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On Hyers–Ulam stability of a multi-order boundary value problems via Riemann–Liouville derivatives and integrals

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Abstract

In this research paper, we introduce a general structure of a fractional boundary value problem in which a 2-term fractional differential equation has a fractional bi-order setting of Riemann–Liouville type. Moreover, we consider the boundary conditions of the proposed problem as mixed Riemann–Liouville integro-derivative conditions with four different orders which cover many special cases studied before. In the first step, we investigate the existence and uniqueness of solutions for the given multi-order boundary value problem, and then the Hyers–Ulam stability is another notion in this regard which we study. Finally, we provide two illustrative examples to support our theoretical findings.

MSC: Primary 34A08; secondary 34A12

Keywords: Boundary value problem; Hyers–Ulam stability; Multi-order fractional differential equation; Riemann–Liouville derivative

1 Introduction and preliminaries

Fractional differential problems have drawn much interest in recent years owing to their extensive utilization in different branches of science such as engineering, mechanics, potential theory, biology, chemistry, etc. (see [1–11]). Many researchers helped in developments on the existence and uniqueness results of fractional differential equations [12–26]. Stability is a notion in physics, and most phenomena include the concept. In fact, stability of physical phenomena has an old history, and one can find a lot of works in the literature not only in the last century but also before it [27–36]. During recent decades, considerable attention has been given to the study of the Hyers–Ulam stability of functional differential and integral equations [37–54].

In 2016, Niyom et al. studied the boundary value problem via four-order fractional Riemann–Liouville derivatives

$$\begin{cases} \lambda \mathcal{D}^k(u(t)) + (1-\lambda)\mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) & (t \in [0, T], k \in (1, 2]), \\ u(0) = 0, \quad \mu_1 \mathcal{D}^{\gamma_1} u(T) + (1-\mu_1)\mathcal{D}^{\gamma_2} u(T) = \delta_1, \end{cases} \tag{1}$$

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under some conditions [55]. In 2017, Ntouyas et al. reviewed a boundary value problem via multiple orders of fractional derivatives and integrals

$$\begin{cases} \lambda \mathcal{D}^k(u(t)) + (1-\lambda)\mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) & (t \in [0, T], k \in (1, 2]), \\ u(0) = 0, \qquad \mu_2 \mathcal{I}^{q_1} u(T) + (1-\mu_2)\mathcal{I}^{q_2} u(T) = \delta_2, \end{cases} \tag{2}$$

under some conditions [15]. In 2018, Xu et al. investigated the existence of solutions and Hyers–Ulam stability for the fractional differential equations

$$\begin{cases} \lambda \mathcal{D}^k(u(t)) + \mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) & (t \in [0, T], k \in (1, 2]), \\ u(0) = 0, \qquad \mu_1 \mathcal{D}^{\gamma_1} u(T) + \mathcal{I}^{q_2} u(\nu) = \delta_2, \end{cases} \tag{3}$$

under some conditions [39]. They considered two-term class of three-point boundary value problems with Riemann–Liouville fractional derivatives and integrals [39].

Now, by using and mixing the idea of the above-mentioned works, we consider a new category of boundary value problem including multi-order Riemann–Liouville fractional equation supplemented with different linear integro-derivative boundary conditions as follows:

$$\begin{cases} \lambda \mathcal{D}^k(u(t)) + (1-\lambda)\mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) & (t \in [0, T], k \in [2, 3)), \\ u(0) = 0, \qquad \mu_1 \mathcal{D}^{\gamma_1} u(T) + (1-\mu_1)\mathcal{D}^{\gamma_2} u(T) = \delta_1, \\ \mu_2 \mathcal{I}^{q_1} u(T) + (1-\mu_2)\mathcal{I}^{q_2} u(T) = \delta_2, \end{cases} \tag{4}$$

where $2 < \theta < k$, $0 < \lambda$, $\mu_1, \mu_2 \le 1$, $0 \le \gamma_1, \gamma_2 < k - \theta$, $q_1, q_2 \in \mathbb{R}^+$, $\mathcal{D}^\beta$ is the Riemann–Liouville fractional derivative of order $\beta \in \{k, \theta, \gamma_1, \gamma_2\}$, $\mathcal{I}^\eta$ denotes the Riemann–Liouville fractional integral of order $\eta \in \{q_1, q_2\}$, and the map $\hat{\Upsilon} : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

As many researchers would like to investigate the stability notion of different boundary value problems, this can be a motivation for us to study the stability of complicated systems supplemented with general boundary conditions. Hence more precisely, our main goal in the present manuscript is to obtain some existence criteria of the solutions for a new general boundary value problem including 2-term fractional differential equation (4) which contains multi-order Riemann–Liouville fractional derivatives and integrals. To fulfil this aim, we use the well-known standard fixed point theorems. Also, in the sequel, we investigate the Hyers–Ulam stability of the proposed problem (4) in the special case $\mu_1 = 1$ and $\mu_2 = 1$. Finally, we present two illustrative examples to show the validity of our theoretical findings. We believe that such proposed boundary value problem is general, and it involves many fractional dynamical systems as special cases in physics and other applied sciences.

2 Preliminaries

Now, let us provide some basic notions. It is known that the Riemann–Liouville fractional integral of order η of a real-valued function $g : (0, \infty) \to \mathbb{R}$ is defined by $\mathcal{I}^\eta g(t) = \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g(s)\, ds$, provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function [1]. The Riemann–Liouville fractional derivative of order k of a func-

tion $g : (0, \infty) \to \mathbb{R}$ is defined by $\mathcal{D}^\beta g(t) = \frac{1}{\Gamma(n-\beta)} (\frac{d}{dt})^n \int_0^t \frac{g(s)}{(t-s)^{\beta-n+1}} \, ds$, where $n = [\beta] + 1$, $[\beta]$ denotes the integer part of real number β provided the right-hand side is point-wise defined on $(0, \infty)$ [1]. We need the next results.

Lemma 1 ([1, 5]) *Let $k > 0$ and $u \in C(0,1)$, where $C(0,1)$ stands for the space of all continuous real-valued functions defined on $(0,1)$. Then the fractional differential equation $\mathcal{D}^k u(t) = 0$ has a general solution $u(t) = C_1 t^{k-1} + C_1 t^{k-2} + \cdots + C_1 t^{k-n}$, where $n - 1 \le k < n$ and $C_1, \dots, C_n$ are some real constants.*

Lemma 2 ([1]) *Let $k > 0$ and $u \in C(0,1)$. Then we have*

$$\mathcal{I}^k \mathcal{D}^k u(t) = u(t) + C_1 t^{k-1} + C_1 t^{k-2} + \cdots + C_1 t^{k-n},$$

where $n - 1 \le k < n$ and $C_1, \dots, C_n$ are some real constants.

Theorem 3 (Krasnoselskii's fixed point theorem, [56]) *Let $\mathcal{M}$ be a closed, bounded, convex, and nonempty subset of a Banach space $\mathcal{X}$. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two operators on $\mathcal{M}$ such that*

(a) $\mathcal{A}_1 u + \mathcal{A}_2 w \in \mathcal{M}$ *for all $u, w \in \mathcal{M}$,*

(b) $\mathcal{A}_1$ *is compact and continuous,*

(c) $\mathcal{A}_2$ *is a contraction.*

Then there exists $z \in \mathcal{M}$ such that $z = \mathcal{A}_1 z + \mathcal{A}_2 z$.

Theorem 4 (Leray–Schauder's nonlinear alternative, [57]) *Let $\mathcal{X}$ be a Banach space, $\mathcal{B}$ be a closed, convex subset of $\mathcal{X}$, $\mathcal{U}$ be an open subset of $\mathcal{B}$, and $0 \in \mathcal{U}$. Assume that $\mathcal{P} : \bar{\mathcal{U}} \to \mathcal{B}$ is a continuous and compact map. Then either*

(a) $\mathcal{P}$ *has a fixed point in $\bar{\mathcal{U}}$, or*

(b) *there is $u \in \partial\mathcal{U}$ (the boundary of $\mathcal{U}$) and $\tau \in (0,1)$ with $u = \tau \mathcal{P}(u)$.*

Lemma 5 ([39]) *Let $\alpha > 0$, $a > 0$, $g(t,s)$ be a nonnegative continuous bounded function defined on $[0,T] \times [0,T]$ and nondecreasing with respect to the first variable and nonincreasing with respect to the second variable. Assume that $u(t)$ is nonnegative and integrable on $[0,T]$ with $u(t) \le a + \int_0^t g(t,s)(t-s)^{\alpha-1} u(s) \, ds$ for $t \in [0,T]$. Then we have*

$$u(t) \le a + a \int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} \, ds.$$

Theorem 6 (Banach contraction principle, [57]) *Let $\mathcal{X}$ be a Banach space and $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a contraction. Then $\mathcal{P}$ has a unique fixed point.*

3 Some existence results

Let $T > 0$, $J = [0,T]$, and $\mathcal{C} = C(J, \mathbb{R})$ be the Banach space of continuous mappings with the sup norm $\|u\| = \sup_{t \in J} |u(t)|$. We first provide our key result.

Lemma 7 *A map u_0 is a solution for boundary value problem* (4) *if and only if u_0 is a solution for the integral equation*

$$
u(t) = \frac{\lambda - 1}{\lambda \Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1} u(s)\, ds + \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} \hat{\Upsilon}\big(t, u(t)\big)\, ds
$$

$$
+ \frac{t^{k-1}}{\Theta} \times \Bigg[\frac{\mu_1 \Lambda_4 (\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_2 \mu_2 (\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T)
$$

$$
+ \frac{\Lambda_4 (1-\mu_1)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_2 (1-\mu_2)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T)
$$

$$
+ \frac{\Lambda_4 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1
$$

$$
+ \frac{\Lambda_4 (1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2 (1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) \Bigg]
$$

$$
- \frac{t^{k-2}}{\Theta} \Bigg[\frac{\mu_1 \Lambda_3 (\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_1 \mu_2 (\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T)
$$

$$
+ \frac{\Lambda_3 (1-\mu_1)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_1 (1-\mu_2)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T)
$$

$$
+ \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1
$$

$$
+ \frac{\Lambda_3 (1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_1 (1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) \Bigg], \tag{5}
$$

where

$$
\Lambda_1 = \frac{\mu_1 \Gamma(k)}{\Gamma(k-\gamma_1)} T^{k-\gamma_1-1} + \frac{(1-\mu_1)\Gamma(k)}{\Gamma(k-\gamma_2)} T^{k-\gamma_2-1},
$$

$$
\Lambda_2 = \frac{\mu_1 \Gamma(k-1)}{\Gamma(k-\gamma_1-1)} T^{k-\gamma_1-2} + \frac{(1-\mu_1)\Gamma(k-1)}{\Gamma(k-\gamma_2-1)} T^{k-\gamma_2-2},
$$

$$
\Lambda_3 = \frac{\mu_2 \Gamma(k)}{\Gamma(k+q_1)} T^{k+q_1-1} + \frac{(1-\mu_2)\Gamma(k)}{\Gamma(k+q_2)} T^{k+q_2-1},
$$

$$
\Lambda_4 = \frac{\mu_2 \Gamma(k-1)}{\Gamma(k+q_1-1)} T^{k+q_1-2} + \frac{(1-\mu_2)\Gamma(k-1)}{\Gamma(k+q_2-1)} T^{k+q_2-2},
$$

$$
\Theta = \Lambda_3 \Lambda_2 - \Lambda_1 \Lambda_4. \tag{6}
$$

Proof First, assume that u_0 is a solution for problem (4). Then we have

$$
\mathcal{D}^k u_0(t) = \frac{\lambda-1}{\lambda} \mathcal{D}^k u_0(t) + \frac{1}{\lambda} \hat{\Upsilon}\big(t, u_0(t)\big). \tag{7}
$$

By taking the Riemann–Liouville fractional integral of order k from both sides of equation (7), we obtain

$$
u_0(t) = \frac{\lambda-1}{\lambda \Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1} u_0(s)\, ds + \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} \hat{\Upsilon}\big(s, u_0(s)\big)\, ds
$$

$$
+ C_1 t^{k-1} + C_2 t^{k-2} + C_3 t^{k-3}
$$

for some real constants C_1, C_2, and C_3. For $2 < k < 3$, the first boundary condition of (4) implies that $C_3 = 0$. Hence,

$$u_0(t) = \frac{\lambda - 1}{\lambda} \mathcal{I}^{k-\theta} u_0(t) + \frac{1}{\lambda} \mathcal{I}^k \hat{\Upsilon}(t, u_0(t)) + C_1 t^{k-1} + C_2 t^{k-2}. \tag{8}$$

By using the Riemann–Liouville fractional derivative and integral of order α and β respectively with $\alpha \in \{\gamma_1, \gamma_2\}$, $\beta \in \{q_1, q_1\}$, $0 < \alpha < k - \theta$, and $2 < \theta < k$, we get

$$\mathcal{D}^\alpha u_0(t) = \frac{\lambda - 1}{\lambda \Gamma(k - \theta - \alpha)} \int_0^t (t-s)^{k-\theta-\alpha-1} u_0(s)\, ds + C_1 \frac{\Gamma(k)}{\Gamma(k-\alpha)} t^{k-\alpha-1}$$
$$+ \frac{1}{\lambda \Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} \hat{\Upsilon}(s, u_0(s))\, ds + C_2 \frac{\Gamma(k-1)}{\Gamma(k-\alpha-1)} t^{k-\alpha-2}$$

and

$$\mathcal{I}^\beta u_0(t) = \frac{\lambda - 1}{\lambda \Gamma(k - \theta + \beta)} \int_0^t (t-s)^{k-\theta+\beta-1} u_0(s)\, ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(k+\beta)} t^{k+\beta-1}$$
$$+ \frac{1}{\lambda \Gamma(k+\beta)} \int_0^t (t-s)^{k+\beta-1} \hat{\Upsilon}(s, u_0(s))\, ds + C_2 \frac{\Gamma(k-1)}{\Gamma(k+\beta-1)} t^{k+\beta-2}.$$

By replacing the values $\alpha = \gamma_1$, $\alpha = \gamma_2$, $\beta = q_1$, and $\beta = q_2$ and using the second condition of (4), we get

$$\frac{\mu_1(\lambda - 1)}{\lambda \Gamma(k - \theta - \gamma_1)} \int_0^T (T-s)^{k-\theta-\gamma_1-1} u_0(s)\, ds$$
$$+ \frac{(1-\mu_1)(\lambda - 1)}{\lambda \Gamma(k - \theta - \gamma_2)} \int_0^T (T-s)^{k-\theta-\gamma_2-1} u_0(s)\, ds$$
$$+ \frac{\mu_1}{\lambda \Gamma(k - \gamma_1)} \int_0^T (T-s)^{k-\gamma_1-1} \hat{\Upsilon}(s, u_0(s))\, ds$$
$$+ \frac{(1-\mu_1)}{\lambda \Gamma(k - \gamma_2)} \int_0^T (T-s)^{k-\gamma_2-1} \hat{\Upsilon}(s, u_0(s))\, ds$$
$$+ C_1 \Lambda_1 + C_2 \Lambda_2 = \delta_1,$$

and

$$\frac{\mu_2(\lambda - 1)}{\lambda \Gamma(k - \theta + q_1)} \int_0^T (T-s)^{k-\theta+q_1-1} u_0(s)\, ds$$
$$+ \frac{(1-\mu_2)(\lambda - 1)}{\lambda \Gamma(k - \theta + q_2)} \int_0^T (T-s)^{k-\theta+q_2-1} u_0(s)\, ds$$
$$+ \frac{\mu_2}{\lambda \Gamma(k + q_1)} \int_0^T (T-s)^{k+q_1-1} \hat{\Upsilon}(s, u_0(s))\, ds$$
$$+ \frac{(1-\mu_2)}{\lambda \Gamma(k + q_2)} \int_0^T (T-s)^{k+q_2-1} \hat{\Upsilon}(s, u_0(s))\, ds$$
$$+ C_1 \Lambda_3 + C_2 \Lambda_4 = \delta_2,$$

which leads to

$$
\begin{aligned}
C_1 = {} & \frac{\mu_1 \Lambda_4 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u_0(T) - \frac{\Lambda_2 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u_0(T) \\
& + \frac{\Lambda_4 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u_0(T) - \frac{\Lambda_2 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u_0(T) \\
& + \frac{\Lambda_4 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u_0(T)\big) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}\big(T, u_0(T)\big) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1 \\
& + \frac{\Lambda_4 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u_0(T)\big) - \frac{\Lambda_2 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u_0(T)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 = {} & \frac{\mu_1 \Lambda_3 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u_0(T) - \frac{\Lambda_1 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u_0(T) \\
& + \frac{\Lambda_3 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u_0(T) - \frac{\Lambda_1 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u_0(T) \\
& + \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u_0(T)\big) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k-q_1} \hat{\Upsilon}\big(T, u_0(T)\big) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1 \\
& + \frac{\Lambda_3 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u_0(T)\big) - \frac{\Lambda_1 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u_0(T)\big).
\end{aligned}
$$

By inserting the values of constants C_1 and C_2 in (8), we find that u_0 satisfies (5). Some calculations show that the converse part holds. This completes the proof.　　□

Based on Lemma 7, define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$
\begin{aligned}
\mathcal{F}u(t) = {} & \frac{\lambda - 1}{\lambda \Gamma(k - \theta)} \int_0^t (t - s)^{k-\theta-1} u(s)\, ds + \frac{1}{\lambda \Gamma(k)} \int_0^t (t - s)^{k-1} \hat{\Upsilon}\big(t, u(t)\big)\, ds \\
& + \frac{t^{k-1}}{\Theta} \times \bigg[\frac{\mu_1 \Lambda_4 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_2 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \\
& + \frac{\Lambda_2 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_2 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T) \\
& + \frac{\Lambda_2 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1 \\
& + \frac{\Lambda_4 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) \bigg] \\
& - \frac{t^{k-2}}{\Theta} \bigg[\frac{\mu_1 \Lambda_3 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_1 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \\
& + \frac{\Lambda_3 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_1 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T) \\
& + \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k-q_1} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1 \\
& + \frac{\Lambda_3 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_1 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) \bigg].
\end{aligned} \tag{9}
$$

Note that boundary value problem (4) has solution u_0 if and only if u_0 is a fixed point of the operator $\mathcal{F}u$. To simplify calculations, we use the notations

$$
\begin{aligned}
\mathcal{W}_1 ={}& \frac{(|\lambda - 1|)(\Lambda_4 + \Lambda_3 T^{-1})}{|\Theta|}\left(\mu_1 \frac{T^{2k-\theta-\gamma_1-1}}{\lambda\Gamma(k-\theta-\gamma_1+1)} + \frac{(1-\mu_1)T^{2k-\theta-\gamma_2-1}}{\lambda\Gamma(k-\theta-\gamma_2+1)}\right) \\
&+ \frac{(|\lambda-1|)(\Lambda_2+\Lambda_1 T^{-1})}{|\Theta|}\left(\mu_2\frac{T^{2k-\theta+q_1-1}}{\lambda\Gamma(k-\theta+q_1+1)} + \frac{(1-\mu_2)T^{2k-\theta+q_2-1}}{\lambda\Gamma(k-\theta+q_2+1)}\right) \\
&+ \frac{(|\lambda-1|)T^{k-\theta}}{\lambda\Gamma(k-\theta+1)}
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\mathcal{W}_2 ={}& \frac{T^k}{\lambda\Gamma(k+1)} + \frac{\Lambda_4 + \Lambda_3 T^{-1}}{|\Theta|}\left(\mu_1\frac{T^{2k-\gamma_1-1}}{\lambda\Gamma(k-\gamma_1+1)} + \frac{(1-\mu_1)T^{2k-\gamma_2-1}}{\lambda\Gamma(k-\gamma_2+1)}\right) \\
&+ \frac{\Lambda_2+\Lambda_1 T^{-1}}{|\Theta|}\left(\mu_2\frac{T^{2k+q_1-1}}{\lambda\Gamma(k+q_1+1)} + \frac{(1-\mu_2)T^{2k+q_2-1}}{\lambda\Gamma(k+q_2+1)}\right).
\end{aligned}
\tag{11}
$$

Theorem 8 *Suppose that $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ is a continuous map and there exists a constant $\mathcal{L} > 0$ such that $|\hat{\Upsilon}(t,u) - \hat{\Upsilon}(t,u')| \leq \mathcal{L}|u - u'|$ for all $t \in J$ and $u, u' \in \mathbb{R}$. If $\mathcal{L}\mathcal{W}_2 + \mathcal{W}_1 < 1$, then problem (4) has a unique solution, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are defined by (10) and (11).*

Proof Put $\sup_{t\in J}|\hat{\Upsilon}(t,0)| = \mathcal{N} < \infty$ and choose

$$
\frac{|\Theta|\mathcal{N}\mathcal{R} + T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)}{|\Theta|(1 - \mathcal{L}\mathcal{W}_2 - \mathcal{W}_1)} \leq \mathcal{R},
$$

where Λ_i $i \in \{1,2,3,4\}$ are defined by (7). Set $\mathcal{B}_\mathcal{R} = \{u \in \mathcal{C} : \|u\| \leq \mathcal{R}\}$. We show that $\mathcal{F}\mathcal{B}_\mathcal{R} \subset \mathcal{B}_\mathcal{R}$. For each $u \in \mathcal{B}_\mathcal{R}$, we have

$$
\begin{aligned}
|\mathcal{F}u(t)| \leq{}& \frac{|\lambda-1|}{\lambda\Gamma(k-\theta)}\int_0^t (t-s)^{k-\theta-1}|u(s)|\,ds \\
&+ \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1}|\hat{\Upsilon}(t,u(t)) - \hat{\Upsilon}(t,0)| + |\hat{\Upsilon}(t,0)|\,ds \\
&+ \frac{T^{k-1}}{|\Theta|}\times\left[\frac{\mu_1\Lambda_4(|\lambda-1|)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}|u(T)| + \frac{\Lambda_2\mu_2(|\lambda-1|)}{\lambda}\mathcal{I}^{k-\theta+q_1}|u(T)|\right. \\
&+ \frac{\Lambda_4(1-\mu_1)(|\lambda-1|)}{\lambda}\mathcal{I}^{k-\theta-\gamma_2}|u(T)| + \frac{\Lambda_2(1-\mu_2)(|\lambda-1|)}{\lambda}\mathcal{I}^{k-\theta+q_2}|u(T)| \\
&+ \frac{\Lambda_4\mu_1}{\lambda}\mathcal{I}^{k-\gamma_1}\left(|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)| + |\hat{\Upsilon}(T,0)|\right) \\
&+ \frac{\Lambda_2\mu_2}{\lambda}\mathcal{I}^{k+q_1}\left(|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)| + |\hat{\Upsilon}(T,0)|\right) + \Lambda_2\delta_2 - \Lambda_4\delta_1 \\
&+ \frac{\Lambda_4(|1-\mu_1|)}{\lambda}\mathcal{I}^{k-\gamma_2}\left(|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)| + |\hat{\Upsilon}(T,0)|\right) \\
&+ \left.\frac{\Lambda_2(|1-\mu_2|)}{\lambda}\mathcal{I}^{k+q_2}\left(|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)| + |\hat{\Upsilon}(T,0)|\right)\right] \\
&+ \frac{T^{k-2}}{|\Theta|}\left[\frac{\mu_1\Lambda_3(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}|u(T)| + \frac{\Lambda_1\mu_2(|\lambda-1|)}{\lambda}\mathcal{I}^{k-\theta+q_1}|u(T)|\right.
\end{aligned}
$$

$$+ \frac{\Lambda_3(1-\mu_1)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2}|u(T)| + \frac{\Lambda_1(1-\mu_2)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_2}|u(T)|$$

$$+ \frac{\Lambda_3\mu_1}{\lambda} \mathcal{I}^{k-\gamma_1}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\right| + \left|\hat{\Upsilon}(T,0)\right|\right)$$

$$+ \frac{\Lambda_1\mu_2}{\lambda} \mathcal{I}^{k-q_1}\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\right| + \left|\hat{\Upsilon}(T,0)\right| + |\Lambda_1\delta_2| + |\Lambda_3\delta_1|$$

$$+ \frac{\Lambda_3(1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\right| + \left|\hat{\Upsilon}(T,0)\right|\right)$$

$$+ \frac{\Lambda_1(1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\right| + \left|\hat{\Upsilon}(T,0)\right|\right) \Bigg]$$

$$\leq \left(\mathcal{L}\|u\| + \mathcal{N}\right)\mathcal{W}_2 + \|u\|\mathcal{W}_1$$

$$+ \frac{1}{|\Theta|}\left[T^{k-1}\left(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|\right) + T^{k-2}\left(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|\right)\right]$$

$$= (\mathcal{L}\mathcal{W}_2 + \mathcal{W}_1)\mathcal{R} + \mathcal{N}\mathcal{W}_2$$

$$+ \frac{1}{|\Theta|}\left[T^{k-1}\left(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|\right) + T^{k-2}\left(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|\right)\right]$$

$$\leq \mathcal{R}.$$

Thus, $\|Fu\| \leq \mathcal{R}$ and so $\mathcal{F}\mathcal{B}_\mathcal{R} \subset \mathcal{B}_\mathcal{R}$. Let $u, u' \in \mathcal{C}$. For each $t \in J$, we have

$$\left|\mathcal{F}u(t) - \mathcal{F}u'(t)\right| \leq \frac{|\lambda-1|}{\lambda\Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1}\left|u(s) - u'(s)\right| ds$$

$$+ \frac{1}{\lambda\Gamma(k)} \int_0^t (t-s)^{k-1}\left|\hat{\Upsilon}(t,u(t)) - \hat{\Upsilon}(t,u'(s))\right| ds$$

$$+ \frac{T^{k-1}}{|\Theta|} \times \left[\frac{\mu_1\Lambda_4(|\lambda|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1}\left|u(T) - u'(T)\right|\right.$$

$$+ \frac{\Lambda_2\mu_2(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_1}\left|u(T) - u'(T)\right|$$

$$+ \frac{\Lambda_4(1-\mu_1)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2}\left|u(T) - u'(T)\right|$$

$$+ \frac{\Lambda_2(1-\mu_2)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_2}\left|u(T) - u'(T)\right|$$

$$+ \frac{\Lambda_4\mu_1}{\lambda} \mathcal{I}^{k-\gamma_1}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\right|\right)$$

$$+ \frac{\Lambda_2\mu_2}{\lambda} \mathcal{I}^{k+q_1}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\right|\right)$$

$$+ \frac{\Lambda_4(|1-\mu_1|)}{\lambda} \times \mathcal{I}^{k-\gamma_2}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\right|\right)$$

$$+ \frac{\Lambda_2(|1-\mu_2|)}{\lambda} \mathcal{I}^{k+q_2}\left(\left|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\right|\right) \Bigg]$$

$$+ \frac{T^{k-2}}{|\Theta|}\left[\frac{\mu_1\Lambda_3(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1}\left|u(T) - u'(T)\right|\right.$$

$$+ \frac{\Lambda_1\mu_2(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_1}\left|u(T) - u'(T)\right|$$

$$+ \frac{\Lambda_3(1-\mu_1)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} |u(T) - u'(T)|$$

$$+ \frac{\Lambda_1(1-\mu_2)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_2} |u(T) - u'(T)|$$

$$+ \frac{\Lambda_3\mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} (|\hat{\Upsilon}(T, u(T)) - \hat{\Upsilon}(T, u'(T))|)$$

$$+ \frac{\Lambda_1\mu_2}{\lambda} \mathcal{I}^{k-q_1} |\hat{\Upsilon}(T, u(T)) - \hat{\Upsilon}(T, u'(T))|$$

$$+ \frac{\Lambda_3(1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} (|\hat{\Upsilon}(T, u(T)) - \hat{\Upsilon}(T, u'(T))|)$$

$$+ \frac{\Lambda_1(1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2} (|\hat{\Upsilon}(T, u(T)) - \hat{\Upsilon}(T, u'(T))|) \Big]$$

$$\leq (\mathcal{L}\|u-u'\|)\mathcal{W}_2 + \|u-u'\|\mathcal{W}_1$$

$$= (\mathcal{L}\mathcal{W}_2 + \mathcal{W}_1)\|u-u'\|.$$

Hence, $\|\mathcal{F}u - \mathcal{F}u'\| \leq (\mathcal{L}\mathcal{W}_2 + \mathcal{W}_1)\|u-u'\|$ and so $\mathcal{F}$ is a contraction. By using the principle of contraction, $\mathcal{F}$ has a unique fixed point which is the unique solution for problem (4). □

Here, by using Krasnoselskii's fixed point theorem, we provide our next existence result.

Theorem 9 *Suppose that* $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ *is a continuous map and there exists a constant* $\mathcal{L} > 0$ *such that* $|\hat{\Upsilon}(t, u) - \hat{\Upsilon}(t, u')| \leq \mathcal{L}|u - u'|$ *for each* $t \in J$ *and* $u, u' \in \mathbb{R}$. *If there is* $\mathcal{V}(t) \in C(J, \mathbb{R}^+)$ *such that* $\hat{\Upsilon}(t, u) \leq \mathcal{V}(t)$ *for all* $(t, u) \in J \times \mathbb{R}$ *and* $\mathcal{W}_1 < 1$, *then problem* (4) *has at least one solution. Here,* $\mathcal{W}_1$ *is given by* (10).

Proof Let $\|\mathcal{V}\| = \sup_{t \in J} |\mathcal{V}(t)|$. Consider the set $\mathcal{B}_r = \{u \in \mathcal{C} : \|u\| \leq r\}$, where

$$\frac{|\Theta|\|v\|\mathcal{W}_2 + T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{\alpha-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)}{|\Theta|(1 - \mathcal{W}_1)} \leq r$$

and Λ_1, Λ_2, Λ_3, Λ_4, and $\mathcal{W}_1$ are given by (7) and (10), respectively. For each $t \in J$, define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{B}_r$ by

$$\mathcal{F}_1 u(t) = \frac{\lambda - 1}{\lambda \Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1} u(s)\, ds$$

$$+ \frac{t^{k-1}}{\Theta} \times \left[\frac{\mu_1 \Lambda_4(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_2\mu_2(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \right.$$

$$\left. + \frac{\Lambda_4(1-\mu_1)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_2(1-\mu_2)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T) \right]$$

$$- \frac{t^{k-2}}{\Theta} \left[\frac{\mu_1 \Lambda_3(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_1\mu_2(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \right.$$

$$\left. + \frac{\Lambda_3(1-\mu_1)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_1(1-\mu_2)(\lambda-1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T) \right]$$

and

$$
\mathcal{F}_2 u(t) = \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} \hat{\Upsilon}\big(t, u(t)\big) \, ds + \frac{t^{k-1}}{\Theta}
$$
$$
\times \left[\frac{\Lambda_4 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1 \right.
$$
$$
+ \frac{\Lambda_4 (1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big) - \left. \frac{\Lambda_2 (1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) \right]
$$
$$
- \frac{t^{k-2}}{\Theta} \left[+ \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k-q_1} \hat{\Upsilon}\big(T, u(T)\big) \right.
$$
$$
+ \frac{\Lambda_3 (1-\mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}\big(T, u(T)\big)
$$
$$
- \left. \frac{\Lambda_1 (1-\mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}\big(T, u(T)\big) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1 \right].
$$

We show that $\mathcal{F}_2 u + \mathcal{F}_2 u' \in \mathcal{B}_r$. Let $u, u' \in \mathcal{B}_r$. Then we have

$$
\big|\mathcal{F}_1 u(t) + F_2 u'(t)\big|
$$
$$
\leq \|\mathcal{V}\| \left[\frac{T^k}{\lambda \Gamma(k+1)} + \frac{\Lambda_4 + \Lambda_3 T^{-1}}{|\Theta|} \left(\frac{\mu_1 T^{2k-\gamma_1-1}}{\lambda \Gamma(k-\gamma_1+1)} \right. \right.
$$
$$
+ \left. \frac{(1-\mu_1) T^{2k-\gamma_2-1}}{\lambda \Gamma(k-\gamma_2+1)} \right) + \frac{\Lambda_2 + \Lambda_1 T^{-1}}{|\Theta|} \left(\frac{\mu_2 T^{2k+q_1-1}}{\lambda \Gamma(k+q_1+1)} + \left. \frac{(1-\mu_2) T^{2k+q_2-1}}{\lambda \Gamma(k+q_2+1)} \right) \right]
$$
$$
+ \|u\| \left[\frac{(|\lambda-1|) T^{\alpha-\theta}}{\lambda \Gamma(\alpha-\theta+1)} + \frac{(|\lambda-1|)(\Lambda_4 + \Lambda_3 T^{-1})}{|\Theta|} \left(\frac{\mu_1 T^{2k-\theta-\gamma_1-1}}{\lambda \Gamma(k-\theta-\gamma_1+1)} \right. \right.
$$
$$
+ \left. \frac{(1-\mu_1) T^{2k-\theta-\gamma_2-1}}{\lambda \Gamma(k-\theta-\gamma_2+1)} \right) + \frac{(|\lambda-1|)(\Lambda_2 + \Lambda_1 T^{-1})}{|\Theta|} \left(\frac{\mu_2 T^{2k-\theta+q_1-1}}{\lambda \Gamma(k-\theta+q_1+1)} \right.
$$
$$
+ \left. \left. \frac{(1-\mu_2) T^{2k-\theta+q_2-1}}{\lambda \Gamma(k-\theta+q_2+1)} \right) \right] + \frac{1}{|\Theta|} \left[T^{k-1}\big(|\Lambda_2 \delta_2| + |\Lambda_4 \delta_1|\big) + T^{k-2}\big(|\Lambda_1 \delta_2| + |\Lambda_3 \delta_1|\big) \right]
$$
$$
\leq \big(\|\mathcal{V}\| \mathcal{W}_2 + r \mathcal{W}_1\big) + \frac{1}{|\Theta|} \left[T^{k-1}\big(|\Lambda_2 \delta_2| + |\Lambda_4 \delta_1|\big) T^{k-2}\big(|\Lambda_1 \delta_2| + |\Lambda_3 \delta_1|\big) \right] \leq r,
$$

and so $\mathcal{F}_1 u + \mathcal{F}_2 u' \in \mathcal{B}_r$. Now, we prove $\mathcal{F}_1$ is a contraction. For every $u, u' \in \mathcal{B}_r$, we have

$$
\big|\mathcal{F}_1 u(t) - \mathcal{F}_1 u'(t)\big|
$$
$$
\leq \frac{|\lambda-1|}{\lambda \Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1} \big|u(s) - u'(s)\big| \, ds
$$
$$
+ \frac{T^{k-1}}{|\Theta|} \times \left[\frac{\mu_1 \Lambda_4 (|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} \big|u(T) - u'(T)\big| \right.
$$
$$
+ \frac{\Lambda_2 \mu_2 (|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_1} \big|u(T) - u'(T)\big| + \frac{\Lambda_4 (1-\mu_1)(|\lambda-1|)}{\lambda}
$$
$$
\times \mathcal{I}^{k-\theta-\gamma_2} \big|u(T) - u'(T)\big| + \left. \frac{\Lambda_2 (1-\mu_2)(|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_2} \big|u(T) - u'(T)\big| \right]
$$
$$
+ \frac{t^{k-2}}{|\Theta|} \left[\frac{\mu_1 \Lambda_3 (\lambda-1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} \big|u(T) - u'(T)\big| + \frac{\Lambda_1 \mu_2 (|\lambda-1|)}{\lambda} \mathcal{I}^{k-\theta+q_1} \big|u(T) - u'(T)\big| \right.
$$

$$+ \frac{\Lambda_3(1 - \mu_1)(|\lambda - 1|)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} |u(T) - u'(T)|$$

$$+ \frac{\Lambda_1(1 - \mu_2)(|\lambda - 1|)}{\lambda} \mathcal{I}^{k-\theta+q_2} |u(T) - u'(T)| \Bigg] \leq \mathcal{W}_1 \|u - u'\|.$$

Since $\mathcal{W}_1 < 1$, $\mathcal{F}_1$ is a contraction. Utilizing the continuity of the function $\hat{\Upsilon}$, we find that the operator $\mathcal{F}_2$ is continuous. If $u \in \mathcal{B}_r$, then

$$\|\mathcal{F}_2 u\| \leq \|\mathcal{V}\| \Bigg(\frac{T^k}{\lambda \Gamma(k+1)} + \frac{\Lambda_4 + \Lambda_3 T^{-1}}{|\Lambda_3 \Lambda_2 - \Lambda_1 \Lambda_4|} \Big(\mu_1 \frac{T^{2k-\gamma_1-1}}{\lambda \Gamma(k-\gamma_1+1)} + \frac{(1-\mu_1)T^{2k-\gamma_2-1}}{\lambda \Gamma(k-\gamma_2+1)} \Big)$$

$$+ \frac{\Lambda_2 + \Lambda_1 T^{-1}}{|\Lambda_3 \Lambda_2 - \Lambda_1 \Lambda_4|} \Big(\mu_2 \frac{T^{2k+q_1-1}}{\lambda \Gamma(k+q_1+1)} + \frac{(1-\mu_2)T^{2k+q_2-1}}{\lambda \Gamma(k+q_2+1)} \Big) \Bigg) = \mathcal{W}_2 \|\mathcal{V}\|.$$

This means that the operator $\mathcal{F}_2$ is uniformly bounded on $\mathcal{B}_r$. Now, we show that $\mathcal{F}_2$ is equicontinuous. Set $\sup_{t \in J, u \in \mathcal{B}_r} |\hat{\Upsilon}(t,u)| = \overline{\mathcal{M}}$. For each t_1, t_2 with $t_2 > t_1$ and $u \in \mathcal{B}_r$, we have

$$\big| \mathcal{F}_2 u(t_2) - \mathcal{F}_2 u(t_1) \big|$$

$$= \Bigg| \frac{1}{\lambda \Gamma(k)} \int_0^{t_2} (t_2 - s)^{k-1} \hat{\Upsilon}(t, u(t)) \, ds - \frac{1}{\lambda \Gamma(k)} \int_0^{t_1} (t_1 - s)^{k-1} \hat{\Upsilon}(t, u(t)) \, ds$$

$$+ \frac{t_2^{k-1} - t_1^{k-1}}{\Theta} \times \Bigg[\frac{\Lambda_4 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}(T, u(T)) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1$$

$$+ \frac{\Lambda_4(1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2(1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}(T, u(T)) \Bigg]$$

$$- \frac{t_2^{k-2} - t_1^{k-2}}{\Theta} \Bigg[+ \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k-q_1} \hat{\Upsilon}(T, u(T))$$

$$+ \frac{\Lambda_3(1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1(1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}(T, u(T)) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1 \Bigg] \Bigg|$$

$$\leq \frac{\overline{\mathcal{M}}(2(t_2 - t_1)^k + |t_2^k - t_1^k|)}{\lambda \Gamma(k+1)}$$

$$+ \frac{t_2^{k-1} - t_1^{k-1}}{|\Theta|} \times \Bigg[\frac{\overline{\mathcal{M}} \Lambda_4 \mu_1 T^{k-\gamma_1}}{\lambda \Gamma(k-\gamma_1+1)} + \frac{\overline{\mathcal{M}} \Lambda_2 \mu_2 T^{k+q_1}}{\lambda \Gamma(k+q_1+1)} + |\Lambda_2 \delta_2| + |\Lambda_4 \delta_1|$$

$$+ \frac{\overline{\mathcal{M}} \Lambda_4(|1 - \mu_1|)T^{k-\gamma_2}}{\lambda \Gamma(k-\gamma_2+1)} + \frac{\overline{\mathcal{M}} \Lambda_2(|1 - \mu_2|)T^{k+q_2}}{\lambda \Gamma(k+q_2+1)} \Bigg]$$

$$+ \frac{t_2^{k-2} - t_1^{k-2}}{|\Theta|} \Bigg[\frac{\overline{\mathcal{M}} \Lambda_3 \mu_1 T^{k-\gamma_1}}{\lambda \Gamma(k-\gamma_1+1)} + \frac{\overline{\mathcal{M}} \Lambda_1 \mu_2 T^{k-q_1}}{\lambda \Gamma(k-q_1+1)}$$

$$+ \frac{\overline{\mathcal{M}} \Lambda_3(|1 - \mu_1|)T^{k-\gamma_2}}{\lambda \Gamma(k-\gamma_2)} + \frac{\overline{\mathcal{M}} \Lambda_1(|1 - \mu_2|)T^{k+q_2}}{\lambda \Gamma(k+q_2+1)} + |\Lambda_1 \delta_2| + |\Lambda_3 \delta_1| \Bigg].$$

The right-hand side of the above inequality tends to zero independently of u as t_2 tends to t_1. Hence, $\mathcal{F}_2$ is equicontinuous, and so $\mathcal{F}_2$ is relatively compact on $\mathcal{B}_r$. Now, by using the Arzela–Ascoli theorem, $\mathcal{F}_2$ is compact on $\mathcal{B}_r$. Now, by using Theorem 3, boundary value problem (4) has at least one solution.   □

Here, by applying the Leray–Schauder theorem, we give another existence result.

Theorem 10 *Suppose that $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ is a continuous map and there are nondecreasing continuous function $\Psi : [0, \infty) \to (0, \infty)$ and $\Phi \in C(J, \mathbb{R}^+)$ such that $|\hat{\Upsilon}(t, u)| \leq \Phi(t)\Psi(\|u\|)$ for all $(t, u) \in J \times \mathbb{R}$. Assume that there exists a constant $\mathcal{Q} > 0$ such that*

$$\frac{\mathcal{Q}|\Theta|}{\mathcal{Q}|\Theta|\mathcal{W}_1 + \Psi(\mathcal{Q})\|\Phi\||\Theta|\mathcal{W}_2 + T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)} > 1,$$

where $\mathcal{W}_1, \mathcal{W}_2$ are defined by (10) and (11), respectively. Then boundary value problem (4) has at least one solution.

Proof Consider the operator $\mathcal{F}$ defined by (9). We show that $\mathcal{F}$ maps bounded sets into bounded sets of $\mathcal{C}$. Let $\rho > 0$ and $\mathcal{B}_\rho = \{u \in \mathcal{C} : \|u\| \leq \rho\}$ be a bounded ball in $\mathcal{C}$ and $t \in J$. Then we have

$$
\begin{aligned}
\left|\mathcal{F}u(t)\right| &\leq \sup_{t \in J}\left|\frac{\lambda - 1}{\lambda - \theta}\int_0^t (t-s)^{k-\theta-1}u(s)\,ds + \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1}\hat{\Upsilon}(t, u(t))\,ds \right. \\
&\quad + \frac{t^{k-1}}{\Theta}\times\left[\frac{\mu_1\Lambda_4(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}u(T) - \frac{\Lambda_2\mu_2(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta+q_1}u(T)\right. \\
&\quad + \frac{\Lambda_4(1-\mu_1)(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_2}u(T) - \frac{\Lambda_2(1-\mu_2)(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta+q_2}u(T) \\
&\quad + \frac{\Lambda_4\mu_1}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2\mu_2}{\lambda}\mathcal{I}^{k+q_1}\hat{\Upsilon}(T, u(T)) + \Lambda_2\delta_2 - \Lambda_4\delta_1 \\
&\quad \left. + \frac{\Lambda_4(1-\mu_1)}{\lambda}\mathcal{I}^{k-\gamma_2}\hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2(1-\mu_2)}{\lambda}\mathcal{I}^{k+q_2}\hat{\Upsilon}(T, u(T))\right] \\
&\quad - \frac{t^{k-2}}{\Theta}\left[\frac{\mu_1\Lambda_3(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}u(T) - \frac{\Lambda_1\mu_2(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta+q_1}u(T)\right. \\
&\quad + \frac{\Lambda_3(1-\mu_1)(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_2}u(T) - \frac{\Lambda_1(1-\mu_2)(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta+q_2}u(T) \\
&\quad + \frac{\Lambda_3\mu_1}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1\mu_2}{\lambda}\mathcal{I}^{k-q_1}\hat{\Upsilon}(T, u(T)) + \Lambda_1\delta_2 - \Lambda_3\delta_1 \\
&\quad \left.\left. + \frac{\Lambda_3(1-\mu_1)}{\lambda}\mathcal{I}^{k-\gamma_2}\hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1(1-\mu_2)}{\lambda}\mathcal{I}^{k+q_2}\hat{\Upsilon}(T, u(T))\right]\right| \\
&\leq \|\Phi\|\Psi(\|u\|)\mathcal{W}_2 + \|u\|\mathcal{W}_1 \\
&\quad + \frac{1}{|\Theta|}\left[T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)\right],
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\left\|\mathcal{F}u(t)\right\| &\leq \|\Phi\|\Psi(\|u\|)\mathcal{W}_2 + \|u\|\mathcal{W}_1 \\
&\quad + \frac{1}{|\Theta|}\left[T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)\right].
\end{aligned}
$$

Now, we prove that the operator $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Assume that $t_1, t_2 \in J$ with $t_1 < t_2$ and $u \in \mathcal{B}_\rho$. Then we have

$$
\begin{aligned}
&\left|\mathcal{F}u(t_2) - \mathcal{F}u(t_1)\right| \\
&\leq \frac{|\lambda - 1|}{\lambda\Gamma(k - \theta)}\left(\left|\int_0^{t_2}(t_2 - s)^{k-\theta-1}u(s)\,ds - \int_0^{t_1}(t_1 - s)^{k-\theta-1}u(s)\,ds\right|\right.
\end{aligned}
$$

$$+ \frac{1}{\lambda \Gamma(k)} \left(\left| \int_0^{t_2} (t_2 - s)^{k-1} \hat{\Upsilon}(t, u(s))\, ds - \int_0^{t_1} (t_1 - s)^{k-1} \hat{\Upsilon}(t, u(s))\, ds \right| \right)$$

$$+ \frac{t_2^{k-1} - t_1^{k-1}}{|\Theta|} \times \left[\left| \left| \frac{\mu_1 \Lambda_4 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_2 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \right. \right. \right.$$

$$+ \frac{\Lambda_4 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_2 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T)$$

$$+ \frac{\Lambda_4 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2 \mu_2}{\lambda} \mathcal{I}^{k+q_1} \hat{\Upsilon}(T, u(T)) + \Lambda_2 \delta_2 - \Lambda_4 \delta_1$$

$$+ \left. \left. \frac{\Lambda_4 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_2 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}(T, u(T)) \right| \right| \right]$$

$$+ \frac{t_2^{k-2} - t_1^{k-2}}{|\Theta|} \left| \left[\frac{\mu_1 \Lambda_3 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_1} u(T) - \frac{\Lambda_1 \mu_2 (\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_1} u(T) \right. \right.$$

$$+ \frac{\Lambda_3 (1 - \mu_1)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta-\gamma_2} u(T) - \frac{\Lambda_1 (1 - \mu_2)(\lambda - 1)}{\lambda} \mathcal{I}^{k-\theta+q_2} u(T)$$

$$+ \frac{\Lambda_3 \mu_1}{\lambda} \mathcal{I}^{k-\gamma_1} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1 \mu_2}{\lambda} \mathcal{I}^{k-q_1} \hat{\Upsilon}(T, u(T)) + \Lambda_1 \delta_2 - \Lambda_3 \delta_1$$

$$+ \left. \left. \frac{\Lambda_3 (1 - \mu_1)}{\lambda} \mathcal{I}^{k-\gamma_2} \hat{\Upsilon}(T, u(T)) - \frac{\Lambda_1 (1 - \mu_2)}{\lambda} \mathcal{I}^{k+q_2} \hat{\Upsilon}(T, u(T)) \right] \right|$$

$$\leq \frac{|\lambda - 1|}{\lambda \Gamma(k - \beta + 1)} \left((t_2^{k-\theta} - t_1^{k-\theta}) + 2(t_2 - t_1)^{k-\theta} \right)$$

$$+ \frac{1}{\lambda \Gamma(k + 1)} \left((t_2^k - t_1^k) + 2(t_2 - t_1)^k \right) + \frac{t_2^{k-1} - t_1^{k-1}}{|\Theta|}$$

$$\times \left\{ \|u\| \left[\frac{T^{k-\theta-\gamma_1}(|\lambda - 1|)}{\lambda \Gamma(k - \theta - \gamma_1 + 1)} + \frac{T^{k-\theta+q_1}(|\lambda - 1|)\Lambda_2 \mu_2}{\lambda \Gamma(k - \theta + q_1 + 1)} \right. \right.$$

$$+ \left. \frac{\Lambda_4 (1 - \mu_1)(\lambda - 1) T^{k-\theta-\gamma_2}}{\lambda \Gamma(k - \theta - \gamma_2 + 1)} + \frac{\Lambda_2 (1 - \mu_2)(|\lambda - 1|) T^{k-\theta+q_2}}{\lambda \Gamma(k - \theta + q_2 + 1)} \right]$$

$$+ \overline{\mathcal{M}} \left[\Lambda_4 \mu_1 \frac{T^{k-\gamma_1}}{\lambda \Gamma(k - \gamma_1 + 1)} + \Lambda_2 \mu_2 \frac{T^{k+q_1}}{\lambda \Gamma(k + q_1 + 1)} \right.$$

$$+ \left. \frac{\Lambda_4 (1 - \mu_1) T^{k-\gamma_2}}{\lambda \Gamma(k - \gamma_2 + 1)} + \frac{\Lambda_2 (1 - \mu_2) T^{k+q_2}}{\lambda \Gamma(k + q_2 + 1)} \right] + |\Lambda_2 \delta_2| + |\Lambda_4 \delta_1| \right\}$$

$$+ \frac{t_2^{k-2} - t_1^{k-2}}{|\Theta|} \left\{ \|u\| \left[\mu_1 \Lambda_3 \frac{T^{k-\theta-\gamma_1}(|\lambda - 1|)}{\lambda \Gamma(k - \theta - \gamma_1 + 1)} + \Lambda_1 \mu_2 \frac{T^{k-\theta+q_1}(|\lambda - 1|)}{\lambda \Gamma(k - \theta + q_1 + 1)} \right. \right.$$

$$+ \left. \frac{\Lambda_3 (1 - \mu_1)(|\lambda - 1|) T^{k-\theta-\gamma_2}}{\lambda \Gamma(k - \theta - \gamma_2 + 1)} + \frac{\Lambda_1 (1 - \mu_2)(|\lambda - 1|) T^{k-\theta+q_2}}{\lambda \Gamma(k - \beta + q_2 + 1)} \right]$$

$$+ \overline{\mathcal{M}} \left[\Lambda_3 \mu_1 \frac{T^{k-\gamma_1}}{\lambda \Gamma(k - \gamma_1 + 1)} + \Lambda_1 \mu_2 \frac{T^{k+q_1}}{\lambda \Gamma(k + q_1 + 1)} \right.$$

$$+ \left. \frac{\Lambda_3 (1 - \mu_1) T^{k-\gamma_2}}{\lambda \Gamma(k - \gamma_2 + 1)} + \frac{\Lambda_1 (1 - \mu_2) T^{k+q_2}}{\lambda \Gamma(k + +1)} \right] + |\Lambda_1 \delta_2| + |\Lambda_3 \delta_1| \right\}.$$

If $t_2 - t_1 \to 0$, then the right-hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_\rho$. Thus, by using the Arzela–Ascoli theorem, the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous. The desired result is deduced from the Leray–Schauder theorem 4 once we prove the boundedness of the set of the solutions for the equation $u = \omega \mathcal{F} u$ for some

$\omega \in (0, 1)$. Let u be a solution of the last equation. For each $t \in J$, we have

$$
\begin{aligned}
\big|u(t)\big| &\le \|\Phi\|\Psi\big(\|u\|\big)\mathcal{W}_2 + \|u\|\mathcal{W}_1 \\
&\quad + \frac{1}{|\Theta|}\Big[T^{k-1}\big(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|\big) + T^{k-2}\big(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|\big)\Big],
\end{aligned}
$$

and so $\dfrac{\|u\|\,|\Theta|}{\|u\|\,|\Theta|\mathcal{W}_1 + \Psi(\|u\|)\|\Phi\|\,|\Theta|\mathcal{W}_2 + T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)} < 1$. Choose the constant $\mathcal{Q}$ with $\|u\| \neq \mathcal{Q}$. Put $\mathcal{U} = \{x \in C : \|x\| < \mathcal{Q}\}$. One can check that the operator $\mathcal{F} : \bar{\mathcal{U}} \to C$ is continuous and completely continuous. In view of the choice of $\mathcal{U}$, there is no $u \in \partial\mathcal{U}$ so that $u = \omega\mathcal{F}u$ for some $\omega \in (0, 1)$. Now, by using the Leray–Schauder theorem, the operator $\mathcal{F}$ has a fixed point $u \in \bar{\mathcal{U}}$ which is a solution of boundary value problem (4).   □

4 Stability analysis

In this section, we study the Hyers–Ulam stability of the boundary value problem

$$
\begin{cases}
\lambda\mathcal{D}^k(u(t)) + (1 - \lambda)\mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) & (t \in [0, T], k \in [2, 3)), \\
u(0) = 0, \qquad \mathcal{D}^{\gamma_1}u(T) = \delta_1, \qquad \mathcal{I}^{q_1}u(T) = \delta_2,
\end{cases}
\tag{12}
$$

which is a special case of problem (4) when we take $\mu_1 = \mu_2 = 1$.

Definition 11 Problem (12) is called Hyers–Ulam stable whenever there exists a real constant $\ell > 0$ such that, for each $\varepsilon > 0$ and $u(t) \in C_\mathbb{R}([0, T])$ satisfying

$$
\big|\lambda\mathcal{D}^k u(t) + (1 - \lambda)\mathcal{D}^\theta\big(u(t)\big) - \hat{\Upsilon}\big(t, u(t)\big)\big| < \varepsilon,
$$

there is a solution $v(t) \in C_\mathbb{R}([0, T])$ of problem (12) such that $|u(t) - v(t)| \le \ell\varepsilon$ for all $t \in [0, T]$.

Theorem 12 *Suppose that $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ is a continuous map and there exists a constant $\mathcal{L} > 0$ such that $|\hat{\Upsilon}(t, u) - \hat{\Upsilon}(t, u')| \le \mathcal{L}|u - u'|$ for all $t \in J$ and $u, u' \in \mathbb{R}$. Then boundary value problem (12) is Hyers–Ulam stable.*

Proof Let $\varepsilon > 0$ and $u(t) \in C_\mathbb{R}([0, T])$ be such that

$$
\big|\lambda\mathcal{D}^k u(t) + (1 - \lambda)\mathcal{D}^\theta\big(u(t)\big) - \hat{\Upsilon}\big(t, u(t)\big)\big| < \varepsilon.
$$

Choose a function g satisfying $\lambda\mathcal{D}^k u(t) + (1 - \lambda)\mathcal{D}^\theta(u(t)) = \hat{\Upsilon}(t, u(t)) + g(t)$ and $|g(t)| \le \varepsilon$ for all t. Then we have

$$
\begin{aligned}
u(t) &= \frac{\lambda - 1}{\lambda}\mathcal{I}^{k-\theta}u(t) + \frac{1}{\lambda}\mathcal{I}^k\hat{\Upsilon}\big(t, u(t)\big) + \frac{1}{\lambda}\mathcal{I}^k g(t) \\
&\quad + \frac{t^{k-1}}{\Theta} \times \bigg[\frac{\Lambda_4(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}u(T) - \frac{\Lambda_2(\lambda - 1)}{\lambda}\mathcal{I}^{k-\theta+q_1}u(T) \\
&\quad + \frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}\big(T, u(T)\big) - \frac{\Lambda_2}{\lambda}\mathcal{I}^{k+q_1}\hat{\Upsilon}\big(T, u(T)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}g(T) - \frac{\Lambda_2}{\lambda}\mathcal{I}^{k+q_1}g(T) + \Lambda_2\delta_2 - \Lambda_4\delta_1 \Bigg] \\
&- \frac{t^{k-2}}{\Theta}\Bigg[\frac{\Lambda_3(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}u(T) - \frac{\Lambda_1(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta+q_1}u(T) \\
&+ \frac{\Lambda_3}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}(T,u(T)) - \frac{\Lambda_1}{\lambda}\mathcal{I}^{k+q_1}\hat{\Upsilon}(T,u(T)) \\
&+ \frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}g(T) - \frac{\Lambda_1}{\lambda}\mathcal{I}^{k+q_1}g(T) + \Lambda_1\delta_2 - \Lambda_3\delta_1 \Bigg].
\end{aligned}
$$

Let $v \in C_{\mathbb{R}}([0,T])$ be the unique solution of (12). Then v is given by

$$
\begin{aligned}
v(t) =\ & \frac{\lambda-1}{\lambda}\mathcal{I}^{k-\theta}v(t) + \frac{1}{\lambda}\mathcal{I}^k\hat{\Upsilon}(t,v(t)) \\
&+ \frac{t^{k-1}}{\Theta} \times \Bigg[\frac{\Lambda_4(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}v(T) - \frac{\Lambda_2(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta+q_1}v(T) \\
&+ \frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}(T,v(T)) - \frac{\Lambda_2}{\lambda}\mathcal{I}^{k+q_1}\hat{\Upsilon}(T,v(T)) + \Lambda_2\delta_2 - \Lambda_4\delta_1 \Bigg] \\
&- \frac{t^{k-2}}{\Theta}\Bigg[\frac{\Lambda_3(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}v(T) - \frac{\Lambda_1(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta+q_1}v(T) \\
&+ \frac{\Lambda_3}{\lambda}\mathcal{I}^{k-\gamma_1}\hat{\Upsilon}(T,v(T)) - \frac{\Lambda_1}{\lambda}\mathcal{I}^{k+q_1}\hat{\Upsilon}(T,v(T)) + \Lambda_1\delta_2 - \Lambda_3\delta_1 \Bigg].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\big| u(t) - v(t) \big| \\
&\leq \frac{|\lambda-1|}{\lambda}\mathcal{I}^{k-\theta}\big| u(t) - v(t) \big| + \frac{1}{\lambda}\mathcal{I}^k\big| \hat{\Upsilon}(t,u(t)) - \hat{\Upsilon}(t,v(t)) \big| \\
&\quad + \frac{T^{k-1}}{|\Theta|} \times \Bigg[\frac{\Lambda_4(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}\big| u(T) - v(T) \big| + \frac{\Lambda_2(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta+q_1}\big| u(T) - v(T) \big| \\
&\quad + \frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}\big| \hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,v(T)) \big| + \frac{\Lambda_2}{\lambda}\mathcal{I}^{k+q_1}\big| \hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,v(T)) \big| \Bigg] \\
&\quad + \frac{T^{k-2}}{|\Theta|}\Bigg[\frac{\Lambda_3(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta-\gamma_1}\big| u(T) - v(T) \big| + \frac{\Lambda_1(\lambda-1)}{\lambda}\mathcal{I}^{k-\theta+q_1}\big| u(T) - v(T) \big| \\
&\quad + \frac{\Lambda_3}{\lambda}\mathcal{I}^{k-\gamma_1}\big| \hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,v(T)) \big| + \frac{\Lambda_1}{\lambda}\mathcal{I}^{k+q_1}\big| \hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,v(T)) \big| \Bigg] \\
&\quad + \frac{1}{\lambda}\mathcal{I}^k\big| g(t) \big| + \frac{T^{k-1}}{\Theta}\Bigg(\frac{\Lambda_4}{\lambda}\mathcal{I}^{k-\gamma_1}\big| g(T) \big| + \frac{\Lambda_2}{\lambda}\mathcal{I}^{k+q_1}\big| g(T) \big| \Bigg) \\
&\quad + \frac{T^{k-2}}{\Theta}\Bigg(\frac{\Lambda_3}{\lambda}\mathcal{I}^{k-\gamma_1}\big| g(T) \big| + \frac{\Lambda_1}{\lambda}\mathcal{I}^{k+q_1}\big| g(T) \big| \Bigg) \\
&\leq \frac{1}{\lambda}\int_0^t \Bigg[|\lambda-1|\frac{(t-s)^{k-\theta-1}}{\Gamma(k-\theta)} + \mathcal{L}\frac{(t-s)^{k-1}}{\Gamma(k)} \Bigg]\big| u(s) - v(s) \big|\,ds + \overline{\mathcal{G}}(\varepsilon)\varepsilon \\
&\quad + \Bigg[\frac{T^k}{\lambda\Gamma(k+1)} + \frac{T^{2k-\gamma_1-1}\Lambda_4}{\lambda\Theta\Gamma(k-\gamma_1+1)} + \frac{T^{k+q_1-1}\Lambda_2}{\lambda\Theta\Gamma(k+q_1+1)} \\
&\quad + \frac{T^{2k-\gamma_1-2}\Lambda_3}{\lambda\Theta\Gamma(k-\gamma_1+1)} + \frac{T^{2k+q_1-2}\Lambda_1}{\lambda\Theta\Gamma(k+q_1+1)} \Bigg]\varepsilon,
\end{aligned}
$$

where

$$\int_0^T \left\{ \frac{\Lambda_4(\lambda-1)T^{k-1}}{\lambda|\Theta|} \frac{(t-s)^{k-\theta-\gamma_1-1}}{\Gamma(k-\theta-\gamma_1)} + \frac{\Lambda_2(\lambda-1)T^{k-1}}{\lambda|\Theta|} + \frac{(t-s)^{k-\theta+q_1-1}}{\Gamma(k-\theta-q_1)} \right.$$

$$+ \frac{\Lambda_4 T^{k-1}}{\lambda|\Theta|} \frac{(t-s)^{k-\gamma_1-1}}{\Gamma(k-\gamma_1)} \mathcal{L} + \frac{\Lambda_2 T^{k-1}}{\lambda|\Theta|} \frac{(t-s)^{k+q_1-1}}{\Gamma(k+q_1)} \mathcal{L}$$

$$+ \frac{\Lambda_3(\lambda-1)T^{k-2}}{\lambda|\Theta|} \frac{(t-s)^{k-\theta-\gamma_1-1}}{\Gamma(k-\theta-\gamma_1)} + \frac{\Lambda_1(\lambda-1)T^{k-2}}{\lambda|\Theta|} + \frac{(t-s)^{k-\theta+q_1-1}}{\Gamma(k-\theta-q_1)}$$

$$\left. + \frac{\Lambda_3 T^{k-2}}{\lambda|\Theta|} \frac{(t-s)^{k-\gamma_1-1}}{\Gamma(k-\gamma_1)} \mathcal{L} + \frac{\Lambda_1 T^{k-2}}{\lambda|\Theta|} \frac{(t-s)^{k+q_1-1}}{\Gamma(k+q_1)} \mathcal{L} \right\} |u(s) - v(t)| \, ds$$

$$\leq \overline{\mathcal{G}}(\varepsilon)\varepsilon$$

and $\overline{\mathcal{G}}(\varepsilon)$ is a constant dependent on ε. Let $g(t,s) = \frac{|\lambda-1|}{\lambda} \frac{1}{\Gamma(k-\theta)} + \mathcal{L} \frac{(t-s)^\theta}{\lambda\Gamma(k)}$ and

$$\Delta = \overline{\mathcal{G}}(\varepsilon) + \left[\frac{T^k}{\lambda\Gamma(k+1)} + \frac{T^{2k-\gamma_1-1}\Lambda_4}{\lambda\Theta\Gamma(k-\gamma_1+1)} + \frac{T^{2k+q_1-1}\Lambda_2}{\lambda\Theta\Gamma(k+q_1+1)} \right.$$

$$\left. + \frac{T^{2k-\gamma_1-2}\Lambda_3}{\lambda\Theta\Gamma(k-\gamma_1+1)} + \frac{T^{2k+qq_1-2}\Lambda_1}{\lambda\Theta\Gamma(k+q_1+1)} \right].$$

Then $|u(t) - v(t)| \leq \Delta\varepsilon + \int_0^t g(t,s)(t-s)^{\alpha-\theta-1}|u(s) - v(t)| \, ds$. Note that

$$g(t,s) \leq \frac{|\lambda-1|}{\lambda} \frac{1}{\Gamma(k-\theta)} + \mathcal{L}\frac{T^\theta}{\lambda\Gamma(k)} = M.$$

In view of Lemma 5, we get

$$|u(t) - v(t)| \leq \Delta\varepsilon + \Delta\varepsilon \int_0^t \sum_{n=1}^\infty \frac{(g(t,s)\Gamma(k-\theta))^n}{\Gamma(n(k-\theta))} (t-s)^{n(k-\theta)-1} \, ds$$

$$\leq \Delta\varepsilon + \Delta\varepsilon \int_0^t \sum_{n=1}^\infty \frac{(M\Gamma(k-\theta))^n}{\Gamma(n(k-\theta))} (t-s)^{n(k-\theta)-1} \, ds$$

$$\leq \Delta\varepsilon + \Delta\varepsilon \sum_{n=1}^\infty \frac{(M\Gamma(k-\theta))^n}{\Gamma(n(k-\theta)+1)} T^{n(k-\theta)}$$

$$\leq \Delta\varepsilon E_{k-\theta}\left(MT^{(k-\theta)}\Gamma(k-\theta)\right).$$

Put $c = E_{k-\theta}(MT^{(k-\theta)}\Gamma(k-\theta))$. Note that the inequality $|u(t) - v(t)| < c\varepsilon$ holds. Thus, boundary value problem (12) is Hyers–Ulam stable. □

5 Examples
Now, we provide two examples to illustrate our main results.

Example 1 Consider the boundary value problem

$$
\begin{cases}
\frac{47}{54}\mathcal{D}^{5/2}u(t) + \frac{7}{54}\mathcal{D}^{2.03}u(t) = t^2\cos(u(t)), & t \in [0, \frac{1}{4}], \\
u(0) = 0, \\
\mu_1\mathcal{D}^{7/15}u(\frac{1}{4}) + (1 - \mu_1)\mathcal{D}^{1/8}u(\frac{1}{4}) = \frac{1}{16}, \\
\mu_2\mathcal{I}^{3/4}u(\frac{1}{4}) + (1 - \mu_2)\mathcal{I}^{5/3}u(\frac{1}{4}) = \frac{5}{12}.
\end{cases}
\tag{13}
$$

Put $\lambda = 47/54$, $k = 5/2$, $\theta = 2.1$, $\gamma_1 = 7/15$, $\gamma_2 = 1/8$, $q_1 = 3/4$, $q_2 = 5/3$, $\delta_1 = 1/16$, $\delta_2 = 5/12$, and $T = 1/4$. Note that $0 < \gamma_1, \gamma_2 < 0.47 = k - \theta$ and

$$
\left|\hat{\Upsilon}\big(t, u(t)\big) - \Upsilon\big(t, u'(t)\big)\right| \leq \left(\frac{1}{4}\right)^2 \left|u(t) - u'(t)\right|
$$

with $\mathcal{L} = 1/16$ and $|\hat{\Upsilon}(t, u(t))| \leq t^2 = \mathcal{V}(t)$. If $\mu_1 = 1/3$ and $\mu_2 = 3/4$, then

$$\Lambda_1 \approx 0.2120, \qquad \Lambda_2 \approx 0.6825, \qquad \Lambda_3 \approx 0.0178, \qquad \Lambda_4 \approx 0.1084,$$

$$\Theta \approx 0.0108, \qquad \mathcal{W}_1 \approx 0.6761, \qquad \mathcal{W}_2 \approx 0.0751.$$

Hence, $\mathcal{L}\mathcal{W}_2 + \mathcal{W}_1 \approx 0.6808 < 1$. Now, by using Theorem 8, problem (13) has a unique solution. If $\mu_1 = 1$ and $\mu_2 = 1$, then $g(t, s) = \frac{|\lambda - 1|}{\lambda}\frac{1}{\Gamma(k-\theta)} + \mathcal{L}\frac{(t-s)^\theta}{\lambda\Gamma(k)}$. Note that

$$
\left|g(t,s)\right| \leq \frac{|\lambda - 1|}{\lambda}\frac{1}{\Gamma(k - \theta)} + \mathcal{L}\frac{T^\theta}{\lambda\Gamma(k)} \approx 0.0823 = M.
$$

Now, by using Theorem 12, problem (2) is Hyers–Ulam stable.

Example 2 Consider the boundary value problem

$$
\begin{cases}
\frac{40}{45}\mathcal{D}^{13/5}u(t) + \frac{5}{45}\mathcal{D}^{2.01}u(t) = \frac{1}{t^2+5}\big(\frac{u^2(t)}{|u(t)|+1} + 4\big), & t \in [0, \frac{1}{4}], \\
u(0) = 0, \\
\frac{11}{33}\mathcal{D}^{7/88}u(\frac{1}{4}) + \frac{22}{33}\mathcal{D}^{3/100}u(\frac{1}{4}) = \frac{21}{156}, \\
\frac{9}{13}\mathcal{I}^{5/4}u(\frac{1}{4}) + \frac{4}{13}\mathcal{I}^{23/33}u(\frac{1}{4}) = \frac{55}{122}.
\end{cases}
\tag{14}
$$

Put $\lambda = 40/45$, $k = 13/5$, $\theta = 2.01$, $\gamma_1 = 7/88$, $\gamma_2 = 3/100$, $q_1 = 5/4$, $q_2 = 23/33$, $\delta_1 = 21/156$, $\delta_2 = 55/122$, and $T = \frac{1}{4}$. Note that $0 < \gamma_1, \gamma_2 < 0.59 = k - \theta$ and

$$
\hat{\Upsilon}\big(t, u(t)\big) = \frac{1}{t^2 + 5}\left(\frac{u^2(t)}{|u(t)| + 1} + 4\right).
$$

Assume that $\mu_1 = 11/33$ and $\mu_2 = 9/13$. Then we have

$$\Lambda_1 \approx 0.1203, \qquad \Lambda_2 \approx 0.4667, \qquad \Lambda_3 \approx 0.0106, \qquad \Lambda_4 \approx 0.0663,$$

$$\Theta \approx 0.0030, \qquad \mathcal{W}_1 \approx 0.6462, \qquad \mathcal{W}_2 \approx 0.0652,$$

and $|\hat{\Upsilon}(t, u(t))| = |\frac{1}{t^2+5}(\frac{u^2(t)}{|u(t)|+1} + 4)| \leq \frac{1}{t^2+5}(|x| + 4)$. Put $\Phi(t) = \frac{1}{t^2+5}$ and $\Psi(|u|) = |x| + 4$ and choose $\mathcal{Q} > 379.5499$ such that

$$\frac{\mathcal{Q}|\Theta|}{\mathcal{Q}|\Theta|\mathcal{W}_1 + \Psi(\mathcal{Q})\|\Phi\||\Theta|\mathcal{W}_2 + T^{k-1}(|\Lambda_2\delta_2| + |\Lambda_4\delta_1|) + T^{k-2}(|\Lambda_1\delta_2| + |\Lambda_3\delta_1|)} > 1.$$

Now, by using Theorem 10, problem (14) has at least one solution.

6 Conclusion

As many researchers would like to investigate the stability notion of different boundary value problems, this can be a motivation for us to study the stability of complicated systems supplemented with general boundary conditions. Hence, our main goal in the present manuscript is to obtain some existence criteria of a new general boundary value problem including 2-term fractional differential equation which contains multi-order Riemann–Liouville fractional derivatives and integrals. In the sequel, we check Hyers–Ulam stability of the proposed problem in the special case $\mu_1 = 1$ and $\mu_2 = 1$. Finally, we provide two illustrative examples to support our theoretical findings. This work can be an introduction for other researchers to study mentioned notions for numerous fractional multi-order modelings in the future.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

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Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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