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On multivalued L-contractions and an application

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Abstract

The aim of this paper is to present several fixed-point results for L-contractive multivalued mappings involving θ-functions in the class of metric spaces. We also give some examples in support of the related concepts and presented results. A homotopy result is also provided.

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1 Introduction

Fifty years ago, Nadler [1] introduced the idea of multivalued contraction mappings and presented his famous result, which generalized the Banach contraction principle [2] for multivalued mappings. In [3], the authors studied a problem of a global optimization using a common best proximity point of a pair of multivalued mappings. Also, Debnath and Srivastava [4] introduced a new and proper extension of Kannan's fixed point theorem to the case of multivalued maps using Wardowski's F-contraction. Going in same direction, several research works in fixed point theory related to multivalued contractions in different areas have appeared. For more details, see [4–23].

Let (X,d) be a metric space and denote by $CB(X)$ the family of nonempty, bounded, and closed subsets of X. For $\Lambda_1,\Lambda_2 \in CB(X)$, define $\mathcal{H} : CB(X) \times CB(X) \to [0,\infty)$ by

$$\mathcal{H}(\Lambda_1,\Lambda_2) = \max\Bigl\{\sup_{a\in\Lambda_1} d(a,\Lambda_2), \sup_{b\in\Lambda_2} d(b,\Lambda_1)\Bigr\},$$

where $d(a,\Lambda_2) = \inf\{d(a,\rho) : \rho \in \Lambda_2\}$. Such a function $\mathcal{H}$ is called the Hausdorff–Pompieu metric induced by the metric d. Also, denote by $CL(X)$ the family of nonempty and closed subsets of X and by $K(X)$ the family of nonempty and compact subsets of X.

On the other hand, a new type of a contraction mapping, known as an θ-contraction, was introduced by Jleli and Samet [24].

Definition 1.1 ([24]) Let (X,d) be a metric space. A map $T:X\to X$ is said to be a θ-contraction whenever there are $k\in(0,1)$ and $\theta\in\Theta$ such that

$$\theta\bigl(d(T\varsigma,T\tau)\bigr)\le\bigl[\theta\bigl(d(\varsigma,\tau)\bigr)\bigr]^k,$$

for all $\varsigma,\tau\in X$ with $d(T\varsigma,T\tau)>0$, where Θ is the set of functions $\theta:(0,\infty)\to(1,\infty)$ verifying the following conditions:

($\theta1$) θ is nondecreasing;

($\theta2$) for each positive sequence $\{t_n\}$, $\lim_{n\to\infty}\theta(t_n)=1$ iff $\lim_{n\to\infty}t_n=0$;

($\theta3$) there are $\nu\in(0,\infty]$ and $\mu\in(0,1)$ such that $\lim_{t\to0^+}\frac{\theta(t)-1}{t^\mu}=\nu$.

Some years later, Vetro [23] introduced multivalued θ-contraction mappings and gave the multivalued version of the main results of Jleli and Samet [24].

Definition 1.2 ([23]) A map $T:X\to CL(X)$ is said to be a weak θ-contraction if there are $k\in(0,1)$ and $\theta\in\Theta$ such that

$$\theta\bigl(\mathcal{H}(T\varsigma,T\tau)\bigr)\le\bigl[\theta\bigl(d(\varsigma,\tau)\bigr)\bigr]^k$$

for all $\varsigma,\tau\in X$ with $\mathcal{H}(T\varsigma,T\tau)>0$.

Theorem 1.3 ([23]) *Let (X,d) be a complete metric space and $T:X\to K(X)$ be a weak θ-contraction mapping. Then T admits a fixed point.*

Later in 2017, Ahmad et al. [25] replaced condition ($\theta3$) by the continuity condition of θ on $(0,\infty)$.

Example 1.4 ([25]) Note that ($\theta3$) and the continuity condition on θ are independent. Indeed, for $s>1$, the function given by $\theta(u)=e^{u^s}$ satisfies conditions ($\theta1$) and ($\theta2$), but it does not satisfy ($\theta3$), while it is continuous. On the other hand, for all $s>1$ and $t\in(0,\frac{1}{s})$, the function defined by $\theta(u)=1+u^t(1+[u])$, where $[u]$ denotes the integer part of u, satisfies conditions ($\theta1$), ($\theta2$), and ($\theta3$) for each $k\in(\frac{1}{s},1)$, but it is not continuous. Also, the function $\theta(u)=e^{\sqrt{u}}$ is continuous and satisfies conditions ($\theta1$), ($\theta2$), and ($\theta3$).

Now, denote by Θ^* the set of continuous functions satisfying ($\theta1$) and ($\theta2$). We observe that $\Theta\nsubseteq\Theta^*$, $\Theta^*\nsubseteq\Theta$, and $\Theta\cap\Theta^*\neq\emptyset$.

In [25], the authors gave the following fixed point theorem, extending the results of Jleli and Samet [24]. In [25] they considered $\theta\in\Theta^*$ instead of $\theta\in\Theta$.

Theorem 1.5 ([25]) *Any θ-contraction mapping $T:X\to X$ (with $\theta\in\Theta^*$) on a complete metric space (X,d) possesses a unique fixed point.*

Recently, Cho [26] initiated the concept L-simulation functions. Let $\xi:[1,\infty)^2\to\mathbb{R}$ verify the following assertions:

(ξ_1) $\xi(1,1)=1$;

(ξ_2) $\xi(\tau,\upsilon)<\frac{\upsilon}{\tau}$ for all $\upsilon,\tau>1$;

(ξ_3) for all sequences $\{\tau_n\}$ and $\{\upsilon_n\}$ in $(1,\infty)$ with $\tau_n < \upsilon_n$ for $n = 1,2,3,\ldots$,

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \upsilon_n > 1 \quad \Longrightarrow \quad \limsup_{n\to\infty} \xi(\tau_n, \upsilon_n) < 1.$$

Any $\xi \in \mathcal{L}$ is said an L-simulation function. Here, $\xi(\iota,\iota) < 1$ for each $\iota > 1$. As examples of L-simulation functions, we state the following:

Example 1.6 ([26]) $\xi(t,s) = \frac{s^k}{t}$ for all $s,t \geq 1$, where $k \in (0,1)$.

Example 1.7 ([26]) $\xi(t,s) = \frac{s}{t\varphi(s)}$ for all $s,t \geq 1$ where $\varphi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{1\}) = \{1\}$.

Example 1.8 ([26])

$$\xi(\upsilon,\eta) = \begin{cases} 1 & \text{if } (\upsilon,\eta) = (1,1), \\ \frac{\upsilon}{2\eta} & \text{if } \upsilon < \eta, \\ \frac{\upsilon^\nu}{\eta} & \text{otherwise,} \end{cases}$$

for all $\upsilon,\eta \geq 1$ where $\nu \in (0,1)$.

This class of L-simulation functions is used as control functions in order to enrich the fixed point theory when dealing with several types of contraction mappings in variant (generalized) metric spaces. In this paper, based on L-simulation functions, we define a new type of multivalued contraction mappings, called multivalued L-contraction mappings via θ-functions. We give some related fixed point results in the context of complete metric spaces. Some consequences are also derived. Moreover, an example to support our results is given. At the end, as an application, a homotopy result is provided.

2 Main results

Now, we introduce the definition of multivalued L-contraction mappings via θ-functions.

Definition 2.1 Let (X,d) be a metric space. A multivalued mapping $T : X \to CB(X)$ is called an L-contraction with respect to ξ whenever there are $\theta \in \Theta^*$ and $\xi \in \mathcal{L}$ such that

$$\xi\big(\theta\big(\mathcal{H}(T\varsigma, T\tau)\big), \theta\big(d(\varsigma,\tau)\big)\big) \geq 1, \tag{2.1}$$

for all $\varsigma,\tau \in X$ with $\mathcal{H}(T\varsigma, T\tau) > 0$.

Lemma 2.2 *If $T : X \to CB(X)$ is an L-contraction with respect to ξ, then T is continuous.*

Proof Let $\upsilon \in X$ and $\{\sigma_n\} \subset X$ be such that

$$\lim_{n\to\infty} d(\sigma_n, \upsilon) = 0, \quad \text{and} \quad \mathcal{H}(T\sigma_n, T\upsilon) > 0, \quad n \geq 0.$$

Since T satisfies condition (2.1), we have

$$1 \leq \xi\big(\theta\big(\mathcal{H}(T\sigma_n, Tx)\big), \theta\big(d(\sigma_n,\upsilon)\big)\big) < \frac{\theta(d(\sigma_n,\upsilon))}{\theta(\mathcal{H}(T\sigma_n, T\upsilon))}, \quad \forall n = 0,1,2,\ldots.$$

This implies that

$$\theta\big(\mathcal{H}(T\sigma_n, T\upsilon)\big) < \theta\big(d(\sigma_n, \upsilon)\big), \quad \forall n = 0, 1, 2, \ldots.$$

From $(\theta 1)$, one gets

$$\mathcal{H}(T\sigma_n, T\upsilon) < d(\sigma_n, \upsilon), \quad \forall n = 0, 1, 2, \ldots.$$

As $n \to \infty$, we find

$$\lim_{n\to\infty} \mathcal{H}(T\sigma_n, T\upsilon) = 0.$$

This proves that T is continuous. □

Theorem 2.3 *Any L-contraction mapping $T : X \to CB(X)$ with respect to ξ on a complete metric space (X, d) possesses a fixed point.*

Proof Let $\sigma_0 \in X$ and $\sigma_1 \in T\sigma_0$. If $\sigma_0 = \sigma_1$, then σ_0 is a fixed point. If $\sigma_1 \in T\sigma_1$, σ_1 is a fixed point. So, assume that $\sigma_0 \neq \sigma_1$ and $\sigma_1 \notin T\sigma_1$. We have

$$0 < d(\sigma_1, T\sigma_1) \leq \mathcal{H}(T\sigma_0, T\sigma_1). \tag{2.2}$$

Since T satisfies condition (2.1) and $\mathcal{H}(T\sigma_0, T\sigma_1) > 0$, we have

$$1 \leq \xi\big(\theta\big(\mathcal{H}(T\sigma_0, T\sigma_1)\big), \theta\big(d(\sigma_0, \sigma_1)\big)\big) < \frac{\theta(d(\sigma_0, \sigma_1))}{\theta(\mathcal{H}(T\sigma_0, T\sigma_1))}.$$

This implies that

$$\theta\big(\mathcal{H}(T\sigma_0, T\sigma_1)\big) < \theta\big(d(\sigma_0, \sigma_1)\big).$$

Using $(\theta 1)$, we get

$$\mathcal{H}(T\sigma_0, T\sigma_1) < d(\sigma_0, \sigma_1). \tag{2.3}$$

Since θ is continuous and nondecreasing, one has

$$\inf_{y\in T\sigma_1} \theta\big(d(\sigma_1, y)\big) = \theta\big(d(\sigma_1, T\sigma_1)\big) \leq \theta\big(\mathcal{H}(T\sigma_0, T\sigma_1)\big).$$

Hence, there exists $\sigma_2 \in T\sigma_1$ such that

$$\theta\big(d(\sigma_1, \sigma_2)\big) \leq \theta\big(\mathcal{H}(T\sigma_0, T\sigma_1)\big). \tag{2.4}$$

By using $(\theta 1)$, (2.3), and (2.4), we obtain

$$d(\sigma_1, \sigma_2) \leq \mathcal{H}(T\sigma_0, T\sigma_1) < d(\sigma_0, \sigma_1).$$

If $\sigma_2 \in T\sigma_2$, then σ_2 is a fixed point. Otherwise, proceeding similarly, there is $\sigma_3 \in T\sigma_2$ such that

$$d(\sigma_2,\sigma_3) \le \mathcal{H}(T\sigma_1, T\sigma_2) < d(\sigma_1,\sigma_2).$$

Similarly,

$$0 < d(\sigma_n,\sigma_{n+1}) \le \mathcal{H}(T\sigma_{n-1}, T\sigma_n) < d(\sigma_{n-1},\sigma_n), \quad \text{for all } n = 1,2,\dots. \tag{2.5}$$

Hence $\{d(\sigma_{n-1},\sigma_n)\}$ is decreasing, and so there is $r \ge 0$ such that

$$\lim_{n\to\infty} d(\sigma_{n-1},\sigma_n) = r.$$

By (2.5), we have

$$r = \lim_{n\to\infty} \mathcal{H}(T\sigma_{n-1}, T\sigma_n) = \lim_{n\to\infty} d(\sigma_{n-1},\sigma_n). \tag{2.6}$$

Assume that $r > 0$. Using $(\theta 1)$ and the continuity of θ, we get

$$\lim_{n\to\infty} \theta\big(\mathcal{H}(T\sigma_{n-1}, T\sigma_n)\big) = \lim_{n\to\infty} \theta\big(d(\sigma_{n-1},\sigma_n)\big) = \theta(r) > 1.$$

Due to $(\xi 3)$, we have

$$1 \le \limsup_{n\to\infty} \xi\Big(\theta\big(\mathcal{H}(T\sigma_{n-1}, T\sigma_n)\big), \theta\big(d(\sigma_{n-1},\sigma_n)\big)\Big) < 1,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(\sigma_n,\sigma_{n+1}) = 0. \tag{2.7}$$

Now, we will show that $\{\sigma_i\}$ is bounded in (X,d).

If it is not the case, then there is a subsequence $\{\sigma_{i(q)}\}$ of $\{\sigma_i\}$ such that for $i(1) = 1$ and for all $q = 1,2,\dots$, we have that $i(q+1)$ is the minimum integer greater than $i(q)$ with

$$d(\sigma_{i(q+1)},\sigma_{i(q)}) > 1 \quad \text{and} \quad d(\sigma_{i(q)},\sigma_l) \le 1, \tag{2.8}$$

for all $i(q) \le l \le i(q+1) - 1$. We have

$$\begin{aligned} 1 &< d(\sigma_{i(q+1)},\sigma_{i(q)}) \le d(\sigma_{i(q+1)},\sigma_{i(q+1)-1}) + d(\sigma_{i(q+1)-1},\sigma_{i(q)}) \\ &\le d(\sigma_{i(q+1)},\sigma_{i(q+1)-1}) + 1. \end{aligned}$$

Letting $q \to \infty$ and using (2.7), we get

$$\lim_{q\to\infty} d(\sigma_{i(q+1)},\sigma_{i(q)}) = 1. \tag{2.9}$$

Also,

$$d(\sigma_{i(q+1)-1},\sigma_{i(q)-1}) \le d(\sigma_{i(q+1)-1},\sigma_{i(q+1)}) + d(\sigma_{i(q+1)},\sigma_{i(q)}) + d(\sigma_{i(q)},\sigma_{i(q)-1}).$$

By taking the limit as $q \to \infty$, from (2.7) and (2.9), we have

$$\lim_{q\to\infty} d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1}) \le 1. \tag{2.10}$$

Since

$$\begin{aligned} 1 &= \lim_{q\to\infty} d(\sigma_{i(q+1)}, \sigma_{i(q)}) \le \lim_{q\to\infty} \bigl[d(\sigma_{i(q+1)}, \sigma_{i(q+1)-1}) + d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1}) + d(\sigma_{i(q)-1}, \sigma_{i(q)})\bigr] \\ &\le \lim_{q\to\infty} d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1}) \le 1, \end{aligned}$$

we obtain

$$\lim_{q\to\infty} d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1}) = 1. \tag{2.11}$$

Again,

$$1 < d(\sigma_{i(q)}, \sigma_{i(q+1)}) \le \mathcal{H}(T\sigma_{i(q)-1}, T\sigma_{i(q+1)-1}), \tag{2.12}$$

so

$$0 < \mathcal{H}(T\sigma_{i(q)-1}, T\sigma_{i(q+1)-1}).$$

Hence, using condition (2.1), we get

$$1 \le \xi\bigl(\theta\bigl(\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)-1})\bigr), \theta\bigl(d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1})\bigr)\bigr) < \frac{d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1})}{\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)-1}))}.$$

One then gets

$$\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)} - 1) < d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1}). \tag{2.13}$$

By the continuity of θ, we have

$$\begin{aligned} \theta\bigl(d(\sigma_{i(q+1)}, T\sigma_{i(q)-1})\bigr) &= \inf_{\sigma_{i(q)} \in T\sigma_{i(q)-1}} \theta\bigl(d(\sigma_{i(q+1)}, \sigma_{i(q)})\bigr) \le \theta\bigl(\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)-1})\bigr) \\ &< \theta\bigl(d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1})\bigr). \end{aligned}$$

By taking the limit as $q \to \infty$, we obtain

$$\lim_{q\to\infty} \theta\bigl(\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)-1})\bigr) = \lim_{q\to\infty} \theta\bigl(d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1})\bigr) = \theta(1) > 1.$$

Due to ($\xi 3$), we get

$$1 \le \limsup_{q\to\infty} \xi\bigl(\theta\bigl(\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{i(q)-1})\bigr), \theta\bigl(d(\sigma_{i(q+1)-1}, \sigma_{i(q)-1})\bigr)\bigr) < 1,$$

which is a contradiction. Hence, $\{\sigma_n\}$ is bounded.

We shall show the Cauchy property of $\{\sigma_n\}$. Let

$$M_i = \sup\{d(\sigma_k,\sigma_m) : k,m \geq i\}. \tag{2.14}$$

It is clear that $0 \leq M_{i+1} \leq M_i < \infty$ for every $i = 1,2,3,\ldots$. Thus, there is $M \geq 0$ such that $\lim_{i\to\infty} M_i = M$. Assume that $M > 0$. Using (2.14), there exist $i(q), j(q) \geq q$ such that

$$M_q - \frac{1}{q} \leq d(\sigma_{i(q)},\sigma_{j(q)}) \leq M_q.$$

Then

$$\lim_{q\to\infty} d(\sigma_{i(q)},\sigma_{j(q)}) = M.$$

By the triangle inequality, we have

$$d(\sigma_{i(q)},\sigma_{j(q)}) \leq d(\sigma_{i(q)},\sigma_{i(q)-1}) + d(\sigma_{i(q)-1},\sigma_{j(q)-1}) + d(\sigma_{j(q)-1},\sigma_{j(q)}).$$

As $q \to \infty$, we find that

$$\lim_{q\to\infty} d(\sigma_{i(q)-1},\sigma_{j(q)-1}) \geq M. \tag{2.15}$$

Also,

$$\lim_{q\to\infty} d(\sigma_{i(q)-1},\sigma_{j(q)-1}) \leq \lim_{q\to\infty}\bigl[d(\sigma_{i(q)-1},\sigma_{i(q)}) + d(\sigma_{i(q)},\sigma_{j(q)}) + d(\sigma_{j(q)},\sigma_{j(q)-1})\bigr] \leq M. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\lim_{q\to\infty} d(\sigma_{i(q)-1},\sigma_{j(q)-1}) = \lim_{q\to\infty} d(\sigma_{i(q)},\sigma_{j(q)}) = M.$$

Due to the continuity of θ, we obtain

$$\theta\bigl(d(\sigma_{i(q)},T\sigma_{j(q)-1})\bigr) = \inf_{\sigma_{j(q)}\in T\sigma_{j(q)-1}} \theta\bigl(d(\sigma_{i(q)},\sigma_{j(q)})\bigr) \leq \theta\bigl(\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1})\bigr).$$

Letting $q \to \infty$, we have

$$\theta(M) \leq \theta\Bigl(\lim_{q\to\infty}\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1})\Bigr).$$

Applying (θ1), one gets

$$0 < M \leq \lim_{q\to\infty}\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1}).$$

This implies that $\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1}) > 0$. So by using condition (2.1), we have

$$1 \leq \xi\bigl(\theta\bigl(\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1})\bigr),\theta\bigl(d(\sigma_{i(q)-1},\sigma_{j(q)-1})\bigr)\bigr) < \frac{d(\sigma_{i(q)-1},\sigma_{j(q)-1})}{\mathcal{H}(T\sigma_{i(q)-1},T\sigma_{j(q)-1})}.$$

Hence,

$$\theta\big(\mathcal{H}(T\sigma_{i(q+1)-1}, T\sigma_{j(q)-1})\big) < \theta\big(d(\sigma_{i(q)-1}, \sigma_{j(q)-1})\big).$$

Letting $q \to \infty$ and using (θ1), we have

$$\lim_{q\to\infty} \mathcal{H}(T\sigma_{i(q)-1}, T\sigma_{j(q)-1}) = \lim_{q\to\infty} d(\sigma_{i(q)-1}, \sigma_{j(q)-1}) = M. \tag{2.17}$$

In view of (ξ3), one gets

$$1 \le \limsup_{q\to\infty} \xi\big(\theta\big(\mathcal{H}(T\sigma_{i(q)-1}, T\sigma_{j(q)-1})\big), \theta\big(d(\sigma_{i(q)-1}, \sigma_{j(q)-1})\big)\big) < 1,$$

which is a contradiction.

Thus, $\lim_{q\to\infty} d(\sigma_{i(q)}, \sigma_{j(q)}) = 0$, and hence $\{\sigma_i\}$ is a Cauchy sequence in the complete metric space X. Therefore, there is $y \in X$ such that

$$\lim_{i\to\infty} d(\sigma_i, y) = 0.$$

By Lemma 2.2, T is continuous, and so

$$0 \le d(y, Ty) = \lim_{i\to\infty} d(\sigma_i, Ty) \le \lim_{i\to\infty} \mathcal{H}(T\sigma_{i-1}, Ty) = \mathcal{H}(Ty, Ty) = 0.$$

This implies that y is a fixed point of T. □

We state some corollaries.

Corollary 2.4 *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a given mapping such that for all $\varsigma, \tau \in X$ with $\mathcal{H}(T\varsigma, T\tau) \neq 0$,*

$$\mathcal{H}(T\varsigma, T\tau) \le d(\varsigma, \tau) - \varphi\big(d(\varsigma, \tau)\big), \tag{2.18}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is lower semicontinuous and nondecreasing such that $\varphi^{-1}(\{0\}) = \{0\}$. Then T admits a unique fixed point.

Proof From condition (2.18), we have

$$e^{\mathcal{H}(T\varsigma, T\tau)} \le e^{d(\varsigma,\tau)-\varphi(d(\varsigma,\tau))}.$$

Putting $\theta(t) = e^t$, we get

$$\theta\big(\mathcal{H}(T\varsigma, T\tau)\big) \le \frac{\theta(d(\varsigma, \tau))}{e^{\varphi(d(\varsigma,\tau))}}.$$

Also, define $\varphi(t) = \ln(\psi(\theta(t)))$, where $\psi : [1, \infty) \to [1, \infty)$ is lower semicontinuous and nondecreasing such that $\psi^{-1}(\{1\}) = \{1\}$. We get

$$\theta\big(\mathcal{H}(T\varsigma, T\tau)\big) \le \frac{\theta(d(\varsigma, \tau))}{\psi(\theta(d(\varsigma, \tau)))}.$$

By putting $\xi(t,s) = \frac{s}{t\psi(s)}$, we get

$$1 \leq \frac{\theta(d(\varsigma,\tau))}{\theta(\mathcal{H}(T\varsigma,T\tau))\psi(\theta(d(\varsigma,\tau)))} = \xi\bigl(\theta\bigl(\mathcal{H}(T\varsigma,T\tau)\bigr),\theta\bigl(d(\varsigma,\tau)\bigr)\bigr).$$

In view of Theorem 2.3, T has a unique fixed point. □

Now, take in Theorem 2.3 the function $\xi(\upsilon,\eta) = \frac{\upsilon^{\kappa}}{\eta}$ for all $\upsilon,\eta \geq 1$, where $\kappa \in (0,1)$.

Corollary 2.5 *Let (X,d) be a complete metric space and $T : X \to K(X)$ be such that for all $\varsigma,\tau \in X$ with $\mathcal{H}(T\varsigma,T\tau) \neq 0$,*

$$\theta\bigl(\mathcal{H}(T\varsigma,T\tau)\bigr) \leq \bigl[\theta\bigl(d(\varsigma,\tau)\bigr)\bigr]^{k}, \tag{2.19}$$

where $\theta \in \Theta^*$. Then T possesses a unique fixed point.

Corollary 2.6 *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be such that for all $\varsigma,\tau \in X$ with $\mathcal{H}(T\varsigma,T\tau) \neq 0$,*

$$\theta\bigl(\mathcal{H}(T\varsigma,T\tau)\bigr) \leq \bigl[\theta\bigl(d(\varsigma,\tau)\bigr)\bigr]^{k}, \tag{2.20}$$

where $\theta \in \Theta^*$ satisfies $(\theta 4)$. Then T admits a unique fixed point.

Remark 2.7 Corollary 2.4 is the multivalued version of Theorem 3.2 in [24], and improves it by replacing the compact range condition by the closed and bounded range, and by considering that φ as lower semicontinuous and not necessary continuous. Corollary 2.5 is the multivalued version of Theorem 1.5. Also, Corollary 2.6 is an extension of Theorem 2.5 in [23] without condition (θ_3).

The following example supports Theorem 2.3. Here, the main result of Vetro [23] is not applicable.

Example 2.8 Let $X = \{0,1,2,4\}$ be a metric space endowed with the metric $d(\varsigma,\tau) = |\varsigma - \tau|$ for all $\varsigma,\tau \in X$. Consider the mapping $T : X \to K(X)$ given by

$$T\varsigma = \begin{cases} \{0\}, & \varsigma = 4, \\ \{0,2\}, & \varsigma \neq 4. \end{cases}$$

Take $\theta(t) = e^{t}$ for each $t > 0$. Choose $\xi(t,s) = \frac{s}{t\phi(s)}$ for all $t,s \geq 1$, where

$$\phi(s) = \begin{cases} 1 & \text{if } s \leq e^2, \\ \sqrt[3]{s} & \text{if } s > e^2. \end{cases}$$

We shall prove that T is a multivalued L-contraction with respect to such ξ. We notice that $\mathcal{H}(T\varsigma,T\tau) > 0$ iff $\varsigma = 4$ and $\tau \neq 4$. In this case, we have $\theta(\mathcal{H}(T\varsigma,T\tau)) = \theta(\mathcal{H}(\{0\},\{0,2\})) = e^2$. We need the following:

Case 1. If $\varsigma = 4$ and $\tau = 0$, then $\theta(d(4,0)) = e^4$ and hence

$$\xi\big(\theta\big(\mathcal{H}(T4,T0)\big),\theta\big(d(4,0)\big)\big) = \frac{e^4}{e^{\frac{10}{3}}} = e^{\frac{2}{3}} > 1.$$

Case 2. If $\varsigma = 4$ and $\tau = 1$, then $\theta(d(4,1)) = e^3$. So,

$$\xi\big(\theta\big(\mathcal{H}(T4,T1)\big),\theta\big(d(4,1)\big)\big) = \frac{e^3}{e^2\sqrt[3]{e^3}} = 1.$$

Case 3. If $\varsigma = 4$ and $\tau = 2$, then $\theta(d(4,2)) = e^2$ and so

$$\xi\big(\theta\big(\mathcal{H}(T4,T2)\big),\theta\big(d(4,2)\big)\big) = \frac{e^2}{e^2(1)} = 1.$$

Then T is a multivalued L-contraction with respect to ξ. Hence all the conditions of Theorem 2.3 hold. Here, T admits a fixed point.

Note that for all $\theta \in \Theta$, we have

$$\theta\big(\mathcal{H}(T4,T2)\big) = \theta(2) > \big[\theta\big(d(4,2)\big)\big]^k = \big[\theta(2)\big]^k \quad \text{for all } \kappa \in [0,1).$$

That is, T is not an θ-contraction of Vetro [23].

3 Application

Now, we present a homotopy result as an application of Corollary 2.4.

Theorem 3.1 *Let (X,d) be a complete metric space, $E \subset X$ be a nonempty open set and let $D \subset X$ be a closed set with $E \subset D$. Also, let $T : D \times [0,1] \to CB(X)$ verify condition (2.18) such that*

(i) $s \notin T(s,u)$, for all $s \in D \setminus E$, $u \in [0,1]$;

(ii) there is a continuous function $\gamma : [0,1] \to \mathbb{R}$ such that for all $u, v \in [0,1]$ and $s \in D$,

$$\mathcal{H}\big(T(s,u),T(s,v)\big) \leq \lambda\big|\gamma(u) - \gamma(v)\big|, \quad \textit{where } \lambda \in (0,1);$$

(iii) if $s \in T(s,u)$, then $T(s,u) = \{s\}$.

Then $T(\cdot,0)$ possesses a fixed point iff $T(\cdot,1)$ possesses a fixed point.

Proof Let $A = \{u \in [0,1]; s \in T(s,u)$, for some $s \in E\}$. Since $T(\cdot,0)$ has a fixed point and from condition (i), we have $0 \in A$, therefore A is nonempty. We claim that A is both open and closed in $[0,1]$, then by the connectedness of $[0,1]$, the proof is completed.

First, we show that A is open in $[0,1]$. Let $u_0 \in A$, then there is $s_0 \in E$ with $s_0 \in T(s_0,u_0)$.

Since E is open in (X,d), there is $r > 0$ such that $B(s_0,r) \subset E$. Take $\epsilon = \frac{\varphi(r)}{\lambda} > 0$, where φ is given in Corollary 2.4. Using the continuity of γ at u_0, there is $\delta(\epsilon) > 0$ such that $|\gamma(u) - \gamma(u_0)| < \epsilon$, for all $u \in (u_0 - \delta, u_0 + \delta)$.

Let $u \in (u_0 - \delta, u_0 + \delta)$. For $s \in B[s_0,r] = \{s \in X; d(s,s_0) \leq r\}$, we get

$$\begin{aligned} d\big(T(s,u),s_0\big) &\leq \mathcal{H}\big(T(s,u),T(s_0,u_0)\big) \\ &\leq \mathcal{H}\big(T(s,u),T(s,u_0)\big) + \mathcal{H}\big(T(s,u_0),T(s_0,u_0)\big). \end{aligned} \tag{3.1}$$

If $\mathcal{H}(T(s,u_0),T(s_0,u_0)) = 0$, then using (iii),

$$d\big(T(s,u),s_0\big) \le \lambda\big|\gamma(u)-\gamma(u_0)\big| < \lambda\epsilon < r. \tag{3.2}$$

In the case that $\mathcal{H}(T(s,u_0),T(s_0,u_0)) > 0$ and since T satisfies the contraction condition (2.18), we obtain

$$\mathcal{H}\big(T(s,u_0),T(s_0,u_0)\big) \le d(s,s_0) - \varphi\big(d(s,s_0)\big). \tag{3.3}$$

By substituting in (3.1) from (3.3) and from (iii), we have

$$\begin{aligned} d\big(T(s,u),s_0\big) &\le \lambda\big|\gamma(u)-\gamma(u_0)\big| + d(s,s_0) - \varphi\big(d(s,s_0)\big) \\ &< \lambda\epsilon + r - \varphi(r) = r. \end{aligned} \tag{3.4}$$

Combining (3.2) and (3.4), and for all $u \in (u_0-\delta,u_0+\delta)$, the operator $T(\cdot,u) : B[s_0,r] \to CB(X)$ satisfies all hypotheses of Corollary 2.5. Hence, $T(\cdot,u_0)$ has a fixed point in $B[s_0,r] \subset D$, and since (i) holds, this fixed point has to lie in E, and $(u_0-\delta,u_0+\delta) \subset A$. Therefore, A is open.

Next, we prove that A is closed. Let $\{u_n\}$ be a sequence in A such that $\lim_{n\to\infty} u_n = \overline{u} \in [0,1]$. Since $\{u_n\} \subset A$, there is $\{s_n\} \subset E$ such that $s_n \in T(s_n,u_n)$. Then, by using condition (iii) and for all $m,n \in \mathbb{Z}^+$, we have

$$\begin{aligned} d(s_n,s_m) &\le \mathcal{H}\big(T(s_n,u_n),T(s_m,u_m)\big) \\ &\le \mathcal{H}\big(T(s_n,u_n),T(s_n,u_m)\big) + \mathcal{H}\big(T(s_n,u_m),T(s_m,u_m)\big). \end{aligned} \tag{3.5}$$

Case 1. If $\mathcal{H}(T(s_n,u_m),T(s_m,u_m)) = 0$ for some $m,n \in \mathbb{Z}^+$, then from condition (ii) we have

$$d(s_n,s_m) \le \lambda\big|\gamma(u_n)-\gamma(u_m)\big|. \tag{3.6}$$

Case 2. If $\mathcal{H}(T(s_n,u_m),T(s_m,u_m)) > 0$, then from the contraction condition (2.18), we have

$$\mathcal{H}\big(T(s_n,u_m),T(s_m,u_m)\big) \le d(s_n,s_m) - \varphi\big(d(s_n,s_m)\big). \tag{3.7}$$

By substituting (3.7) and condition (iii) in (3.5), we have

$$d(s_n,s_m) \le \lambda\big|\gamma(u_n)-\gamma(u_m)\big| + d(s_n,s_m) - \varphi\big(d(s_n,s_m)\big). \tag{3.8}$$

From (3.6) and (3.8), we have

$$d(s_n,s_m) \le \lambda\big|\gamma(u_n)-\gamma(u_m)\big| + d(s_n,s_m) - \varphi\big(d(s_n,s_m)\big),$$

for each $m,n \in \mathbb{Z}^+$. Hence

$$\varphi\big(d(s_n,s_m)\big) \le \lambda\big|\gamma(u_n)-\gamma(u_m)\big|.$$

By taking the limit as $n, m \to \infty$ and from the lower semicontinuity of φ and continuity of γ, we get

$$\lim_{n,m\to\infty} d(s_n, s_m) = 0.$$

Hence, the sequence $\{s_n\}$ is Cauchy in (X, d), which is complete, so there is $\bar{s} \in D$ with $\lim_{n\to\infty} d(s_n, \bar{s}) = 0$. We have

$$\begin{aligned} d\bigl(s_n, T(\bar{s}, \overline{u})\bigr) &\leq \mathcal{H}\bigl(T(s_n, u_n), T(\bar{s}, \overline{u})\bigr) \\ &\leq \mathcal{H}\bigl(T(s_n, u_n), T(s_n, \overline{u})\bigr) + \mathcal{H}\bigl(T(s_n, \overline{u}), T(\bar{s}, \overline{u})\bigr). \end{aligned} \tag{3.9}$$

Case 1. If $\mathcal{H}(T(s_n, \overline{u}), T(\bar{s}, \overline{u})) = 0$, for some n then from condition (ii) we have

$$d\bigl(s_n, T(\bar{s}, \overline{u})\bigr) \leq \lambda \bigl|\gamma(u_n) - \gamma(\overline{u})\bigr|. \tag{3.10}$$

Case 2. If $\mathcal{H}(T(s_n, \overline{u}), T(\bar{s}, \overline{u})) > 0$ for each n, then from the contraction condition (2.18), we have

$$\mathcal{H}\bigl(T(s_n, \overline{u}), T(\bar{s}, \overline{u})\bigr) \leq d(s_n, \bar{s}) - \varphi\bigl(d(s_n, \bar{s})\bigr). \tag{3.11}$$

By substituting from (3.11) and condition (iii) in (3.9), we have

$$d\bigl(s_n, T(\bar{s}, \overline{u})\bigr) \leq \lambda \bigl|\gamma(u_n) - \gamma(\overline{u})\bigr| + d(s_n, \bar{s}) - \varphi\bigl(d(s_n, \bar{s})\bigr). \tag{3.12}$$

By (3.10) and (3.12), we get that

$$d\bigl(s_n, T(\bar{s}, \overline{u})\bigr) \leq \lambda \bigl|\gamma(u_n) - \gamma(\overline{u})\bigr| + d(s_n, \bar{s}),$$

for each $n \in \mathbb{Z}^+$. By taking the limit as $n \to \infty$ and from continuity of γ, we obtain that

$$\lim_{n\to\infty} d\bigl(s_n, T(\bar{s}, \overline{u})\bigr) = d\bigl(\bar{s}, T(\bar{s}, \overline{u})\bigr) = 0. \tag{3.13}$$

We deduce that $\bar{s} \in T(\bar{s}, \overline{u})$, and by using condition (i), we have $\bar{s} \in E$. Thus, $\overline{u} \in A$, and so A is a closed subset of $[0, 1]$.

Similarly, we can deduce the reverse implication. □

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