Saker et al. *Advances in Difference Equations* (2020) 2020:549
https://doi.org/10.1186/s13662-020-03006-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Half-linear dynamic equations and investigating weighted Hardy and Copson inequalities

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Abstract

In this paper, we employ some algebraic equations due to Hardy and Littlewood to establish some conditions on weights in dynamic inequalities of Hardy and Copson type. For illustrations, we derive some dynamic inequalities of Wirtinger, Copson and Hardy types and formulate the classical integral and discrete inequalities with sharp constants as particular cases. The results improve some results obtained in the literature.

MSC: 26A15; 26A51; 26D10; 26D15; 39A13; 34A40

Keywords: Hardy's dynamic inequality; Copson's dynamic inequality; Half-linear equations; Time scales

1 Introduction

Hardy in [12] proved the discrete inequality

$$\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{i=1}^{s}a(i)\right)^{q} \leq \left(\frac{q}{q-1}\right)^{q}\sum_{s=1}^{\infty}a^{q}(s), \quad \text{for } q>1, s\geq 1, \tag{1}$$

where $a(s)$ is a positive sequence for $s\geq 1$. In [13] Hardy proved the integral form

$$\int_{0}^{\infty}\left(\frac{1}{y}\int_{0}^{y}f(s)\,ds\right)^{q}dy \leq \left(\frac{q}{q-1}\right)^{q}\int_{0}^{\infty}f^{q}(y)\,dy, \quad \text{for } q>1, \tag{2}$$

where f is a positive function. In [10] Copson considered a new type of inequalities of the form

$$\int_{0}^{\infty}\left(\int_{x}^{\infty}f(y)\,dy\right)^{q}dx \leq q^{q}\int_{0}^{\infty}y^{q}f^{q}(y)\,dy, \tag{3}$$

where f is a positive function and $q > 1$. In [11] Copson (see also [14, Theorem 344]) proved the discrete version of (3), which is given by

$$\sum_{s=0}^{\infty}\left(\sum_{k=s}^{\infty} a(k)\right)^q \leq q^q \sum_{s=0}^{\infty} k^q a^q(s), \tag{4}$$

where $q > 1$ and $a(s) > 0$ for $s \geq 0$. In [4] Beesack proved an inequality of the form

$$\int_a^b \omega(y)\left(\int_a^y f(\tau)\,d\tau\right)^q dy \leq \int_a^b \gamma(y) f^q(y)\,dy, \tag{5}$$

where γ and ω satisfy the differential equation of Euler–Lagrange type

$$\left(\gamma(y)\left(z'(y)\right)^{q-1}\right)' + \omega(y) z^{q-1}(y) = 0. \tag{6}$$

The method of the proofs in [4] depends on the solution of (6) when the first derivative $z' > 0$ on the interval (a, b). The approach of Beesack extended to generalized Hardy's type inequalities; see, e.g., Beesack [5] and Shum [23]. Some of the conditions on z, γ and ω were removed by Tomaselli [26]. In particular Tomaselli followed up the papers by Talenti [24] and [25] and proved some inequalities with some special weighted functions.

The discrete analogues for the continuous results have been considered by some authors, we refer to the articles by Chen [8, 9] and Liao [15]. It is worth to mention here that some parts of the proofs of Liao's results are based on the technique used in [7] and [14], which are based on the application of the variational principle which is not an easy task to apply on time scales and then we did not consider in our proofs of characterizations of weights of the inequalities that we will consider in our paper. There is thus an urgent need of a new technique that helps us in studying such problems on time scales, which is our main aim in this paper.

The dynamic equations and inequalities have been introduced by Hilger in 1988 and considered by a lot of authors, we refer to Refs. [1–3, 16–20, 27]. One of the applications of Hardy-type inequalities in dynamic equations was demonstrated in [19]. In particular, in [19] the author established a time-scale version of the Hardy inequality, which unifies and extends well-known Hardy inequalities in the continuous and in the discrete setting, and presented an application in the oscillation theory of half-linear dynamic equations to obtain sharp oscillation results. Recently, in [21] the authors established some conditions on the weights of dynamic inequalities of Hardy's type to be hold. More precisely it has been proved that, if $1 < p \leq q < \infty$,

$$\int_c^{\infty} v^{-\frac{1}{p-1}}(y)\Delta y = \infty, \int_c^{x} v^{1-p^*}(y)\Delta y < \infty, \quad \text{for } y \in [c, d]_{\mathbb{T}}, \tag{7}$$

where p^* is the conjugate of $p > 1$, then

$$\left(\int_c^d w(x)\left(\int_c^y g(s)\,ds\right)^q \Delta y\right)^{\frac{1}{q}} \leq C\left(\int_c^d v(y) g^p(y)\Delta y\right)^{\frac{1}{p}}, \tag{8}$$

where w is a positive function, if and only if there is a number $\eta > 0$, such that the equation

$$\eta\left(v^{\frac{q}{p}}(s)\left(u^{\triangle}(s)\right)^{\frac{q}{p^*}}\right)^{\triangle} + w(s) u^{\frac{q}{p^*}}\left(\sigma(s)\right) = 0 \tag{9}$$

has a solution $u(s)$ that satisfies

$$u(s), \qquad u^{\triangle}(s) > 0 \quad \text{and} \quad u^{\triangle\triangle}(s) < 0, \quad \text{for } s \in [c,d]_{\mathbb{T}}. \tag{10}$$

In this paper, we establish some relations between the weights in generalized inequalities of Hardy and Copson type by using the solutions of dynamic equations of half-linear types. The technique in our paper allows us to cover the inequalities with tails of Copson's type with weights and improve the above results, since our results do not need the restrictive condition (7). In Sect. 2, we are concerned with the presentation of some basic definitions and preliminaries regarding the time-scale calculus. The dynamic Hardy-type inequality of the form

$$\int_{\alpha}^{\beta} \omega(t)\left(\int_{\alpha}^{\sigma(t)} \Gamma(\tau)\Delta\tau\right)^{q} \Delta t \leq \int_{\alpha}^{\beta} \gamma(t)\Gamma^{q}(t)\Delta t, \tag{11}$$

will be proved in Sect. 3, where the method reduces the proofs to the solvability of dynamic equation

$$\left(\gamma(y)\left(u^{\Delta}(y)\right)^{q-1}\right))^{\Delta} + \omega(y)\left(u^{\sigma}(y)\right)^{q-1} = 0, \quad y \in [\alpha,\beta]_{\mathbb{T}}, \tag{12}$$

where $u > 0, u^{\Delta} > 0$. Next, we prove new conditions on weights in the dynamic Copson-type inequality with tail of the form

$$\int_{\alpha}^{\beta} \omega(t)\left(\int_{\sigma(t)}^{\infty} \Gamma(y)\Delta y\right)^{q} \Delta t \leq \int_{\alpha}^{\beta} \gamma(y)\Gamma^{q}(y)\Delta y, \tag{13}$$

and we prove that the conditions on the weights reduces to the solvability of the dynamic equation

$$\left(\gamma(s)\left(-u^{\Delta}(s)\right)^{q-1}\right)^{\Delta} - \omega(s)\left(u^{\sigma}(s)\right)^{q-1} = 0, \quad s \in [\alpha,\beta]_{\mathbb{T}}, \tag{14}$$

where $u > 0, u^{\Delta} < 0$. To the best of the authors' knowledge the results in this case are essentially new. For illustration, we derive some dynamic inequalities as special cases and from them we formulate some classical and discrete inequalities.

2 Preliminaries

In this section, we present some basic definitions that will be used in the sequel and for more details see [6]. The derivative on time scales of $\Gamma\Upsilon$ and Γ/Υ of two functions Γ and Υ are given by

$$(\Gamma\Upsilon)^{\Delta} = \Gamma\Upsilon^{\Delta} + \Gamma^{\Delta}\Upsilon^{\sigma} \quad \text{and} \quad \left(\frac{\Gamma}{\Upsilon}\right)^{\Delta} = \frac{\Gamma^{\Delta}\Upsilon - \Gamma\Upsilon^{\Delta}}{\Upsilon\Upsilon^{\sigma}}. \tag{15}$$

The forward jump operator $\sigma(t)$ on a time scale $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the graininess function μ is defined by $\mu(t) := \sigma(t) - t$, and for any function $\Gamma : \mathbb{T} \to \mathbb{R}$ the notation $\Gamma^{\sigma}(t)$ denotes $\Gamma(\sigma(t))$. The Cauchy (delta)-integral is defined by $\int_{a}^{t} \Gamma^{\Delta}(\omega)\Delta\omega =$

$\Gamma(t) - \Gamma(a)$. The integration formula on a discrete time scale reads

$$\int_a^b \Gamma(t)\Delta t = \sum_{t\in[a,b)} \mu(t)\Gamma(t),$$

while the infinite integral is defined as $\int_a^\infty \Gamma(t)\Delta t = \lim_{b\to\infty} \int_a^b \Gamma(t)\Delta t$. The chain rule for functions $\Theta : \mathbb{R} \to \mathbb{R}$, which is continuously differentiable, and $\Upsilon : \mathbb{T} \to \mathbb{R}$, which is delta-differentiable, is given by

$$(\Theta \circ \Upsilon)^\Delta(t) = \Theta'\bigl(\Upsilon(c)\bigr)\Upsilon^\Delta(t), \quad \text{for } c \in \bigl[t, \sigma(t)\bigr],$$

this rule leads to the useful form

$$\bigl(\Upsilon^q(t)\bigr)^\Delta = q\Upsilon^{q-1}(c)\Upsilon^\Delta(t), \quad \text{for } c \in \bigl[t, \sigma(t)\bigr]. \tag{16}$$

Another formula pertaining to the chain rule states that

$$(\Theta \circ \Upsilon)^\Delta(t) = \int_0^1 \Theta'\bigl(\Upsilon(t) + s\mu(t)\Upsilon^\Delta(t)\bigr)\, ds\Upsilon^\Delta(t),$$

which provides us with the following useful form:

$$\bigl(\Upsilon^q(t)\bigr)^\Delta = q\int_0^1 \bigl(s\Upsilon^\sigma(t) + (1-s)\Upsilon(t)\bigr)^{q-1}\, ds\Upsilon^\Delta(t). \tag{17}$$

For $a, b \in \mathbb{T}$ and $\Gamma, \Theta \in C_{rd}(\mathbb{T})$, the integration by parts formula is given by

$$\int_a^b \Gamma^\sigma(t)\Theta^\Delta(t)\Delta t = \Gamma\Theta(b) - \Gamma\Theta(a) - \int_a^b \Gamma^\Delta(t)\Theta(t)\Delta t. \tag{18}$$

The Hölder inequality on time scales is given by

$$\int_a^b \bigl|\Gamma(t)\Theta(t)\bigr|\Delta t \le \left(\int_a^b \bigl|\Gamma(t)\bigr|^q \Delta t\right)^{\frac{1}{q}} \left(\int_a^b \bigl|\Theta(t)\bigr|^q \Delta t\right)^{\frac{1}{q}},$$

where $q > 1$ and $1/q + 1/q = 1$. We define the time-scale interval $[a,\infty)_\mathbb{T}$ by $[a,\infty)_\mathbb{T} := [a,\infty) \cap \mathbb{T}$.

3 Main results

In this section, we will prove the main results and we begin with inequalities of Hardy's type with heads. In what follows, all functions in the statements of the theorems are assumed to be rd-continuous, and positive functions (see [6]). The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. Let $\gamma, \omega \in \mathrm{C}_{rd}([\alpha,\beta]_\mathbb{T}, \mathbb{R}^+)$, and $\mathcal{S} \in \mathrm{C}^1_{rd}([\alpha,\beta]_\mathbb{T}, \mathbb{R}^+)$. Suppose that $\gamma(t)$ and $\omega(t)$ satisfy the half-linear dynamic equation of Euler–Lagrange type

$$\bigl(\gamma(t)\bigl(\mathcal{S}^\Delta(t)\bigr)^{\theta-1}\bigr)^\Delta + \omega(t)\bigl(\mathcal{S}^\sigma(t)\bigr)^{\theta-1} = 0, \quad \text{for } t \in [\alpha,\beta]_\mathbb{T}, \tag{19}$$

for any real number $\theta > 1$, and $\mathcal{S}(t) > 0, \mathcal{S}^{\Delta}(t) > 0$ for $t \in [\alpha, \beta]_{\mathbb{T}}$, and define

$$\nu := \frac{\gamma(\mathcal{S}^{\Delta})^{\theta-1}}{\mathcal{S}^{\theta-1}}. \tag{20}$$

Definition 3.1 Suppose that $z \in C_{rd}([\alpha, \beta]_{\mathbb{T}}, \mathbb{R}^{+})$. The function z is said to be in the class $\hat{H}_1$ if

$$\int_{\alpha}^{\beta} \omega(s)\big(z^{\sigma}(s)\big)^{\theta}\Delta s > -\infty, \qquad \int_{\alpha}^{\beta} \gamma(s)\big(z^{\Delta}(s)\big)^{\theta}\Delta s < \infty, \tag{21}$$

and

$$\liminf_{t\to\alpha}\big(\nu z^{\theta}\big)(t) < \infty, \qquad \limsup_{t\to\beta}\big(\nu z^{\theta}\big)(t) > -\infty. \tag{22}$$

For convenience sometimes in the computations we skip the argument t.

Lemma 3.1 *Assume that α, $\beta \in \mathbb{T}$ and suppose that $\mathcal{S}(t)$ and $\Upsilon(t)$ are nondecreasing. If $z \in \hat{H}_1$, and γ and ω satisfy Eq.* (19), *then*

$$\big(\nu z^{\theta}\big)^{\Delta} = -\omega\big(z^{\sigma}\big)^{\theta} + \gamma\big(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\big)^{\theta} + \gamma\mathcal{S}\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\Upsilon^{\theta}\big)^{\Delta}. \tag{23}$$

Proof From the definition of ν and by using the rules of derivative on time scales (15), we have, for $t \geq \alpha$,

$$\nu^{\Delta} = \big(\gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big)^{\Delta}\big(\mathcal{S}^{\sigma}\big)^{1-\theta} + \gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\mathcal{S}^{1-\theta}\big)^{\Delta}. \tag{24}$$

Since

$$\big(\mathcal{S}^{1-\theta}\big)^{\Delta} = \big(\mathcal{S}\cdot\mathcal{S}^{-\theta}\big)^{\Delta} = \mathcal{S}^{\Delta}\big(\mathcal{S}^{\sigma}\big)^{-\theta} + \mathcal{S}\big(\mathcal{S}^{-\theta}\big)^{\Delta}, \tag{25}$$

we have from (19) and (25)

$$\begin{aligned}
\nu^{\Delta} &= \big(\gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big)^{\Delta}\big(\mathcal{S}^{\sigma}\big)^{1-\theta} + \gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\mathcal{S}^{1-\theta}\big)^{\Delta} \\
&= \big(\gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big)^{\Delta}\big(\mathcal{S}^{\sigma}\big)^{1-\theta} + \gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\mathcal{S}\mathcal{S}^{-\theta}\big)^{\Delta} \\
&= -\omega + \gamma\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\mathcal{S}^{\Delta}\big(\mathcal{S}^{\sigma}\big)^{-\theta} + \mathcal{S}\big(\mathcal{S}^{-\theta}\big)^{\Delta}\big) \\
&= -\omega + \gamma\big(\mathcal{S}^{\Delta}\big)^{\theta}\big(\mathcal{S}^{\sigma}\big)^{-\theta} + \gamma\mathcal{S}\big(\mathcal{S}^{\Delta}\big)^{\theta-1}\big(\mathcal{S}^{-\theta}\big)^{\Delta}.
\end{aligned} \tag{26}$$

Let $z = \mathcal{S}\Upsilon$, then we have by employing the product rule of derivative

$$\big(z^{\theta}\big)^{\Delta} = \big(\mathcal{S}^{\theta}\Upsilon^{\theta}\big)^{\Delta} = \big(\mathcal{S}^{\theta}\big)^{\Delta}\big(\Upsilon^{\sigma}\big)^{\theta} + \mathcal{S}^{\theta}\big(\Upsilon^{\theta}\big)^{\Delta}. \tag{27}$$

Also by the product rule of derivative, we have

$$\big(\nu z^{\theta}\big)^{\Delta} = \nu^{\Delta}\big(z^{\sigma}\big)^{\theta} + \nu\big(z^{\theta}\big)^{\Delta}. \tag{28}$$

Substituting (26) and (27) into (28), we obtain

$$\begin{aligned}
\left(\nu z^{\theta}\right)^{\Delta} &= \left(-\omega+\gamma\left(\mathcal{S}^{\Delta}\right)^{\theta}\left(\mathcal{S}^{\sigma}\right)^{-\theta}+\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{-\theta}\right)^{\Delta}\right)\left(z^{\sigma}\right)^{\theta} \\
&\quad+\left(\gamma\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\mathcal{S}^{1-\theta}\right)\left(\left(\mathcal{S}^{\theta}\right)^{\Delta}\left(\Upsilon^{\sigma}\right)^{\theta}+\mathcal{S}^{\theta}\left(\Upsilon^{\theta}\right)^{\Delta}\right) \\
&= -\omega\left(z^{\sigma}\right)^{\theta}+\gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta}+\gamma\mathcal{S}\Upsilon^{\sigma}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{\theta} \\
&\quad+\gamma\mathcal{S}\Upsilon^{\sigma}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta}+\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \\
&= -\omega\left(z^{\sigma}\right)^{\theta}+\gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta}+\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \\
&\quad+\gamma\mathcal{S}\Upsilon^{\sigma}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{\theta}+\mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta}\right).
\end{aligned} \tag{29}$$

The product rule of derivative now yields

$$\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\theta}\right)^{\sigma}+\mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta}=\left(\mathcal{S}^{-\theta}\cdot\mathcal{S}^{\theta}\right)^{\Delta}=(1)^{\Delta}=0. \tag{30}$$

Substituting (30) into (29), we have

$$\left(\nu z^{\theta}\right)^{\Delta}=-\omega\left(z^{\sigma}\right)^{\theta}+\gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta}+\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta},$$

which is the desired equation, Eq. (23). The proof is complete. □

Lemma 3.2 *Let $\alpha,\beta\in\mathbb{T}$ and suppose that $\mathcal{S}(t)$ and $\Upsilon(t)$ are nondecreasing.* If$z\in\hat{H}_1$, *and γ and ω satisfy Eq.* (19), *then*

$$\gamma\left(z^{\Delta}\right)^{\theta}=\omega\left(z^{\sigma}\right)^{\theta}+G+\left(\nu z^{\theta}\right)^{\Delta}, \tag{31}$$

with $G\geq 0$. Furthermore, $G=0$ if and only if $z=c\mathcal{S}$ with $c=\mathrm{const}\geq 0$.

Proof From (31), we see that

$$G=\gamma\left(z^{\Delta}\right)^{\theta}-\omega\left(z^{\sigma}\right)^{\theta}-\left(\nu z^{\theta}\right)^{\Delta}. \tag{32}$$

Let $z=\mathcal{S}\Upsilon$, then $z^{\Delta}=\mathcal{S}^{\Delta}\Upsilon^{\sigma}+\mathcal{S}\Upsilon^{\Delta}\geq 0$, and

$$\gamma\left(z^{\Delta}\right)^{\theta}=\gamma\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma}+\mathcal{S}\Upsilon^{\Delta}\right)^{\theta}. \tag{33}$$

By using Lemma 3.1, we get

$$\left(\nu z^{\theta}\right)^{\Delta}+\omega\left(z^{\sigma}\right)^{\theta}=\gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta}+\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}. \tag{34}$$

Substituting (33) and (34) into (32), we obtain

$$\begin{aligned}
G &= \gamma\left(z^{\Delta}\right)^{\theta}-\omega\left(z^{\sigma}\right)^{\theta}-\left(\nu z^{\theta}\right)^{\Delta} \\
&= \gamma\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma}+\mathcal{S}\Upsilon^{\Delta}\right)^{\theta}-\gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta}-\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}.
\end{aligned} \tag{35}$$

Since $\mathcal{S}$ and Υ are nondecreasing for $t \in [\alpha, \beta]_{\mathbb{T}}$, then $\mathcal{S}^{\Delta} \geq 0$, and by (17), we get

$$\left(\Upsilon^{\theta}(t)\right)^{\Delta} = \theta \left\{ \int_0^1 \left(s\Upsilon^{\sigma}(t) + (1-s)\Upsilon(t)\right)^{\theta-1} ds \right\} \Upsilon^{\Delta}(t) \leq \theta\left(\Upsilon^{\sigma}(t)\right)^{\theta-1}\Upsilon^{\Delta}(t),$$

and then

$$-\gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \geq -\gamma\theta\mathcal{S}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\Upsilon^{\Delta}. \tag{36}$$

Substituting (36) into (35), we obtain

$$\begin{aligned} G &= \gamma\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)^{\theta} - \gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \gamma\mathcal{S}\left(\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \\ &\geq \gamma\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)^{\theta} - \gamma\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \gamma\theta\mathcal{S}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\Upsilon^{\Delta} \\ &= \gamma\left(\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)^{\theta} - \left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \theta\mathcal{S}\left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\Upsilon^{\Delta}\right). \end{aligned} \tag{37}$$

By employing the inequality (see [14])

$$a^{\theta} - b^{\theta} \geq \theta a^{\theta-1}(a-b), \quad \text{for } a > b > 0 \text{ and } \theta > 1, \tag{38}$$

we have

$$\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)^{\theta} - \left(\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} \geq \theta\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)^{\theta-1}\mathcal{S}\Upsilon^{\Delta} \geq \theta\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma}\right)^{\theta-1}\mathcal{S}\Upsilon^{\Delta}.$$

This and (37) imply that $G \geq 0$ where $\gamma > 0$ on $[\alpha, \beta]_{\mathbb{T}}$. Furthermore $G = 0$ if and only if $\Upsilon^{\Delta} = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. This in fact gives us that $\Upsilon^{\Delta} = (z\mathcal{S}^{-1})^{\Delta} = 0$, and then $z = c\mathcal{S}$ with $c = \text{const} \geq 0$. The proof is complete. □

Theorem 3.1 *Let α, $\beta \in \mathbb{T}$ and $\mathcal{S}(t)$ and $\Upsilon(t)$ are nondecreasing. If $z \in \hat{H}_1$, and γ and ω satisfy Eq.* (19), *then*

$$\int_{\alpha}^{\beta} \omega(t)\left(z^{\sigma}(s)\right)^{\theta} \Delta s + \lim_{s\to\beta} vz^{\theta}(s) - \lim_{s\to\alpha} vz^{\theta}(s) \leq \int_{\alpha}^{\beta} \gamma(s)\left(z^{\Delta}(s)\right)^{\theta} \Delta s. \tag{39}$$

Furthermore the inequality (39) *becomes an equality if and only if $z = c\mathcal{S}$ with $c = \text{const} \geq 0$.*

Proof From Lemma 3.2, we have

$$\omega\left(z^{\sigma}\right)^{\theta} + G(s) + \left(vz^{\theta}\right)^{\Delta} = \gamma\left(z^{\Delta}\right)^{\theta}.$$

Then

$$\int_{\alpha}^{\beta} \omega\left(z^{\sigma}\right)^{\theta} \Delta s + \int_{\alpha}^{\beta} G(s)\Delta s + \int_{\alpha}^{\beta} \left(vz^{\theta}\right)^{\Delta} \Delta s = \int_{\alpha}^{\beta} \gamma\left(z^{\Delta}\right)^{\theta} \Delta s.$$

Since $G(s) \geq 0$, on $[\alpha, \beta]_{\mathbb{T}}$, we get from the last equation

$$\int_{\alpha}^{\beta} \omega\left(z^{\sigma}\right)^{\theta} \Delta s + \lim_{s\to\beta} vz^{\theta}(s) - \lim_{s\to\alpha} vz^{\theta}(s) \leq \int_{\alpha}^{\beta} \gamma\left(z^{\Delta}\right)^{\theta} \Delta s,$$

which is the desired inequality (39). Furthermore the inequality (39) becomes equality if and only if $G(s) = 0$, this implies that $\Upsilon^{\Delta}(s) = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. This in fact gives $\Upsilon^{\Delta} = (z\mathcal{S}^{-1})^{\Delta} = 0$, and then $z = c\mathcal{S}$ with $c = \text{const} \geq 0$. The proof is complete. □

Definition 3.2 Let $z \in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ such that $z \geq 0$ on $[\alpha,\beta]_{\mathbb{T}}$. The function z belongs to the class $\mathcal{H}_1$ if

$$\int_{\alpha}^{\beta} \omega(s)\bigl(z^{\sigma}(s)\bigr)^{\theta}\Delta s > -\infty, \qquad \int_{\alpha}^{\beta} \gamma(s)\bigl(z^{\Delta}(s)\bigr)^{\theta}\Delta s < \infty, \tag{40}$$

and

$$\lim_{s\to\alpha}\bigl(vz^{\theta}\bigr)(s) \leq 0, \qquad \lim_{s\to\beta}\bigl(vz^{\theta}\bigr)(s) \geq 0. \tag{41}$$

Remark 3.1 Obviously $\mathcal{H}_1 \subset \hat{H}_1$.

Theorem 3.2 *Suppose that γ and ω satisfy Eq. (19) and $\mathcal{S}(t)$ and $\Upsilon(t)$ are nondecreasing for $t \in [\alpha,\beta]_{\mathbb{T}}$. If $z \in \mathcal{H}_1$ then*

$$\int_{\alpha}^{\beta} \omega(t)\bigl(z^{\sigma}(t)\bigr)^{\theta}\Delta t \leq \int_{\alpha}^{\beta} \gamma(t)\bigl(z^{\Delta}(t)\bigr)^{\theta}\Delta t. \tag{42}$$

If $z \neq 0$, then inequality (42) becomes equality if and only if $z\mathcal{S}^{-1} = \text{const} \neq 0$, $\mathcal{S} \in \hat{H}_1$, and

$$\lim_{t\to\alpha}\bigl(\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S}\bigr)(t) \leq 0 \quad \textit{and} \quad \lim_{t\to\beta}\bigl(\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S}\bigr)(t) \geq 0. \tag{43}$$

Proof From (41), we have $\lim_{t\to\beta} vz^{\theta}(t) - \lim_{t\to\alpha} vz^{\theta}(t) \geq 0$. From Theorem 3.1, we see that

$$\int_{\alpha}^{\beta} \omega\bigl(z^{\sigma}\bigr)^{\theta}\Delta t \leq \int_{\alpha}^{\beta} \gamma\bigl(z^{\Delta}\bigr)^{\theta}\Delta t,$$

which is the desired inequality (42). Furthermore inequality (42) becomes an equality if and only if $G(t) = 0$, this implies that $\Upsilon^{\Delta} = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. This leads to $\Upsilon^{\Delta} = (z\mathcal{S}^{-1})^{\Delta} = 0$, and then $z = c\mathcal{S}$ with $c = \text{const} \geq 0$. Since $v = \gamma(\mathcal{S}^{\Delta})^{\theta-1}\mathcal{S}^{1-\theta}$, and $z = c\mathcal{S}$ with $c = \text{const} \geq 0$, we have

$$\lim_{t\to\beta}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S} = \lim_{t\to\beta}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S}^{1-\theta}(c\mathcal{S})^{\theta} = \lim_{t\to\beta} vz^{\theta}(t), \tag{44}$$

and

$$\lim_{t\to\alpha}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S} = \lim_{t\to\alpha}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S}^{1-\theta}(c\mathcal{S})^{\theta} = \lim_{t\to\alpha} vz^{\theta}(t). \tag{45}$$

Now by using (41), and (44) and (45), we have

$$\lim_{t\to\alpha}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S} \leq 0 \quad \text{and} \quad \lim_{t\to\beta}\gamma\bigl(\mathcal{S}^{\Delta}\bigr)^{\theta-1}\mathcal{S} \geq 0,$$

and hence $\mathcal{S} \in \hat{H}_1$. The proof is complete. □

When $\Upsilon = 1$, we have from Theorem 3.2 the result.

Theorem 3.3 *Let γ and ω satisfy*

$$\left(\gamma(y)\left(\mathcal{S}^{\Delta}(y)\right)^{\theta-1}\right)^{\Delta}+\omega(y)\left(\mathcal{S}^{\sigma}(y)\right)^{\theta-1}=0,\quad \textit{for } y\in[\alpha,\beta]_{\mathbb{T}},\tag{46}$$

such that $\mathcal{S}(y)>0,\mathcal{S}^{\Delta}(y)>0$. *If* $z\in\Upsilon_1$, *then*

$$\int_{\alpha}^{\beta}\omega(s)\left(z^{\sigma}(s)\right)^{\theta}\Delta ts\leq\int_{\alpha}^{\beta}\gamma(s)\left(z^{\Delta}(s)\right)^{\theta}\Delta s.\tag{47}$$

Remark 3.2 If $\mathbb{T}=\mathbb{R}$, then we obtain from Theorem 3.3

$$\int_{\alpha}^{\beta}\omega(t)\left(\int_{0}^{t}u(\tau)\,d\tau\right)^{\theta}dt\leq\int_{\alpha}^{\beta}\gamma(t)u^{\theta}(t)\,dt,$$

where γ and ω verify the differential equation

$$\left(\gamma(t)\left(\mathcal{S}'(t)\right)^{\theta-1}\right)'+\omega(t)\left(\mathcal{S}(t)\right)^{\theta-1}=0,\quad \text{for } t\in[\alpha,\beta]_{\mathbb{R}},\tag{48}$$

where $\mathcal{S}(t)>0,\mathcal{S}'(t)>0$.

Remark 3.3 For the differential form, we get from Remark 3.2 the Wirtinger inequality

$$\int_{\alpha}^{\beta}\omega(s)\left(u(s)\right)^{\theta}ds\leq\int_{\alpha}^{\beta}\gamma(s)\left(u'(s)\right)^{\theta}ds,$$

where γ and ω satisfy the differential equation (48) and $u(0)=0$.

Now, we give some examples for illustration.

Example 3.1 Let $\mathbb{T}=\mathbb{R}$ and $\alpha,\beta\in\mathbb{R}$, such that $0\leq\alpha<\beta$. Assume that $\gamma(s)=1$, and $\mathcal{S}(s)=s^{\frac{\theta-1}{\theta}}$ for $s\in[\alpha,\beta]$ where $\theta>1$. It is clear that $\gamma,\mathcal{S}>0$, and

$$\mathcal{S}'(s)=\frac{\theta-1}{\theta}s^{\frac{-1}{\theta}}\geq 0.$$

From (48), we have

$$\begin{aligned}\omega(s)&=-\left(\gamma(s)\left(\mathcal{S}'(s)\right)^{\theta-1}\right)'\mathcal{S}^{1-\theta}=-\left(\left(\frac{\theta-1}{\theta}s^{\frac{-1}{\theta}}\right)^{\theta-1}\right)'\left(s^{\frac{\theta-1}{\theta}}\right)^{1-\theta}\\&=-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\left(s^{\frac{-(\theta-1)}{\theta}}\right)'\left(s^{\frac{\theta-1}{\theta}}\right)^{1-\theta}=\left(\frac{\theta-1}{\theta}\right)^{\theta}s^{\frac{1-2\theta}{\theta}}\left(s^{\frac{\theta-1}{\theta}}\right)^{1-\theta}\\&=\left(\frac{\theta-1}{\theta}\right)^{\theta}s^{\frac{1-2\theta+2\theta-1-\theta^2}{\theta}}=\left(\frac{\theta-1}{\theta}\right)^{\theta}s^{-\theta}.\end{aligned}$$

Now, by applying Theorem 3.2 when $\mathbb{T}=\mathbb{R}$, we have

$$\int_{\alpha}^{\beta}\left(\frac{1}{s}\int_{0}^{s}u(\tau)\,d\tau\right)^{\theta}ds\leq\left(\frac{\theta}{\theta-1}\right)^{\theta}\int_{\alpha}^{\beta}u^{\theta}(s)\,ds,$$

which is the Hardy inequality (5) with a sharp constant $(\theta/(\theta-1))^{\theta}$.

Example 3.2 Let $\mathbb{T} = \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, such that $0 \leq \alpha < \beta$. Assume that $\gamma(s) = s^{\theta-\eta}$, and $\mathcal{S} = s^{\frac{\eta-1}{\theta}}$ for $s \in [\eta, \beta]$ where $\eta > 1$ is an arbitrary constant. Then $\gamma, \mathcal{S} > 0$, and

$$\mathcal{S}' = \frac{\eta-1}{\theta} s^{\frac{\eta-1-\theta}{\theta}} \geq 0, \quad \text{for } \theta > 1 \text{ and } s > 0.$$

From (48), we have

$$\begin{aligned}
\omega(s) &= -\big(\gamma(s)\big(\mathcal{S}'(s)\big)^{\theta-1}\big)' \mathcal{S}^{1-\theta} \\
&= -\left(s^{\theta-\eta}\left(\frac{\eta-1}{\theta} s^{\frac{\eta-1-\theta}{\theta}}\right)^{\theta-1}\right)' \big(s^{\frac{\eta-1}{\theta}}\big)^{1-\theta} \\
&= -\left(\frac{\eta-1}{\theta}\right)^{\theta-1} \big(s^{\frac{-(\eta-1)}{\theta}}\big)' \big(s^{\frac{\eta-1}{\theta}}\big)^{1-\theta} \\
&= \left(\frac{\eta-1}{\theta}\right)^{\theta} s^{\frac{1-\eta-\theta}{\theta}} \big(s^{\frac{\eta-1}{\theta}}\big)^{1-\theta} = \left(\frac{\eta-1}{\theta}\right)^{\theta} s^{-\eta}.
\end{aligned}$$

By applying Theorem 3.2 when $\mathbb{T} = \mathbb{R}$, we see that

$$\int_{\eta}^{\beta} \frac{1}{s^{\eta}} \left(\int_{0}^{s} u(\tau)\, d\tau\right)^{\theta} ds \leq \left(\frac{\theta}{\eta-1}\right)^{\theta} \int_{\eta}^{\beta} s^{\theta-\eta} u^{\theta}(s)\, ds,$$

which is the Hardy–Littlewood inequality with a sharp constant $(\theta/(\eta-1))^{\theta}$ (see [22]).

Remark 3.4 When $\mathbb{T} = \mathbb{N}$, we obtain from Theorem 3.3 the following inequality:

$$\sum_{n=0}^{N} \omega(n) \left(\sum_{\omega=0}^{n} u(\omega)\right)^{\theta} \leq \sum_{n=0}^{N} \gamma(n) u^{\theta}(n),$$

where u is a positive summable sequence, and the sequences γ and ω satisfy the difference equation

$$\Delta\big(\gamma(n)\big(\Delta\mathcal{S}(n)\big)^{\theta-1}\big) + \omega(n)\big(\mathcal{S}(n+1)\big)^{\theta-1} = 0, \quad \text{for } n \in [1, N]_{\mathbb{N}}, \tag{49}$$

where $\mathcal{S}(n) > 0, \Delta\mathcal{S}(n) > 0$.

Example 3.3 Let $\mathbb{T} = \mathbb{N}$ with $[1, N] \subset \mathbb{N}$ such that $1 \leq N < \infty$. Assume that $\gamma(k) = 1$, and $\mathcal{S}(k) = k^{\frac{\eta-1}{\eta}}$ for $k \in [1, k]$ where $\eta > 1$. It is clear that $\gamma, \mathcal{S} > 0$, and by using the inequality

$$\gamma y^{\gamma-1}(y-z) \leq y^{\gamma} - z^{\gamma} \leq \gamma z^{\gamma-1}(y-z), \quad \text{for } y \geq z > 0 \text{ and } 0 < \gamma < 1, \tag{50}$$

with $\gamma = (\eta-1)/\eta < 1$, we have

$$\Delta\mathcal{S}(k) = (k+1)^{\frac{\eta-1}{\eta}} - k^{\frac{\eta-1}{\eta}} \leq \frac{\eta-1}{\eta} k^{\frac{-1}{\eta}}, \quad \text{for } \eta > 1 \text{ and } k > 0.$$

From (49), we have

$$\omega(k) = -\Delta\big[\gamma(k)\big(\Delta\mathcal{S}(k)\big)^{\eta-1}\big]\mathcal{S}^{1-\eta}(k+1)$$

$$\geq -\left(\frac{\eta-1}{\eta}\right)^{\eta-1}\Delta\bigl(k^{\frac{-(\eta-1)}{\eta}}\bigr)\bigl((k+1)^{\frac{\eta-1}{\eta}}\bigr)^{1-\eta}. \tag{51}$$

By applying the inequality

$$\gamma z^{\gamma-1}(y-z) \leq y^{\gamma} - z^{\gamma} \leq \gamma y^{\gamma-1}(y-z), \quad \text{for } y \geq z > 0, \gamma \geq 1 \text{ or } \gamma < 0, \tag{52}$$

with $\gamma = -(\eta-1)/\eta < 0$, we get

$$\Delta\bigl(k^{\frac{-(\eta-1)}{\eta}}\bigr) \leq \frac{-(\eta-1)}{\eta}k^{\frac{-(2\eta-1)}{\eta}}.$$

Substituting the last inequality into (51), we get

$$\begin{aligned}\omega(k) &= -\Delta\bigl(\gamma(k)\bigl(\Delta\mathcal{S}(k)\bigr)^{\eta-1}\bigr)\mathcal{S}^{1-\eta}(k+1)\\ &\geq \left(\frac{\eta-1}{\eta}\right)^{\eta}k^{\frac{-(2\eta-1)}{\eta}}\bigl((k+1)^{\frac{\eta-1}{\eta}}\bigr)^{1-\eta} \geq \left(\frac{\eta-1}{\eta}\right)^{\eta}(k+1)^{-\eta}.\end{aligned}$$

Now, by applying Remark 3.4, we have

$$\sum_{k=1}^{N}\left(\frac{\eta-1}{\eta}\right)^{\eta}(k+1)^{-\eta}\left(\sum_{\omega=1}^{k+1}u(\omega)\right)^{\eta} \leq \sum_{k=1}^{N}u^{\eta}(k),$$

That is,

$$\sum_{k=1}^{N}\frac{1}{(k+1)^{\eta}}\left(\sum_{\omega=1}^{k+1}u(\omega)\right)^{\eta} \leq \left(\frac{\eta}{\eta-1}\right)^{\eta}\sum_{k=1}^{N}u^{\eta}(k),$$

which is the Hardy inequality (1) with a best constant $(\eta/(\eta-1))^{\eta}$.

Example 3.4 Let $\mathbb{T} = \mathbb{N}$ with $k \in \mathbb{N}$, and $1 \leq k < \infty$. Assume that $\gamma(k) = k^{\eta-\alpha}$, and $\mathcal{S}(k) = k^{\frac{\alpha-1}{\eta}}$ for $k \in [1, N]$ where $\alpha > 1$ is an arbitrary constant. Then $\gamma, \mathcal{S} > 0$ and

$$\Delta\mathcal{S}(k) \leq \frac{\alpha-1}{\eta}k^{\frac{\alpha-1-\eta}{\eta}} \geq 0.$$

As in Example 3.3, we have from (48)

$$\omega(k) \geq \left(\frac{\alpha-1}{\eta}\right)^{\eta}(k+1)^{-\alpha}.$$

By applying Remark 3.4, we see that

$$\sum_{k=1}^{N}\frac{1}{(k+1)^{\alpha}}\left(\sum_{\omega=1}^{k+1}u(\omega)\right)^{\eta} \leq \left(\frac{\eta}{\alpha-1}\right)^{\eta}\sum_{k=1}^{N}(k+1)^{\eta-\alpha}u^{\eta}(k),$$

which is the discrete Hardy–Littlewood inequality with a best constant $(\eta/(\alpha-1))^{\eta}$ (see [3]).

Now, we will prove some characterizations of the weights of the inequalities of Copson's type with tails on time scales. Let $\mathbb{T}$ be a time scale with $\alpha,\beta \in \mathbb{T}$. Assume that $\gamma,\omega \in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R}^+)$, and $\mathcal{S} \in C^1_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R}^+)$. Let θ be any real number such that $\theta > 1$, and let γ and ω satisfy dynamic equation of the Euler–Lagrange type

$$\left(\gamma(t)\left(-\mathcal{S}^{\Delta}(t)\right)^{\theta-1}\right)^{\Delta} - \omega(t)\left(\mathcal{S}^{\sigma}(t)\right)^{\theta-1} = 0, \quad \text{for } t \in [\alpha,\beta]_{\mathbb{T}}, \tag{53}$$

where $\mathcal{S}(t) > 0, \mathcal{S}^{\Delta}(t) < 0$ for $t \in [\alpha,\beta]_{\mathbb{T}}$, and define

$$\nu := \frac{-\gamma(-\mathcal{S}^{\Delta})^{\theta-1}}{\mathcal{S}^{\theta-1}}. \tag{54}$$

Definition 3.3 Assume that $z \in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ such that $z \geq 0$ on $[\alpha,\beta]_{\mathbb{T}}$. We say that $z \in \hat{H}_2$ if

$$\int_{\alpha}^{\beta} \omega(t)\left(z^{\sigma}(t)\right)^{\theta} \Delta t > -\infty, \qquad \int_{\alpha}^{\beta} \gamma(t)\left|z^{\Delta}(t)\right|^{\theta} \Delta t < \infty, \tag{55}$$

and

$$\liminf_{t\to\alpha}\left(\nu z^{\theta}\right)(t) < \infty, \qquad \limsup_{t\to\beta}\left(\nu z^{\theta}\right)(t) > -\infty. \tag{56}$$

Lemma 3.3 *Suppose that* α, $\beta \in \mathbb{T}$ *and* $\mathcal{S}(t)$ *and* $\Upsilon(t)$ *are nonincreasing. If* $z \in \hat{H}_2$, *and* γ *and* ω *satisfy the dynamic equation* (53) *then*

$$\left(\nu z^{\theta}\right)^{\Delta} = -\omega\left(z^{\sigma}\right)^{\theta} + \gamma\left|\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right|^{\theta} - \gamma\mathcal{S}\left|\mathcal{S}^{\Delta}\right|^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}. \tag{57}$$

Proof From the definition of ν and by using the rules (15) for $t \geq \alpha$, we have

$$\nu^{\Delta} = -\left(\gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\right)^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{1-\theta} - \gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{1-\theta}\right)^{\Delta}. \tag{58}$$

Since

$$\left(\mathcal{S}^{1-\theta}\right)^{\Delta} = \left(\mathcal{S}\cdot\mathcal{S}^{-\theta}\right)^{\Delta} = \mathcal{S}^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{-\theta} + \mathcal{S}\left(\mathcal{S}^{-\theta}\right)^{\Delta}, \tag{59}$$

from (53) and (59), we see that (note $\mathcal{S}^{\Delta} < 0$)

$$\begin{aligned}
\nu^{\Delta} &= -\left(\gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\right)^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{1-\theta} - \gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{1-\theta}\right)^{\Delta} \\
&= -\omega - \gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{-\theta} + \mathcal{S}\left(\mathcal{S}^{-\theta}\right)^{\Delta}\right) \\
&= -\omega + \gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta}\left(\mathcal{S}^{\sigma}\right)^{-\theta} - \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{-\theta}\right)^{\Delta}.
\end{aligned} \tag{60}$$

Let $z = \mathcal{S}\Upsilon$, then we have

$$\left(z^{\theta}\right)^{\Delta} = \left(\mathcal{S}^{\theta}\Upsilon^{\theta}\right)^{\Delta} = \left(\mathcal{S}^{\theta}\right)^{\Delta}\left(\Upsilon^{\sigma}\right)^{\theta} + \mathcal{S}^{\theta}\left(\Upsilon^{\theta}\right)^{\Delta}. \tag{61}$$

Substituting (60) and (61) into

$$\left(\nu z^{\theta}\right)^{\Delta} = \nu^{\Delta}\left(z^{\sigma}\right)^{\theta} + \nu\left(z^{\theta}\right)^{\Delta},$$

we obtain

$$\begin{aligned}
\left(vz^{\theta}\right)^{\Delta} &= \left(-\omega + \gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta}\left(\mathcal{S}^{\sigma}\right)^{-\theta} - \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{-\theta}\right)^{\Delta}\right)\left(z^{\sigma}\right)^{\theta} \\
&\quad + \left(-\gamma\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\mathcal{S}^{1-\theta}\right)\left(\left(\mathcal{S}^{\theta}\right)^{\Delta}\left(\Upsilon^{\sigma}\right)^{\theta} + \mathcal{S}^{\theta}\left(\Upsilon^{\theta}\right)^{\Delta}\right) \\
&= -\omega\left(z^{\sigma}\right)^{\theta} + \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \gamma\mathcal{S}\left(\Upsilon^{\sigma}\right)^{\theta}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\sigma}\right)^{\theta} \\
&\quad - \gamma\mathcal{S}\left(\Upsilon^{\sigma}\right)^{\theta}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta} - \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \\
&= -\omega\left(z^{\sigma}\right)^{\theta} + \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} \\
&\quad - \gamma\mathcal{S}\left(\Upsilon^{\sigma}\right)^{\theta}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\theta}\right)^{\sigma} + \mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta}\right) \\
&\quad - \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}.
\end{aligned} \tag{62}$$

By using the product rule, we see that

$$\left(\left(\mathcal{S}^{-\theta}\right)^{\Delta}\left(\mathcal{S}^{\theta}\right)^{\sigma} + \mathcal{S}^{-\theta}\left(\mathcal{S}^{\theta}\right)^{\Delta}\right) = \left(\mathcal{S}^{-\theta}\cdot\mathcal{S}^{\theta}\right)^{\Delta} = (1)^{\Delta} = 0. \tag{63}$$

Substituting (63) into (62), we have

$$\left(vz^{\theta}\right)^{\Delta} = -\omega\left(z^{\sigma}\right)^{\theta} + \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}.$$

Since $z = \mathcal{S}\Upsilon \geq 0$, we get (note that $|\mathcal{S}^{\Delta}| = -\mathcal{S}^{\Delta}$)

$$\left(vz^{\theta}\right)^{\Delta} = -\omega\left(z^{\sigma}\right)^{\theta} + \gamma\left|\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right|^{\theta} - \gamma\mathcal{S}\left|\mathcal{S}^{\Delta}\right|^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta},$$

which is the desired equation, Eq. (57). The proof is complete. □

Lemma 3.4 *Suppose that $\alpha, \beta \in \mathbb{T}$ and $\mathcal{S}(t)$ and $\Upsilon(t)$ are nonincreasing. If $z \in \hat{H}_2$, and γ and ω satisfy the dynamic equation (53) then*

$$\gamma\left|z^{\Delta}\right|^{\theta} = \omega\left(z^{\sigma}\right)^{\theta} + G + \left(vz^{\theta}\right)^{\Delta}, \tag{64}$$

with $G \geq 0$. Furthermore, $G = 0$ if and only if $z = c\mathcal{S}$ with $c = \text{const} \geq 0$.

Proof From (64), we see that

$$G = \gamma\left|z^{\Delta}\right|^{\theta} - \omega\left(z^{\sigma}\right)^{\theta} - \left(vz^{\theta}\right)^{\Delta}. \tag{65}$$

Let $z = \mathcal{S}\Upsilon$, then $z^{\Delta} = \mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta} \leq 0$, we have

$$\gamma\left(-z^{\Delta}\right)^{\theta} = \gamma\left(-\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)\right)^{\theta}. \tag{66}$$

By using Lemma 3.3, we get

$$\left(vz^{\theta}\right)^{\Delta} + \omega\left(z^{\sigma}\right)^{\theta} = \gamma\left|\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right|^{\theta} - \gamma\mathcal{S}\left|\mathcal{S}^{\Delta}\right|^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}. \tag{67}$$

Substituting (66) and (67) into (65), we obtain

$$\begin{aligned} G &= \gamma \left|z^{\Delta}\right|^{\theta} - \omega\left(z^{\sigma}\right)^{\theta} - \left(\nu z^{\theta}\right)^{\Delta} \\ &= \gamma\left(-\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)\right)^{\theta} - \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} + \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta}. \end{aligned} \tag{68}$$

Since $\mathcal{S}$ and Υ are nonincreasing for $t \in [\alpha,\beta]_{\mathbb{T}}$, then, by using the chain rule (17), we see that

$$\left(\Upsilon^{\theta}\right)^{\Delta} = \theta\left\{\int_0^1 \left(s\Upsilon^{\sigma}(t) + (1-s)\Upsilon(t)\right)^{\theta-1} ds\right\}\Upsilon^{\Delta} \geq \theta\left(\Upsilon^{\sigma}(t)\right)^{\theta-1}\Upsilon^{\Delta}. \tag{69}$$

Substituting (69) into (68), we obtain

$$\begin{aligned} G &= \gamma\left(-\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)\right)^{\theta} - \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} + \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\left(\Upsilon^{\theta}\right)^{\Delta} \\ &\geq \gamma\left(-\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)\right)^{\theta} - \gamma\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} + \gamma\mathcal{S}\left(-\mathcal{S}^{\Delta}\right)^{\theta-1}\theta\left(\Upsilon^{\sigma}\right)^{\theta-1}\Upsilon^{\Delta} \\ &= \gamma\left(\left(-\left(\mathcal{S}^{\Delta}\Upsilon^{\sigma} + \mathcal{S}\Upsilon^{\Delta}\right)\right)^{\theta} - \left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta} - \theta\left(-\mathcal{S}\Upsilon^{\Delta}\right)\left(-\Upsilon^{\sigma}\mathcal{S}^{\Delta}\right)^{\theta-1}\right). \end{aligned}$$

By using the inequality (38) and proceeding as in the proof of Lemma 3.2 we can prove that $G \geq 0$, since $\gamma > 0$ on $[\alpha,\beta]_{\mathbb{T}}$. Furthermore, $G = 0$ if and only if $\Upsilon^{\Delta} = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. This implies that $\Upsilon^{\Delta} = (z\mathcal{S}^{-1})^{\Delta} = 0$, and then $z = c\mathcal{S}$ with $c = \text{const} \geq 0$. The proof is complete. □

Theorem 3.4 *Suppose that $\alpha, \beta \in \mathbb{T}$ and $\mathcal{S}(t)$ and $\Upsilon(t)$ are nonincreasing. If $z \in \hat{H}_2$, and γ and ω satisfy the dynamic equation (53) then*

$$\int_{\alpha}^{\beta} \omega(t)\left(z^{\sigma}(t)\right)^{\theta}\Delta t + \lim_{t\to\beta}\left(\nu z^{\theta}\right)(t) - \lim_{t\to\alpha}\left(\nu z^{\theta}\right)(t) \leq \int_{\alpha}^{\beta} \gamma(t)\left|z^{\Delta}(t)\right|^{\theta}\Delta t \tag{70}$$

holds on $[\alpha,\beta]_{\mathbb{T}}$. Furthermore, inequality (70) becomes an equality if and only if $z = c\mathcal{S}$ with $c = \text{const} \geq 0$.

Proof From Lemma 3.4, we see that

$$\omega\left(z^{\sigma}\right)^{\theta} + G(t) + \left(\nu z^{\theta}\right)^{\Delta} = \gamma\left|z^{\Delta}\right|^{\theta}.$$

Then

$$\int_{\alpha}^{\beta} \omega\left(z^{\sigma}\right)^{\theta}\Delta t + \int_{\alpha}^{\beta} G(t)\Delta t + \int_{\alpha}^{\beta}\left(\nu z^{\theta}\right)^{\Delta}\Delta t = \int_{\alpha}^{\beta}\gamma\left|z^{\Delta}\right|^{\theta}\Delta t.$$

Since $G \geq 0$, on $[\alpha,\beta]_{\mathbb{T}}$,

$$\int_{\alpha}^{\beta} \omega(t)\left(z^{\sigma}(t)\right)^{\theta}\Delta t + \lim_{t\to\beta}\nu z^{\theta}(t) - \lim_{t\to\alpha}\nu z^{\theta}(t) \leq \int_{\alpha}^{\beta}\gamma(t)\left|z^{\Delta}(t)\right|^{\theta}\Delta t,$$

which is the desired inequality (70). Furthermore, the inequality (70) becomes an equality if and only if $G(t) = 0$, this implies that $\Upsilon^{\Delta} = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. This yields $\Upsilon^{\Delta} = (z\mathcal{S}^{-1})^{\Delta} = 0$, and then $z = c\mathcal{S}$ with $c = \text{const} \geq 0$. The proof is complete. □

Definition 3.4 Let $z \in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R})$ such that $z \geq 0$ on $[\alpha,\beta]_{\mathbb{T}}$. We say that $z \in \mathcal{H}_2$ if

$$\int_\alpha^\beta \omega(s)\big(z^\sigma(s)\big)^\theta \Delta s > -\infty, \qquad \int_\alpha^\beta \gamma(s)\big|z^\Delta(s)\big|^\theta \Delta s < \infty, \tag{71}$$

and

$$\lim_{t\to\alpha}\big(\nu z^\theta\big)(t) \leq 0, \qquad \lim_{t\to\beta}\big(\nu z^\theta\big)(t) \geq 0. \tag{72}$$

Remark 3.5 Obviously $\mathcal{H}_2 \subset \hat{H}_2$.

Theorem 3.5 *Suppose that $\alpha, \beta \in \mathbb{T}$ and $\mathcal{S}(t)$ and $\Upsilon(t)$ are nonincreasing. If $z \in \mathcal{H}_2$, then*

$$\int_\alpha^\beta \omega(s)\big(z^\sigma(s)\big)^\theta \Delta s \leq \int_\alpha^\beta \gamma(s)\big|z^\Delta(s)\big|^\theta \Delta s. \tag{73}$$

If $z \neq 0$, then inequality (73) becomes an equality if and only if $z\mathcal{S}^{-1} = \mathrm{const} \neq 0$, $\mathcal{S} \in \hat{H}_2$, and

$$\lim_{t\to\alpha}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t) \leq 0 \quad \textit{and} \quad \lim_{t\to\beta}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t) \geq 0. \tag{74}$$

Proof From (72), we have $\lim_{t\to\beta}(\nu z^\theta)(t) - \lim_{t\to\alpha}(\nu z^\theta)(t) \geq 0$. By applying Theorem (3.4), we obtain

$$\int_\alpha^\beta \omega(t)\big(z^\sigma(t)\big)^\theta \Delta t \leq \int_\alpha^\beta \gamma(t)\big|z^\Delta(t)\big|^\theta \Delta t,$$

which is the desired inequality (73). Furthermore, inequality (73) reduces to an equality if and only if $G(t) = 0$, this implies that $\Upsilon^\Delta = 0$, where $\Upsilon = z\mathcal{S}^{-1}$. Thus $\Upsilon^\Delta = (z\mathcal{S}^{-1})^\Delta = 0$, and then $z = c\mathcal{S}$ with $c = \mathrm{const} \geq 0$. Since $\nu = -\gamma(-\mathcal{S}^\Delta)^{\theta-1}\mathcal{S}^{1-\theta}$, and $z = c\mathcal{S}$ with $c = \mathrm{const} \geq 0$, this implies that

$$\lim_{t\to\beta}\nu z^\theta(t) = \lim_{t\to\beta}\big(-\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}^{1-\theta}(c\mathcal{S})\big)^\theta(t) = c\lim_{t\to\beta}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t)$$

and

$$\lim_{t\to\alpha}\nu z^\theta(t) = \lim_{t\to\alpha}\big(-\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}^{1-\theta}(c\mathcal{S})^\theta\big)(t) = c\lim_{t\to\alpha}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t).$$

Now, by using (72), we have

$$\lim_{t\to\alpha}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t) \leq 0 \quad \text{and} \quad \lim_{t\to\beta}\big(\gamma\big(-\mathcal{S}^\Delta\big)^{\theta-1}\mathcal{S}\big)(t) \geq 0,$$

and then $\mathcal{S} \in \hat{H}$. The proof is complete. □

Remark 3.6 When $\mathbb{T} = \mathbb{R}$, we have from Theorem 3.5

$$\int_\alpha^\beta \omega(t)\bigg(\int_t^\beta u(\tau)\,d\tau\bigg)^\theta dt \leq \int_\alpha^\beta \gamma(t)\big|u(t)\big|^\theta dt, \tag{75}$$

where γ and ω satisfy

$$\big(\gamma(t)\big(-x'(t)\big)^{\theta-1}\big)' - \omega(t)x^{\theta-1}(t) = 0 \quad \text{for } t \in [\alpha,\beta], \tag{76}$$

where $x(t) > 0, x'(t) < 0$.

Remark 3.7 As a special case of (75), we get the Wirtinger type inequality

$$\int_\alpha^\beta \omega(t)\big(u(t)\big)^\theta \, dt \le \int_\alpha^\beta \gamma(t)\big|u'(t)\big|^\theta \, dt,$$

where γ and ω satisfy the differential equation (76) and $u(\beta) = 0$.

We give some examples for illustration.

Example 3.5 Let $\mathbb{T} = \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$, and $0 \le \alpha < \beta \le \infty$ and assume that $\gamma(s) = s^{\theta-\gamma}$, and $\mathcal{S} = s^{\frac{\gamma-1}{\theta}}$, where $0 \le \gamma < 1 < \theta$ is an arbitrary constant. Then $\gamma, \mathcal{S} > 0$ for $s \in [\alpha,\beta]$, and

$$\mathcal{S}' = \frac{\gamma-1}{\theta} s^{\frac{\gamma-1-\theta}{\theta}} \le 0.$$

From (76), we have

$$\begin{aligned}
\omega(s) &= \big(\gamma(s)\big(-\mathcal{S}'(s)\big)^{\theta-1}\big)'\mathcal{S}^{1-\theta} \\
&= \left(s^{\theta-\gamma}\left(\frac{1-\gamma}{\theta}s^{\frac{\gamma-1-\theta}{\theta}}\right)^{\theta-1}\right)'\big(s^{\frac{\gamma-1}{\theta}}\big)^{1-\theta} \\
&= \left(\frac{1-\gamma}{\theta}\right)^{\theta-1}\big(s^{\frac{-(\gamma-1)}{\theta}}\big)'\big(s^{\frac{\gamma-1}{\theta}}\big)^{1-\theta} \\
&= \left(\frac{1-\gamma}{\theta}\right)^{\theta} s^{\frac{1-\gamma-\theta}{\theta}}\big(s^{\frac{\gamma-1}{\theta}}\big)^{1-\theta} = \left(\frac{1-\gamma}{\theta}\right)^{\theta} s^{-\gamma}.
\end{aligned}$$

By applying Remark 3.6, we see that

$$\int_\alpha^\beta \frac{1}{s^\gamma}\left(\int_s^\beta u(\tau)\,d\tau\right)^\theta \Delta s \le \left(\frac{\theta}{1-\gamma}\right)^\theta \int_\alpha^\beta s^{\theta-\gamma}u^\theta(s)\Delta s,$$

which is the Hardy–Littlewood inequality with a sharp constant $(\theta/(1-\gamma))^\theta$ (see [22]).

Example 3.6 Let $\mathbb{T} = \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$, and $0 \le \alpha < \beta \le \infty$ and assume that $\gamma(s) = s^\theta$, and $\mathcal{S}(s) = s^{\frac{-1}{\theta}}$ where $\theta > 1$. It is clear that $\gamma, \mathcal{S} > 0$, and

$$\mathcal{S}'(s) = \frac{-1}{\theta}s^{\frac{-1}{\theta}-1} = \frac{-1}{\theta}s^{\frac{-(1+\theta)}{\theta}} \le 0.$$

From (76), we have

$$\omega(s) = \big(\gamma(s)\big(-\mathcal{S}'(s)\big)^{\theta-1}\big)'\mathcal{S}^{1-\theta} = \left(s^\theta\left(\frac{1}{\theta}s^{\frac{-(1+\theta)}{\theta}}\right)^{\theta-1}\right)'\big(s^{\frac{-1}{\theta}}\big)^{1-\theta}$$

$$= \left(\frac{1}{\theta}\right)^{\theta-1} \left(s^{\frac{1}{\theta}}\right)' \left(s^{\frac{-1}{\theta}}\right)^{1-\theta} = \left(\frac{1}{\theta}\right)^{\theta} s^{\frac{1-\theta}{\theta}} \left(s^{\frac{-1}{\theta}}\right)^{1-\theta}$$
$$= \left(\frac{1}{\theta}\right)^{\theta} s^{\frac{1-\theta+\theta-1}{\theta}} = \left(\frac{1}{\theta}\right)^{\theta}.$$

Now, by applying Remark 3.6, we have

$$\int_\alpha^\beta \left(\int_s^\beta u(\tau)\,d\tau\right)^\theta ds \le \theta^\theta \int_\alpha^\beta \left(su(s)\right)^\theta ds,$$

which is the Hardy inequality (6) with a best constant θ^θ.

Remark 3.8 When $\mathbb{T} = \mathbb{N}$, we have from Theorem 3.5 the inequality

$$\sum_{k=1}^{N} \omega(k) \left(\sum_{i=k}^{N} u(i)\right)^\theta \le \sum_{k=1}^{N} \gamma(k) u^\theta(k),$$

where u is a positive summable sequence over any finite interval $(1,k)$,and the sequences γ and ω satisfy the difference equation

$$\Delta\left(\gamma(n)\left(-\Delta\mathcal{S}(n)\right)^{\theta-1}\right) - \omega(n)\left(\mathcal{S}(n+1)\right)^{\theta-1} = 0, \quad \text{for } n \in [1,N]_{\mathbb{N}}, \tag{77}$$

where $\mathcal{S}(n) > 0, \Delta\mathcal{S}(n) < 0$.

Example 3.7 Let $\mathbb{T} = \mathbb{N}$ with $N \in \mathbb{N}$, and $1 < N \le \infty$ and assume that $\gamma(\varrho) = (\varrho+1)^{\theta-\gamma}$, and $\mathcal{S}(\varrho) = \varrho^{\frac{\gamma-1}{\theta}}$, where $\theta > 1$, and $1 > \gamma \ge 0$ is an arbitrary constant. Then $\gamma, \mathcal{S} > 0$, and by using the inequality

$$x^\alpha - y^\alpha \le \alpha x^{\alpha-1}(x-y), \quad \text{for } x \ge y > 0 \ (\alpha \ge 1 \text{ or } \alpha < 0),$$

with $\alpha = (\gamma-1)/\theta < 0$, we have

$$\Delta\mathcal{S}(\varrho) = (\varrho+1)^{\frac{\gamma-1}{\theta}} - \varrho^{\frac{\gamma-1}{\theta}} \le \frac{\gamma-1}{\theta}(\varrho+1)^{\frac{\gamma-1-\theta}{\theta}} \le 0.$$

This implies that

$$\gamma(\varrho)\left(-\Delta\mathcal{S}(\varrho)\right)^{\theta-1} \ge (\varrho+1)^{\theta-\gamma}\left(\frac{1-\gamma}{\theta}(\varrho+1)^{\frac{\gamma-1-\theta}{\theta}}\right)^{\theta-1}$$
$$= \left(\frac{1-\gamma}{\theta}\right)^{\theta-1} (\varrho+1)^{\theta-\gamma}\left((\varrho+1)^{\frac{\gamma-1-\theta}{\theta}}\right)^{\theta-1}$$
$$= \left(\frac{1-\gamma}{\theta}\right)^{\theta-1} (\varrho+1)^{\frac{1-\gamma}{\theta}}$$

and

$$\Delta\left[\gamma(\varrho)\left(-\Delta\mathcal{S}(\varrho)\right)^{\theta-1}\right] \ge \Delta\left(\frac{1-\gamma}{\theta}\right)^{\theta-1} (\varrho+1)^{\frac{1-\gamma}{\theta}}$$

$$= \left(\frac{1-\gamma}{\theta}\right)^{\theta-1} \Delta(\varrho+1)^{\frac{1-\gamma}{\theta}}.$$

By using the inequality

$$x^{\alpha} - y^{\alpha} \geq \alpha x^{\alpha-1}(x-y), \quad \text{for } x \geq y > 0 \text{ and } (0 < \alpha < 1),$$

with $\alpha = (1-\gamma)/\theta < 1$, we have

$$\begin{aligned} \Delta\left[\gamma(\varrho)\left(-\Delta\mathcal{S}(\varrho)\right)^{\theta-1}\right] &\geq \left(\frac{1-\gamma}{\theta}\right)^{\theta-1} \Delta(\varrho+1)^{\frac{1-\gamma}{\theta}} \\ &= \left(\frac{1-\gamma}{\theta}\right)^{\theta-1} \left((\varrho+2)^{\frac{1-\gamma}{\theta}} - (\varrho+1)^{\frac{1-\gamma}{\theta}}\right) \\ &\geq \left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+2)^{\frac{1-\gamma-\theta}{\theta}}. \end{aligned}$$

From (77), we have

$$\begin{aligned} \omega(\varrho) &= \Delta\left[\gamma(\varrho)\left(-\Delta\mathcal{S}(\varrho)\right)^{\theta-1}\right]\mathcal{S}^{1-\theta}(\varrho+1) \\ &\geq \left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+2)^{\frac{1-\gamma-\theta}{\theta}} \left((\varrho+1)^{\frac{\gamma-1}{\theta}}\right)^{1-\theta} \\ &= \left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+2)^{\frac{1-\gamma-\theta}{\theta}} (\varrho+1)^{\frac{\gamma-1-\theta\gamma+\theta}{\theta}} \\ &= \left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+2)^{\frac{1-\gamma-\theta}{\theta}} (\varrho+1)^{\frac{-(1-\gamma-\theta)}{\theta}} (\varrho+1)^{-\gamma} \\ &= A(\varrho)\left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+1)^{-\gamma}, \end{aligned}$$

where

$$A(\varrho) = \left(\frac{\varrho+2}{\varrho+1}\right)^{\frac{1-\gamma-\theta}{\theta}} < 1.$$

Now, by applying Remark 3.8, we have

$$\sum_{\varrho=0}^{N} A(\varrho)\left(\frac{1-\gamma}{\theta}\right)^{\theta} (\varrho+1)^{-\gamma} \left(\sum_{i=\varrho}^{N} u(i)\right)^{\theta} \leq \sum_{\varrho=0}^{N} (\varrho+1)^{\theta-\gamma} u^{\theta}(\varrho),$$

That is,

$$\sum_{\varrho=0}^{N} \frac{A(\varrho)}{(\varrho+1)^{\gamma}} \left(\sum_{i=\varrho}^{N} u(i)\right)^{\theta} \leq \left(\frac{\theta}{1-\gamma}\right)^{\theta} \sum_{\varrho=0}^{N} (\varrho+1)^{\theta-\gamma} u^{\theta}(\varrho),$$

which is the discrete Hardy–Littlewood inequality with a sharp constant $(\theta/(1-\gamma))^{\theta}$.

Acknowledgements
This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Funding
This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

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Publisher's Note

Received: 14 February 2020 Accepted: 24 September 2020 Published online: 02 October 2020

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