{S}(u_n, v^*) \tag{2.19}$$

where

$$\begin{aligned}
\mathcal{E}(u_n, v^*) &= \max\left\{\begin{array}{l} d(u_n, v^*) + |d(u_n, \hbar u_n) - d(v^*, \hbar v^*)| \\ \frac{d(u_n,\hbar u_n)+d(v^*,\hbar v^*)+|d(u_n,\hbar u_n)-d(v^*,\hbar v^*)|}{2}, \\ \frac{d(u_n,\hbar v^*)+d(v^*,\hbar u_n)+|d(u_n,\hbar u_n)-d(v^*,\hbar v^*)|}{2} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} d(u_n, v^*) + |d(u_n, u_{n+1}) - d(v^*, \hbar v^*)|, \\ \frac{d(u_n,u_{n+1})+d(v^*,\hbar v^*)+|d(u_n,u_{n+1})-d(v^*,\hbar v^*)|}{2}, \\ \frac{d(u_n,\hbar v^*)+d(v^*,u_{n+1})+|d(u_n,u_{n+1})-d(v^*,\hbar v^*)|}{2} \end{array}\right\}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(u_n,\nu^*) &= \Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+\|u_n\|+\|\nu^*\|+\|\hbar u_n\|+\|\hbar\nu^*\|\right]^\beta \\ &= \Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+\|u_n\|+\|\nu^*\|+\|u_{n+1}\|+\|\hbar\nu^*\|\right]^\beta \\ &= \Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+\kappa_n+\|\nu^*\|+\kappa_{n+1}+\|\hbar\nu^*\|\right]^\beta.\end{aligned}$$

Taking into account the boundedness of the sequence $\{\kappa_n\}$, we have

$$\mathcal{S}(u_n,\nu^*) \le \Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+2\mathcal{K}+\|\nu^*\|+\|\hbar\nu^*\|\right]^\beta.$$

On the other hand,

$$\lim_{n\to\infty}\mathcal{E}(u_n,\nu^*) = d(\nu^*,\hbar\nu^*).$$

Letting $n\to\infty$ in the inequality (2.19), we find

$$d(\nu^*,\hbar\nu^*) \le (1-\varepsilon)d(\nu^*,\hbar\nu^*) + \Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+2\mathcal{K}+\|\nu^*\|+\|\hbar\nu^*\|\right]^\beta,$$

which is equivalent to

$$d(\nu^*,\hbar\nu^*) \le \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)\left[1+2\mathcal{K}+\|\nu^*\|+\|\hbar\nu^*\|\right]^\beta.$$

Obviously, we obtain for $\varepsilon=0$ that $d(\nu^*,\hbar\nu^*)\le 0$, so $\nu^*=\hbar\nu^*$. Thus, ν^* is a fixed point of $\hbar$. Finally, to prove the uniqueness of the fixed point, we suppose that there exist two fixed points $\nu^*,\omega^*\in Fix_{\mathcal{X}}(\hbar)$ such that $\nu^*\neq\omega^*$. We have

$$\begin{aligned}0 &\le \tilde{\zeta}\left(\alpha(\nu^*,\omega^*)d(\hbar\nu^*,\hbar\omega^*),(1-\varepsilon)\mathcal{E}(\nu^*,\omega^*)+\mathcal{S}(\nu^*,\omega^*)\right) \\ &\le (1-\varepsilon)\mathcal{E}(\nu^*,\omega^*)+\mathcal{S}(\nu^*,\omega^*)-\alpha(\nu^*,\omega^*)d(\hbar\nu^*,\hbar\omega^*).\end{aligned}$$

Taking into account (iv), we obtain

$$\begin{aligned}d(\nu^*,\omega^*) &\le \alpha(\nu^*,\omega^*)d(\hbar\nu^*,\hbar\omega^*) \le (1-\varepsilon)\mathcal{E}(\nu^*,\omega^*)+\mathcal{S}(\nu^*,\omega^*) \\ &= (1-\varepsilon)d(\nu^*,\omega^*)+\Lambda\varepsilon^\lambda\psi(\varepsilon)\left[1+2\|\nu^*\|+2\|\omega^*\|\right]^\beta,\end{aligned}$$

which leads to

$$d(\nu^*,\omega^*) \le \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)\left[1+2\|\nu^*\|+2\|\omega^*\|\right]^\beta.$$

In the limit $\varepsilon\to 0$, we get $d(\nu^*,\omega^*)\le 0$, that is, $\nu^*=\omega^*$, which is a contradiction. Therefore, the fixed point of $\hbar$ is unique. □

In the following, we present an example that supports our statement, that is, Theorem 2.3 is a generalization of Theorem 1.8.

Example 2.4 Take $\mathcal{X} = A \times A$, where $A = [0, 11]$ and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is the usual distance. Define the mapping $\hbar : \mathcal{X} \to \mathcal{X}$ by

$$\hbar\nu = \begin{cases} (2,0), & \text{if } \nu \in B, \\ (11,9), & \text{if } \nu = (11,0), \\ (5,0), & \text{otherwise,} \end{cases}$$

where $B = \{(\chi, 0) | \chi \in [0, 11)\}$. For $\nu_1 = (11, 0)$ and $\nu_2 = (2, 0)$, we have

$$\begin{aligned} &d(\nu_1, \nu_2) = 9, \qquad d\big(\hbar(\nu_1), \hbar(\nu_2)\big) = d\big((11,9),(2,0)\big) = 9\sqrt{2}, \\ &d\big(\nu_2, \hbar(\nu_2)\big) = d(\nu_2, \nu_2) = 0, \qquad d\big(\nu_1, \hbar(\nu_1)\big) = d\big((11,0),(11,9)\big) = 9, \\ &d\big(\nu_1, \hbar(\nu_2)\big) = d\big((11,0),(2,0)\big) = 9, \qquad d\big(\nu_2, \hbar(\nu_1)\big) = d\big((2,0),(11,9)\big) = 9\sqrt{2}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(\nu_1, \nu_2) &= \max\left\{ d(\nu_1, \nu_2), \frac{d(\nu_1, \hbar(\nu_1)) + d(\nu_2, \hbar(\nu_2))}{2}, \frac{d(\nu_1, \hbar(\nu_2)) + d(\nu_1, \hbar(\nu_1))}{2} \right\} \\ &= \max\left\{ 9, \frac{9}{2}, \frac{9(1+\sqrt{2})}{2} \right\} = \frac{9(1+\sqrt{2})}{2}. \end{aligned}$$

Thus,

$$d\big(\hbar(\nu_1), \hbar(\nu_2)\big) = 9\sqrt{2} > \frac{9(1+\sqrt{2})}{2} = \mathcal{M}(\nu_1, \nu_2),$$

so that the inequality (1.6) does not hold for $\varepsilon = 0$. That is, $\hbar$ is not a Pata type Zamfirescu mapping.

Consider the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ given as

$$\alpha(\nu, \omega) = \begin{cases} 2, & \text{if } \nu, \omega \in B, \\ 1, & \text{if } \nu = (11,0), \omega = (2,0), \\ 0, & \text{otherwise.} \end{cases}$$

Since the assumptions (i)–(iv) are obviously satisfied, we have to prove that $\hbar$ is an α-$\tilde{\zeta}$-$\mathcal{E}$-Pata contraction. Take $\alpha = \beta = 1$, $\Lambda = 6$ and the functions $\Psi(t) = \frac{t}{2}$, $\tilde{\zeta}(\chi, y) = y - \chi$.

For $\nu, \omega \in B$, we have $d(\hbar(\nu), \hbar(\omega)) = 0$, so that (2.1) holds.

For $\nu = (11, 0)$ and $\omega = (2, 0)$ we have

$$\begin{aligned} &\alpha(\nu, \omega) d\big(\hbar(\nu), \hbar(\omega)\big) \\ &\quad = 9\sqrt{2} \le \frac{3}{4} \cdot 18 = \frac{3}{4}\big(9 + |9 - 0|\big) \\ &\quad = \frac{3}{4}\big(d(\nu, \omega) + \big|d(\nu, \hbar\nu) - d(\omega, \hbar\omega)\big|\big) \\ &\quad \le (1 - \varepsilon)\big(d(\nu, \omega) + \big|d(\nu, \hbar\nu) - d(\omega, \hbar\omega)\big|\big) \\ &\qquad + \left(\frac{3}{4} + \varepsilon - 1\right)\big(d(\nu, \omega) + \big|d(\nu, \hbar\nu) - d(\omega, \hbar\omega)\big|\big) \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\varepsilon)(\mathcal{E}(\nu,\omega) + \frac{3}{4}\left(1+\frac{4(\varepsilon-1)}{3}\right)\bigl(d(\nu,\omega)+d(\nu,\hbar\nu)+d(\omega,\hbar\omega)\bigr) \\
&\leq (1-\varepsilon)\mathcal{E}(\nu,\omega) + \frac{3}{2}\varepsilon^2\bigl(2d(\nu,\nu_0)+2d(\nu_0,\omega)+d(\nu_0,\hbar\nu)+d(\nu_0,\hbar\omega)\bigr) \\
&\leq (1-\varepsilon)(\mathcal{E}(\nu,\omega) + 3\varepsilon^2\bigl(1+\|\nu\|+\|\omega\|+\|\hbar\nu\|+\|\hbar\omega\|\bigr) \\
&= (1-\varepsilon)\mathcal{E}(\nu,\omega) + \mathcal{S}(\nu,\omega).
\end{aligned}
$$

Due to the way the function α was defined, we omit the other cases.

3 An application on a fractional boundary value problem

In this section, we ensure the existence of a solution of a nonlinear fractional differential equation (for more related details, see [17–23]). Denote by $\mathcal{X} = C[0,1]$ the set of all continuous functions defined on $[0,1]$. We endow $\mathcal{X}$ with the metric given as

$$d(\rho,\omega) = \|\rho-\omega\|_\infty = \max_{s\in[0,1]}\bigl|\rho(s)-\omega(s)\bigr|.$$

Consider the fractional differential equation

$$^{c}D^{\mu}\rho(t) = f\bigl(t,\rho(t)\bigr), \quad 0<t<1, 1<\mu\leq 2, \tag{3.1}$$

with boundary conditions

$$\begin{cases} \rho(0)=0, \\ I\rho(1)=\rho'(0). \end{cases} \tag{3.2}$$

Here, $^{c}D^{\mu}$ corresponds for the Caputo fractional derivative of order μ, given as

$$D^{\mu}f(t) = \frac{1}{\Gamma(n-\mu)}\int_0^1 (t-s)^{n-\mu-1}f^{n}(s)\,ds, \tag{3.3}$$

where $n-1<\mu<n$ and $n=[\mu]+1$, and $I^{\mu}f$ is the Riemann–Liouville fractional integral of order μ of a continuous function f, defined by

$$I^{\mu}f(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}f(s)\,ds, \quad \mu>0. \tag{3.4}$$

In [24], it is showed that the problem (3.1) and (3.2) can be written in the following integral form:

$$\rho(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}f\bigl(s,\rho(s)\bigr)\,ds + \frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1}f\bigl(r,\rho(r)\bigr)\,dr\,ds. \tag{3.5}$$

Theorem 3.1 *Assume that*

1. $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous;*

2. *for all* $\rho,\omega \in \mathcal{X}$, *we have*

$$\left|f\big(s,\rho(s)\big) - f\big(s,\omega(s)\big)\right| \le \frac{\varepsilon^2}{4}\Gamma(\mu+1)\left|\rho(s)-\omega(s)\right|, \tag{3.6}$$

for each $s \in [0,1]$, *where* $\varepsilon \in [0,1]$.

Then the problem 3.1 *and* 3.2 *possesses a unique solution.*

Proof Consider the functional

$$T\rho(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f\big(s,\rho(s)\big)\,ds + \frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\rho(r)\big)\,dr\,ds. \tag{3.7}$$

Note that a solution of (3.5) is also a fixed point of T. We mention that T is well posed. For all $\rho,\omega \in \mathcal{X}$ and $s \in [0,1]$, we have

$$\begin{aligned}
&\left|T\rho(t) - T(\omega(t)\right| \\
&= \bigg|\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f\big(s,\rho(s)\big)\,ds + \frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\rho(r)\big)\,dr\,ds \\
&\quad - \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f\big(s,\omega(s)\big)\,ds - \frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\omega(r)\big)\,dr\,ds\bigg| \\
&\le \bigg|\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f\big(s,\rho(s)\big)\,ds - \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f\big(s,\omega(s)\big)\,ds\bigg| \\
&\quad + \bigg|\frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\rho(r)\big)\,dr\,ds - \frac{2t}{\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\omega(r)\big)\,dr\,ds\bigg| \\
&\le \frac{1}{\Gamma(\mu)}\bigg|\int_0^t (t-s)^{\mu-1} f\big(s,\rho(s)\big)\,ds - \int_0^t (t-s)^{\mu-1} f\big(s,\omega(s)\big)\,ds\bigg| \\
&\quad + \frac{2}{\Gamma(\mu)}\bigg|\int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\rho(r)\big)\,dr\,ds - \int_0^1\int_0^s (s-r)^{\mu-1} f\big(r,\omega(r)\big)\,dr\,ds\bigg| \\
&\le \frac{\varepsilon^2\Gamma(\mu+1)}{4\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left|\rho(s)-\omega(s)\right|ds \\
&\quad + \frac{2\varepsilon^2\Gamma(\mu+1)}{4\Gamma(\mu)}\int_0^1\int_0^s (s-r)^{\mu-1}\left|\rho(r)-\omega(r)\right|dr\,ds \\
&\le \frac{\varepsilon^2\Gamma(\mu+1)}{4\Gamma(\mu)}d(x,y)\int_0^t (t-s)^{\mu-1}\,ds \\
&\quad + \frac{2\varepsilon^2\Gamma(\mu+1)}{4\Gamma(\mu)}d(x,y)\int_0^1\int_0^s (s-r)^{\mu-1}\,dr\,ds \\
&\le \frac{\varepsilon^2\Gamma(\mu)\Gamma(\mu+1)}{4\Gamma(\mu)\Gamma(\mu+1)}d(\rho,\omega) \\
&\quad + 2\varepsilon^2 B(\mu+1,1)\frac{\Gamma(\mu)\Gamma(\mu+1)}{4\Gamma(\mu)\Gamma(\mu+1)}d(\rho,\omega) \\
&\le \frac{\varepsilon^2}{4}d(\rho,\omega) + \frac{\varepsilon^2}{2}d(\rho,\omega) \\
&\le \varepsilon^2 d(\rho,\omega),
\end{aligned}$$

where B is the beta function. Consequently, one has

$$\begin{aligned} d(T\rho, T\omega) &\leq \varepsilon^2 d(\rho,\omega) \\ &= \varepsilon d(\rho,\omega) - \varepsilon^2 d(\rho,\omega) + 2\varepsilon^2 d(\rho,\omega) \\ &\leq (1-\varepsilon)\mathcal{E}(\rho,\omega) + 2\varepsilon^2 d(\rho,\omega) \\ &\leq (1-\varepsilon)\mathcal{E}(\rho,\omega) + 2\varepsilon^2 \bigl[d(\rho,0) + d(0,\omega)\bigr] \\ &= (1-\varepsilon)\mathcal{E}(\rho,\omega) + 2\varepsilon^2 \bigl[\|\rho\| + \|\omega\|\bigr] \\ &\leq (1-\varepsilon)\mathcal{E}(\rho,\omega) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\bigl[1 + \|\rho\| + \|\omega\| + \|T\rho\| + \|T\omega\|\bigr]^{\beta}, \end{aligned}$$

where $\psi(\varepsilon) = \varepsilon$, $\beta = \lambda = 1$ and $\Lambda = 2$. Applying Theorem 2.3, the functional T admits a unique fixed point, that is, the problem (3.1) and (3.2) possesses a unique solution. □

4 Conclusion and remarks

Our results merged from and generalized several existing results in the related literature. First of all, as underlined in Remark 2.2, the main result of [16] is a consequence of our given theorem. On the other hand, by choosing the auxiliary functions in a proper way, we may state a long list of corollaries. More precisely, by choosing the mapping α in a proper way, we can get the analogue of our result in the setting of partially ordered metric spaces, or in the set-up of cyclic mappings. Note that, if we take $\alpha(x,y) = 1$ for all x, y, we get the standard fixed point theorems in the context of complete metric spaces; see [25–29]. In addition, by choosing the appropriate simulation function, one can get several more results; see [30–35].

Acknowledgements
The authors thank the anonymous referees for their remarkable comments, suggestions, and ideas, which helped to improve this paper. The authors also thanks to their universities.

Funding
We declare that funding is not applicable for our paper.

Availability of data and materials
No data were used to support this study. It is not applicable for our paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Writing-review and editing were done by HA, AF and EK. All authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

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Publisher's Note

Received: 28 April 2020 Accepted: 21 September 2020 Published online: 01 October 2020

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