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Investigation of linear difference equations with random effects

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Abstract

In this study, random linear difference equations obtained by transforming the components of deterministic difference equations to random variables are investigated. Uniform, Bernoulli, binomial, negative binomial (or Pascal), geometric, hypergeometric and Poisson distributions have been used for the random effects for obtaining the random behavior of linear difference equations. The random version of the Z-transform, the RZ-transform, has been used to obtain an approximation for the random linear difference equation. Approximate expected values and variances are calculated by using the RZ-transform. The results have been obtained with Maple and are shown in graphs. It is shown that the random Z-transform is an effective tool for the investigation of random linear difference equations.

Keywords: Linear difference equations; Expected value; Variance; Z-transform method

1 Introduction

Difference equations are known as the first theory to emerge with the systematical development of mathematics. Having emerged as the discrete analogues of differential equations, difference equations are a field of mathematics with a rich application area. Difference equations found in economy, biology, signal processing, computer engineering, genetics, medicine, ecology and digital control some of its application fields [1–9].

Today, difference equations are used for system analysis and design with the use of the Z-transform. The Z-transform has been introduced in mid-20th century [10]. The z-transform, together with probability theory, was first introduced by de Moivre in 1730 and is known among mathematicians as the "generator function method". The Z-transform of an array is the generating function of this array, where the independent variable Z is replaced by a mutual $1/Z$. It has been used within probability theory [11] and to treat data control systems [12]. In 1952, it was called the Z-transform by Ragazzini and Zadeh [13]. Recently, the Z-transform has been highly developed as a result of its use in digital computer systems. In these systems, discrete system theory is developed due to the fact that information and signals are discrete. The Z-transformation method is one of the transformation methods that can be applied to the solution of linear difference equations. It reduces the solutions of such equations to algebraic solutions and resembles the Laplace transform method commonly used in solving differential equations [12]. Some of the studies in this area are given in [14–23].

In this study the random version of the Z-transform, the RZ-transform, has been applied to obtain an approximation to the solution of random linear difference equation. Difference equations are transformed into random difference equations through the use of several probability distributions. The approximate solutions obtained by Z-transform are used to obtain the approximate expected values and variances of random linear difference equations, which are shown in graphs. Sections 2 and 3 contain introductory information on linear difference equations and Z-transform method. Section 4 contains numerical examples with uniform, geometric, Poisson and Bernoulli distributions.

2 Difference equations

Definition 1 The equation with $n \in \mathbb{N} = \{0, 1, \ldots\}$ independent variables and unknown x such that

$$F\bigl(n, x(n), x(n+1), \ldots, x(n+k)\bigr) = 0$$

is called a difference equation [1–9].

Definition 2 Let x be a continuous variable and $h_k(x)$ and $g(x)$ be real valued functions with $n \geq n_0$ such that $h_k \neq 0$, then the difference equation

$$h_0(x)f(x+n) + h_1 f(x+n-1) + \cdots + h_n(x)f(x) = g(x) \tag{1}$$

is called a linear difference equation of order n.

Equation (1) is called a linear homogeneous difference equation of order n for $g(x) = 0$ and a non-homogeneous linear difference equation of order n for $g(x) \neq 0$ [13].

Z-transform

Definition 3 ([10–13, 24–27]; Z-transform) Let the sequence $x(n)$ defined for the negative integers $n = -1, -2, \ldots$ be given. The Z-transform for the sequence $x(n)$ is given as

$$\tilde{x}(z) = Z\bigl(x(n)\bigr) = \sum_{i=0}^{\infty} x(i) z^{-i}. \tag{2}$$

Theorem 1 *The set of numbers z on the complex plane is called the convergence region $x(z)$ for the convergence of the sequence* (2). *The ratio test is the most used method for finding the convergence region of the sequence* (2). *Assume that*

$$\lim_{i\to\infty} \left| \frac{x(i+1)}{x(i)} \right| = R.$$

The convergence of (2) with ratio test is

$$\lim_{i\to\infty} \left| \frac{x(i+1)z^{-i-1}}{x(i)z^{-i}} \right| < 1$$

and its divergence

$$\lim_{i\to\infty}\left|\frac{x(i+1)z^{-i-1}}{x(i)z^{-i}}\right| > 1.$$

Hence, the sequence (2) is convergent for $|z| > R$ and divergent for $|z| < R$. The number R is called the radius of convergence. If $R = 0$, the sequence $\tilde{x}(z)$ is everywhere convergent. On the other side, if $R = \infty$, the Z-transform is everywhere divergent.

Properties of Z-transform

1. Linearity: Let $\tilde{x}(z)$ be the Z-transform of $x(n)$ with a radius of convergence of R_1 and $\tilde{y}(z)$ be the Z-transform of $y(n)$ with a radius of convergence of R_2. For complex numbers λ, β

$$Z\big[\lambda x(n) + \beta y(n)\big] = \lambda\tilde{x}(z) + \beta\tilde{y}(z), \quad |z| > \max(R_1, R_2) \tag{3}$$

2. Shifting: Let R be the radius of convergence of $\tilde{x}(z)$.

a. Right shifting: If $x(-j) = 0$, $j = 1, 2, \ldots, k$, then

$$Z\big[x(n-k)\big] = z^{-k}\tilde{x}(z), \quad |z| > R. \tag{4}$$

b. Left shifting:

$$Z\big[x(n+k)\big] = z^{k}\tilde{x}(z) - \sum_{t=0}^{k-1} x(t)z^{k-t}, \quad |z| > R. \tag{5}$$

The most used versions of (5) are

$$Z\big[x(n+1)\big] = z^{k}\tilde{x}(z) - zx(0), \quad |z| > R,$$
$$Z\big[x(n+2)\big] = z^{2}\tilde{x}(z) - z^{2}x(0) - zx(1), \quad |z| > R.$$

3. Initial and final values:

a. Initial value theorem:

$$\lim_{|z|\to\infty} \tilde{x}(z) = x(0). \tag{6}$$

b. Final value theorem:

$$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\tilde{x}(z). \tag{7}$$

Proof a. The proof of (6) follows from the definition of $\tilde{x}(z)$.

b. To prove (7)

$$Z\big[x(n+1) - x(n)\big] = \sum_{i=0}^{\infty}\big[x(i+1) - x(i)\big]z^{-i}.$$

If (5) is applied to the left of this equation

$$(z-1)\tilde{x}(z) = x(0) + \sum_{i=0}^{\infty}\big[x(i+1) - x(i)\big]z^{-i}.$$

Thus,

$$\lim_{z\to 1}(z-1)\tilde{x}(z) = x(0) + \sum_{i=0}^{\infty}\bigl[x(i+1) - x(i)\bigr]z^{-i} = \lim_{n\to\infty} x(n). \qquad \square$$

4. Multiplication by a^n: Let $\tilde{x}(z)$ be the Z-transform of $x(n)$ with a radius of convergence of R. Then

$$Z\bigl[a^n x(n)\bigr] = \tilde{x}\left(\frac{z}{a}\right), \quad |z| > |a|R. \tag{8}$$

The proof of (8) follows from the definition.

5. Multiplication by n^k:

$$Z\bigl(na^n\bigr) = -z\frac{d}{dz}Z\bigl(a^n\bigr).$$

Similarly, if the order is increased

$$Z\bigl(n^2a^n\bigr) = -z\frac{d}{dz}\left[-z\frac{d}{dz}Z\bigl(a^n\bigr)\right]$$

and hence

$$Z\bigl(n^2a^n\bigr) = \left(-z\frac{d}{dz}\right)^2 Z\bigl(a^n\bigr).$$

Its generalization gives

$$\left(-z\frac{d}{dz}\right)^k \tilde{x}(z) = \left(-z\frac{d}{dz}\left(-z\frac{d}{dz}\left(\dots\left(-z\frac{d}{dz}\right)\dots\right)\right)\right).$$

Thus

$$Z\bigl[n^k x(n)\bigr] = \left(-z\frac{d}{dz}\right)^k Z\bigl(x(n)\bigr). \tag{9}$$

Definition (Inverse Z-transform) A Z-transform transforms the difference equation of an unknown $x(n)$ sequence to an algebraic equation in $\tilde{x}(z)$. Afterwards, the $x(n)$ sequence is obtained from $\tilde{x}(z)$ through an operation known as inverse Z-transform. Symbolically, this operation can be shown as

$$Z^{-1}\bigl[\tilde{x}(z)\bigr] = x(n). \tag{10}$$

The uniqueness of the inverse Z-transform can be obtained as follows. Assume two sequences $x(n)$, $y(n)$ have the same Z-transform, i.e.

$$\sum_{i=0}^{\infty} x(i)z^{-i} = \sum_{i=0}^{\infty} y(i)z^{-i}, \quad |z| > R.$$

Table 1 Z-transform

$x(n), n = 0,1,2,3,\ldots$	$\tilde{x}(z) = \sum_{i=0}^{\infty} x(i)z^{-i}$
1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$
a^{n-1}	$\frac{1}{z-a}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
n^k	$(-1)^k D^k(\frac{z}{z-1}); D = z\frac{d}{dz}$
na^n	$\frac{az}{(z-a)^2}$
n^2a^n	$\frac{az(z+a)}{(z-a)^3}$
n^ka^n	$(-1)^k D^k(\frac{z}{z-a}); D = z\frac{d}{dz}$
$x(n-k)$	$z^{-k}\tilde{x}(z)$
$x(n+k)$	$z^{-k}\tilde{x}(z) - \sum_{r=0}^{k-1} x(r)z^{k-r}$

Hence,

$$\sum_{i=0}^{\infty}\bigl[x(i) - y(i)\bigr]z^{-i} = 0, \quad |z| > R$$

is used to obtain the inverse Z-transform [25].

3 Discrete time probability distributions

3.1 Discrete uniform distribution

Definition For a random variable, the value of the probability function is a discrete probability distribution if all the values between the lower and upper bounds of the integer are constant. For any random positive n in the form of $k_1, k_2, \ldots, k_n$

$$P(k,n) = \begin{cases} 1/n, & k = 1,2,3,\ldots,n, \\ 0, & \text{otherwise}, \end{cases}$$

if each of them shows equal probability, then this random variable is called a (discrete) uniform distributed random variable [28, 29].

Theorem *If X has a discrete uniform distribution, then*

a. $E(X) = \frac{k+1}{2}$,

b. $V(X) = \frac{k^2-1}{12}$,

c. $M_x(t) = \frac{1}{k}\sum_{x=1}^{k} e^{tx}$.

3.2 Bernoulli distribution

Definition For the random variable X, p takes a value of 1 with probability of success, and X, takes a value of 0 with probability of failure where $q = 1 - p$, then this random variable is called the Bernoulli random variable. The Bernoulli probability mass function is given as [28, 29]:

$$f(x,p) = p^x(1-p)^{1-x}, \quad x = 0,1.$$

Theorem *If X has a Bernoulli distribution,*

a. $E(X) = p$,

b. $V(X) = p(1-p)$,

c. $M_x(t) = e^t p + (1-p)$.

3.3 Binomial distribution

Definition Let n be the total number of independent Bernoulli successful trials and X be the random variable. If the probability of success for a single experiment is p and the probability of failure $(1-p)$, then X is called the binomial random variable X and the probability function of X is

$$f(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,\dots,n.$$

Calculation of consecutive binomial probabilities:

$$f(x+1;n,p) = \frac{(n-x)p}{(x+1)(1-p)} f(x;n,p), \quad x = 0,1,\dots,n-1.$$

Theorem *If X has a binomial distribution,*

a. $E(X) = np$,

b. $V(X) = np(1-p)$,

c. $M_x(t) = [e^t p + (1-p)]^n$.

3.4 Negative binomial (Pascal) distribution

Definition Let X be the random variable of the number of trials required to achieve success $K \geq 1$, with the probability of success p in each experiment for independent Bernoulli trials. In this case, X is called a negative binomial random variable and its probability function is given as

$$f(x) = \binom{x-1}{K-1} p^K (1-p)^{x-K}, \quad x = K, K+1, \dots.$$

Theorem *If X has a negative binomial distribution,*

a. $E(X) = \frac{k}{p}$,

b. $V(X) = \frac{k(1-p)}{p^2}$,

c. $M_x(t) = \frac{p^k e^{tk}}{(1-e^t q)^k}$.

3.5 Geometric distribution

Definition If the number of Bernoulli trials is repeated n times, the number of tests performed to achieve the first success is called geometric distribution and if X is the probability of success for each trial, p is the probability function for the number of trials required to achieve a single success [28, 29],

$$f(x) = P(X = k) = q^{k-1} p, \quad x = 1,2,3,\dots.$$

Theorem *If X has a geometric distribution,*

a. $E(X) = \frac{1}{p}$,
b. $V(X) = \frac{(1-p)}{p^2}$,
c. $M_x(t) = pe^t \frac{1}{1-[e^t(1-p)]}$.

3.6 Hypergeometric distribution

Definition Let a be the number of elements of a given Type A in a mass consisting of a finite number of N elements. Let X be the number of elements of its type in a sample of n units that are randomly drawn without replacing them again. X is a random hypergeometric variable and the hypergeometric probability mass function is given as [28, 29]

$$f(x; N, M, n) = \frac{\binom{M}{X}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, \ldots, n.$$

Theorem *If X has a hypergeometric distribution,*
a. $E(X) = \frac{nM}{N}$,
b. $V(X) = \frac{M(M-1)n(n-1)}{N(N-1)} + M\frac{n}{N}$,
c. $M_x(t) = \frac{N-n}{N-1} n \frac{M}{N}(1 - \frac{M}{N})$.

3.7 Poisson distribution

Definition $f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots, \lambda > 0$. The Taylor expansion of the function e^y and the probability function gives ($e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!}$):

$$\sum_{x=0}^{\infty} f(X = x; \lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

Theorem *If X has a Poisson distribution,*
a. $E(X) = \lambda$,
b. $V(X) = \lambda$,
c. $M_x = e^{\lambda(e^t-1)}$.

4 Numerical examples

Some numerical examples are given for random linear difference equations through the use of various probability distributions.

Example 1 Let A, B be random variables with uniform distribution such that

$$x(n+2) + 5x(n+1) + 4x(n) = 0, \quad x(0) = A, x(1) = B. \tag{11}$$

We investigate the behavior of the solution of (11) with the Z-transform method.
Solution. The Z-transform of both sides of (11) gives

$$\begin{aligned} &z^2\tilde{x}(z) - z^2x(0) - zx(1) + 5\tilde{x}(z) - 5zx(0) + 4\tilde{x}(z) = 0, \\ &\tilde{x}(z)\left[z^2 + 5z + 4\right] - z^2A - zB - 5zA = 0. \end{aligned} \tag{12}$$

Partial fractions of $\tilde{x}(z)/z$ gives

$$\frac{\tilde{x}(z)}{z} = \frac{z^2A + B + 5A}{(z+1)(z+4)} = \frac{a_1}{z+1} + \frac{a_2}{z+4}. \tag{13}$$

It can be found that $a_1 = \frac{4A+B}{3}$, $a_2 = \frac{-A-B}{3}$. Hence,

$$\tilde{x}(z) = \frac{(\frac{4A+B}{3})z}{z+1} + \frac{(\frac{-A-B}{3})z}{z+4}. \tag{14}$$

The inverse Z-transform $Z^{-1}[\tilde{x}(z)] = x(n)$ gives

$$x(n) = \left(\frac{4A+B}{3}\right)(-1)^n + \left(\frac{-A-B}{3}\right)(-4)^n. \tag{15}$$

Higher moments of the random variables are needed to obtain the approximate expected values and variances. The moment generating function of a uniformly distributed random variable $X \sim U(\alpha, \beta)$ is given as [29]

$$M_X(t) = E\big[e^{tX}\big] = \frac{e^{\beta t} - e^{\alpha t}}{(\beta - \alpha)t}.$$

Hence, the expected value and variance of the random variable X are

$$E[X] = \frac{\alpha + \beta}{2}, \qquad \operatorname{Var}[X] = \frac{(\alpha - \beta)^2}{12}.$$

To find the numerical characteristics of (5), we start with the expectation:

$$E\big[x(n)\big] = E\left[\left(\frac{4A+B}{3}\right)(-1)^n\right] + E\left[\left(\frac{-A-B}{3}\right)(-4)^n\right] \tag{16}$$

which gives

$$= \left(\frac{4}{3}(-1)^n - \frac{1}{3}(-4)^n\right)E[A] + \left(-\frac{1}{3}(-4)^n + \frac{1}{3}(-1)^n\right)E[B].$$

The variance is obtained as follows:

$$\operatorname{Var}\big[x(n)\big] = \operatorname{Var}\left[\left(\frac{4A+B}{3}\right)(-1)^n\right] + \operatorname{Var}\left[\left(\frac{-A-B}{3}\right)(-4)^n\right] \tag{17}$$

which is found to be

$$= \operatorname{Var}[A]\left(\frac{1}{9}(16)^n + \frac{16}{9}\right) + \operatorname{Var}[B]\left(\frac{1}{9}(16)^n + \frac{1}{9}\right).$$

Let us investigate the case $\alpha = 3$ and $\beta = 1$ for $A, B \sim U(\alpha = 3, \beta = 1)$. The numerical characteristics of the approximate solution of the random linear difference equation obtained from the random Z-transform are obtained as follows [30–44]:

$$\begin{aligned} E\big[x(n)\big] &= E\left[\left(\frac{4A+B}{3}\right)(-1)^n\right] + E\left[\left(\frac{-A-B}{3}\right)(-4)^n\right] \\ &= \left(\frac{4}{3}(-1)^n - \frac{1}{3}(-4)^n\right)E[A] + \left(-\frac{1}{3}(-4)^n + \frac{1}{3}(-1)^n\right)E[B] \end{aligned}$$

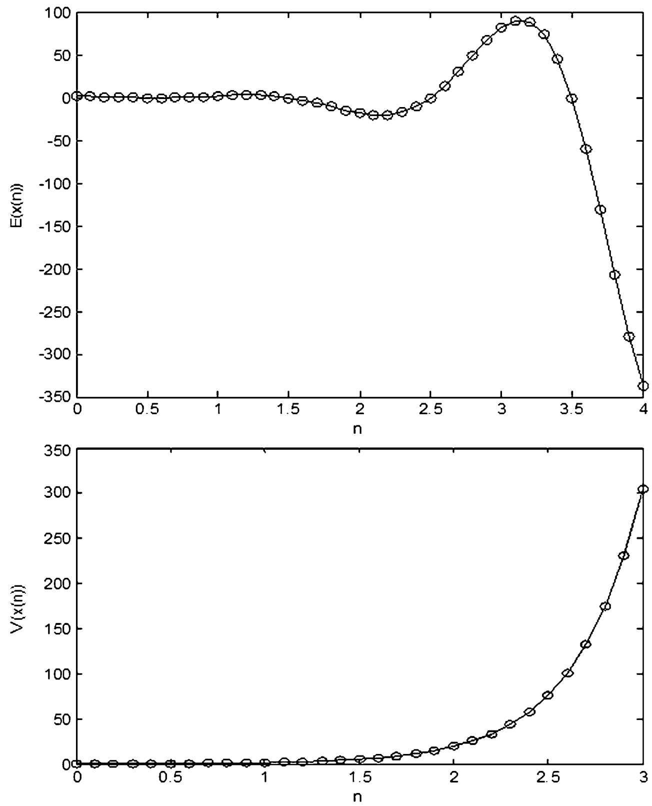

Figure 1 Expected value and variance obtained from the Z-transform of (11)

$$
\begin{aligned}
&= \left(\frac{4}{3}(-1)^n - \frac{1}{3}(-4)^n\right)2 + \left(-\frac{1}{3}(-4)^n + \frac{1}{3}(-1)^n\right)2 \\
&= \left(\frac{10}{3}\right)(-1)^n + \left(\frac{-4}{3}\right)(-4)^n, \\
\operatorname{Var}\bigl[x(n)\bigr] &= \operatorname{Var}\left[\left(\frac{4A+B}{3}\right)(-1)^n\right] + \operatorname{Var}\left[\left(\frac{-A-B}{3}\right)(-4)^n\right] \\
&= \operatorname{Var}[A]\left(\frac{1}{9}(16)^n + \frac{16}{9}\right) + \operatorname{Var}[B]\left(\frac{1}{9}(16)^n + \frac{1}{9}\right) \\
&= \frac{1}{3}\left(\frac{1}{9}(16)^n + \frac{16}{9}\right) + \frac{1}{3}\left(\frac{1}{9}(16)^n + \frac{1}{9}\right) \\
&= \left(\frac{2}{27}\right)(16)^n + \left(\frac{17}{27}\right).
\end{aligned}
$$

Example 2 Let A, B be random variables with geometric distribution such that $A, B \sim G(p,q)$ where $p = \frac{1}{3}$ and $q = \frac{2}{3}$.

$$x(n+2) - 2x(n+1) - 3x(n) = 2^n, \quad x(0) = A, x(1) = B. \tag{18}$$

We investigate the behavior of the solution of (18) with the Z-transform method.

Solution. The Z-transform of both sides gives

$$\begin{aligned} &z^2\tilde{x}(z) - z^2x(0) - zx(1) - 2\tilde{x}(z) + 2zx(0) - 3\tilde{x}(z) = \frac{z}{z-2}, \\ &\tilde{x}(z)\bigl[z^2 - 2z - 3\bigr] - z^2A - zB + 2zA = \frac{z}{z-2}. \end{aligned} \tag{19}$$

Partial fraction of $\tilde{x}(z)/z$ gives

$$\frac{\tilde{x}(z)}{z} = \frac{1 + (z-2)[A(z-2) + B]}{(z-2)(z-3)(z+1)} = \frac{a_1}{z-2} + \frac{a_2}{z-3} + \frac{a_3}{z+1}. \tag{20}$$

This equation gives $a_1 = \frac{-1}{3}$, $a_2 = \frac{1+A+B}{4}$, $a_3 = \frac{1+9A-3B}{12}$. Hence

$$\tilde{x}(z) = \frac{(\frac{-1}{3})z}{z-2} + \frac{(\frac{1+A+B}{4})z}{z-3} + \frac{(\frac{1+9A-3B}{12})z}{z+1}. \tag{21}$$

The inverse Z-transform $Z^{-1}[\tilde{x}(z)] = x(n)$ gives

$$x(n) = \left(\frac{-1}{3}\right)(2)^n + \left(\frac{1}{4}\right)(3)^n + \left(\frac{1}{12}\right)(-1)^n + \left(\frac{A}{4} + \frac{B}{4}\right)(3)^n + \left(\frac{3A}{4} - \frac{B}{4}\right)(-1)^n. \tag{22}$$

The moment generating function of a geometrically distributed random variable $X \sim G(p,q)$ is given as [29]

$$M_X(t) = E\bigl[e^{tX}\bigr] = \frac{pe^t}{1 - qe^t}.$$

Hence, the expected value and variance of the random variable X are

$$E[X] = \frac{1}{p}, \qquad \operatorname{Var}[X] = \frac{q}{p^2}.$$

To find the numerical characteristics of (22), we start with the expectation:

$$\begin{aligned} E\bigl[x(n)\bigr] = {}& E\left[\left(\frac{-1}{3}\right)(2)^n\right] + E\left[\left(\frac{1}{4}\right)(3)^n\right] + E\left[\left(\frac{1}{12}\right)(-1)^n\right] \\ & + E\left[\left(\frac{A}{4} + \frac{B}{4}\right)(3)^n\right] + E\left[\left(\frac{3A}{4} - \frac{B}{4}\right)(-1)^n\right] \end{aligned} \tag{23}$$

which gives

$$= \frac{-1}{3}(2)^n + \frac{1}{4}(3)^n + \frac{1}{12}(-1)^n + E[A]\left(\frac{1}{4}(3)^n + \frac{3}{4}(-1)^n\right) + E[B]\left(\frac{1}{4}(3)^n - \frac{1}{4}(-1)^n\right).$$

The variance is obtained as follows:

$$\begin{aligned} \operatorname{Var}\bigl[x(n)\bigr] = {}& \operatorname{Var}\left[\left(\frac{-1}{3}\right)(2)^n\right] + \operatorname{Var}\left[\left(\frac{1}{4}\right)(3)^n\right] + \operatorname{Var}\left[\left(\frac{1}{12}\right)(-1)^n\right] \\ & + \operatorname{Var}\left[\left(\frac{A}{4} + \frac{B}{4}\right)(3)^n\right] + \operatorname{Var}\left[\left(\frac{3A}{4} - \frac{B}{4}\right)(-1)^n\right] \end{aligned} \tag{24}$$

which is found to be

$$= \mathrm{Var}[A]\left(\frac{9}{16}+\frac{1}{16}(9)^n\right)+\mathrm{Var}[B]\left(\frac{1}{16}+\frac{1}{16}(9)^n\right).$$

Let the parameters be $A, B \sim G(p=\frac{1}{3}, q=\frac{2}{3})$. The numerical characteristics of the approximate solution of the random linear difference equation obtained from the random Z-transform are as follows [30–44]:

$$\begin{aligned}
E\bigl[x(n)\bigr] &= E\left[\left(\frac{-1}{3}\right)(2)^n\right]+E\left[\left(\frac{1}{4}\right)(3)^n\right]+E\left[\left(\frac{1}{12}\right)(-1)^n\right]\\
&\quad + E\left[\left(\frac{A}{4}+\frac{B}{4}\right)(3)^n\right]+E\left[\left(\frac{3A}{4}-\frac{B}{4}\right)(-1)^n\right]\\
&= \frac{-1}{3}(2)^n+\frac{1}{4}(3)^n+\frac{1}{12}(-1)^n+E[A]\left(\frac{1}{4}(3)^n+\frac{3}{4}(-1)^n\right)\\
&\quad + E[B]\left(\frac{1}{4}(3)^n-\frac{1}{4}(-1)^n\right)\\
&= \frac{-1}{3}(2)^n+\frac{1}{4}(3)^n+\frac{1}{12}(-1)^n+3\left(\frac{1}{4}(3)^n+\frac{3}{4}(-1)^n\right)+3\left(\frac{1}{4}(3)^n-\frac{1}{4}(-1)^n\right)\\
&= \frac{-1}{3}(2)^n+\frac{7}{4}(3)^n+\frac{19}{12}(-1)^n,\\
\mathrm{Var}\bigl[x(n)\bigr] &= \mathrm{Var}\left[\left(\frac{-1}{3}\right)(2)^n\right]+\mathrm{Var}\left[\left(\frac{1}{4}\right)(3)^n\right]+\mathrm{Var}\left[\left(\frac{1}{12}\right)(-1)^n\right]\\
&\quad + \mathrm{Var}\left[\left(\frac{A}{4}+\frac{B}{4}\right)(3)^n\right]+\mathrm{Var}\left[\left(\frac{3A}{4}-\frac{B}{4}\right)(-1)^n\right]\\
&= \mathrm{Var}[A]\left(\frac{9}{16}+\frac{1}{16}(9)^n\right)+\mathrm{Var}[B]\left(\frac{1}{16}+\frac{1}{16}(9)^n\right)\\
&= 6\left(\frac{9}{16}+\frac{1}{16}(9)^n\right)+6\left(\frac{1}{16}+\frac{1}{16}(9)^n\right)\\
&= \frac{15}{4}+\frac{3}{4}(9)^n.
\end{aligned}$$

Example 3 Let A, B be random variables with a Poisson distribution such that

$$\begin{aligned}
&x(n+3)-2x(n+2)-x(n+1)+2x(n)\\
&\quad = A(-3)^n+B2^n, \quad x(0)=-1, x(1)=1, x(2)=2.
\end{aligned} \tag{25}$$

We investigate the behavior of the solution of (25) with the Z-transform method.

Solution. The Z-transform of both sides gives

$$\begin{aligned}
&z^3\tilde{x}(z)-z^3x(0)-z^2x(1)-zx(2)-2z^2\tilde{x}(z)+2z^2x(0)+2zx(1)-z\tilde{x}(z)+zx(0)+2\tilde{x}(z)\\
&\quad = \frac{Az}{z+3}+\frac{Bz}{z-2},\\
&\tilde{x}(z)\bigl[z^3-2z^2-z+2\bigr]-z\bigl(-z^2+3z+1\bigr)=\frac{Az}{z+3}+\frac{Bz}{z-2}.
\end{aligned} \tag{26}$$

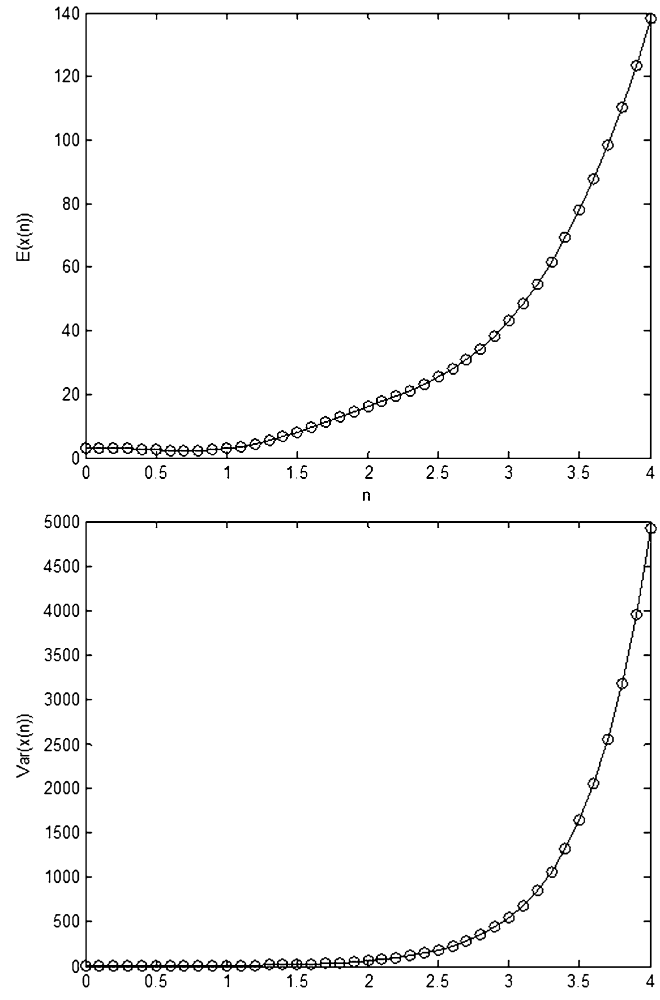

Figure 2 Expected value and variance obtained from the Z-transform of (18)

Partial fraction of $\tilde{x}(z)/z$ gives

$$\begin{aligned}\frac{\tilde{x}(z)}{z} &= \frac{A(z-2)+B(z+3)+(1+3z-z^2)(z+3)(z-2)}{(z+3)(z+1)(z-1)(z-2)^2} \\ &= \frac{b_1}{z+3}+\frac{a_1}{z+1}+\frac{a_2}{z-1}+\frac{a_3}{z-2}+\frac{a_4}{(z-2)^2}.\end{aligned} \tag{27}$$

Here,

$$\begin{aligned}&b_1 = \frac{-A}{40}, \qquad a_1 = \frac{A}{12}-\frac{B}{18}-\frac{1}{2}, \qquad a_2 = \frac{-A}{8}+\frac{B}{2}-\frac{3}{2}, \\ &a_3 = \frac{A}{15}-\frac{8B}{18}+1, \qquad a_4 = \frac{B}{3}.\end{aligned}$$

Hence,

$$\tilde{x}(z) = \frac{\frac{-A}{40}z}{z+3}+\frac{(\frac{A}{12}-\frac{B}{18}-\frac{1}{2})z}{z+1}+\frac{(\frac{-A}{8}+\frac{B}{2}-\frac{3}{2})z}{z-1}+\frac{(\frac{A}{15}-\frac{8B}{18}+1)z}{z-2}+\frac{\frac{B}{3}z}{(z-2)^2}.$$

The inverse Z-transform $Z^{-1}[\tilde{x}(z)] = x(n)$ gives

$$x(n) = \left(\frac{-A}{40}\right)(-3)^n + (2)^n - \left(\frac{1}{2}\right)(-1)^n - \frac{3}{2}(1)^n + \left(\frac{A}{12} - \frac{B}{18}\right)(-1)^n + \left(\frac{B(n+1)}{6}\right)(2)^n + \left(-\frac{A}{15} + \frac{11B}{18}\right)(2)^n - \frac{A}{8} + \frac{B}{2}. \tag{28}$$

The moment generating function of a Poisson distributed random variable $X \sim P(\lambda)$ is given as [29]

$$M_x = e^{\lambda(e^t-1)}.$$

Hence, the expected value and variance of the random variable X are

$$E(X) = \lambda, \qquad V(X) = \lambda.$$

To find the numerical characteristics of (28), we start with the expectation:

$$\begin{aligned} E\big[x(n)\big] &= E\left[\left(\frac{-A}{40}\right)(-3)^n\right] + E\big[(2)^n\big] + E\left[-\left(\frac{1}{2}\right)(-1)^n\right] + E\left[-\frac{3}{2}\right] \\ &+ E\left[\left(\frac{A}{12} - \frac{B}{18}\right)(-1)^n\right] + E\left[\left(\frac{B(n+1)}{6}\right)(2)^n\right] + E\left[\left(-\frac{A}{15} + \frac{11B}{18}\right)(2)^n\right] \\ &+ E\left[\frac{-A}{8} + \frac{B}{2}\right] \end{aligned} \tag{29}$$

which gives

$$\begin{aligned} &= (2)^n - \left(\frac{1}{2}\right)(-1)^n - \frac{3}{2} + E[A]\left(\left(\frac{-1}{40}\right)(-3)^n + \frac{1}{12}(-1)^n - \frac{1}{15}(2)^n - \frac{1}{8}\right) \\ &+ E[B]\left(-\frac{1}{18}(-1)^n + \frac{11}{18}(2)^n + \frac{n+1}{6}(2)^n + \frac{1}{2}\right). \end{aligned}$$

The variance is obtained as follows:

$$\begin{aligned} \operatorname{Var}\big[x(n)\big] = {} & \operatorname{Var}\left[\left(\frac{-A}{40}\right)(-3)^n\right] + \operatorname{Var}\big[(2)^n\big] + \operatorname{Var}\left[-\left(\frac{1}{2}\right)(-1)^n\right] + \operatorname{Var}\left[-\frac{3}{2}\right] \\ & + \operatorname{Var}\left[\left(\frac{A}{12} - \frac{B}{18}\right)(-1)^n\right] + \operatorname{Var}\left[\left(\frac{B(n+1)}{6}\right)(2)^n\right] \\ & + \operatorname{Var}\left[\left(-\frac{A}{15} + \frac{11B}{18}\right)(2)^n\right] + \operatorname{Var}\left[\frac{-A}{8} + \frac{B}{2}\right] \end{aligned} \tag{30}$$

which is found to be

$$= \operatorname{Var}[A]\left(\frac{1}{900}(9)^n + \frac{1}{144} + \frac{1}{225}(4)^n + \frac{1}{64}\right) + \operatorname{Var}[B]\left(\frac{1}{324} + \frac{121}{324}(4)^n + \frac{(n+1)^2}{36}(4)^n + \frac{1}{4}\right).$$

Let the parameters be $A, B \sim P(\lambda = 2)$. The numerical characteristics of the approximate solution of the random linear difference equation obtained from the random Z-transform are obtained as follows [30–44]:

$$\begin{aligned}
E\big[x(n)\big] &= E\left[\left(\frac{-A}{40}\right)(-3)^n\right] + E\big[(2)^n\big] + E\left[-\left(\frac{1}{2}\right)(-1)^n\right] + E\left[-\frac{3}{2}\right] \\
&\quad + E\left[\left(\frac{A}{12} - \frac{B}{18}\right)(-1)^n\right] + E\left[\left(\frac{B(n+1)}{6}\right)(2)^n\right] + E\left[\left(-\frac{A}{15} + \frac{11B}{18}\right)(2)^n\right] \\
&\quad + E\left[\frac{-A}{8} + \frac{B}{2}\right] \\
&= (2)^n - \left(\frac{1}{2}\right)(-1)^n - \frac{3}{2} + E[A]\left(\left(\frac{-1}{40}\right)(-3)^n + \frac{1}{12}(-1)^n - \frac{1}{15}(2)^n - \frac{1}{8}\right) \\
&\quad + E[B]\left(-\frac{1}{18}(-1)^n + \frac{11}{18}(2)^n + \frac{n+1}{6}(2)^n + \frac{1}{2}\right) \\
&= \frac{-1}{20}(-3)^n - \frac{4}{9}(-1)^n + \frac{109+15n}{45} - \frac{3}{4}, \\
\operatorname{Var}\big[x(n)\big] &= \operatorname{Var}[A]\left(\frac{1}{900}(9)^n + \frac{1}{144} + \frac{1}{225}(4)^n + \frac{1}{64}\right) \\
&\quad + \operatorname{Var}[B]\left(\frac{1}{324} + \frac{121}{324}(4)^n + \frac{(n+1)^2}{36}(4)^n + \frac{1}{4}\right) \\
&= 2\left(\frac{1}{900}(9)^n + \frac{1}{144} + \frac{1}{225}(4)^n + \frac{1}{64}\right) \\
&\quad + 2\left(\frac{1}{324} + \frac{121}{324}(4)^n + \frac{(n+1)^2}{36}(4)^n + \frac{1}{4}\right) \\
&= \frac{1}{450}(9)^n + \left(\frac{2}{225} + \frac{121}{162} + \frac{(n+1)^2}{18}\right)(4)^n + \frac{1}{72} + \frac{1}{32} + \frac{1}{162} + \frac{1}{2}.
\end{aligned}$$

Example 4 Let A, B be random variables with Bernoulli distribution such that $p = \frac{1}{5}$ and $q = \frac{4}{5}$ in

$$x(n+2) + 2x(n+1) + 5x(n) = 0, \quad x(0) = A, x(1) = B, A, B \sim B(p,q). \tag{31}$$

We investigate the behavior of the solution of (31) with the Z-transform method.
Solution. The Z-transform of both sides gives

$$\begin{aligned}
&z^2\tilde{x}(z) - z^2x(0) - zx(1) + 2z\tilde{x}(z) + 2zx(0) + 5\tilde{x}(z) = 0, \\
&\tilde{x}(z)\big[z^2 + 2z + 5\big] - z^2A - zB - 2zA = 0.
\end{aligned} \tag{32}$$

Partial fraction of $\tilde{x}(z)/z$ gives

$$\frac{\tilde{x}(z)}{z} = \frac{z(zA + B + 2A)}{z^2 + 2z + 5} = \frac{a_1}{|z - (-1+2i)|} + \frac{a_2}{|z - (-1-2i)|}. \tag{33}$$

Here,

$$a_1 = \frac{(1+2i)A + B}{4i}, \qquad a_2 = -\frac{(1-2i)A + B}{4i}, \qquad a_2 = \overline{a_1}.$$

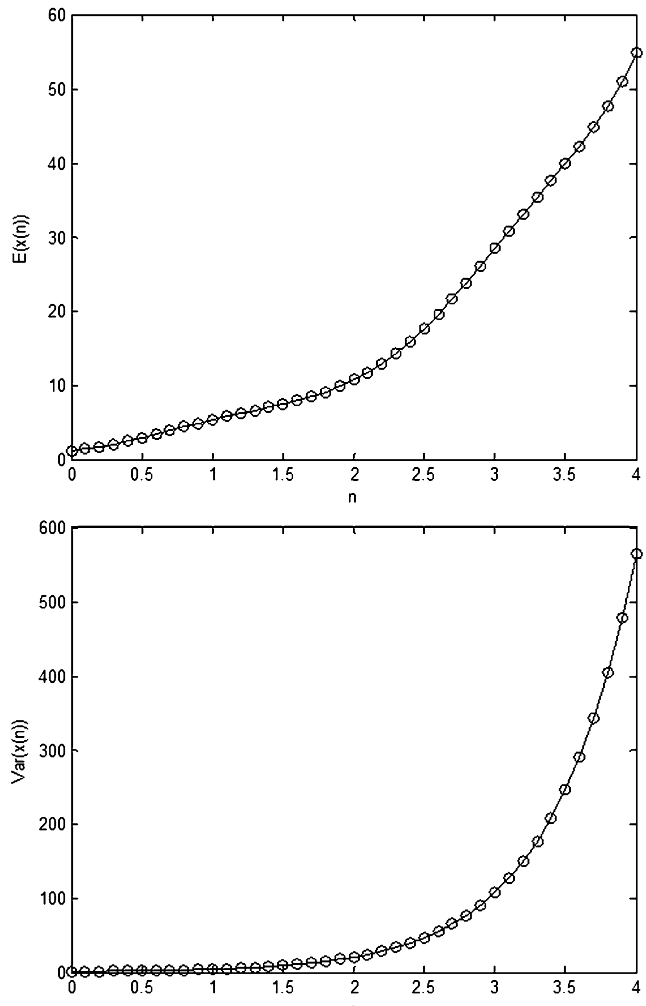

Figure 3 Expected value and variance obtained from the Z-transform of (25)

Hence,

$$\tilde{x}(z) = \frac{a_1 z}{|z - (-1 + 2i)|} + \frac{\overline{a_1} z}{|z - (-1 - 2i)|}. \tag{34}$$

The inverse Z-transform $Z^{-1}[\tilde{x}(z)] = x(n)$ gives

$$x(n) = \left(\frac{(1 + 2i)A + B}{4i}\right)(-1 + 2i)^n + \left(-\frac{(1 - 2i)A + B}{4i}\right)(-1 - 2i)^n. \tag{35}$$

The moment generating function of a Bernoulli distributed random variable $X \sim B(p, q)$ is given as [29]

$$M_x(t) = e^t p + (1 - p).$$

Hence, the expected value and variance of the random variable X are

$$E(X) = p, V(X) = p(1 - p).$$

To find the numerical characteristics of (35), we start with the expectation:

$$E\big[x(n)\big] = E\left[\left(\frac{(1+2i)A+B}{4i}\right)(-1+2i)^n\right] + E\left[\left(-\frac{(1-2i)A+B}{4i}\right)(-1-2i)^n\right] \tag{36}$$

which gives

$$= E[A]\left(\frac{(1+2i)}{4i}(-1+2i)^n - \frac{(1-2i)}{4i}(-1-2i)^n\right) + E[B]\left(\frac{1}{4i}(-1+2i)^n - \frac{1}{4i}(-1-2i)^n\right).$$

The variance is obtained as follows:

$$\operatorname{Var}\big[x(n)\big] = \operatorname{Var}\left[\left(\frac{(1+2i)A+B}{4i}\right)(-1+2i)^n\right] + \operatorname{Var}\left[\left(-\frac{(1-2i)A+B}{4i}\right)(-1-2i)^n\right] \tag{37}$$

which is found to be

$$\begin{aligned} &= \operatorname{Var}[A]\left(\frac{1}{16}(-1+2i)^{2n}(1+2i)^2 + \frac{1}{16}(-1-2i)^{2n}(1-2i)^2\right) \\ &\quad + \operatorname{Var}[B]\left(\frac{1}{16}(-1+2i)^{2n} + \frac{1}{16}(-1-2i)^{2n}\right). \end{aligned}$$

Let the parameters be $A, B \sim B(p = \frac{1}{5}, q = \frac{4}{5})$. The numerical characteristics of the approximate solution of the random linear difference equation obtained from the random Z-transform are obtained as follows [30–44]:

$$\begin{aligned} E\big[x(n)\big] &= E\left[\left(\frac{(1+2i)A+B}{4i}\right)(-1+2i)^n\right] + E\left[\left(-\frac{(1-2i)A+B}{4i}\right)(-1-2i)^n\right] \\ &= E[A]\left(\frac{(1+2i)}{4i}(-1+2i)^n - \frac{(1-2i)}{4i}(-1-2i)^n\right) \\ &\quad + E[B]\left(\frac{1}{4i}(-1+2i)^n - \frac{1}{4i}(-1-2i)^n\right) \\ &= \frac{1}{5}\left(\frac{(1+2i)}{4i}(-1+2i)^n - \frac{(1-2i)}{4i}(-1-2i)^n\right) \\ &\quad + \frac{1}{5}\left(\frac{1}{4i}(-1+2i)^n - \frac{1}{4i}(-1-2i)^n\right) \\ &= \frac{(1+i)}{10i}(-1+2i)^n - \frac{(1-i)}{10i}(-1-2i)^n, \\ \operatorname{Var}\big[x(n)\big] &= \operatorname{Var}\left[\left(\frac{(1+2i)A+B}{4i}\right)(-1+2i)^n\right] + \operatorname{Var}\left[\left(-\frac{(1-2i)A+B}{4i}\right)(-1-2i)^n\right] \\ &= \operatorname{Var}[A]\left(\frac{1}{16}(1+2i)^2(-1+2i)^{2n} + \frac{1}{16}(1-2i)^2(-1-2i)^{2n}\right) \\ &\quad + \operatorname{Var}[B]\left(\frac{1}{16}(-1+2i)^{2n} + \frac{1}{16}(-1-2i)^{2n}\right) \\ &= \frac{4}{25}\left(\frac{1}{16}(-1+2i)^{2n}(1+2i)^2 + \frac{1}{16}(-1-2i)^{2n}(1-2i)^2\right) \\ &\quad + \frac{4}{25}\left(\frac{1}{16}(-1+2i)^{2n} + \frac{1}{16}(-1-2i)^{2n}\right) \end{aligned}$$

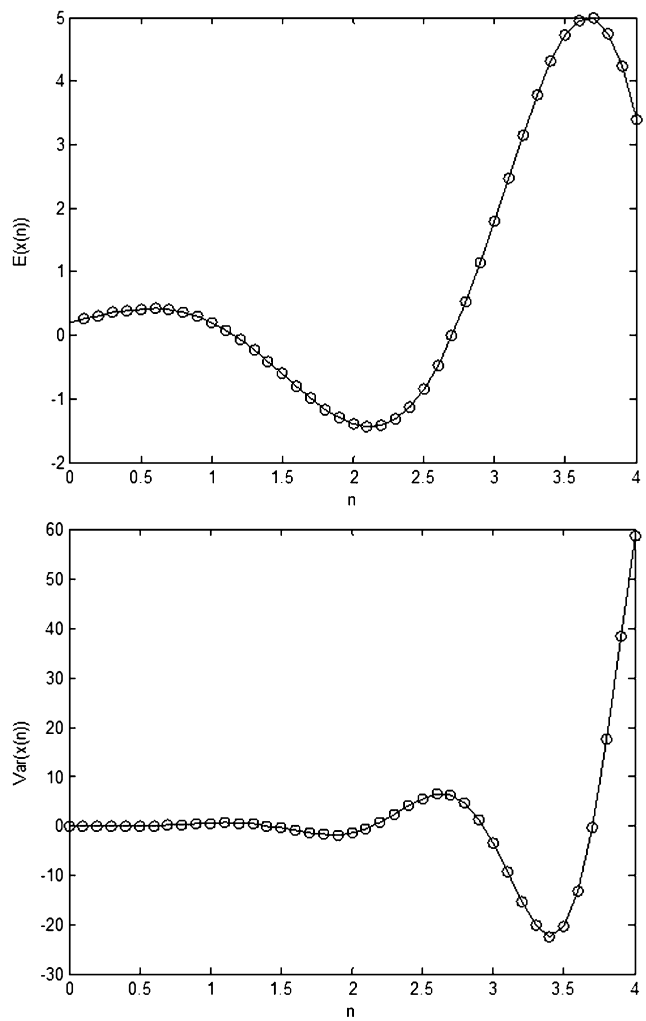

Figure 4 Expected value and variance obtained from the Z-transform of (31)

$$= \left(\frac{-3+4i}{100}\right)(-1+2i)^{2n} + \left(\frac{-3-4i}{100}\right)(-1-2i)^{2n}.$$

5 Conclusion

The application of the Z-transform for obtaining solutions to random linear difference equations is examined in this study and random behavior of the solutions have been investigated with uniform, geometric, Poisson and Bernoulli distributions for the random behavior of linear difference equations. Expected values and variances of the solutions have been obtained and are shown in graphs. Hence, it has been shown that the Z-transform is the most suitable method for the solutions of random linear difference equations.

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Authors' contributions
MM is responsible for the model formulation and study planning. ŞŞ has done the calculation, and the application. All authors have read and approved the final manuscript.

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