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On the Hermite–Hadamard inequalities for interval-valued coordinated convex functions

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Abstract
In this work, we introduce the notion of interval-valued coordinated convexity and demonstrate Hermite–Hadamard type inequalities for interval-valued convex functions on the co-ordinates in a rectangle from the plane. Moreover, we prove Hermite–Hadamard inequalities for the product of interval-valued convex functions on coordinates. Our results generalize several other well-known inequalities given in the existing literature on this subject.

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1 Introduction

The classical Hermite–Hadamard inequality is one of the most well-established inequalities in the theory of convex functions with geometrical interpretation, and it has many applications. The Hermite–Hadamard inequality states that, if $f: I \rightarrow R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities in (1.1) hold in the reversed direction if f is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity, and it is implied easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years, and a remarkable variety of refinements and generalizations have been studied. In [7], Dragomir demonstrated the subsequent inequality of Hadamard type for coordinated convex functions.

Theorem 1 *Let $f: \Delta = [a,b] \times [c,d] \rightarrow \mathbb{R}$ be convex on coordinates Δ. Then the following inequalities hold*:

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\quad \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\
&\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
\tag{1.2}
$$

For more results related to (1.2), we refer the readers to [1, 9, 15] and the references therein.

On the other hand, interval analysis is a notable case of set-valued analysis, which is the discussion of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle the interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of an interval enclosure is Archimede's method, which is related to computing the circumference of a circle. In 1966, the first book related to interval analysis was given by Moore, who is known as the first user of intervals in computational mathematics, see [11]. After his book, several scientists started to investigate the theory and application of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various areas which are interested intensely in uncertain data. You can see applications in computer graphics, experimental and computational physics, error analysis, robotics, and many others.

What is more, several important inequalities (Hermite–Hadamard, Ostrowski, etc.) have been studied for interval-valued functions in recent years. In [3, 4], Chalco-Cano et al. obtained Ostrowski type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. In [16], Román-Flores et al. established Minkowski and Beckenbach's inequality for interval-valued functions. For other related results, we refer the readers to [5, 6, 8, 10, 13, 17, 18].

2 Preliminaries and known results

In this section, we review some basic definitions, results, notions, and properties which are used throughout the paper. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$, and closed negative intervals of $\mathbb{R}$ are denoted by $\mathbb{R}_{\mathcal{I}}$, $\mathbb{R}_{\mathcal{I}}^{+}$, $\mathbb{R}_{\mathcal{I}}^{-}$, respectively. The Hausdorff distance between $[\underline{X},\overline{X}]$ and $[\underline{Y},\overline{Y}]$ is defined as

$$
d\big([\underline{X},\overline{X}],[\underline{Y},\overline{Y}]\big) = \max\big\{|\underline{X}-\underline{Y}|,|\overline{X}-\overline{Y}|\big\}.
$$

The metric space $(\mathbb{R}_{\mathcal{I}},d)$ is a complete metric space. For more in-depth notations on interval-valued functions, see [12, 19].

In [11], Moore gave the notion of the Riemann integral for interval-valued functions. The sets of all Riemann integrable interval-valued functions and real-valued functions on $[a,b]$ are denoted by $IR_{([a,b])}$and $R_{([a,b])}$, respectively. The following theorem gives a relation between (IR)-integrable functions and Riemann integrable (R-integrable) functions (see, [12, p. 131]).

Theorem 2 *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function with the property that $F(t) = [\underline{F}(t), \overline{F}(t)]$. $F \in IR_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in R_{([a,b])}$ and*

$$(IR) \int_a^b F(t)\,dt = \left[(R)\int_a^b \underline{F}(t)\,dt, (R)\int_a^b \overline{F}(t)\,dt\right].$$

In [19, 21], Zhao et al. introduced a kind of convex interval-valued function as follows.

Definition 1 Let $h : [c,d] \to \mathbb{R}$ be a nonnegative function, $(0,1) \subseteq [c,d]$ and $h \neq 0$. We say that $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ is an h-convex interval-valued function if, for all $x, y \in [a,b]$ and $t \in (0,1)$, we have

$$h(t)F(x) + h(1-t)F(y) \subseteq F\big(tx + (1-t)y\big). \tag{2.1}$$

With $\mathrm{SX}(h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, we will show the set of all h-convex interval-valued functions.

The usual notion of convex interval-valued function matches a relation (2.1) with $h(t) = t$ (see [18]). Moreover, if we take $h(t) = t^s$ in (2.1), then Definition 1 gives the s-convex interval-valued function defined by Breckner (see [2]).

In [19], Zhao et al. obtained the following Hermite–Hadamard inequality for interval-valued functions by using h-convexity.

Theorem 3 *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in IR_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h(\frac{1}{2}) \neq 0$. If $F \in \mathrm{SX}(h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$\frac{1}{2h(\frac{1}{2})} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)\,dx \supseteq \big[F(a) + F(b)\big]\int_0^1 h(t)\,dt. \tag{2.2}$$

Remark 1 (i) If $h(t) = t$, then (2.2) reduces to the following result:

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)\,dx \supseteq \frac{F(a)+F(b)}{2}, \tag{2.3}$$

which was obtained by Sadowska in [18].

(ii) If $h(t) = t^s$, then (2.2) reduces to the following result:

$$2^{s-1} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)\,dx \supseteq \frac{F(a)+F(b)}{s+1},$$

which was obtained by Osuna-Gómez et al. in [14].

Theorem 4 *Let $F, G : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ be two interval-valued functions such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, where $F, G \in IR_{([a,b])}$, $h_1, h_2 : [0,1] \to \mathbb{R}$ are two nonnegative functions and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. If $F, G \in \mathrm{SX}(h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right)$$

$$\supseteq \frac{1}{b-a}(IR)\int_a^b F(x)G(x)\,dx + M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt$$
$$+ N(a,b)\int_0^1 h_1(t)h_2(t)\,dt \tag{2.4}$$

and

$$\frac{1}{b-a}(IR)\int_a^b F(x)G(x)\,dx \supseteq M(a,b)\int_0^1 h_1(t)h_2(t)\,dt$$
$$+ N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt, \tag{2.5}$$

where

$$M(a,b) = F(a)G(a) + F(b)G(b) \quad \textit{and} \quad N(a,b) = F(a)G(b) + F(b)G(a).$$

Remark 2 If $h(t) = t$, then (2.4) reduces to the following result:

$$\frac{1}{b-a}(IR)\int_a^b F(x)G(x)\,dx \supseteq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b). \tag{2.6}$$

Remark 3 If $h(t) = t$, then (2.5) reduces to the following result:

$$2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)G(x)\,dx$$
$$+ \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b). \tag{2.7}$$

3 Interval-valued double integral

A set of numbers $\{t_{i-1},\xi_i,t_i\}_{i=1}^m$ is called a tagged partition P_1 of $[a,b]$ if

$$P_1 : a = t_0 < t_1 < \cdots < t_n = b$$

with $t_{i-1} \leq \xi_i \leq t_i$ for all $i = 1,2,3,\ldots,m$. Moreover, if we have $\Delta t_i = t_i - t_{i-1}$, then P_1 is said to be δ-fine if $\Delta t_i < \delta$ for all i. Let $P(\delta,[a,b])$ denote the set of all δ-fine partitions of $[a,b]$. If $\{t_{i-1},\xi_i,t_i\}_{i=1}^m$ is a δ-fine P_1 of $[a,b]$ and $\{s_{j-1},\eta_j,t_j\}_{j=1}^n$ is δ-fine P_2 of $[c,d]$, then the rectangles

$$\Delta_{i,j} = [t_{i-1},t_i] \times [s_{j-1},s_j]$$

partition rectangle $\Delta = [a,b] \times [c,d]$ with the points (ξ_i,η_j) are inside the rectangles $[t_{i-1},t_i] \times [s_{j-1},s_j]$. Furthermore, by $P(\delta,\Delta)$ we denote the set of all δ-fine partitions P of Δ with $P_1 \times P_2$, where $P_1 \in P(\delta,[a,b])$ and $P_2 \in P(\delta,[c,d])$. Let $\Delta A_{i,j}$ be the area of the rectangle $\Delta_{i,j}$. In each rectangle $\Delta_{i,j}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, choose arbitrary (ξ_i,η_j) and get

$$S(F,P,\delta,\Delta) = \sum_{i=1}^m \sum_{j=1}^n F(\xi_i,\eta_j)\Delta A_{i,j}.$$

We call $S(F,P,\delta,\Delta)$ an integral sum of F associated with $P \in P(\delta,\Delta)$.

Now, we review the concepts and notations of interval-valued double integral given by Zhao et al. in [20].

Theorem 5 ([20]) *Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then F is called ID-integrable on Δ with ID-integral $U = (ID)\iint_{\Delta} F(t,s)\,dA$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d\big(S(F,P,\delta,\Delta),U\big) < \epsilon$$

for any $P \in P(\delta,\Delta)$. The collection of all ID-integrable functions on Δ will be denoted by $ID_{(\Delta)}$.

Theorem 6 ([20]) *Let $\Delta = [a,b] \times [c,d]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is ID-integrable on Δ, then we have*

$$(ID)\iint_{\Delta} F(s,t)\,dA = (IR)\int_a^b (IR)\int_c^d F(s,t)\,ds\,dt.$$

Example 1 Let $F : \Delta = [0,1] \times [1,2] \to \mathbb{R}_{\mathcal{I}}^{+}$ be defined by

$$F(s,t) = [st, s+t],$$

then $F(s,t)$ is integrable on Δ and $(ID)\iint_{\Delta} F(t,s)\,dA = [\frac{3}{4}, 2]$.

4 Crucial results

In this section, we define interval-valued coordinated convex functions and then demonstrate some inequalities of Hermite–Hadamard type by using the new definition. Throughout this section, we use $\Delta = [a,b] \times [c,d]$, where $a < b$ and $c < d$, $a,b,c,d \in \mathbb{R}$.

Definition 2 A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is said to be interval-valued coordinated convex function if the following inequality holds:

$$\begin{aligned} &F\big(tx + (1-t)y, su + (1-s)w\big) \\ &\quad \supseteq tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w) \end{aligned}$$

for all $(x,y), (u,w) \in \Delta$ and $s,t \in [0,1]$.

Lemma 1 *A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued convex on coordinates if and only if there exist two functions $F_x : [c,d] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_x(w) = F(x,w)$ and $F_y : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_y(u) = F(y,u)$ are interval-valued convex.*

Proof The proof of this lemma follows immediately by the definition of interval-valued coordinated convex function. □

It is easy to prove that an interval-valued convex function is interval-valued coordinated convex, but the converse may not be true. For this, we can see the following example.

Example 2 An interval-valued function $F : [0,1]^2 \to \mathbb{R}_{\mathcal{I}}^{+}$ defined as $F(x,y) = [xy, (6-e^x)(6-e^y)]$ is interval-valued convex on coordinates, but it is not interval-valued convex on $[0,1]^2$.

Proposition 1 *If $F, G : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ are two interval-valued coordinated convex functions on Δ and $\alpha \geq 0$, then $F + G$ and αF are interval-valued coordinated convex functions.*

Proposition 2 *If $F, G : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ are two interval-valued coordinated convex functions on Δ, then (FG) is interval-valued coordinated convex function on Δ.*

In what follows, without causing confusion, we will delete the notations of (R), (IR), and (ID). We start with the following theorem.

Theorem 7 *If $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued coordinated convex function on Δ such that $F(t) = [\underline{F}(t), \overline{F}(t)]$, then the following inequalities hold:*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad \supseteq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2}, y\right) dy\right] \\
&\quad \supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx \\
&\quad \supseteq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b F(x,c)\,dx + \frac{1}{b-a}\int_a^b F(x,d)\,dx\right. \\
&\qquad \left. + \frac{1}{d-c}\int_c^d F(a,y)\,dy + \frac{1}{d-c}\int_c^d F(b,y)\,dy\right] \\
&\quad \supseteq \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned} \tag{4.1}$$

Proof Since F is an interval-valued coordinated convex function on coordinates Δ, then $F_x : [c,d] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_x(y) = F(x,y)$ is an interval-valued convex function on $[c,d]$ and for all $x \in [a,b]$. From inequality (2.3), we have

$$F_x\left(\frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d F_x(y)\,dy \supseteq \frac{F_x(c)+F_x(d)}{2},$$

which can be written as

$$F\left(x, \frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d F(x,y)\,dy \supseteq \frac{F(x,c)+F(x,d)}{2}. \tag{4.2}$$

Integrating (4.2) with respect to x over $[a,b]$ and dividing both sides by $b-a$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F\left(x, \frac{c+d}{2}\right) dx \\
&\quad \supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx \\
&\quad \supseteq \frac{1}{2(b-a)}\left[\int_a^b F(x,c)\,dx + \int_a^b F(x,d)\,dx\right].
\end{aligned} \tag{4.3}$$

Similarly, $F_y : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$, $F_y(x) = F(x,y)$ is an interval-valued convex function on $[a,b]$ and $y \in [c,d]$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy \\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_a^b F(x,y)\,dy\,dx \\
&\supseteq \frac{1}{2(d-c)}\left[\int_c^d F(a,y)\,dy + \int_c^d F(b,y)\,dy\right].
\end{aligned}\tag{4.4}$$

By adding (4.3) and (4.4) and using Theorem 2, we have the second and third inequality in (4.1). From (2.3) we also have

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)dx,\tag{4.5}$$

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy.\tag{4.6}$$

By adding (4.5) and (4.6) and using Theorem 2, we have the first inequality in (4.1). In the end, again from (2.2) and Theorem 2, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F(x,c)\,dx \supseteq \frac{F(a,c)+F(b,c)}{2}, \\
&\frac{1}{b-a}\int_a^b F(x,d)\,dx \supseteq \frac{F(a,d)+F(b,d)}{2}, \\
&\frac{1}{d-c}\int_c^d F(a,y)\,dy \supseteq \frac{F(a,c)+F(a,d)}{2}, \\
&\frac{1}{d-c}\int_c^d F(b,y)\,dy \supseteq \frac{F(b,c)+F(b,d)}{2},
\end{aligned}$$

and the proof is completed. □

Example 3 Suppose that $[a,b] = [0,1]$ and $[c,d] = [0,1]$. Let $F : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{I}}^+$ be given as $F(x,y) = [xy,(6-e^x)(6-e^y)]$ for all $x \in [a,b]$ and $y \in [c,d]$. We have

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right) = \left[\frac{1}{4},(6-\sqrt{e})^2\right], \\
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy\right] = \left[\frac{1}{4},(7-e)(6-\sqrt{e})\right], \\
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx = \left[\frac{1}{4},(7-e)^2\right], \\
&\frac{1}{4}\left[\frac{1}{b-a}\int_a^b F(x,c)\,dx + \frac{1}{b-a}\int_a^b F(x,d)\,dx\right. \\
&\qquad\left. + \frac{1}{d-c}\int_c^d F(a,y)\,dy + \frac{1}{d-c}\int_c^d F(b,y)\,dy\right]
\end{aligned}$$

$$= \left[\frac{1}{4}, \frac{(7-e)(11-e)}{2}\right],$$

$$\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4} = \left[\frac{1}{4}, \frac{(6-e)(16-e)+25}{4}\right].$$

Consequently, Theorem 7 is verified.

Remark 4 If $\overline{F} = \underline{F}$, then Theorem 7 reduces to Theorem 1.

Theorem 8 *If $F, G : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ are two interval-valued coordinated convex functions such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, then the following inequality holds*:

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x,y)G(x,y)\,dy\,dx \\ &\quad \supseteq \frac{1}{9}P(a,b,c,d) + \frac{1}{18}M(a,b,c,d) + \frac{1}{36}N(a,b,c,d), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} P(a,b,c,d) &= F(a,c)G(a,c) + F(a,d)G(a,d) + F(b,c)G(b,c) + F(b,d)G(b,d), \\ M(a,b,c,d) &= F(a,c)G(a,d) + F(a,d)G(a,c) + F(b,c)G(b,d) + F(b,d)G(b,c) \\ &\quad + F(b,c)G(a,c) + F(a,c)G(b,c) + F(b,d)G(a,d) + F(a,d)G(b,d), \\ N(a,b,c,d) &= F(b,c)G(a,d) + F(a,d)G(b,c) + F(b,d)G(a,c) + F(a,c)G(b,d). \end{aligned}$$

Proof Since F and G are interval-valued coordinated convex functions on Δ, therefore

$$F_x(y) : [c,d] \to \mathbb{R}^+_{\mathcal{I}}, \qquad F_x(y) = F(x,y), \qquad G_x(y) : [c,d] \to \mathbb{R}^+_{\mathcal{I}}, \qquad G_x(y) = G(x,y),$$

and

$$F_y(x) : [a,b] \to \mathbb{R}^+_{\mathcal{I}}, \qquad F_y(x) = F(x,y), \qquad G_y(x) : [a,b] \to \mathbb{R}^+_{\mathcal{I}}, \qquad G_y(x) = G(x,y)$$

are interval-valued convex functions on $[c,d]$ and $[a,b]$, respectively, for all $x \in [a,b]$, $y \in [c,d]$.

Now, from inequality (2.6), we have

$$\begin{aligned} \frac{1}{d-c} \int_c^d F_x(y)G_x(y)\,dy \supseteq{}& \frac{1}{3}\bigl[F_x(c)G_x(c) + F_x(d)G_x(d)\bigr] \\ &+ \frac{1}{6}\bigl[F_x(c)G_x(d) + F_x(d)G_x(c)\bigr], \end{aligned}$$

which can be written as

$$\begin{aligned} \frac{1}{d-c} \int_c^d F(x,y)G(x,y)\,dy \supseteq{}& \frac{1}{3}\bigl[F(x,c)G(x,c) + F(x,d)G(x,d)\bigr] \\ &+ \frac{1}{6}\bigl[F(x,c)G(x,d) + F(x,d)G(x,c)\bigr]. \end{aligned}$$

Integrating the above inequality with respect to x over $[a,b]$ and dividing both sides by $b-a$, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)G(x,y)\,dy\,dx \\
&\quad\supseteq \frac{1}{3(b-a)}\int_a^b\bigl[F(x,c)G(x,c)+F(x,d)G(x,d)\bigr]\,dx \\
&\qquad+\frac{1}{6(b-a)}\int_a^b\bigl[F(x,c)G(x,d)+F(x,d)G(x,c)\bigr]\,dx.
\end{aligned}\tag{4.8}$$

Now, using inequality (2.6) for each integral on the right-hand side of (4.8), we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F(x,c)G(x,c)\,dx \\
&\quad\supseteq \frac{1}{3}\bigl[F(a,c)G(a,c)+F(b,c)G(b,c)\bigr]+\frac{1}{6}\bigl[F(a,c)G(b,c)+F(b,c)G(a,c)\bigr],
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F(x,d)G(x,d)\,dx \\
&\quad\supseteq \frac{1}{3}\bigl[F(a,d)G(a,d)+F(b,d)G(b,d)\bigr]+\frac{1}{6}\bigl[F(a,d)G(b,d)+F(b,d)G(a,d)\bigr],
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F(x,c)G(x,d)\,dx \\
&\quad\supseteq \frac{1}{3}\bigl[F(a,c)G(a,d)+F(b,c)G(b,d)\bigr]+\frac{1}{6}\bigl[F(a,c)G(b,d)+F(b,c)G(a,d)\bigr],
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b F(x,d)G(x,c)\,dx \\
&\quad\supseteq \frac{1}{3}\bigl[F(a,d)G(a,c)+F(b,d)G(b,c)\bigr]+\frac{1}{6}\bigl[F(a,d)G(b,c)+F(b,d)G(a,c)\bigr].
\end{aligned}\tag{4.12}$$

Substituting (4.9)–(4.12) in (4.8), we have our desired inequality (4.7). Similarly, we can find the same inequality by using $F_y(x)G_y(x)$ on $[a,b]$. □

Remark 5 If $\overline{F}=\underline{F}$, then Theorem 8 reduces to [9, Theorem 4].

Theorem 9 *If $F,G:\Delta\to\mathbb{R}_{\mathcal{I}}^{+}$ are two interval-valued coordinated convex functions such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ and $G(t)=[\underline{G}(t),\overline{G}(t)]$, then we have the following inequality*:

$$\begin{aligned}
&4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\quad\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)G(x,y)\,dy\,dx \\
&\qquad+\frac{5}{36}P(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d),
\end{aligned}\tag{4.13}$$

where $P(a,b,c,d)$, $M(a,b,c,d)$, and $N(a,b,c,d)$ are defined in Theorem 8.

Proof Since F and G are interval-valued coordinated convex functions, from (2.7) we have

$$\begin{aligned}
&2F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\supseteq \frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx+\frac{1}{6}\left[F\left(a,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\right.\\
&\quad\left.+F\left(b,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\right]+\frac{1}{3}\left[F\left(a,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\right.\\
&\quad\left.+F\left(b,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\right]
\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}
&2F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\supseteq \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy+\frac{1}{6}\left[F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},c\right)\right.\\
&\quad\left.+F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},d\right)\right]+\frac{1}{3}\left[F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},d\right)\right.\\
&\quad\left.+F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},c\right)\right].
\end{aligned}\tag{4.15}$$

Adding (4.14) and (4.15), then multiplying both sides of the resultant one by 2, we get

$$\begin{aligned}
&8F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\supseteq \frac{2}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx+\frac{2}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\\
&\quad+\frac{1}{6}\left[2F\left(a,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)+2F\left(b,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\right]\\
&\quad+\frac{1}{6}\left[2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},c\right)+2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},d\right)\right]\\
&\quad+\frac{1}{3}\left[2F\left(a,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)+2F\left(b,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\right]\\
&\quad+\frac{1}{3}\left[2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},d\right)+2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},c\right)\right].
\end{aligned}\tag{4.16}$$

Now, from (2.7), we have

$$\begin{aligned}
&2F\left(a,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\\
&\supseteq \frac{1}{d-c}\int_c^d F(a,y)G(a,y)\,dy+\frac{1}{6}\left[F(a,c)G(a,c)+F(a,d)G(a,d)\right]\\
&\quad+\frac{1}{3}\left[F(a,c)G(a,d)+F(a,d)G(a,c)\right],
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
&2F\left(b,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\\
&\quad\supseteq \frac{1}{d-c}\int_c^d F(b,y)G(b,y)\,dy+\frac{1}{6}\bigl[F(b,c)G(b,c)+F(b,d)G(b,d)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(b,c)G(b,d)+F(b,d)G(b,c)\bigr], \qquad (4.18)\\
&2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},c\right)\\
&\quad\supseteq \frac{1}{b-a}\int_a^b F(x,c)G(x,c)\,dx+\frac{1}{6}\bigl[F(a,c)G(a,c)+F(b,c)G(b,c)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(a,c)G(b,c)+F(b,c)G(a,c)\bigr], \qquad (4.19)\\
&2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},d\right)\\
&\quad\supseteq \frac{1}{b-a}\int_a^b F(x,d)G(x,d)\,dx+\frac{1}{6}\bigl[F(a,d)G(a,d)+F(b,d)G(b,d)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(a,d)G(b,d)+F(b,d)G(a,d)\bigr], \qquad (4.20)\\
&2F\left(a,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\\
&\quad\supseteq \frac{1}{d-c}\int_c^d F(a,y)G(b,y)\,dy+\frac{1}{6}\bigl[F(a,c)G(b,c)+F(a,d)G(b,d)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(a,c)G(b,d)+F(a,d)G(b,c)\bigr], \qquad (4.21)\\
&2F\left(b,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\\
&\quad\supseteq \frac{1}{d-c}\int_c^d F(b,y)G(a,y)\,dy+\frac{1}{6}\bigl[F(b,c)G(a,c)+F(b,d)G(a,d)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(b,c)G(a,d)+F(b,d)G(a,c)\bigr], \qquad (4.22)\\
&2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},d\right)\\
&\quad\supseteq \frac{1}{b-a}\int_a^b F(x,c)G(x,d)\,dx+\frac{1}{6}\bigl[F(a,c)G(a,d)+F(b,c)G(b,d)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(a,c)G(b,d)+F(b,c)G(a,d)\bigr], \qquad (4.23)\\
&2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},c\right)\\
&\quad\supseteq \frac{1}{b-a}\int_a^b F(x,d)G(x,c)\,dx+\frac{1}{6}\bigl[F(a,d)G(a,c)+F(b,d)G(b,c)\bigr]\\
&\qquad+\frac{1}{3}\bigl[F(a,d)G(b,c)+F(b,d)G(a,c)\bigr]. \qquad (4.24)
\end{aligned}$$

Using (4.17)–(4.24) in (4.16), we have

$$8F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$
\begin{aligned}
&\supseteq \frac{2}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx+\frac{2}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\\
&\quad+\frac{1}{6(d-c)}\int_c^d F(a,y)G(a,y)\,dy+\frac{1}{6(b-a)}\int_a^b F(x,c)G(x,c)\,dx\\
&\quad+\frac{1}{6(d-c)}\int_c^d F(b,y)G(b,y)\,dy+\frac{1}{6(b-a)}\int_a^b F(x,d)G(x,d)\,dx\\
&\quad+\frac{1}{3(d-c)}\int_c^d F(a,y)G(b,y)\,dy+\frac{1}{3(d-c)}\int_c^d F(b,y)G(a,y)\,dy\\
&\quad+\frac{1}{3(b-a)}\int_a^b F(x,c)G(x,d)\,dx+\frac{1}{3(b-a)}\int_a^b F(x,d)G(x,c)\,dx\\
&\quad+\frac{1}{18}P(a,b,c,d)+\frac{1}{9}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d).
\end{aligned}
\tag{4.25}
$$

Again from (2.7), we have the following relation:

$$
\begin{aligned}
&\frac{2}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)G(x,y)\,dy\,dx+\frac{1}{6(d-c)}\int_c^d F(a,y)G(a,y)\,dy\\
&\quad+\frac{1}{6(d-c)}\int_c^d F(b,y)G(b,y)\,dy+\frac{1}{3(d-c)}\int_c^d F(a,y)G(b,y)\,dy\\
&\quad+\frac{1}{3(d-c)}\int_c^d F(b,y)G(a,y)\,dy,
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
&\frac{2}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx\\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)G(x,y)\,dx\,dy+\frac{1}{6(b-a)}\int_a^b F(x,c)G(x,c)\,dx\\
&\quad+\frac{1}{6(b-a)}\int_a^b F(x,d)G(x,d)\,dx+\frac{1}{3(b-a)}\int_a^b F(x,c)G(x,d)\,dx\\
&\quad+\frac{1}{3(b-a)}\int_a^b F(x,d)G(x,c)\,dx.
\end{aligned}
\tag{4.27}
$$

Using (4.26) and (4.27) in (4.25), we have

$$
\begin{aligned}
&8F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\supseteq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)G(x,y)\,dy\,dx\\
&\quad+\frac{1}{3(d-c)}\int_c^d F(a,y)G(a,y)\,dy+\frac{1}{3(d-c)}\int_c^d F(b,y)G(b,y)\,dy\\
&\quad+\frac{1}{3(b-a)}\int_a^b F(x,c)G(x,c)\,dx+\frac{1}{3(b-a)}\int_a^b F(x,d)G(x,d)\,dx\\
&\quad+\frac{2}{3(d-c)}\int_c^d F(a,y)G(b,y)\,dy+\frac{2}{3(d-c)}\int_c^d F(b,y)G(a,y)\,dy
\end{aligned}
$$

$$+\frac{2}{3(b-a)}\int_a^b F(x,c)G(x,d)\,dx+\frac{2}{3(b-a)}\int_a^b F(x,d)G(x,c)\,dx$$
$$+\frac{1}{18}P(a,b,c,d)+\frac{1}{9}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d), \qquad (4.28)$$

and by using (2.6) for each integral in (4.28), we have our required result. □

Remark 6 If $\overline{F}=\underline{F}$, then Theorem 9 reduces to [9, Theorem 5].

5 Conclusion

In this study, coordinated convexity for interval-valued functions is introduced, and some new Hermite–Hadamard type inequalities are established. It is also shown that the results derived in this article are the potential generalization of the existing comparable results in the literature. As future directions, one may find similar inequalities through different types of convexities.

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