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On some Hermite–Hadamard type inequalities for $\mathcal{T}$-convex interval-valued functions

Zehao Sha[1], Guoju Ye[1*], Dafang Zhao[2] and Wei Liu[1]

*Correspondence: ygjhhu@163.com
[1]College of Science, Hohai University, Nanjing, 210098, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the concepts of map $\mathcal{T}$ and interval-valued $\mathcal{T}$-convex, and give some basic properties. Further, we extend fractional Hermite–Hadamard inequalities in the case of $\mathcal{T}$-convex and Ostrowski type inequalities for interval-valued functions. Several examples are presented to illustrate the results.

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1 Introduction

Interval analysis was first proposed in order to reduce errors during mathematical computation. Also, it has been widely used in engineering, economics, statistics, and many other fields. Especially in engineering field, the dynamics model can solve many dynamic problems which always involve multiple uncertain parameters or interval coefficients. The monograph written by Moore [13] is the first systematic review of relevant studies on this theory. In 2009, Stefanini introduced the concepts of gH-difference and gH-derivative in [18] which addressed the problems in subtraction between two intervals. The theory of interval analysis has been continuously developed since then. In 2015, Lupulescu developed a theory of the fractional calculus for interval-valued functions in [11]. In 2019, Chalco-Cano dealt with the algebra of gH-differentiable interval-valued functions in [5]. More and more attention has been paid to the research of interval analysis, interval differential equations, interval optimization, and other related problems.

The importance of convexity is reflected in all fields of mathematics. Over the past years, the classical convex has been generalized to other different types such as harmonically convex, h-convex, p-convex, etc. In 2017, Costa gave the concepts of interval-valued convex in [6]. Based on the works of Costa, many convexities related to real functions have been gradually generalized to the case of interval-valued functions. Meanwhile, some inequalities have been also expanded, such as Hermite–Hadamard inequality, Gauss inequality, Ostrowski inequality (for more details, see [7, 10, 12, 14–17, 19]). Based on the

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work of Lupulescu, in 2019, Budak and Tunç presented the right-hand side Riemann–Liouville fractional integral for interval-valued functions and studied fractional Hermite–Hadamard inequalities. In the year 2020, Dragomir gave a new generalization of convexity for real functions which related different composite functions [9]. Motivated by the above works, we define a map $\mathcal{T}$ and introduce the concept of $\mathcal{T}$-convex interval-valued functions in this paper. Some important properties are also given. After this, we present some new generalizations of Hermite–Hadamard inequalities, fractional Hermite–Hadamard inequalities, and Ostrowski type inequalities. In addition, the concepts of map $\mathcal{T}$ and $\mathcal{T}$-convex can be used as a powerful tool in fuzzy-valued function, interval optimization, and interval-valued differential equations.

In Sect. 2, we provide the basic theory of interval analysis and some properties. The definitions of $\mathcal{T}$ and $\mathcal{T}$-convex are proposed in Sect. 3. Based on these, the statements of Hermite–Hadamard type inequalities, fractional Hermite–Hadamard type inequalities, and fractional Ostrowski type inequalities are given in Sects. 3 and 4.

2 Preliminaries

First, let $\mathcal{K}_c = \{I = [a,b] \mid a,b \in \mathbb{R}, a \leq b\}$ denote the set of all nonempty intervals belonging to $\mathbb{R}$. The length of any interval $I = [a,b] \in \mathcal{K}_c$ can be defined by $\ell(I) := b - a$.

For two intervals $A = [a^-,a^+]$ and B= $[b^-,b^+]$ belonging to $\mathcal{K}_c$, $A \subseteq B$ if and only if $a^- \geq b^-$ and $a^+ \leq b^+$. Some basic properties of algebra operations between two intervals can be found in [13]. Now we give the following important properties.

Definition 2.1 ([18]) For any $A, B \in \mathcal{K}_c$, the gH-difference between A, B can be defined as follows:

$$A \ominus_g B = \begin{cases} [a^- - b^-, a^+ - b^+], & \text{if } \ell(A) \geq \ell(B), \\ [a^+ - b^+, a^- - b^-], & \text{if } \ell(A) < \ell(B). \end{cases} \tag{2.1}$$

Specially, if $B = b \in \mathbb{R}$ is a constant, then

$$A \ominus_g B = \left[a^- - b, a^+ - b\right].$$

More properties of gH-difference can be found in [18].

The Hausdorff–Pompeiu distance $\mathcal{H} : \mathcal{K}_c \times \mathcal{K}_c \to [0,\infty)$ between A and B is defined by $\mathcal{H}(A,B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$. Then $(\mathcal{K}_c, \mathcal{H})$ is a complete and separable metric space (see [8]).

Based on this, the map $\|\cdot\| : \mathcal{K}_c \to [0,\infty)$ defined by $\|A\| := \max\{|a^-|, |a^+|\} = \mathcal{H}(A, \{0\})$ is a norm on $\mathcal{K}_c$. Hence, $(\mathcal{K}_c, \|\cdot\|)$ is a normed quasi-linear space (see [11]).

In this paper, we use symbols F and G to refer to interval-valued functions. Let I be any finite interval for any $F : I \to \mathcal{K}_c$ such that $F = [f^-, f^+]$, we say that F is ℓ-increasing (or ℓ-decreasing) on I if $\ell(F) : I \to [0,\infty)$ is increasing(or decreasing) on I. If $\ell(F)$ is monotone on I, then we say F is ℓ-monotone on $[a,b]$.

Definition 2.2 ([11]) Let $F : I \to \mathcal{K}_c$. F is said to be continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} \left\| F(x) \ominus_g F(x_0) \right\| = 0. \tag{2.2}$$

We denote by $C(I,\mathcal{K}_c)$ the set of all continuous interval-valued functions on $[a,b]$. Then $C(I,\mathcal{K}_c)$ is a complete normed space with respect to the norm $\|F\|_c := \max_{x\in I}\|F(x)\|$.

Definition 2.3 ([18]) Let $F : I \to \mathcal{K}_c$, we say that $F(x)$ is gH-differential at $x_0 \in I$ if there exists $\mathcal{D} \in \mathcal{K}_c$ such that

$$\lim_{h\to 0}\frac{F(x_0+h)\ominus_g F(x_0)}{h} = \mathcal{D}. \tag{2.3}$$

$\mathcal{D}$ defined in this fashion is denoted by the symbol $F'(x_0)$, and $F'(x_0)$ is said to be the gH-derivative of $F(x)$ at x_0.

For more details about gH-derivative, please see [5]. Let $F : I \to \mathcal{K}_c$ such that $F = [f^-,f^+]$. We say that F is (i)-gH-differentiable on I if F is ℓ-increasing on I and $F' = [(f^-)',(f^+)']$. Similarly, we say that F is (ii)-gH-differentiable on I if F is ℓ-decreasing on I and $F' = [(f^+)',(f^-)']$.

The introduction of Lebesgue integral for interval-valued function can be found in [1] and [2]. For $1 \le p \le \infty$, let $L^p(I,\mathcal{K}_c)$ be the set of p-times Lebesgue integrable interval-valued functions on I. The next statement of fundamental theorem of calculus can be found in [11].

Theorem 2.4 *Let $F : [s,t] \to \mathcal{K}_c$ be* gH*-differentiable on $[s,t]$. If $F' \in L^1([s,t],\mathcal{K}_c)$ and F is ℓ-monotone on $[s,t]$, then*

$$F(x)\ominus_g F(s) = \int_s^x F'(\tau)\,d\tau, \quad x \in [s,t]. \tag{2.4}$$

The following results are easily obtained through the basic properties of interval-valued Lebesgue integral, so we omit the proof.

Corollary 2.5 *Let $F,G : [s,t] \to \mathcal{K}_c$. If $F,G \in L^1([s,t],\mathcal{K}_c)$, then*

$$\left\|\int_s^t F \ominus_g \int_s^t G\right\| \le \int_s^t \|F\ominus_g G\|. \tag{2.5}$$

3 $\mathcal{T}$-Convex for interval-valued functions

In this section, we define the map $\mathcal{T}$ and introduce the concept of $\mathcal{T}$-convex which can be regard as a sense of "weak" convex compared with classical convex. Let $I \subset \mathbb{R}$ be any finite interval.

Consider the map $\mathcal{T} : \mathcal{K}_c \to \mathcal{K}_c$ satisfying $\mathcal{T}(A) \subseteq A, \forall A \in \mathcal{K}_c$. We denote $\mathfrak{T}$ by the family of all these maps, i.e., $\mathfrak{T} = \{\mathcal{T} \mid \mathcal{T} : \mathcal{K}_c \to \mathcal{K}_c \text{ and } \mathcal{T}(A) \subseteq A, \forall A \in \mathcal{K}_c\}$.

We define the partial order relationship "$\preceq$" between $\mathcal{T}_1,\mathcal{T}_2 \in \mathfrak{T}$ such that, for all $A \in \mathcal{K}_c$,

$$\mathcal{T}_1 \preceq \mathcal{T}_2 \quad \text{if and only if} \quad \mathcal{T}_1(A) \supseteq \mathcal{T}_2(A).$$

We now give a new generalization of convexity for interval-valued function.

Definition 3.1 Let $\mathcal{T} \in \mathfrak{T}$ and $F : I \to \mathcal{K}_c$. We say that F is $\mathcal{T}$-convex if, for any $0 \le \theta \le 1$ and $x, y \in I$,

$$\theta\mathcal{T}\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}\bigl(F(y)\bigr) \subseteq F\bigl(\theta x + (1-\theta)y\bigr). \tag{3.1}$$

For any $\mathcal{T} \in \mathfrak{T}$, we denote

$$CX_{\mathcal{T}}(I, \mathcal{K}_c) = \{F : I \to \mathcal{K}_c, F \text{ is } \mathcal{T}\text{-convex}\}.$$

Remark 3.2 For the case $\mathcal{T} = \mathcal{ID} \in \mathfrak{T}$ is an identity map, i.e., $\mathcal{ID}(A) = A$ for any $A \in \mathcal{K}_c$, then we get the classical convex of interval-valued functions in [6], and we denote

$$CX(I, \mathcal{K}_c) = \{F : I \to \mathcal{K}_c, F \text{ is convex}\}.$$

Example 3.3 We consider $\mathcal{T}$ defined by

$$\mathcal{T}(A) = \begin{cases} [e^{a^-}, a^+] & \text{if } e^{a^-} < a^+, \\ A & \text{if } e^{a^-} \ge a^+, \end{cases}$$

for any $A = [a^-, a^+] \in \mathcal{K}_c$. Obviously, $\mathcal{T} \in \mathfrak{T}$ and $\mathcal{T}$ is well-defined. Next, let $F : [1, 2] \to \mathcal{K}_c$ given by $F(x) = [\ln(x^2), x + 2]$. We can verify $F(x)$ is $\mathcal{T}$-convex but never convex on $[1, 2]$.

Let $F : I \to \mathcal{K}_c$ and $a, b \in I$ with $a < b$, we say that F is convex if $\theta F(a) + (1-\theta)F(b) \subseteq F(\theta a + (1-\theta)b)$ for any $\theta \in [0, 1]$. The left part $\theta F(a) + (1-\theta)F(b)$ can be regarded as a contraction of $F(x)$ on $[a, b]$ (see Fig. 1). Actually, any convex $F(x)$ can be controlled on I, which means the length of $F(x)$ cannot always be increasing. Thus, convex interval-valued functions have great advantages in interval optimization, interval data processing, and other fields.

For any $F = [f^-, f^+]$ is convex if and only if f^- is convex and f^+ is concave (see [6]). So, we can find a suitable map $\mathcal{T}$ as a contraction of any F with nice properties but never convex in the classical sense and treat these F as a "convex" function.

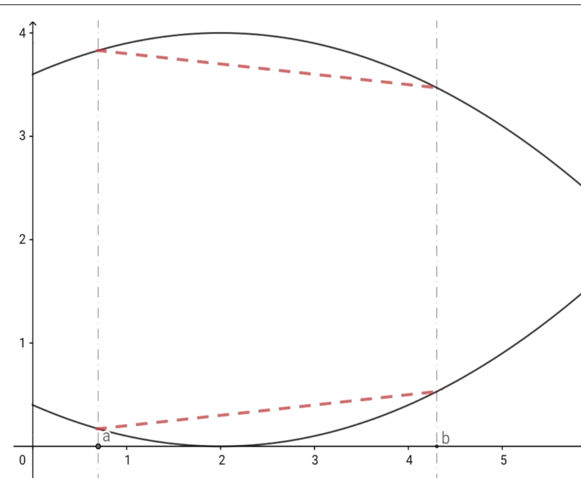

Figure 1 Convex interval-valued function $F = [f^-, f^+]$. The red dotted line represents $\theta F(a) + (1-\theta)F(b)$

We now give some fundamental properties for the map $\mathcal{T}$.

Theorem 3.4 *Let* $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{T}$, *then* $\mathcal{T}_1 \preceq \mathcal{T}_2$ *if and only if*

$$CX_{\mathcal{T}_1}(I, \mathcal{K}_c) \subseteq CX_{\mathcal{T}_2}(I, \mathcal{K}_c).$$

Moreover,

$$CX(I, \mathcal{K}_c) \subseteq CX_{\mathcal{T}}(I, \mathcal{K}_c) \quad \text{for any } \mathcal{T} \in \mathfrak{T}.$$

Proof Assume that $F \in CX_{\mathcal{T}_1}(I, \mathcal{K}_c)$. Since $\mathcal{T}_1 \preceq \mathcal{T}_2$, for any $A \in \mathcal{K}_c$, we have $\mathcal{T}_1(A) \supseteq \mathcal{T}_2(A)$. Then

$$\begin{aligned} F\bigl(\theta x + (1-\theta)y\bigr) &\supseteq \theta \mathcal{T}_1\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}_1\bigl(F(y)\bigr) \\ &\supseteq \theta \mathcal{T}_2\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}_2\bigl(F(y)\bigr), \end{aligned}$$

which implies $F \in CX_{\mathcal{T}_2}(I, \mathcal{K}_c)$.

Conversely, for any $F \in CX_{\mathcal{T}_1}(I, \mathcal{K}_c)$, we have $F \in CX_{\mathcal{T}_2}(I, \mathcal{K}_c)$. It shows that if $\theta \mathcal{T}_1(F(x)) + (1-\theta)\mathcal{T}_1(F(y)) \subseteq F(\theta x + (1-\theta)y)$, then $\theta \mathcal{T}_2(F(x)) + (1-\theta)\mathcal{T}_2(F(y)) \subseteq F(\theta x + (1-\theta)y)$. Thus, we have

$$\theta \mathcal{T}_2\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}_2\bigl(F(y)\bigr) \subseteq \theta \mathcal{T}_1\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}_1\bigl(F(y)\bigr)$$

and $\mathcal{T}_2(F(x)) \subseteq \mathcal{T}_1(F(x))$. Otherwise, we may assume $\mathcal{T}_2(F(x)) \supset \mathcal{T}_1(F(x))$. For the case $\theta \mathcal{T}_1(F(x)) + (1-\theta)\mathcal{T}_1(F(y)) = F(\theta x + (1-\theta)y)$, that is contradiction. By the arbitrariness of F and I, we obtain $\mathcal{T}_1 \preceq \mathcal{T}_2$.

Finally, $\mathcal{T}(A) \subseteq A = \mathcal{ID}(A)$ and $\mathcal{ID} \preceq \mathcal{T}$ for any $\mathcal{T} \in \mathfrak{T}$. Consequently,

$$CX(I, \mathcal{K}_c) \subseteq CX_{\mathcal{T}}(I, \mathcal{K}_c). \qquad \square$$

Theorem 3.5 *Consider* $\mathcal{T} \in \mathfrak{T}$. *For* $n = 0, 1, 2, \dots$, *let* $\mathcal{T}^0 = \mathcal{ID}$ *and* $\mathcal{T}^{n+1} = \mathcal{T} \circ \mathcal{T}^n$, *then* $\mathcal{T}^n \in \mathfrak{T}$. *Moreover,*

$$CX_{\mathcal{T}^n}(I, \mathcal{K}_c) \subseteq CX_{\mathcal{T}^{n+1}}(I, \mathcal{K}_c).$$

Proof For all $A \in \mathcal{K}_c$, take $\mathcal{T} \in \mathfrak{T}$, we have $\mathcal{T}(A) \subseteq A$. Since $\mathcal{T}(A) \in \mathcal{K}_c$, it is obvious that

$$\mathcal{T}^2(A) = \mathcal{T}\bigl(\mathcal{T}(A)\bigr) \subseteq \mathcal{T}(A) \subseteq A.$$

Repeating this step, we obtain

$$\mathcal{T}^n(A) \subseteq \mathcal{T}^{n-1}(A) \subseteq \cdots \subseteq \mathcal{T}(A) \subseteq \mathcal{T}^0(A) = A,$$

which implies $\mathcal{T}^n \in \mathfrak{T}$ and $\mathcal{T}^n \preceq \mathcal{T}^{n+1}$. By Theorem 3.4, we have

$$CX_{\mathcal{T}^n}(I, \mathcal{K}_c) \subseteq CX_{\mathcal{T}^{n+1}}(I, \mathcal{K}_c). \qquad \square$$

Corollary 3.6 *Let $\mathcal{T} \in \mathfrak{T}$ and $F : I \to \mathcal{K}_c$. If $\mathcal{T} \circ F \in CX(I, \mathcal{K}_c)$, then $F \in CX_{\mathcal{T}}(I, \mathcal{K}_c)$.*

Proof Assume that $\mathcal{T} \circ F \in CX(I, \mathcal{K}_c)$, then for any $x, y \in I$ and $\theta \in [0, 1]$, we have

$$\begin{aligned} F\bigl(\theta x + (1-\theta)y\bigr) &\supseteq \mathcal{T}\bigl(F\bigl(\theta x + (1-\theta)y\bigr)\bigr) \\ &\supseteq \theta \mathcal{T}\bigl(F(x)\bigr) + (1-\theta)\mathcal{T}\bigl(F(y)\bigr). \end{aligned}$$

Consequently, $F \in CX_{\mathcal{T}}(I, \mathcal{K}_c)$. □

The next example shows that the converse of Corollary 3.6 is not true.

Example 3.7 Take $F(x) = [\ln(x^2), x + 2]$ in Example 3.3, and let

$$\mathcal{T}(A) = \begin{cases} [(a^-) + 1, a^+] & \text{if } \ell(A) > 1, \\ \{a^+\} & \text{if } \ell(A) \leq 1. \end{cases}$$

We can verify that $F \in CX_{\mathcal{T}}(I, \mathcal{K}_c)$, but $\mathcal{T} \circ F \notin CX(I, \mathcal{K}_c)$.

Now, we give Hermite–Hadamard type inequalities for $\mathcal{T}$-convex interval-valued functions. Take $I = [s, t]$ where $s \leq t$ and $s, t \in \mathbb{R}$.

Theorem 3.8 *Let $\mathcal{T} \in \mathfrak{T}$ and $F \in CX_{\mathcal{T}}([s, t], \mathcal{K}_c)$. If $F, \mathcal{T} \circ F \in L^1([s, t], \mathcal{K}_c)$, then*

$$\frac{\mathcal{T}(F(s)) + \mathcal{T}(F(t))}{2} \subseteq \frac{1}{t-s} \int_s^t F(x)\,dx \tag{3.2}$$

and

$$\frac{1}{t-s} \int_s^t \mathcal{T}\bigl(F(x)\bigr)\,dx \subseteq F\left(\frac{s+t}{2}\right). \tag{3.3}$$

Proof Since $F \in CX_{\mathcal{T}}([s, t], \mathcal{K}_c)$, we have

$$\theta \mathcal{T}\bigl(F(s)\bigr) + (1-\theta)\mathcal{T}\bigl(F(t)\bigr) \subseteq F\bigl(\theta s + (1-\theta)t\bigr), \quad \text{where } \theta \in [0, 1].$$

Integrating θ over $[0, 1]$, then

$$\frac{\mathcal{T}(F(s)) + \mathcal{T}(F(t))}{2} \subseteq \int_0^1 F\bigl(\theta s + (1-\theta)t\bigr)\,d\theta.$$

Taking $x = \theta s + (1-\theta)t$, we obtain (3.2).

For the last part, let us consider

$$\frac{s+t}{2} = \frac{\theta s + (1-\theta)t}{2} + \frac{(1-\theta)s + \theta t}{2}.$$

Then

$$F\left(\frac{s+t}{2}\right) \supseteq \frac{\mathcal{T}(F(\theta s + (1-\theta)t)) + \mathcal{T}(F((1-\theta)s + \theta t))}{2}.$$

Integrating θ over $[0,1]$, we have

$$F\left(\frac{s+t}{2}\right) \supseteq \frac{1}{2}\left[\int_0^1 \mathcal{T}\big(F\big(\theta s+(1-\theta)t\big)\big)\,d\theta + \int_0^1 \mathcal{T}\big(F\big((1-\theta)s+\theta t\big)\big)\,d\theta\right].$$

By changing variate $\tau = 1-\theta$, we obtain

$$\int_0^1 \mathcal{T}\big(F\big((1-\theta)s+\theta t\big)\big)\,d\theta = \int_0^1 \mathcal{T}\big(F\big(\tau s+(1-\tau)t\big)\big)\,d\tau,$$

which implies

$$F\left(\frac{s+t}{2}\right) \supseteq \int_0^1 \mathcal{T}\big(F\big(\theta s+(1-\theta)t\big)\big)\,d\theta.$$

Taking $x = \theta s + (1-\theta)t$, we obtain (3.3). □

Theorem 3.9 *Let $\mathcal{T} \in \mathfrak{T}$ and $F \in CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$. If $\mathcal{T}\circ F \in L^1([s,t],\mathcal{K}_c) \cap CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$, then*

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2} \subseteq \frac{1}{t-s}\int_s^t \mathcal{T}\big(F(x)\big)\,dx \subseteq F\left(\frac{s+t}{2}\right). \tag{3.4}$$

Proof Since $\mathcal{T}\circ F$ is $\mathcal{T}$-convex, we easily obtain that

$$\theta\mathcal{T}^2\big(F(s)\big) + (1-\theta)\mathcal{T}^2\big(F(t)\big) \subseteq \mathcal{T}\big(F\big(\theta s+(1-\theta)t\big)\big).$$

Integrating θ over $[0,1]$,

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2} \subseteq \int_0^1 \mathcal{T}\big(F\big(\theta s+(1-\theta)t\big)\big)\,d\theta.$$

Thus,

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2} \subseteq \frac{1}{t-s}\int_s^t \mathcal{T}\big(F(x)\big)\,dx.$$

Combine with (3.3), the result follows directly. □

Example 3.10 Consider $\mathcal{T} \in \mathfrak{T}$ and $F : [1,2] \to \mathcal{K}_c$ in Example 3.3. We obtain

$$F\left(\frac{s+t}{2}\right) = F\left(\frac{3}{2}\right) = \left[\ln\frac{9}{4}, \frac{7}{2}\right],$$
$$\frac{1}{t-s}\int_s^t \mathcal{T}\big(F(x)\big)\,dx = \int_1^2 \big[x^2, x+2\big]\,dx = \left[\frac{7}{3}, \frac{7}{2}\right],$$

and

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2} = \frac{\mathcal{T}^2(F(1))+\mathcal{T}^2(F(2))}{2} = \left[\frac{e+4}{2}, \frac{7}{2}\right].$$

Since

$$\left[\frac{e+4}{2},\frac{7}{2}\right]\subseteq\left[\frac{7}{3},\frac{7}{2}\right]\subseteq\left[\ln\frac{9}{4},\frac{7}{2}\right],$$

which implies

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2}\subseteq\frac{1}{t-s}\int_s^t\mathcal{T}\bigl(F(x)\bigr)\,dx\subseteq F\left(\frac{s+t}{2}\right).$$

Consequently, Theorem 3.9 is verified.

Corollary 3.11 *Let $\mathcal{T}\in\mathfrak{T}$ and $F\in CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$. If $\mathcal{T}^n\circ F,\mathcal{T}^{n-1}\circ F\in L^1([s,t],\mathcal{K}_c)\cap CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$, then*

$$\begin{aligned}&\frac{\mathcal{T}^{n+1}(F(s))+\mathcal{T}^{n+1}(F(t))}{2}\\&\quad\subseteq\frac{1}{t-s}\int_s^t\mathcal{T}^n\bigl(F(x)\bigr)\,dx\subseteq\mathcal{T}^{n-1}F\left(\frac{s+t}{2}\right),\quad n=1,2,\ldots.\end{aligned}\tag{3.5}$$

Remark 3.12 Let $n=1$ in Corollary 3.11, we obtain Theorem 3.9. Also, for the case $\mathcal{T}\equiv\mathcal{ID}$, we obtain the classic Hermite–Hadamard inequality.

4 Fractional integral inequalities

In this section, we introduce fractional Hermite–Hadamard type inequalities and Ostrowski type inequality for $\mathcal{T}$-convex interval-valued functions. Not so long ago, Budak introduced the fractional Hermite–Hadamard type inequalities for interval-valued functions in [3].

Let $F\in L^1([s,t],\mathcal{K}_c)$, the left-hand side Riemann–Liouville fractional integral of F is defined by (see [11])

$$\mathfrak{J}_{s^+}^{\alpha}F(x)=\frac{1}{\Gamma(\alpha)}\int_s^x(x-\tau)^{\alpha-1}F(\tau)\,d\tau,\quad x>s,\alpha>0,\tag{4.1}$$

where Γ is the Euler gamma function. Meanwhile, the right-hand side Riemann–Liouville fractional integral of F is defined by (see [3])

$$\mathfrak{J}_{t^-}^{\alpha}F(x)=\frac{1}{\Gamma(\alpha)}\int_x^t(\tau-x)^{\alpha-1}F(\tau)\,d\tau,\quad x<t,\alpha>0.\tag{4.2}$$

More properties of Riemann–Liouville fractional integral can be found in [11]. Now we can give the following statement for $\mathcal{T}$-convex interval-valued function.

Theorem 4.1 *Let $\mathcal{T}\in\mathfrak{T}$ and $F\in CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$. If $F,\mathcal{T}\circ F\in L^1([s,t],\mathcal{K}_c)$, then*

$$\frac{\mathcal{T}(F(s))+\mathcal{T}(F(t))}{2}\subseteq\frac{\Gamma(\alpha+1)}{2(t-s)^{\alpha}}\bigl[\mathfrak{J}_{s+}^{\alpha}F(t)+\mathfrak{J}_{t-}^{\alpha}F(s)\bigr]\tag{4.3}$$

and

$$\frac{\Gamma(\alpha+1)}{2(t-s)^{\alpha}}\bigl[\mathfrak{J}_{s+}^{\alpha}\mathcal{T}\bigl(F(t)\bigr)+\mathfrak{J}_{t-}^{\alpha}\mathcal{T}\bigl(F(s)\bigr)\bigr]\subseteq F\left(\frac{s+t}{2}\right).\tag{4.4}$$

Proof Since $F \in CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$, we have

$$\theta\mathcal{T}\big(F(s)\big) + (1-\theta)\mathcal{T}\big(F(t)\big) \subseteq F\big(\theta s + (1-\theta)t\big)$$

and

$$(1-\theta)\mathcal{T}\big(F(s)\big) + \theta\mathcal{T}\big(F(t)\big) \subseteq F\big((1-\theta)s + \theta t\big),$$

where $\theta \in [0,1]$. Adding the above two inequalities, we obtain

$$\mathcal{T}\big(F(s)\big) + \mathcal{T}\big(F(t)\big) \subseteq F\big(\theta s + (1-\theta)t\big) + F\big((1-\theta)s + \theta t\big).$$

Multiplying by $\theta^{\alpha-1}$, with $\alpha > 0$ and integrating θ over $[0,1]$, then

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1}\big[\mathcal{T}\big(F(s)\big) + \mathcal{T}\big(F(t)\big)\big]\,d\theta\\ &\quad\subseteq \int_0^1 \theta^{\alpha-1}\big[F\big(\theta s + (1-\theta)t\big) + F\big((1-\theta)s + \theta t\big)\big]\,d\theta.\end{aligned}$$

For the left-hand side,

$$\int_0^1 \theta^{\alpha-1}\big[\mathcal{T}\big(F(s)\big) + \mathcal{T}\big(F(t)\big)\big]\,d\theta = \frac{1}{\alpha}\big[\mathcal{T}\big(F(s)\big) + \mathcal{T}\big(F(t)\big)\big].$$

For the right-hand side, taking $x = \theta s + (1-\theta)t$ in the first integral and taking $y = (1-\theta)s + \theta t$ in the second integral, we get

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1}F\big(\theta s + (1-\theta)t\big)\,d\theta + \int_0^1 \theta^{\alpha-1}F\big((1-\theta)s + \theta t\big)\,d\theta\\ &\quad= \frac{1}{(t-s)^\alpha}\int_s^t (t-x)^{\alpha-1}F(x)\,dx + \frac{1}{(t-s)^\alpha}\int_s^t (y-s)^{\alpha-1}F(y)\,dy\\ &\quad= \frac{\Gamma(\alpha)}{(t-s)^\alpha}\big[\mathfrak{J}_{s+}^{\alpha}F(t) + \mathfrak{J}_{t-}^{\alpha}F(s)\big].\end{aligned}$$

Combining the both parts, we obtain the first inequality.

Next, since

$$\frac{s+t}{2} = \frac{\theta s + (1-\theta)t}{2} + \frac{(1-\theta)s + \theta t}{2},$$

we have

$$F\left(\frac{s+t}{2}\right) \supseteq \frac{\mathcal{T}(F(\theta s + (1-\theta)t)) + \mathcal{T}(F((1-\theta)s + \theta t))}{2}.$$

Multiplying by $\theta^{\alpha-1}$, with $\alpha > 0$ and integrating θ over $[0,1]$, then

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1}F\left(\frac{s+t}{2}\right)d\theta\\ &\quad\supseteq \frac{1}{2}\left[\int_0^1 \theta^{\alpha-1}\mathcal{T}\big(F\big(\theta s + (1-\theta)t\big)\big)\,d\theta + \int_0^1 \theta^{\alpha-1}\mathcal{T}\big(F\big((1-\theta)s + \theta t\big)\big)\,d\theta\right].\end{aligned}$$

Same as before, for the left-hand side,

$$\int_0^1 \theta^{\alpha-1} F\left(\frac{s+t}{2}\right) d\theta = \frac{1}{\alpha} F\left(\frac{s+t}{2}\right).$$

For the right-hand side,

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1} \mathcal{T}\big(F\big(\theta s + (1-\theta)t\big)\big)\, d\theta + \int_0^1 \theta^{\alpha-1} \mathcal{T}\big(F\big((1-\theta)s + \theta t\big)\big)\, d\theta \\ &\quad = \frac{\Gamma(\alpha)}{(t-s)^\alpha}\big[\mathfrak{J}_{s+}^{\alpha} \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^{\alpha} \mathcal{T}\big(F(s)\big)\big].\end{aligned}$$

Combining the both parts, we obtain the second inequality. □

Theorem 4.2 *Let $\mathcal{T} \in \mathfrak{T}$ and $F \in CX_{\mathcal{T}}([s,t], \mathcal{K}_c)$. If $\mathcal{T} \circ F \in L^1([s,t], \mathcal{K}_c) \cap CX_{\mathcal{T}}([s,t], \mathcal{K}_c)$, then*

$$\begin{aligned}&\frac{\mathcal{T}^2(F(s)) + \mathcal{T}^2(F(t))}{2} \\ &\quad \subseteq \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\big[\mathfrak{J}_{s+}^{\alpha} \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^{\alpha} \mathcal{T}\big(F(s)\big)\big] \subseteq F\left(\frac{s+t}{2}\right).\end{aligned} \tag{4.5}$$

Proof Since $\mathcal{T} \circ F$ is $\mathcal{T}$-convex, we easily obtain that

$$\theta \mathcal{T}^2\big(F(s)\big) + (1-\theta)\mathcal{T}^2\big(F(t)\big) \subseteq \mathcal{T}\big(F\big(\theta s + (1-\theta)t\big)\big).$$

Similarly, we get

$$(1-\theta)\mathcal{T}^2\big(F(s)\big) + \theta \mathcal{T}^2\big(F(t)\big) \subseteq \mathcal{T}\big(F\big((1-\theta)s + \theta t\big)\big).$$

By adding the above two inequalities, we have

$$\mathcal{T}^2\big(F(s)\big) + \mathcal{T}^2\big(F(t)\big) \subseteq \mathcal{T}\big(F\big(\theta s + (1-\theta)t\big)\big) + \mathcal{T}\big(F\big((1-\theta)s + \theta t\big)\big).$$

Multiplying by $\theta^{\alpha-1}$, with $\alpha > 0$ and integrating θ over $[0,1]$, then

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1}\big[\mathcal{T}^2\big(F(s)\big) + \mathcal{T}^2\big(F(t)\big)\big]\, d\theta \\ &\quad \subseteq \int_0^1 \theta^{\alpha-1} \mathcal{T}\big(F\big(\theta s + (1-\theta)t\big)\big)\, d\theta + \int_0^1 \theta^{\alpha-1} \mathcal{T}\big(F\big((1-\theta)s + \theta t\big)\big)\, d\theta.\end{aligned}$$

For the left-hand side,

$$\int_0^1 \theta^{\alpha-1} \mathcal{T}^2\big(F(s)\big) + \mathcal{T}^2\big(F(t)\big)\, d\theta = \frac{1}{\alpha}\big[\mathcal{T}^2\big(F(s)\big) + \mathcal{T}^2\big(F(t)\big)\big].$$

For the right-hand side,

$$\begin{aligned}&\int_0^1 \theta^{\alpha-1}\mathcal{T}\big(F\big(\theta s+(1-\theta)t\big)\big)\,d\theta+\int_0^1 \theta^{\alpha-1}\mathcal{T}\big(F\big((1-\theta)s+\theta t\big)\big)\,d\theta\\&\quad=\frac{\Gamma(\alpha)}{(t-s)^\alpha}\big[\mathfrak{J}^\alpha_{s+}\mathcal{T}\big(F(t)\big)+\mathfrak{J}^\alpha_{t-}\mathcal{T}\big(F(s)\big)\big].\end{aligned}$$

Thus, we obtain

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2}\subseteq\frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\big[\mathfrak{J}^\alpha_{s+}\mathcal{T}\big(F(t)\big)+\mathfrak{J}^\alpha_{t-}\mathcal{T}\big(F(s)\big)\big].$$

Combining with (4.4), we finish the proof. □

Example 4.3 Let $F:[1,2]\to\mathcal{K}_c$ be given by $F(x)=[\sqrt{x},6]$, and let

$$\mathcal{T}(A)=\begin{cases}[(a^-)^2,(a^+)-1] & \text{if } (a^-)^2+1\le a^+,\\ \{\frac{a^++a^-}{2}\} & \text{otherwise,}\end{cases}$$

for any $A\in\mathcal{K}_c$. First, we remark that $F\in CX_{\mathcal{T}}([1,2],\mathcal{K}_c)$ and $\mathcal{T}\circ F\in L^1([1,2],\mathcal{K}_c)\cap CX_{\mathcal{T}}([1,2],\mathcal{K}_c)$. Then we have

$$\begin{aligned}\mathfrak{J}^{\frac{1}{2}}_{1^+}\mathcal{T}\big(F(2)\big)&=\frac{1}{\Gamma(1/2)}\int_1^2(2-\tau)^{-\frac{1}{2}}\mathcal{T}\big(F(\tau)\big)\,d\tau=\frac{1}{\sqrt{\pi}}\left[\frac{10}{3},10\right],\\ \mathfrak{J}^{\frac{1}{2}}_{2^-}\mathcal{T}\big(F(1)\big)&=\frac{1}{\Gamma(1/2)}\int_1^2(\tau-1)^{-\frac{1}{2}}\mathcal{T}\big(F(\tau)\big)\,d\tau=\frac{1}{\sqrt{\pi}}\left[\frac{8}{3},10\right],\end{aligned}$$

and

$$\frac{\Gamma(3/2)}{2(2-1)^{1/2}}\big[\mathfrak{J}^{\frac{1}{2}}_{1^+}\mathcal{T}\big(F(2)\big)+\mathfrak{J}^{\frac{1}{2}}_{2^-}\mathcal{T}\big(F(1)\big)\big]=\left[\frac{3}{2},5\right].$$

Meanwhile, we have

$$F\left(\frac{s+t}{2}\right)=F\left(\frac{1+2}{2}\right)=\left[\frac{\sqrt{6}}{2},6\right]$$

and

$$\frac{\mathcal{T}^2(F(s))+\mathcal{T}^2(F(t))}{2}=\frac{\mathcal{T}^2(F(1))+\mathcal{T}^2(F(2))}{2}=\left[\frac{5}{2},4\right].$$

Since

$$\left[\frac{5}{2},4\right]\subseteq\left[\frac{3}{2},5\right]\subseteq\left[\frac{\sqrt{6}}{2},6\right],$$

which illustrates

$$\frac{\mathcal{T}^2(F(1))+\mathcal{T}^2(F(2))}{2}\subseteq\frac{\Gamma(3/2)}{2(2-1)^{1/2}}\big[\mathfrak{J}^{\frac{1}{2}}_{1^+}\mathcal{T}\big(F(2)\big)+\mathfrak{J}^{\frac{1}{2}}_{2^-}\mathcal{T}\big(F(1)\big)\big]\subseteq F\left(\frac{1+2}{2}\right).$$

Consequently, Theorem 4.2 is verified.

Corollary 4.4 *Let $\mathcal{T} \in \mathfrak{T}$ and $F \in CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$. If $\mathcal{T}^n \circ F, \mathcal{T}^{n-1} \circ F \in L^1([s,t],\mathcal{K}_c) \cap CX_{\mathcal{T}}([s,t],\mathcal{K}_c)$, then*

$$\begin{aligned} \frac{\mathcal{T}^{n+1}(F(s)) + \mathcal{T}^{n+1}(F(t))}{2} &\subseteq \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\left[\mathfrak{J}_{s+}^\alpha \mathcal{T}^n\big(F(t)\big) + \mathfrak{J}_{t-}^\alpha \mathcal{T}^n\big(F(s)\big)\right] \\ &\subseteq \mathcal{T}^{n-1}\left(F\left(\frac{s+t}{2}\right)\right), \quad n = 1,2,\ldots. \end{aligned} \tag{4.6}$$

Remark 4.5 Let $n = 1$ in Corollary 4.4, we obtain Theorem 4.2. Also, for the case $\mathcal{T} \equiv \mathcal{ID}$, we obtain the classic fractional Hermite–Hadamard inequality.

Next, we give the Ostrowski type inequality for interval-valued function.

Lemma 4.6 *Let $\mathcal{T} \in \mathfrak{T}$ and $F : [s,t] \to \mathcal{K}_c$. If $\mathcal{T} \circ F$ is ℓ-monotone and differentiable on $[s,t]$ with $(\mathcal{T} \circ F)' \in C([s,t],\mathcal{K}_c)$, then*

$$\big\|\mathcal{T}\big(F(t)\big) \ominus_g \mathcal{T}\big(F(s)\big)\big\| \le \big\|(\mathcal{T} \circ F)'\big\|_c (t-s). \tag{4.7}$$

Theorem 4.7 *Let $\mathcal{T} \in \mathfrak{T}$ and $F : [s,t] \to \mathcal{K}_c$. If $\mathcal{T} \circ F$ is ℓ-monotone and differentiable on $[s,t]$ with $(\mathcal{T} \circ F)' \in C([s,t],\mathcal{K}_c)$, then for any $x \in [s,t]$,*

$$\begin{aligned} &\left\| \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\left[\mathfrak{J}_{s+}^\alpha \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^\alpha \mathcal{T}\big(F(s)\big)\right] \ominus_g \mathcal{T}\big(F(x)\big) \right\| \\ &\quad \le \big\|(\mathcal{T} \circ F)'\big\|_c \left(\frac{(\alpha-1)(t-s)^{\alpha+1} + 2(x-s)^{\alpha+1} + 2(t-x)^{\alpha+1}}{2(\alpha+1)(t-s)^\alpha} \right). \end{aligned} \tag{4.8}$$

Proof Since we have

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\left[\mathfrak{J}_{s+}^\alpha \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^\alpha \mathcal{T}\big(F(s)\big)\right] \\ &\quad = \frac{\alpha}{2(t-s)^\alpha}\left(\int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau + \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \right) \end{aligned}$$

and

$$\mathcal{T}\big(F(x)\big) = \frac{\alpha}{2(t-s)^\alpha}\left(\int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau + \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right).$$

Then

$$\begin{aligned} &\left\| \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\left[\mathfrak{J}_{s+}^\alpha \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^\alpha \mathcal{T}\big(F(s)\big)\right] \ominus_g F(x) \right\| \\ &\quad = \frac{\alpha}{2(t-s)^\alpha}\left\| \left(\int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau + \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \right) \right. \\ &\qquad \left. \ominus_g \left(\int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau + \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right) \right\| \\ &\quad \le \frac{\alpha}{2(t-s)^\alpha}\left(\left\| \int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \ominus_g \int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right\| \right. \\ &\qquad \left. + \left\| \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \ominus_g \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right\| \right). \end{aligned}$$

By Corollary 2.5 and Lemma 4.6, we obtain

$$\begin{aligned}
&\left\| \int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \ominus_g \int_s^t (t-\tau)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right\| \\
&\quad\leq \int_s^t (t-\tau)^{\alpha-1}\big\|\mathcal{T}\big(F(\tau)\big)\ominus_g \mathcal{T}\big(F(x)\big)\big\|\,d\tau \\
&\quad\leq \big\|(\mathcal{T}\circ F)'\big\|_c \int_s^t (t-\tau)^{\alpha-1}|\tau - x|\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(\tau)\big)\,d\tau \ominus_g \int_s^t (\tau-s)^{\alpha-1}\mathcal{T}\big(F(x)\big)\,d\tau \right\| \\
&\quad\leq \int_s^t (\tau-s)^{\alpha-1}\big\|\mathcal{T}\big(F(\tau)\big)\ominus_g \mathcal{T}\big(F(x)\big)\big\|\,d\tau \\
&\quad\leq \big\|(\mathcal{T}\circ F)'\big\|_c \int_s^t (\tau-s)^{\alpha-1}|\tau - x|\,d\tau.
\end{aligned}$$

Combining the above results, then

$$\begin{aligned}
&\left\| \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha}\big[\mathfrak{J}_{s+}^\alpha \mathcal{T}\big(F(t)\big) + \mathfrak{J}_{t-}^\alpha \mathcal{T}\big(F(s)\big)\big] \ominus_g \mathcal{T}\big(F(x)\big)\right\| \\
&\quad\leq \frac{\alpha\|(\mathcal{T}\circ F)'\|_c}{2(t-s)^\alpha}\left(\int_s^t (t-\tau)^{\alpha-1}|\tau-x|\,d\tau + \int_s^t (\tau-s)^{\alpha-1}|\tau-x|\,d\tau\right) \\
&\quad= \big\|(\mathcal{T}\circ F)'\big\|_c \left(\frac{(\alpha-1)(t-s)^{\alpha+1} + 2(x-s)^{\alpha+1} + 2(t-x)^{\alpha+1}}{2(\alpha+1)(t-s)^\alpha}\right). \qquad \square
\end{aligned}$$

Remark 4.8 If we take $\mathcal{T} = \mathcal{ID}$ and $\alpha = 1$ in (4.8), then we get Theorem 4.1 in [4].

Example 4.9 Consider $F : [1,2] \to \mathcal{K}_c$ and $\mathcal{T} \in \mathfrak{T}$ in Example 4.3. For the left, we check that

$$\left\| \frac{\Gamma(3/2)}{2(2-1)^{1/2}}\big[\mathfrak{J}_{1+}^{\frac{1}{2}} \mathcal{T}\big(F(2)\big) + \mathfrak{J}_{2-}^{\frac{1}{2}} \mathcal{T}\big(F(1)\big)\big] \ominus_g \mathcal{T}\big(F(x)\big)\right\| = \left| x - \frac{3}{2}\right|$$

for any $x \in [1,2]$. Also, we have

$$\begin{aligned}
&\big\|(\mathcal{T}\circ F)'\big\|_c \left(\frac{(-1/2)(2-1)^{3/2} + 2(x-1)^{3/2} + 2(2-x)^{3/2}}{3}\right) \\
&\quad= \frac{4(x-1)^{\frac{3}{2}} + 4(2-x)^{\frac{3}{2}} - 1}{6}.
\end{aligned}$$

Obviously, for any $x \in [1,2]$, we have

$$\left| x - \frac{3}{2}\right| \leq \frac{4(x-1)^{\frac{3}{2}} + 4(2-x)^{\frac{3}{2}} - 1}{6}.$$

Consequently, Theorem 4.7 is verified (see Fig. 2).

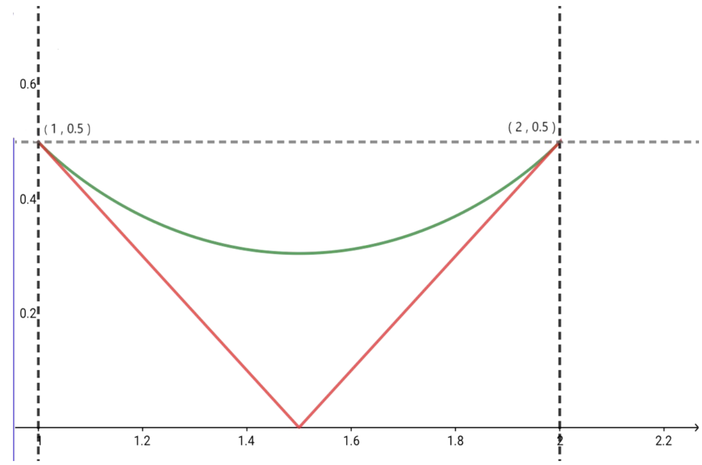

Figure 2 The graph of the result in Example 4.9

Corollary 4.10 *Let $\mathcal{T} \in \mathfrak{T}$ and $F : [s,t] \to \mathcal{K}_c$. If $\mathcal{T} \circ F$ is ℓ-monotone and differentiable on $[s,t]$ with $(\mathcal{T} \circ F)' \in C([s,t], \mathcal{K}_c)$, then*

$$\begin{aligned} &\left\| \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha} \left[\mathfrak{J}^\alpha_{s+} \mathcal{T}\big(F(t)\big) + \mathfrak{J}^\alpha_{t-} \mathcal{T}\big(F(s)\big) \right] \ominus_g \mathcal{T}\left(F\left(\frac{s+t}{2} \right) \right) \right\| \\ &\quad \le \left\| (\mathcal{T} \circ F)' \right\|_c \left(\frac{\alpha(t-s)^{\alpha+1}}{2^\alpha(\alpha+1)(t-s)^\alpha} \right). \end{aligned} \tag{4.9}$$

Remark 4.11 If we take $\mathcal{T} = \mathcal{ID}$ and $\alpha = 1$, then we can estimate the gap between two intervals and construct a suitable map $\mathcal{T}$ such that $F \in CX_{\mathcal{T}}([s,t], \mathcal{K}_c)$.

5 Conclusions

This article presents a new map $\mathcal{T}$ on $\mathcal{K}_c$ and studies some basic properties of $\mathcal{T}$-convex interval-valued functions. Meanwhile, some new Hermite–Hadamard type inequalities and Ostrowski type inequality based on these concepts are established. In the field of mathematical optimization, comparing convex optimization with other methods, its merit is to make every local minimum a global minimum. A good deal of problem classes, such as least squares, linear programming, geometric programming, are all convex optimization problems, or can be switched to convex optimization problems via simple transformations. The research results of this paper can expand the application range of convex function. Thereby, in the field of interval convex optimizations, our results are more applicable than ever. Next, we intend to study fuzzy-valued functions and some applications in interval and fuzzy convex optimizations by using $\mathcal{T}$-convex. Also, we may research more details about the map $\mathcal{T}$ and discuss other maps on $\mathcal{K}_c$ in the future.

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Author details
[1]College of Science, Hohai University, Nanjing, 210098, P.R. China. [2]School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, P.R. China.

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