Ali et al. *Advances in Difference Equations* (2020) 2020:572
https://doi.org/10.1186/s13662-020-03032-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Study of fractional order pantograph type impulsive antiperiodic boundary value problem

Arshad Ali[1], Kamal Shah[1], Thabet Abdeljawad[2,3,4*], Hasib Khan[5*] and Aziz Khan[2]

*Correspondence:
tabdeljawad@psu.edu.sa;
hasibkhan13@yahoo.com
[2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[5]Department of Mathematics, Shaheed BB University, Dir Uppr, Khyber Pakhtunkhwa, Pakistan
Full list of author information is available at the end of the article

Abstract

In this paper, we study existence and stability results of an anti-periodic boundary value problem of nonlinear delay (pantograph) type implicit fractional differential equations with impulsive conditions. Using Schaefer's fixed point theorem and Banach's fixed point theorem, we have established results of at least one solution and uniqueness. Also, using the Hyers–Ulam concept, we have derived various kinds of Ulam stability results for the considered problem. Finally, we have applied our obtained results to a numerical problem.

Keywords: Delay differential equation; Impulsive conditions; Schaefer's fixed point theorem; Hyers–Ulam stability

1 Introduction

The study of non-integer order differential equations has emerged as one of the most active research fields in modern mathematics. The non-integer order derivative is also known as fractional order derivative. The main advantage of non-integer order derivative is that it is a global operator and produces accurate and stable results, while the integer order derivative is a local operator. The non-integer order differential equations have multi-dimensional applications in various fields of modern sciences. For example, the non-integer order oscillators are used to control the phase difference and to achieve independently the high frequency oscillation. In electrical engineering the non-integer order DC-DC converter models are used to get best estimation of the power conversion efficiency. Similarly, the non-integer order bio-impedance models give the best fitting to the measured data obtained from vegetables and fruits. For historical background and some applications of non-integer order derivatives, we refer the readers to study [1–9].

The impulsive differential equations have impulsive conditions at points of discontinuity. Various physical and evolutionary phenomena which have discontinuous jumps and sudden changes can be modeled via these equations, and therefore, these equations constitute an important class of differential equations. For some recent work, we refer the readers to study work in [10–14].

Some processes and phenomena cannot be described at the current time and depend on their previous states. For this purpose various types of delay differential equations are used. There are mainly three types of delay differential equations: discrete delay differential equations, continuous delay differential equations, and proportional delay differential equations. The proportional delay differential equations are known as pantograph differential equations. The pantograph differential equations are used to model numerous processes and phenomena. More specifically these equations have applications in electrodynamic, quantum mechanics, number theory, biology, etc. We refer the readers to study [15–17].

Existence theory and stability analysis are two important aspects of qualitative theory. The Hyers–Ulam stability concept is one of the well-known methods used to study the stability of functional and differential problems. This concept was given by Ulam and Hyers in 1940–41 [18, 19]. Rassias was the first who established Hyers–Ulam stability of linear mapping [20]. Jung studied Hyers–Ulam–Rassias stability results for functional equations in nonlinear analysis [21]. Due to its simple procedure, Hyers–Ulam concept has got a great deal of attention from researchers. Using this concept, they investigated stability of various systems of functional and differential equations. We refer the readers to study the recent work in [22–27]. Also, for more related results about the existence, uniqueness, and stability, the readers may consider the work in [28–35].

The antiperiodic boundary value problems are of great interest as these problems appear in various fields of science. In [36], Ahmad et al. investigated a class of fractional integro-differential equations with dual anti-periodic boundary conditions. In [37], Agarwal et al. studied a problem of fractional-order differential equations with anti-periodic boundary conditions.

In [38], Wang et al. studied the existence of solution to the following antiperiodic problem with impulsive conditions:

$$\begin{cases} {}^{C}_{0}\mathbb{D}^{\varsigma}_{t} w(t) = \mathrm{g}(t, w(t)), \quad t \in [0,1], t \neq t_k, 0 < \delta < 1, \varsigma \in (2,3], \\ \Delta w(t_k) = \mathcal{I}_k(w(t_k)), \quad k = 1,2,\dots,n, \\ \Delta w'(t_k) = \hat{\mathcal{I}}_k(w(t_k)), \qquad \Delta w''(t_k) = \bar{\mathcal{I}}_k(w(t_k)), \quad k = 1,2,\dots,n, \\ w(0) = -w(1), \qquad w'(0) = -w'(1), \qquad w''(0) = -w''(1), \end{cases} \tag{1}$$

where $\mathrm{g} : [0,1] \times \mathbb{R} \to [0,\infty)$ and $\mathcal{I}_k, \hat{\mathcal{I}}_k, \bar{\mathcal{I}}_k : \mathbb{R} \to \mathbb{R}$ are continuous functions. In this paper, we generalize problem 1 to an anti-periodic boundary value problem of nonlinear pantograph implicit fractional differential equations with given impulsive conditions as follows:

$$\begin{cases} {}^{C}_{0}\mathbb{D}^{\varsigma}_{t} w(t) = \mathrm{g}(t, w(t), w(\delta t), {}^{C}_{0}\mathbb{D}^{\varsigma}_{t} w(t)), \quad t \in [0,1], t \neq t_k, 0 < \delta < 1, \varsigma \in (2,3], \\ \Delta w(t_k) = \mathcal{I}_k(w(t_k)), \quad k = 1,2,\dots,n, \\ \Delta w'(t_k) = \hat{\mathcal{I}}_k(w(t_k)), \qquad \Delta w''(t_k) = \bar{\mathcal{I}}_k(w(t_k)), \quad k = 1,2,\dots,n, \\ w(0) = -w(1), \qquad w'(0) = -w'(1), \qquad w''(0) = -w''(1), \end{cases} \tag{2}$$

where $\mathrm{g} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ and $\mathcal{I}_k, \hat{\mathcal{I}}_k, \bar{\mathcal{I}}_k : \mathbb{R} \to \mathbb{R}$ are continuous functions. ${}^{C}_{0}\mathbb{D}^{\varsigma}_{t}$ is a standard notion for Caputo type fractional differential operator of order ς. $\Delta w(t_k) = w(t_k^+) - w(t_k^-)$, $\Delta w'(t_k) = w'(t_k^+) - w'(t_k^-)$, $\Delta w''(t_k) = w''(t_k^+) - w''(t_k^-)$; $w(t_k^+), w'(t_k^+), w''(t_k^+)$ are right-hand limits and $w(t_k^-), w'(t_k^-), w''(t_k^-)$ are left-hand limits of the function $w(t)$ at $t = t_k$. And the sequence t_k satisfies that $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1}, n \in \mathbb{N}$. We study the existence, uniqueness, and Hyers–Ulam stability of the generalized problem (2).

Let $I = [0,1]$, $I_0 = [0,t_1]$ and $I_k = (t_k, t_{k+1}]$. We define the space of piecewise continuous functions by

$$\begin{aligned}\mathscr{X} = PC(I,\mathbb{R}) = \bigl\{ & w : I \to \mathbb{R} | w \in C(I), \quad k = 1,2,\ldots,n, \\ & w(t_k^+), w(t_k^-) \text{ exist for } k = 1,2,\ldots,n\bigr\}.\end{aligned}$$

$\mathscr{X}$ is a Banach space with the norm defined by $\|w\|_{\mathscr{X}} = \max_{t\in I} |w(t)|$.

2 Preliminaries

Definition 1 ([24]) The fractional order integral of function $h \in L^1([0,1],\mathbb{R}^+)$ of order $\varsigma \in \mathbb{R}^+$ is defined by

$$ {}_0\mathrm{I}_t^{\varsigma} h(t) = \int_0^t \frac{(t-s)^{\varsigma-1}}{\Gamma(\varsigma)} h(s)\, ds, \tag{3}$$

provided that integral on the right-hand side exists.

Definition 2 ([25]) For a function $h \in C^n[0,+\infty)$, the Caputo fractional derivative of order ς is defined as

$$ {}_0^C\mathbb{D}_t^{\varsigma} h(t) = \frac{1}{\Gamma(n-\varsigma)} \int_0^t (t-s)^{n-\varsigma-1} h^{(n)}(s)\, ds, \quad n-1 < \varsigma < n, \tag{4}$$

where $n = [\varsigma] + 1$; $[\varsigma]$ denotes the integer part of ς.

Lemma 1 ([24]) *Let* $\varsigma > 0$, *then* ${}_0\mathrm{I}_t^{\varsigma}\, {}_0^C\mathbb{D}_t^{\varsigma} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}, i = 0,1,\ldots,n-1, n = [\varsigma] + 1$.

Let $\phi \in C(I,\mathbb{R}_+)$ be a nondecreasing function, $\varphi \geq 0$, $\nu \in \mathscr{X}$ such that, for $t \in I_k$, $k = 1,2,\ldots,n$, the following sets of inequalities are satisfied:

$$\begin{cases} {}_0^C\mathbb{D}_t^{\varsigma} \nu(t) - \mathrm{g}(t, \nu(t), \nu(\delta t), {}_0^C\mathbb{D}_t^{\varsigma} \nu(t)) \leq \epsilon, \quad \varsigma \in (2,3], 0 < \delta < 1, \\ \Delta \nu(t_k) - \mathcal{I}_k(\nu(t_k)) \leq \epsilon, \\ \Delta \nu'(t_k) - \hat{\mathcal{I}}_k(\nu(t_k)) \leq \epsilon, \\ \Delta \nu''(t_k) - \bar{\mathcal{I}}_k(\nu(t_k)) \leq \epsilon; \end{cases} \tag{5}$$

$$\begin{cases} {}_0^C\mathbb{D}_t^{\varsigma} \nu(t) - \mathrm{g}(t, \nu(t), \nu(\delta t), {}_0^C\mathbb{D}_t^{\varsigma} \nu(t)) \leq \phi(t), \quad \varsigma \in (2,3], 0 < \delta < 1, \\ \Delta \nu(t_k) - \mathcal{I}_k(\nu(t_k)) \leq \varphi, \\ \Delta \nu'(t_k) - \hat{\mathcal{I}}_k(\nu(t_k)) \leq \varphi, \\ \Delta \nu''(t_k) - \bar{\mathcal{I}}_k(\nu(t_k)) \leq \varphi; \end{cases} \tag{6}$$

$$\begin{cases} {}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t) - \mathrm{g}(t,\nu(t),\nu(\delta t),{}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t)) \le \epsilon\phi(t), & \varsigma \in (2,3], 0 < \delta < 1, \\ \Delta\nu(t_k) - \mathcal{I}_k(\nu(t_k)) \le \epsilon\varphi, \\ \Delta\nu'(t_k) - \hat{\mathcal{I}}_k(\nu(t_k)) \le \epsilon\varphi, \\ \Delta\nu''(t_k) - \bar{\mathcal{I}}_k(\nu(t_k)) \le \epsilon\varphi. \end{cases} \tag{7}$$

Definition 3 ([39]) If for $\epsilon > 0$ there exists a constant $C_{\mathrm{g}} > 0$ such that, for any solution $\nu \in \mathscr{Z}$ of inequality (5), there is a unique solution $w \in \mathscr{Z}$ of system (2) that satisfies

$$\left|\nu(t) - w(t)\right| \le C_{\mathrm{g}}\epsilon, \quad t \in I,$$

then system (2) is Hyers–Ulam stable.

Definition 4 If for $\epsilon > 0$ and set of positive real numbers $\mathbb{R}^+$ there exists $\phi \in C(\mathbb{R}^+,\mathbb{R}^+)$, with $\phi(0) = 0$ such that, for any solution $\nu \in \mathscr{Z}$ of inequality (6), there exist $\epsilon > 0$ and a unique solution $w \in \mathscr{Z}$ of system (2) that satisfy

$$\left|\nu(t) - w(t)\right| \le \phi(\epsilon), \quad t \in I,$$

then system (2) is generalized Hyers–Ulam stable.

Definition 5 ([39]) If for $\epsilon > 0$ there exists a real number $C_{\mathrm{g}} > 0$ such that, for any solution $\nu \in \mathscr{Z}$ of inequality (7), there is a unique solution $w \in \mathscr{Z}$ of system (2) that satisfies

$$\left|\nu(t) - w(t)\right| \le C_{\mathrm{g}}\epsilon\bigl(\varphi + \phi(t)\bigr), \quad t \in I,$$

then system (2) is Hyers–Ulam–Rassias stable with respect to (φ,ϕ).

Definition 6 ([39]) If there exists a constant $C_{\mathrm{g}} > 0$ such that, for any solution $\nu \in \mathscr{Z}$ of inequality (6), there is a unique solution $w \in \mathscr{Z}$ of system (2) that satisfies

$$\left|\nu(t) - w(t)\right| \le C_{\mathrm{g}}\bigl(\varphi + \phi(t)\bigr), \quad t \in I,$$

then system (2) is generalized Hyers–Ulam–Rassias stable with respect to (φ,ϕ).

Remark 1 The function $\nu \in \mathscr{Z}$ is called a solution for inequality (5) if there exists a function $\psi \in \mathscr{Z}$ together with a sequence ψ_k, $k = 1,2\ldots,n$ (which depends on ν) such that

(i) $|\psi(t)| \le \epsilon$, $|\psi_k| \le \epsilon$, $t \in I$,
(ii) ${}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t) = \mathrm{g}(t,\nu(t),\nu(\delta t),{}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t)) + \psi(t)$, $\varsigma \in (2,3]$, $0 < \delta < 1$, $t \in I$,
(iii) $\Delta\nu(t_k) = \mathcal{I}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(iv) $\Delta\nu'(t_k) = \hat{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(v) $\Delta\nu''(t_k) = \bar{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$.

Remark 2 A function $\nu \in \mathscr{Z}$ is a solution of inequality (6) if there exist a function $\psi \in \mathscr{Z}$ and a sequence ψ_k, $k = 1,2\ldots,n$ (which depends on ν) such that

(i) $|\psi(t)| \le \phi(t)$, $|\psi_k| \le \varphi$, $t \in I$,
(ii) ${}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t) = \mathrm{g}(t,\nu(t),\nu(\delta t),{}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t)) + \psi(t)$, $\varsigma \in (2,3]$, $0 < \delta < 1$, $t \in I$,

(iii) $\Delta\nu(t_k) = \mathcal{I}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(iv) $\Delta\nu'(t_k) = \hat{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(v) $\Delta\nu''(t_k) = \bar{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$.

Remark 3 A function $\nu \in \mathscr{X}$ is a solution of inequality (7) if there exist a function $\psi \in \mathscr{X}$ and a sequence ψ_k, $k = 1, 2 \dots, n$ (which depends on ν) such that
(i) $|\psi(t)| \le \epsilon\phi(t)$, $|\psi_k| \le \epsilon\varphi$, $t \in I$,
(ii) ${}_0^C\mathbb{D}_t^\varsigma \nu(t) = \mathrm{g}(t, \nu(t), \nu(\delta t), {}_0^C\mathbb{D}_t^\varsigma \nu(t)) + \psi(t)$, $\varsigma \in (2,3]$, $0 < \delta < 1$, $t \in I$,
(iii) $\Delta\nu(t_k) = \mathcal{I}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(iv) $\Delta\nu'(t_k) = \hat{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$,
(v) $\Delta\nu''(t_k) = \bar{\mathcal{I}}_k(\nu(t_k)) + \psi_k$, $t \in I$.

Lemma 2 *For a given function $\vartheta \in C[0,1]$, function w is the solution of the linear BVP of impulsive differential equations*

$$\begin{cases} {}_0^C\mathbb{D}_t^\varsigma w(t) = \vartheta(t), \quad t \in I', \varsigma \in (2,3], k = 1, 2, \dots, n, \\ \Delta w(t_k) = \mathcal{I}_k(w(t_k)), \quad k = 1, 2, \dots, n, \\ \Delta w'(t_k) = \hat{\mathcal{I}}_k(w(t_k)), \qquad \Delta w''(t_k) = \bar{\mathcal{I}}_k(w(t_k)), \quad k = 1, 2, \dots, n, \\ w(0) = -w(1), \qquad w'(0) = -w'(1), \qquad w''(0) = -w''(1) \end{cases} \tag{8}$$

if and only if w satisfies the following fractional integral equation:

$$w(t) = \begin{cases} \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1}\vartheta(s)\,ds - \mathcal{A}, \quad t \in I_0; \\ \frac{1}{\Gamma(\varsigma)} \int_{t_k}^t (t-s)^{\varsigma-1}\vartheta(s)\,ds + \frac{1}{\Gamma(\varsigma)} \sum_{m=1}^k \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1}\vartheta(s)\,ds \\ \quad + \sum_{m=1}^{k-1} \frac{(t_k-t_m)}{\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\vartheta(s)\,ds \\ \quad + \sum_{m=1}^{k-1} \frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\vartheta(s)\,ds \\ \quad + \sum_{m=1}^{k} \frac{(t-t_k)}{\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\vartheta(s)\,ds \\ \quad + \sum_{m=1}^{k-1} \frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\vartheta(s)\,ds \\ \quad + \sum_{m=1}^{k} \frac{(t-t_k)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\vartheta(s)\,ds + \sum_{m=1}^k \mathcal{I}_m(w(t_m)) \\ \quad + \sum_{m=1}^{k-1} (t_k - t_m)\hat{\mathcal{I}}_m(w(t_m)) \\ \quad + \sum_{m=1}^{k-1} \frac{(t_k-t_m)^2}{2} \bar{\mathcal{I}}_m(w(t_m)) + \sum_{m=1}^{k} (t-t_k)\mathcal{I}_m(w(t_m)) \\ \quad + \sum_{m=1}^{k-1} (t-t_k)(t_k-t_m)\bar{\mathcal{I}}_m(w(t_m)) \\ \quad + \sum_{m=1}^{k} \frac{(t-t_k)^2}{2} \bar{\mathcal{I}}_m(w(t_m)) - \mathcal{A}, \quad t \in I_k, k = 1, 2, \dots, n, \end{cases} \tag{9}$$

where

$$\begin{aligned} \mathcal{A} = {} & \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1}\vartheta(s)\,ds + \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\vartheta(s)\,ds \\ & + \sum_{m=1}^{n-1} \frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\vartheta(s)\,ds + \sum_{m=1}^{n} \frac{(1-2t_n)}{4\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\vartheta(s)\,ds \\ & + \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\vartheta(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \vartheta(s)\,ds \\
&- \frac{1}{4\Gamma(\varsigma-1)} \int_{t_n}^{1} (1-s)^{\varsigma-2} \vartheta(s)\,ds + \frac{1}{2} \sum_{m=1}^{n} \mathcal{I}_m\big(w(t_m)\big) + \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2} \hat{\mathcal{I}}_m\big(w(t_m)\big) \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4} \bar{\mathcal{I}}_m\big(w(t_m)\big) + \sum_{m=1}^{n} \frac{(1-2t_n)}{4} \mathcal{I}_m\big(w(t_m)\big) \\
&+ \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n - t_m)}{4} \bar{\mathcal{I}}_m\big(w(t_m)\big) \\
&- \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4} \bar{\mathcal{I}}_m\big(w(t_m)\big) + \frac{t}{2\Gamma(\varsigma-1)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \vartheta(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{t(t_n - t_m)}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \vartheta(s)\,ds + \sum_{m=1}^{n} \frac{t(1-2t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \vartheta(s)\,ds \\
&- \frac{t}{4\Gamma(\varsigma-2)} \int_{t_n}^{1} (1-s)^{\varsigma-3} \vartheta(s)\,ds + \frac{t}{2} \sum_{m=1}^{n} \mathcal{I}_m\big(w(t_m)\big) + \sum_{m=1}^{n-1} \frac{t(t_n - t_m)}{2} \bar{\mathcal{I}}_m\big(w(t_m)\big) \\
&+ \sum_{m=1}^{n} \frac{t(1-2t_n)}{4} \bar{\mathcal{I}}_m\big(w(t_m)\big) + \frac{t^2}{4\Gamma(\varsigma-2)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \vartheta(s)\,ds \\
&- \frac{t^2}{4} \sum_{m=1}^{n} \bar{\mathcal{I}}_m\big(w(t_m)\big).
\end{aligned}
$$

Proof For proof, see [38]. □

3 Main results

Corollary 1 *As a result of Lemma 2, system (2) has the following solution:*

$$
w(t) = \begin{cases}
\frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} \beta_w(s)\,ds - \mathcal{M}, \quad t \in I_0; \\
\frac{1}{\Gamma(\varsigma)} \int_{t_k}^t (t-s)^{\varsigma-1} \beta_w(s)\,ds + \frac{1}{\Gamma(\varsigma)} \sum_{m=1}^{k} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k-1} \frac{(t_k - t_m)}{\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k-1} \frac{(t_k - t_m)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k} \frac{(t-t_k)}{\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k-1} \frac{(t-t_k)(t_k - t_m)}{\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k} \frac{(t-t_k)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \beta_w(s)\,ds \\
\quad + \sum_{m=1}^{k} \mathcal{I}_m(w(t_m)) + \sum_{m=1}^{k-1} (t_k - t_m) \hat{\mathcal{I}}_m(w(t_m)) \\
\quad + \sum_{m=1}^{k-1} \frac{(t_k - t_m)^2}{2} \bar{\mathcal{I}}_m(w(t_m)) \\
\quad + \sum_{m=1}^{k} (t - t_k) \mathcal{I}_m(w(t_m)) + \sum_{m=1}^{k-1} (t - t_k)(t_k - t_m) \bar{\mathcal{I}}_m(w(t_m)) \\
\quad + \sum_{m=1}^{k} \frac{(t-t_k)^2}{2} \bar{\mathcal{I}}_m(w(t_m)) - \mathcal{M}, \quad t \in I_k, k = 1, 2, \ldots, n,
\end{cases} \tag{10}
$$

where

$$
\beta_w = \mathrm{g}\big(t, w(t), w(\delta t), {}_0^C\mathbb{D}_t^{\varsigma} w(t)\big),
$$

and

$$
\begin{aligned}
\mathcal{M} ={}& \frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\beta_w(s)\,ds + \sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds + \sum_{m=1}^{n}\frac{(1-2t_n)}{4\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&- \sum_{m=1}^{n}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&- \frac{1}{4\Gamma(\varsigma-1)}\int_{t_n}^{1}(1-s)^{\varsigma-2}\beta_w(s)\,ds + \frac{1}{2}\sum_{m=1}^{n}\mathcal{I}_m\bigl(w(t_m)\bigr) + \sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2}\hat{\mathcal{I}}_m\bigl(w(t_m)\bigr) \\
&+ \sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr) + \sum_{m=1}^{n}\frac{(1-2t_n)}{4}\mathcal{I}_m\bigl(w(t_m)\bigr) \\
&+ \sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr) \\
&- \sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr) + \frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds + \sum_{m=1}^{n}\frac{t(1-2t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&- \frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\beta_w(s)\,ds + \frac{t}{2}\sum_{m=1}^{n}\mathcal{I}_m\bigl(w(t_m)\bigr) + \sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr) \\
&+ \sum_{m=1}^{n}\frac{t(1-2t_n)}{4}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr) + \frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&- \frac{t^2}{4}\sum_{m=1}^{n}\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr).
\end{aligned}
$$

For $w\in\mathscr{X}$, we define an operator $\mathcal{N}:\mathscr{X}\to\mathscr{X}$ by

$$
\begin{aligned}
\mathcal{N}w(t) ={}& \frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\beta_w(s)\,ds + \frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&+ \sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{k-1}\frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds\\
&+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds\\
&+\sum_{m=1}^{k}\mathcal{I}_m\big(w(t_m)\big)+\sum_{m=1}^{k-1}(t_k-t_m)\hat{\mathcal{I}}_m\big(w(t_m)\big)\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\bar{\mathcal{I}}_m\big(w(t_m)\big)+\sum_{m=1}^{k}(t-t_k)\mathcal{I}_m\big(w(t_m)\big)\\
&+\sum_{m=1}^{k-1}(t-t_k)(t_k-t_m)\bar{\mathcal{I}}_m\big(w(t_m)\big)\\
&+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2}\bar{\mathcal{I}}_m\big(w(t_m)\big)-\frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n}\frac{(1-2t_n)}{4\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds\\
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds+\frac{1}{4\Gamma(\varsigma-1)}\int_{t_n}^{1}(1-s)^{\varsigma-2}\beta_w(s)\,ds\\
&-\frac{1}{2}\sum_{m=1}^{n}\mathcal{I}_m\big(w(t_m)\big)-\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2}\hat{\mathcal{I}}_m\big(w(t_m)\big)-\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4}\bar{\mathcal{I}}_m\big(w(t_m)\big)\\
&-\sum_{m=1}^{n}\frac{(1-2t_n)}{4}\mathcal{I}_m\big(w(t_m)\big)-\sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4}\bar{\mathcal{I}}_m\big(w(t_m)\big)\\
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\bar{\mathcal{I}}_m\big(w(t_m)\big)\\
&-\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds\\
&-\sum_{m=1}^{n}\frac{t(1-2t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\beta_w(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{t}{2}\sum_{m=1}^{n}\mathcal{I}_m\big(w(t_m)\big)-\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\bar{\mathcal{I}}_m\big(w(t_m)\big)-\sum_{m=1}^{n}\frac{t(1-2t_n)}{4}\bar{\mathcal{I}}_m\big(w(t_m)\big)\\
&-\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds+\frac{t^2}{4}\sum_{m=1}^{n}\bar{\mathcal{I}}_m\big(w(t_m)\big),\\
&t\in I_k, k=1,2,\ldots,n. \qquad (11)
\end{aligned}
$$

We take the following assumptions:

(H_1) Let $\mathrm{g}:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be a jointly continuous function.

(H_2) For any $x,y,z,\bar{x},\bar{y},\bar{z}\in C(I,\mathbb{R}), let the following inequality$

$$
\big|\mathrm{g}(t,x,y,z)-\mathrm{g}(t,\bar{x},\bar{y},\bar{z})\big|\le L_{\mathrm{g}}\big(|x-\bar{x}|+|y-\bar{y}|\big)+N_{\mathrm{g}}|z-\bar{z}|
$$

hold, where $L_{\mathrm{g}}>0$ and $0<N_{\mathrm{g}}<1$.

(H_3) There exist $C_1,C_2,C_3>0$ such that the following relations hold true:

$$
\begin{aligned}
&\big|\mathcal{I}_m\big(w(t_m)\big)-\mathcal{I}_m\big(\bar{w}(t_m)\big)\big|\le C_1\big|w(t_m)-\bar{w}(t_m)\big|,\\
&\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\le C_2\big|w(t_m)-\bar{w}(t_m)\big|,\\
&\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)-\hat{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\le C_3\big|w(t_m)-\bar{w}(t_m)\big|.
\end{aligned}
$$

(H_4) There exist functions $\theta_1,\theta_2,\theta_3\in C(I,\mathbb{R}^+)$, with

$$
\begin{aligned}
&\big|\mathrm{g}\big(t,w(t),w(\delta t),{}_0^C D_{t_i}^{\varsigma}w(t)\big)\big|\le\theta_1(t)+\theta_2(t)\big(|w|+\big|w(\delta t)\big|\big)+\theta_3(t)\big|{}_0^C D_{t_i}^{\varsigma}w(t)\big|,\\
&\quad\text{for } t\in I, w\in \mathrm{E},
\end{aligned}
$$

such that $\theta_3^*=\max_{t\in I}|\theta_3(t)|<1$.

(H_5) If $\mathrm{g},\mathcal{I}_m,\hat{\mathcal{I}}_m,\bar{\mathcal{I}}_m$ are continuous functions and there exist constants C_4, C_5, $C_6>0$ such that, for all $w\in\mathbb{R}$, the following inequalities are satisfied:

$$
\big|\mathcal{I}_m(w)(t)\big|\le C_4,\qquad \big|\hat{\mathcal{I}}_m(w)(t)\big|\le C_5,\qquad \big|\bar{\mathcal{I}}_m(w)(t)\big|\le C_6.
$$

Theorem 1 *If assumptions (H_1)–(H_3) and the following inequality*

$$
\left[\frac{L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\left(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)}\right)+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\right]<1
$$

are satisfied, then problem (2) has a unique solution.

Proof We take $w,\bar{w}\in\mathscr{X}$ and consider

$$
\begin{aligned}
&\big|\mathcal{N}w(t)-\mathcal{N}\bar{w}(t)\big|\\
&\quad\le\frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\\
&\qquad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\sum_{m=1}^{k}\big|\mathcal{I}_m\big(w(t_m)\big)-\mathcal{I}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{k-1}(t_k-t_m)\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)-\hat{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{k}(t-t_k)\big|\mathcal{I}_m\big(w(t_m)\big)-\mathcal{I}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{k-1}(t-t_k)(t_k-t_m)\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\\
&+\frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{n}\frac{|1-2t_n|}{4\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\sum_{m=1}^{n-1}\frac{|1-2t_n|(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds+\frac{1}{4\Gamma(\varsigma-1)}\int_{t_n}^{1}(1-s)^{\varsigma-2}\big|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\big|\,ds\\
&+\frac{1}{2}\sum_{m=1}^{n}\big|\mathcal{I}_m\big(w(t_m)\big)-\mathcal{I}_m\big(\bar{w}(t_m)\big)\big|+\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2}\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)-\hat{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|+\sum_{m=1}^{n}\frac{|1-2t_n|}{4}\big|\mathcal{I}_m\big(w(t_m)\big)-\mathcal{I}_m\big(\bar{w}(t_m)\big)\big|\\
&+\sum_{m=1}^{n-1}\frac{|1-2t_n|(t_n-t_m)}{4}\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)-\bar{\mathcal{I}}_m\big(\bar{w}(t_m)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)-\bar{\mathcal{I}}_m\bigl(\bar{w}(t_m)\bigr)\bigr|\\
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\bigl|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\bigr|\,ds+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\bigr|\,ds\\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\bigr|\,ds\\
&+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\bigl|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\bigr|\,ds+\frac{t}{2}\sum_{m=1}^{n}\bigl|\mathcal{I}_m\bigl(w(t_m)\bigr)-\mathcal{I}_m\bigl(\bar{w}(t_m)\bigr)\bigr|\\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)-\bar{\mathcal{I}}_m\bigl(\bar{w}(t_m)\bigr)\bigr|\\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)-\bar{\mathcal{I}}_m\bigl(\bar{w}(t_m)\bigr)\bigr|\\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)-\bar{\beta}_{\bar{w}}(s)\bigr|\,ds\\
&+\frac{t^2}{4}\sum_{m=1}^{n}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)-\bar{\mathcal{I}}_m\bigl(\bar{w}(t_m)\bigr)\bigr|,\\
t\in I_k,\ &k=1,2,\ldots,n, \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_w(t)&=\mathrm{g}\bigl(t,w(t),w(\delta t),\beta_w(t)\bigr),\\
\bar{\beta}_{\bar{w}}(t)&=\mathrm{g}\bigl(t,\bar{w}(t),\bar{w}(\delta t),\bar{\beta}_{\bar{w}}(t)\bigr).
\end{aligned}
$$

By (H_2) we have

$$
\begin{aligned}
\bigl|\beta_w(t)-\bar{\beta}_{\bar{w}}(t)\bigr|&=\bigl|\mathrm{g}\bigl(t,w(t),w(\delta t),\beta_w(t)\bigr)-\mathrm{g}\bigl(t,\bar{w}(t),\bar{w}(\delta t),\bar{\beta}_{\bar{w}}(t)\bigr)\bigr|\\
&\le L_{\mathrm{g}}\bigl(\bigl|w(t)-\bar{w}(t)\bigr|+\bigl|w(\delta t)-\bar{w}(\delta t)\bigr|\bigr)+N_{\mathrm{g}}\bigl|\beta_w(t)-\bar{\beta}_{\bar{w}}(t)\bigr|\\
&\le\frac{2L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\bigl|w(t)-\bar{w}(t)\bigr|.
\end{aligned}
$$

Hence using the last inequality and assumption (H_3), from (12), we get

$$
\begin{aligned}
\|\mathcal{N}w-\mathcal{N}\bar{w}\|_{\mathscr{X}}\le\biggl[&\frac{L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\biggl(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)}\biggr)\\
&+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\biggr]\|w-\bar{w}\|_{\mathscr{X}}.
\end{aligned}
$$

Since

$$\left[\frac{L_g}{(1-N_g)}\left(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)}\right)\right.$$
$$\left.+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\right]<1,$$

therefore, by the Banach contraction principle, problem (2) has a unique solution. □

Theorem 2 *If assumptions* (H_1)–(H_4) *and the inequality are satisfied, then system* (2) *has at least one solution.*

Proof The proof is given in the following four steps.

Step 1: We show that the operator $\mathcal{N}$ defined in (11) is continuous. We take a sequence $w_n \in \mathscr{Z}$ such that $w_n \to w \in \mathscr{Z}$. Consider

$$\begin{aligned}
&\left|\mathcal{N}w_n(t)-\mathcal{N}w(t)\right| \\
&\quad\leq \frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds \\
&\qquad+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds \\
&\qquad+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)} \\
&\qquad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds \\
&\qquad+\sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds \\
&\qquad+\sum_{m=1}^{k-1}\frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\varsigma-2)} \\
&\qquad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_{w,n}(s)-\beta_w(s)\right|ds+\sum_{m=1}^{k}\left|\mathcal{I}_m\big(w_n(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\right| \\
&\qquad+\sum_{m=1}^{k-1}(t_k-t_m)\left|\hat{\mathcal{I}}_m\big(w_n(t_m)\big)-\hat{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&\qquad+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\left|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&\qquad+\sum_{m=1}^{k}(t-t_k)\left|\mathcal{I}_m\big(w_n(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\right| \\
&\qquad+\sum_{m=1}^{k-1}(t-t_k)(t_k-t_m)\left|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{k}\frac{(t-t_k)^2}{2}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds+\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)} \\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n}\frac{(1-2t_n)}{4\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)} \\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds+\frac{1}{4\Gamma(\varsigma-1)}\int_{t_n}^{1}(1-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\frac{1}{2}\sum_{m=1}^{n}\big|\mathcal{I}_m\big(w_n(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2}\big|\hat{\mathcal{I}}_m\big(w_n(t_m)\big)-\hat{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n}\frac{(1-2t_n)}{4}\big|\mathcal{I}_m\big(w_n(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)} \\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n}\frac{t(1-2t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds \\
&+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds+\frac{t}{2}\sum_{m=1}^{n}\big|\mathcal{I}_m\big(w_n(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n}\frac{t(1-2t_n)}{4}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|\,ds\\
&+\frac{t^2}{4}\sum_{m=1}^{n}\big|\bar{\mathcal{I}}_m\big(w_n(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|,\\
&t\in I_k, k=1,2,\ldots,n;
\end{aligned}
\tag{13}
$$

where $\beta_{w,n}(t),\beta_w(t)\in C(I,\mathbb{R})$, which satisfy the following function equations:

$$
\begin{aligned}
&\beta_{w,n}(t)=\mathrm{g}\big(t,w_n(t),w_n(\delta t),\beta_{w,n}(t)\big),\\
&\beta_w(t)=\mathrm{g}\big(t,w(t),w(\delta t),\beta_w(t)\big).
\end{aligned}
$$

By using (H_2), we obtain

$$
\|\beta_{w,n}-\beta_w\|_{\mathscr{X}}\le\frac{2L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\|w_n-w\|_{\mathscr{X}}.
$$

$w_n\to w$ as $n\to\infty$, this implies that $\beta_{w,n}\to\beta_w$ as $n\to\infty$. Moreover, every convergent sequence is bounded, hence let for each $t\in PC(I,\mathbb{R})$ there exist $\ell>0$ such that $|\beta_{w,n}(t)|\le\ell$ and $|\beta_w(t)|\le\ell$. Then

$$
\begin{aligned}
(t-s)^{\varsigma-1}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(t-s)^{\varsigma-1}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(t-s)^{\varsigma-1},\\
(t_m-s)^{\varsigma-1}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(t_m-s)^{\varsigma-1}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(t_m-s)^{\varsigma-1},\\
(t_m-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(t_m-s)^{\varsigma-2}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(t_m-s)^{\varsigma-2},\\
(t_m-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(t_m-s)^{\varsigma-3}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(t_m-s)^{\varsigma-3},\\
(1-s)^{\varsigma-2}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(1-s)^{\varsigma-2}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(1-s)^{\varsigma-2},\\
(1-s)^{\varsigma-3}\big|\beta_{w,n}(s)-\beta_w(s)\big|&\le(1-s)^{\varsigma-3}\big(\big|\beta_{w,n}(s)\big|+\big|\beta_w(s)\big|\big)\\
&\le 2\ell(1-s)^{\varsigma-3}.
\end{aligned}
$$

For $t\in PC(I,\mathbb{R})$, the functions $s\to 2\ell(t_m-s)^{\varsigma-1}$, $s\to 2\ell(t_m-s)^{\varsigma-1}$, $s\to 2\ell(t_m-s)^{\varsigma-2}$, $s\to 2\ell(t_m-s)^{\varsigma-3}$, $s\to 2\ell(1-s)^{\varsigma-2}$, $s\to 2\ell(1-s)^{\varsigma-3}$ are integrable on $[0,t]$. Thus, using the Lebesgue dominated convergent theorem, from (13) we have $|\mathcal{N}w_n(t)-\mathcal{N}w(t)|\to 0$ as $t\to\infty$, which implies $\|\mathcal{N}w_n-\mathcal{N}w\|_{\mathscr{X}}\to 0$ as $t\to\infty$. Therefore, the operator $\mathcal{N}$ is continuous.

Step 2: Next we show that $\mathcal{N}$ is a bounded operator. For each $w \in \mathbf{S}_{\varrho} = \{w \in \mathscr{X} : \|w\|_{\mathscr{X}} \leq \varrho\}$, $\|\mathcal{N}w\|_{\mathscr{X}} \leq \Bbbk$. For $t \in \mathrm{J_k}$,

$$
\begin{aligned}
\bigl|\mathcal{N}w(t)\bigr| \leq{}& \frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\bigl|\beta_w(s)\bigr|\,ds + \frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k-1}\frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{k}\bigl|\mathcal{I}_m\bigl(w(t_m)\bigr)\bigr| + \sum_{m=1}^{k-1}(t_k-t_m)\bigl|\hat{\mathcal{I}}_m\bigl(w(t_m)\bigr)\bigr| \\
&+ \sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)\bigr| + \sum_{m=1}^{k}(t-t_k)\bigl|\mathcal{I}_m\bigl(w(t_m)\bigr)\bigr| \\
&+ \sum_{m=1}^{k-1}(t-t_k)(t_k-t_m)\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)\bigr| \\
&+ \sum_{m=1}^{k}\frac{(t-t_k)^2}{2}\bigl|\bar{\mathcal{I}}_m\bigl(w(t_m)\bigr)\bigr| + \frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{n}\frac{(1-2t_n)}{4\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \sum_{m=1}^{n}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\bigl|\beta_w(s)\bigr|\,ds \\
&+ \frac{1}{4\Gamma(\varsigma-1)}\int_{t_n}^{1}(1-s)^{\varsigma-2}\bigl|\beta_w(s)\bigr|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{m=1}^{n}\left|\mathcal{I}_m\big(w(t_m)\big)\right| + \sum_{m=1}^{n-1}\frac{(t_n-t_m)}{2}\left|\hat{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\sum_{m=1}^{n-1}\frac{(t_n-t_m)^2}{4}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\sum_{m=1}^{n}\frac{(1-2t_n)}{4}\left|\mathcal{I}_m\big(w(t_m)\big)\right| + \sum_{m=1}^{n-1}\frac{(1-2t_n)(t_n-t_m)}{4}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\left|\beta_w(s)\right|\,ds \\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_w(s)\right|\,ds \\
&+\sum_{m=1}^{n}\frac{t(1-2t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_w(s)\right|\,ds \\
&+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\left|\beta_w(s)\right|\,ds \\
&+\frac{t}{2}\sum_{m=1}^{n}\left|\mathcal{I}_m\big(w(t_m)\big)\right| + \sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\sum_{m=1}^{n}\frac{t(1-2t_n)}{4}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right| \\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\left|\beta_w(s)\right|\,ds \\
&+\frac{t^2}{4}\sum_{m=1}^{n}\left|\bar{\mathcal{I}}_m\big(w(t_m)\big)\right|, \quad t\in I_k, k=1,2,\ldots,n.
\end{aligned}
\tag{14}
$$

Using (H_4) with $\theta_1^* = \max_{t\in I}|\theta_1(t)|, \theta_2^* = \max_{t\in I}|\theta_2(t)|$, we have

$$
\begin{aligned}
\left|\beta_w(t)\right| &= \left|\mathrm{g}\big(t, w(t), w(\delta t), \beta_w(t)\big)\right| \\
&\leq \theta_1(t) + \theta_2(t)\big(|w| + \left|w(\delta t)\right|\big) + \theta_3(t)\left|\beta_w(t)\right|.
\end{aligned}
$$

Taking maximum, we have

$$
\max_{t\in I}\left|\beta_w(t)\right| \leq \theta_1^* + 2\theta_2^*\varrho + \theta_3^*\left|\beta_w(t)\right|,
$$

which implies

$$
\max_{t\in I}\left|\beta_w(t)\right| \leq \frac{\theta_1^* + 2\theta_2^*\varrho}{1-\theta_3^*} =: \mu. \tag{15}
$$

Using the result (15), we obtain from (14)

$$\|\mathcal{N}w\|_{\mathscr{X}} \leq \mu\left[\frac{2k+n+3}{2\Gamma(\varsigma+1)}+\frac{8k+5n-3}{4\Gamma(\varsigma)}+\frac{8k+7n-8}{4\Gamma(\varsigma-1)}\right] + \left(\frac{4k+5n}{4}\right)C_4+\left(\frac{2k+n-3}{2}\right)C_5+\left(\frac{2k+7n-6}{4}\right)C_6 := \mathbb{k}.$$

Thus $\mathcal{N}$ is a bounded operator.

Step 3: To show that $\mathcal{N}$ is equicontinuous, we take $w \in \mathbf{S}_\xi = \{w \in \mathscr{X} : \|w\|_{\mathscr{X}} \leq \xi\}$ and $t_1, t_2 \in I_k$ such that $t_2 > t_1$. We have

$$\begin{aligned}
&\big|\mathcal{N}w(t_2)-\mathcal{N}w(t_1)\big| \\
&\leq \frac{1}{\Gamma(\varsigma)}\int_{t_1}^{t_2}(t_2-s)^{\varsigma-1}\big|\beta_w(s)\big|\,ds+\frac{1}{\Gamma(\varsigma)}\sum_{0<t_k<t_2-t_1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_w(s)\big|\,ds \\
&\quad+\sum_{0<t_{k-1}<t_2-t_1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)\big|\,ds+\sum_{0<t_{k-1}<t_2-t_1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)} \\
&\quad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds+\sum_{0<t_k<t_2-t_1}\frac{(t_2-t_1)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)\big|\,ds \\
&\quad+\sum_{0<t_{k-1}<t_2-t_1}\frac{(t_2-t_1)(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_w(s)\,ds \\
&\quad+\sum_{0<t_k<t_2-t_1}\frac{|(t_2-t_1)(t_2+t_1-2t_k)|}{2\Gamma(\varsigma-2)} \\
&\quad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds+\sum_{0<t_k<t_2-t_1}\big|\mathcal{I}_m\big(w(t_m)\big)\big| \\
&\quad+\sum_{0<t_{k-1}<t_2-t_1}(t_k-t_m)\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&\quad+\sum_{0<t_{k-1}<t_2-t_1}\frac{(t_k-t_m)^2}{2}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{0<t_k<t_2-t_1}(t_2-t_1)\big|\mathcal{I}_m\big(w(t_m)\big)\big| \\
&\quad+\sum_{0<t_{k-1}<t_2-t_1}(t_k-t_m)(t_2-t_1)\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&\quad+\sum_{0<t_k<t_2-t_1}\frac{|(t_2-t_1)(t_2+t_1-2t_k)|}{2}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&\quad+\frac{1}{2\Gamma(\varsigma)}\sum_{0<t_{n+1}<t_2-t_1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_w(s)\big|\,ds \\
&\quad+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_n-t_m)}{2\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)\big|\,ds \\
&\quad+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds+\sum_{0<t_n<t_2-t_1}\frac{(1-2t_n)}{4\Gamma(\varsigma-1)} \\
&\quad\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)\big|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_{n-1}<t_2-t_1}\frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds\\
&+\sum_{0<t_n<t_2-t_1}\frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds+\frac{1}{2}\sum_{0<t_n<t_2-t_1}\big|\mathcal{I}_m\big(w(t_m)\big)\big|\\
&+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_n-t_m)}{2}\big|\hat{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_n-t_m)^2}{4}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{0<t_n<t_2-t_1}\frac{(1-2t_n)}{4}\big|\mathcal{I}_m\big(w(t_m)\big)\big|+\sum_{0<t_{n-1}<t_2-t_1}\frac{(1-2t_n)(t_n-t_m)}{4}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{0<t_n<t_2-t_1}\frac{t_n(1-t_n)}{4}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{0<t_{n+1}<t_2-t_1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_w(s)\big|\,ds\\
&+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_2-t_1)(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds\\
&+\sum_{0<t_n<t_2-t_1}\frac{(t_2-t_1)(1-2t_n)}{4\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds\\
&+\frac{(t_2-t_1)}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds+\frac{|t_2-t_1|}{2}\sum_{0<t_n<t_2-t_1}\big|\mathcal{I}_m\big(w(t_m)\big)\big|\\
&+\sum_{0<t_{n-1}<t_2-t_1}\frac{(t_2-t_1)(t_n-t_m)}{2}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{0<t_n<t_2-t_1}\frac{(t_2-t_1)(1-2t_n)}{4}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\frac{({t_2}^2-{t_1}^2)}{4\Gamma(\varsigma-2)}\sum_{0<t_{n+1}<t_2-t_1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_w(s)\big|\,ds\\
&+\frac{({t_2}^2-{t_1}^2)}{4}\sum_{0<t_n<t_2-t_1}\big|\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|.
\end{aligned}
\tag{16}
$$

Using assumptions (H_1), (H_4)–(H_5), inequality (15) in (16) and evaluating, we can easily show that as t_1 tends to t_2 the right-hand side of (16) tends to 0. Thus by the Arzelà–Ascoli theorem the operator $\mathcal{N}$ is completely continuous.

Step 4: In the final step we define a set $\mathbf{S}_{\varrho^*} = \{w \in \mathscr{X} : \varrho^*\mathcal{N}w, for 0 < \varrho^* < 1\}$. We need to show that $\mathbf{S}_{\varrho^*}$ is bounded. Let $w \in \mathbf{S}_{\varrho^*}$, then by definition, we have $w = \varrho^*\mathcal{N}w$. From Step 2, we obtain

$$
\begin{aligned}
\|\mathcal{N}w\|_{\mathscr{X}} \le{}& \varrho^*\mu\left[\frac{2k+n+3}{2\Gamma(\varsigma+1)}+\frac{8k+5n-3}{4\Gamma(\varsigma)}+\frac{8k+7n-8}{4\Gamma(\varsigma-1)}\right]\\
&+\left(\frac{4k+5n}{4}\right)C_4+\left(\frac{2k+n-3}{2}\right)C_5+\left(\frac{2k+7n-6}{4}\right)C_6=\varrho^*\Bbbk\le\Bbbk.
\end{aligned}
$$

Thus set $\mathcal{E}$ is bounded. Therefore, by Schaefer's fixed point theorem, system (2) has at least one solution. □

4 Stability results

In this section we investigate the results related to Hyers–Ulam stability of system (2).

Theorem 3 *If assumptions* (H_1)–(H_3) *and the inequality*

$$\left[\frac{L_g}{(1-N_g)}\left(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)}\right)+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\right]<1$$

are satisfied, then system (2) *is Hyers–Ulam stable.*

Proof Let ν be any solution of inequality (5). Then, by Remark 1, we write

$$\begin{cases} {}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t)=g(t,\nu(t),\nu(\delta t),{}^{C}_{0}\mathbb{D}^{\varsigma}_{t}\nu(t))+\psi(t)\varsigma\in(2,3], \quad 0<\delta<1,\\ \Delta\nu(t_k)=\mathcal{I}_k(\nu(t_k))+\psi_k\\ \Delta\nu'(t_k)-\hat{\mathcal{I}}_k(\nu(t_k))+\psi_k\\ \Delta\nu''(t_k)-\bar{\mathcal{I}}_k(\nu(t_k))+\psi_k\\ \nu(0)=-\nu(1), \qquad \nu'(0)=-\nu'(1), \qquad \nu''(0)=-\nu''(1). \end{cases} \tag{17}$$

By Corollary 1, the solution of (17) for $t\in I_0$ is given by

$$\nu(t)=\frac{1}{\Gamma(\varsigma)}\int_0^t(t-s)^{\varsigma-1}\beta_\nu(s)\,ds+\frac{1}{\Gamma(\varsigma)}\int_0^t(t-s)^{\varsigma-1}\psi(s)\,ds-\mathcal{M}^*$$

and the solution of (17) for $t\in I_k$, $k=1,2,\dots,n$, is given by

$$\begin{aligned}\nu(t)={}&\frac{1}{\Gamma(\varsigma)}\int_{t_k}^t(t-s)^{\varsigma-1}\beta_\nu(s)\,ds+\frac{1}{\Gamma(\varsigma)}\int_{t_k}^t(t-s)^{\varsigma-1}\psi(s)\,ds\\ &+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\beta_\nu(s)\,ds\\ &+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\psi(s)\,ds+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_\nu(s)\,ds\\ &+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\\ &\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\psi(s)\,ds+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\beta_\nu(s)\,ds\\ &+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\\ &\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\psi(s)\,ds+\sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\beta_\nu(s)\,ds+\sum_{m=1}^{k}\frac{(t-t_k)}{\Gamma(\varsigma-1)}\end{aligned}$$

$$\begin{aligned}
&\times \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2}\psi(s)\,ds + \sum_{m=1}^{k-1} \frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\beta_\nu(s)\,ds \\
&+ \sum_{m=1}^{k-1} \frac{(t-t_k)(t_k-t_m)}{\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\psi(s)\,ds \\
&+ \sum_{m=1}^{k} \frac{(t-t_k)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\beta_\nu(s)\,ds \\
&+ \sum_{m=1}^{k} \frac{(t-t_k)^2}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\psi(s)\,ds + \sum_{m=1}^{k} \mathcal{I}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{k} \psi_k + \sum_{m=1}^{k-1} (t_k - t_m)\hat{\mathcal{I}}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{k-1} (t_k - t_m)\psi_k + \sum_{m=1}^{k-1} \frac{(t_k-t_m)^2}{2} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{k-1} \frac{(t_k-t_m)^2}{2} \psi_k + \sum_{m=1}^{k} (t-t_k)\mathcal{I}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{k} (t-t_k)\psi_k + \sum_{m=1}^{k-1} (t-t_k)(t_k-t_m)\bar{\mathcal{I}}_m\big(\nu(t_m)\big) + \sum_{m=1}^{k-1} (t-t_k)(t_k-t_m)\psi_k \\
&+ \sum_{m=1}^{k} \frac{(t-t_k)^2}{2} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) + \sum_{m=1}^{k} \frac{(t-t_k)^2}{2} \psi_k - \mathcal{M}^*,
\end{aligned}$$

where

$$\beta_\nu = \mathrm{g}\big(t, \nu(t), \nu(\delta t), {}_0^C\mathbb{D}_t^\varsigma \nu(t)\big),$$

and

$$\begin{aligned}
\mathcal{M}^* = {}& \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1}\beta_\nu(s)\,ds + \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1}\psi(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n-t_m)}{2\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2}\beta_\nu(s)\,ds + \sum_{m=1}^{n-1} \frac{(t_n-t_m)}{2\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2}\psi(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\beta_\nu(s)\,ds + \sum_{m=1}^{n-1} \frac{(t_n-t_m)^2}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\psi(s)\,ds \\
&+ \sum_{m=1}^{n} \frac{(1-2t_n)}{4\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2}\beta_\nu(s)\,ds + \sum_{m=1}^{n} \frac{(1-2t_n)}{4\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2}\psi(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\beta_\nu(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n-t_m)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3}\psi(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\beta_\nu(s)\,ds - \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\psi(s)\,ds \\
&- \frac{1}{4\Gamma(\varsigma-1)} \int_{t_n}^{1} (1-s)^{\varsigma-2}\beta_\nu(s)\,ds - \frac{1}{4\Gamma(\varsigma-1)} \int_{t_n}^{1} (1-s)^{\varsigma-2}\psi(s)\,ds \\
&+ \frac{1}{2}\sum_{m=1}^{n} \mathcal{I}_m\big(\nu(t_m)\big) + \frac{1}{2}\sum_{m=1}^{n} \psi_k \\
&+ \sum_{m=1}^{n-1} \frac{(t_n-t_m)}{2} \hat{\mathcal{I}}_m\big(\nu(t_m)\big) + \sum_{m=1}^{n-1} \frac{(t_n-t_m)}{2} \psi_k \\
&+ \sum_{m=1}^{n-1} \frac{(t_n-t_m)^2}{4} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) + \sum_{m=1}^{n-1} \frac{(t_n-t_m)^2}{4} \psi_k \\
&+ \sum_{m=1}^{n} \frac{(1-2t_n)}{4} \mathcal{I}_m\big(\nu(t_m)\big) + \sum_{m=1}^{n} \frac{(1-2t_n)}{4} \psi_k + \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n-t_m)}{4} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{n-1} \frac{(1-2t_n)(t_n-t_m)}{4} \psi_k - \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \sum_{m=1}^{n} \frac{t_n(1-t_n)}{4} \psi_k \\
&+ \frac{t}{2\Gamma(\varsigma-1)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\beta_\nu(s)\,ds + \frac{t}{2\Gamma(\varsigma-1)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-2}\psi(s)\,ds \\
&+ \sum_{m=1}^{n-1} \frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\beta_\nu(s)\,ds + \sum_{m=1}^{n-1} \frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\psi(s)\,ds \\
&+ \sum_{m=1}^{n} \frac{t(1-2t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\beta_\nu(s)\,ds + \sum_{m=1}^{n} \frac{t(1-2t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\psi(s)\,ds \\
&- \frac{t}{4\Gamma(\varsigma-2)} \int_{t_n}^{1} (1-s)^{\varsigma-3}\beta_\nu(s)\,ds - \frac{t}{4\Gamma(\varsigma-2)} \int_{t_n}^{1} (1-s)^{\varsigma-3}\psi(s)\,ds \\
&+ \frac{t}{2}\sum_{m=1}^{n} \mathcal{I}_m\big(\nu(t_m)\big) \\
&+ \frac{t}{2}\sum_{m=1}^{n} \psi_k + \sum_{m=1}^{n-1} \frac{t(t_n-t_m)}{2} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) + \sum_{m=1}^{n-1} \frac{t(t_n-t_m)}{2} \psi_k \\
&+ \sum_{m=1}^{n} \frac{t(1-2t_n)}{4} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) \\
&+ \sum_{m=1}^{n} \frac{t(1-2t_n)}{4} \psi_k + \frac{t^2}{4\Gamma(\varsigma-2)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\beta_\nu(s)\,ds + \frac{t^2}{4\Gamma(\varsigma-2)} \sum_{m=1}^{n+1} \\
&\times \int_{t_{m-1}}^{t_m} (t_m-s)^{\varsigma-3}\psi(s)\,ds - \frac{t^2}{4}\sum_{m=1}^{n} \bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \frac{t^2}{4}\sum_{m=1}^{n} \psi_k.
\end{aligned}
$$

We consider, for $t \in I_k$,

$$
\begin{aligned}
&\big|\nu(t) - w(t)\big| \\
&\quad \le \frac{1}{\Gamma(\varsigma)} \int_{t_k}^{t} (t-s)^{\varsigma-1}\big|\beta_\nu(s) - \beta_w(s)\big|\,ds + \frac{1}{\Gamma(\varsigma)} \int_{t_k}^{t} (t-s)^{\varsigma-1}\big|\psi(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|t-t_k|}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|t-t_k|}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{|t-t_k|(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\sum_{m=1}^{k-1}\frac{|t-t_k|(t_k-t_m)}{\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k}\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k}|\psi_k|\\
&+\sum_{m=1}^{k-1}(t_k-t_m)\big|\hat{\mathcal{I}}_m\big(\nu(t_m)\big)-\hat{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k-1}(t_k-t_m)|\psi_k|+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\\
&\times\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}|\psi_k|\\
&+\sum_{m=1}^{k}|t-t_k|\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{k}|t-t_k||\psi_k|+\sum_{m=1}^{k-1}|t-t_k|(t_k-t_m)\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{k-1}|t-t_k|(t_k-t_m)|\psi_k|\\
&+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2}|\psi_k|+\frac{1}{2\Gamma(\varsigma)}\sum_{m=1}^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds + \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2\Gamma(\varsigma - 1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2\Gamma(\varsigma - 1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4\Gamma(\varsigma - 2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4\Gamma(\varsigma - 2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4\Gamma(\varsigma - 1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4\Gamma(\varsigma - 1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)}{4\Gamma(\varsigma - 2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds + \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)}{4\Gamma(\varsigma - 2)} \\
&\times \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds + \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4\Gamma(\varsigma - 2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4\Gamma(\varsigma - 2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds \\
&+ \frac{1}{4\Gamma(\varsigma - 1)} \int_{t_n}^{1} (1 - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \frac{1}{4\Gamma(\varsigma - 1)} \int_{t_n}^{1} (1 - s)^{\varsigma-2} \big|\psi(s)\big|\, ds + \frac{1}{2} \sum_{m=1}^{n} \big|\mathcal{I}_m\big(\nu(t_m)\big) - \mathcal{I}_m\big(w(t_m)\big)\big| + \frac{1}{2} \sum_{m=1}^{n} |\psi_k| \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2} \big|\hat{\mathcal{I}}_m\big(\nu(t_m)\big) - \hat{\mathcal{I}}_m\big(w(t_m)\big)\big| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2} |\psi_k| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4} \\
&\times \big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \bar{\mathcal{I}}_m\big(w(t_m)\big)\big| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4} |\psi_k| \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4} \big|\mathcal{I}_m\big(\nu(t_m)\big) - \mathcal{I}_m\big(w(t_m)\big)\big| \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4} |\psi_k| + \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)|}{4} \big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+ \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)}{4} |\psi_k| + \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4} \big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+ \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4} |\psi_k|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\frac{t}{2}\sum_{m=1}^{n}\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|+\frac{t}{2}\sum_{m=1}^{n}|\psi_k|\\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}|\psi_k|+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4}|\psi_k|\\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\frac{t^2}{4}\sum_{m=1}^{n}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\frac{t^2}{4}\sum_{m=1}^{n}|\psi_k|. \qquad (18)
\end{aligned}
$$

Taking into account assumptions (H_1)–(H_3) and taking maximum value, we obtain

$$
\begin{aligned}
&\|\nu-w\|_{\mathscr{X}}\\
&\quad\le\bigg[\frac{L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\bigg(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}\\
&\qquad+\frac{15n-8}{2\Gamma(\varsigma-1)}\bigg)+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\bigg]\|\nu-w\|_{\mathscr{X}}\\
&\qquad+\bigg(\frac{3(n+1)}{2\Gamma(\varsigma+1)}+\frac{13n-3}{4\Gamma(\varsigma)}+\frac{15n-8}{4\Gamma(\varsigma-1)}+\frac{34n-16}{4}\bigg)\epsilon.
\end{aligned}
$$

This inequality implies

$$\|\nu - w\|_{\mathscr{X}} \leq \frac{(\frac{3(n+1)}{2\Gamma(\varsigma+1)} + \frac{13n-3}{4\Gamma(\varsigma)} + \frac{15n-8}{4\Gamma(\varsigma-1)} + \frac{34n-16}{4})\epsilon}{1 - [\frac{L_g}{(1-N_g)}(\frac{3(n+1)}{\Gamma(\varsigma+1)} + \frac{13n-3}{2\Gamma(\varsigma)} + \frac{15n-8}{2\Gamma(\varsigma-1)}) + \frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}]}.$$

Or

$$\|\nu - w\|_{\mathscr{X}} \leq C_g\epsilon,$$

where

$$C_g = \frac{(\frac{3(n+1)}{2\Gamma(\varsigma+1)} + \frac{13n-3}{4\Gamma(\varsigma)} + \frac{15n-8}{4\Gamma(\varsigma-1)} + \frac{34n-16}{4})}{1 - [\frac{L_g}{(1-N_g)}(\frac{3(n+1)}{\Gamma(\varsigma+1)} + \frac{13n-3}{2\Gamma(\varsigma)} + \frac{15n-8}{2\Gamma(\varsigma-1)}) + \frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}]}$$

with

$$\left[\frac{L_g}{(1-N_g)}\left(\frac{3(n+1)}{\Gamma(\varsigma+1)} + \frac{13n-3}{2\Gamma(\varsigma)} + \frac{15n-8}{2\Gamma(\varsigma-1)}\right) + \frac{13nC_1 + 5(3n-2)C_2 + 6(n-1)C_3}{4}\right] < 1.$$

Therefore, problem (2) is Hyers–Ulam stable. □

Corollary 2 *In Theorem 3, if we set* $\phi(\epsilon) = C_g(\epsilon)$ *such that* $\phi(0) = 0$, *then problem* (2) *becomes generalized Hyers–Ulam stable.*

For the next coming result, we assume that

(H_6) There exist a non-decreasing function $\phi \in C(I,\mathbb{R})$ and constants $\lambda_\phi > 0, \epsilon > 0$ such that the following inequality holds:

$${}_0I_t{}^{\varsigma}\phi(t) \leq \lambda_\phi\phi(t).$$

Theorem 4 *If assumptions* (H_1)–(H_3), (H_6) *and the inequality*

$$\left[\frac{L_g}{(1-N_g)}\left(\frac{3(n+1)}{\Gamma(\varsigma+1)} + \frac{13n-3}{2\Gamma(\varsigma)} + \frac{15n-8}{2\Gamma(\varsigma-1)}\right) + \frac{13nC_1 + 5(3n-2)C_2 + 6(n-1)C_3}{4}\right] < 1$$

are satisfied, then system (2) *is Hyers–Ulam–Rassias stable with respect to* (φ,ϕ), *where* ϕ *is a nondecreasing function and* $\varphi \geq 0$.

Proof Let ν be any solution of inequality (7) and w be the unique solution of problem (2). Then, from the proof of 3, we have the following inequality:

$$\begin{aligned}&\left|\nu(t) - w(t)\right| \\ &\quad\leq \frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\left|\beta_\nu(s) - \beta_w(s)\right|ds + \frac{1}{\Gamma(\varsigma)}\int_{t_k}^{t}(t-s)^{\varsigma-1}\left|\psi(s)\right|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\frac{1}{\Gamma(\varsigma)}\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-1}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|t-t_k|}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|t-t_k|}{\Gamma(\varsigma-1)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k-1}\frac{|t-t_k|(t_k-t_m)}{\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\sum_{m=1}^{k-1}\frac{|t-t_k|(t_k-t_m)}{\Gamma(\varsigma-2)}\\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds\\
&+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds\\
&+\sum_{m=1}^{k}\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k}|\psi_k|\\
&+\sum_{m=1}^{k-1}(t_k-t_m)\big|\hat{\mathcal{I}}_m\big(\nu(t_m)\big)-\hat{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k-1}(t_k-t_m)|\psi_k|+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}\\
&\times\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{k-1}\frac{(t_k-t_m)^2}{2}|\psi_k|\\
&+\sum_{m=1}^{k}|t-t_k|\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{k}|t-t_k||\psi_k|+\sum_{m=1}^{k-1}|t-t_k|(t_k-t_m)\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{k-1}|t-t_k|(t_k-t_m)|\psi_k|+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|\\
&+\sum_{m=1}^{k}\frac{|(t-t_k)^2|}{2}|\psi_k|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \frac{1}{2\Gamma(\varsigma)} \sum_{m=1}^{n+1} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-1} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4\Gamma(\varsigma-1)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-2} \big|\psi(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)}{4\Gamma(\varsigma-2)} \\
&\times \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds + \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \sum_{m=1}^{n} \frac{t_n(1 - t_n)}{4\Gamma(\varsigma-2)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\varsigma-3} \big|\psi(s)\big|\, ds \\
&+ \frac{1}{4\Gamma(\varsigma-1)} \int_{t_n}^{1} (1 - s)^{\varsigma-2} \big|\beta_\nu(s) - \beta_w(s)\big|\, ds \\
&+ \frac{1}{4\Gamma(\varsigma-1)} \int_{t_n}^{1} (1 - s)^{\varsigma-2} \big|\psi(s)\big|\, ds + \frac{1}{2} \sum_{m=1}^{n} \big|\mathcal{I}_m\big(\nu(t_m)\big) - \mathcal{I}_m\big(w(t_m)\big)\big| + \frac{1}{2} \sum_{m=1}^{n} |\psi_k| \\
&+ \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2} \big|\hat{\mathcal{I}}_m\big(\nu(t_m)\big) - \hat{\mathcal{I}}_m\big(w(t_m)\big)\big| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)}{2} |\psi_k| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4} \\
&\times \big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \bar{\mathcal{I}}_m\big(w(t_m)\big)\big| + \sum_{m=1}^{n-1} \frac{(t_n - t_m)^2}{4} |\psi_k| \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4} \big|\mathcal{I}_m\big(\nu(t_m)\big) - \mathcal{I}_m\big(w(t_m)\big)\big| \\
&+ \sum_{m=1}^{n} \frac{|1 - 2t_n|}{4} |\psi_k| + \sum_{m=1}^{n-1} \frac{|1 - 2t_n|(t_n - t_m)|}{4} \big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big) - \bar{\mathcal{I}}_m\big(w(t_m)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{n-1}\frac{|1-2t_n|(t_n-t_m)}{4}|\psi_k| \\
&+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{n}\frac{t_n(1-t_n)}{4}|\psi_k|+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1} \\
&\times\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\frac{t}{2\Gamma(\varsigma-1)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-2}\big|\psi(s)\big|\,ds \\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds \\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds \\
&+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4\Gamma(\varsigma-2)}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds \\
&+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds+\frac{t}{4\Gamma(\varsigma-2)}\int_{t_n}^{1}(1-s)^{\varsigma-3}\big|\psi(s)\big|\,ds \\
&+\frac{t}{2}\sum_{m=1}^{n}\big|\mathcal{I}_m\big(\nu(t_m)\big)-\mathcal{I}_m\big(w(t_m)\big)\big| \\
&+\frac{t}{2}\sum_{m=1}^{n}|\psi_k|+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big| \\
&+\sum_{m=1}^{n-1}\frac{t(t_n-t_m)}{2}|\psi_k|+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\sum_{m=1}^{n}\frac{t|1-2t_n|}{4}|\psi_k| \\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\beta_\nu(s)-\beta_w(s)\big|\,ds \\
&+\frac{t^2}{4\Gamma(\varsigma-2)}\sum_{m=1}^{n+1}\int_{t_{m-1}}^{t_m}(t_m-s)^{\varsigma-3}\big|\psi(s)\big|\,ds \\
&+\frac{t^2}{4}\sum_{m=1}^{n}\big|\bar{\mathcal{I}}_m\big(\nu(t_m)\big)-\bar{\mathcal{I}}_m\big(w(t_m)\big)\big|+\frac{t^2}{4}\sum_{m=1}^{n}|\psi_k|.
\end{aligned}
\tag{19}
$$

Using assumptions (H_1)–(H_3) and (H_6), we get the following result in its simplified form:

$$
\begin{aligned}
&\|\nu-w\|_{\mathscr{X}} \\
&\quad\le\Bigg[\frac{L_{\mathrm{g}}}{(1-N_{\mathrm{g}})}\bigg(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)} \\
&\qquad+\frac{15n-8}{2\Gamma(\varsigma-1)}\bigg)+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}\Bigg]\|\nu-w\|_{\mathscr{X}} \\
&\qquad+\epsilon\big(\phi(t)+\varphi\big)\bigg(\frac{\lambda_\phi(34n-9)+2(17n-8)}{4}\bigg),
\end{aligned}
$$

which implies

$$\|\nu - w\|_{\mathscr{X}} \leq \frac{\epsilon(\phi(t)+\varphi)(\frac{\lambda_\phi(34n-9)+2(17n-8)}{4})}{1-[\frac{L_g}{(1-N_g)}(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)})+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}]},$$

or

$$\|\nu - w\|_{\mathscr{X}} \leq C_g\epsilon\big(\phi(t)+\varphi\big), \tag{20}$$

where

$$C_g = \frac{(\frac{\lambda_\phi(34n-9)+2(17n-8)}{4})}{1-[\frac{L_g}{(1-N_g)}(\frac{3(n+1)}{\Gamma(\varsigma+1)}+\frac{13n-3}{2\Gamma(\varsigma)}+\frac{15n-8}{2\Gamma(\varsigma-1)})+\frac{13nC_1+5(3n-2)C_2+6(n-1)C_3}{4}]}.$$

Therefore, problem (2) is Hyers–Ulam–Rassias stable. □

5 Application

In this section we provide a numerical problem to verify the applications of our main results.

Example 1 Consider the following problem:

$$\begin{cases} {}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t) = \frac{e^{-\pi t}}{12} + \frac{e^{-t}}{44+t^2}(\sin(|w(t)|) + w(\frac{1}{3}t) + \sin(|{}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t)|)), \\ \quad t \in [0,1], t \neq \frac{1}{3}, k = 1, \\ \Delta w(\frac{1}{3}) = \mathcal{I}_1(w(\frac{1}{3})) = \frac{|w(\frac{1}{3})|}{26+|w(\frac{1}{3})|}, \\ \Delta w'(\frac{1}{3}) = \hat{\mathcal{I}}_1(w(\frac{1}{3})) = \frac{|w(\frac{1}{3})|}{25+|w(\frac{1}{3})|}, \\ \Delta w''(\frac{1}{3}) = \bar{\mathcal{I}}_1(w(\frac{1}{3})) = \frac{|w(\frac{1}{3})|}{20+|w(\frac{1}{3})|}, \\ w(0) = -w(1), \\ w'(0) = -w'(1), \\ w''(0) = -w''(1), \end{cases} \tag{21}$$

where e is an exponential function.

Here,

$$g\big(t, w(t), w(\delta t), {}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t)\big) = \frac{e^{-\pi t}}{12} + \frac{\exp(-t)}{44+t^2}\left(\sin\big(|w(t)|\big) + w\left(\frac{1}{3}t\right) + \sin\big(\big|{}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t)\big|\big)\right),$$

with $\varsigma = \frac{5}{2}$, $\delta = \frac{1}{3}$. The continuity of g is obvious.

By hypothesis (H_2), for any $w, \bar{w} \in \mathbb{R}$, we have

$$\begin{aligned} &\big|g\big(t, w(t), w(\lambda_\phi t), {}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t)\big) - g\big(t, \bar{w}(t), \bar{w}(\lambda_\phi t), {}_0^C\mathbb{D}_t^{\frac{5}{2}} \bar{w}(t)\big)\big| \\ &\quad \leq \frac{1}{44}\big[2\big|w(t) - \bar{w}(t)\big| + \big|{}_0^C\mathbb{D}_t^{\frac{5}{2}} w(t) - {}_0^C\mathbb{D}_t^{\frac{5}{2}} \bar{w}(t)\big|\big]. \end{aligned}$$

Hence g satisfies hypothesis (H_2) with $L_g = N_g = \frac{1}{44}$. Also hypothesis (H_4) holds with $\theta_0(t) = \frac{\exp(-\pi t)}{12}$, $\theta_1(t) = \theta_2(t) = \frac{\exp(-t)}{44+t}$, where $\theta_0^*(t) = \frac{1}{12}$, $\theta_1^*(t) = \theta_2^*(t) = \frac{1}{44}$.

At $t_1 = \frac{1}{3}$ the impulsive conditions are given as follows:

$$\begin{aligned} \mathcal{I}_1 w\left(\frac{1}{3}\right) &= \frac{|w(\frac{1}{3})|}{26 + |w(\frac{1}{3})|}, \\ \hat{\mathcal{I}}_1 w'\left(\frac{1}{3}\right) &= \frac{|w(\frac{1}{3})|}{25 + |w(\frac{1}{3})|}, \\ \bar{\mathcal{I}}_1 w''\left(\frac{1}{3}\right) &= \frac{|w(\frac{1}{3})|}{20 + |w(\frac{1}{3})|}. \end{aligned}$$

For any $w, \bar{w} \in \mathrm{E}$, we have

$$\begin{aligned} \left|\mathcal{I}_1\left(w\left(\frac{1}{3}\right)\right) - \mathcal{I}_1\left(\bar{w}\left(\frac{1}{3}\right)\right)\right| &= \left|\frac{|w(\frac{1}{3})|}{26 + |w(\frac{1}{3})|} - \frac{|w(\frac{1}{3})|}{26 + |\bar{w}(\frac{1}{3})|}\right| \le \frac{1}{26}\left|w\left(\frac{1}{3}\right) - \bar{w}\left(\frac{1}{3}\right)\right|, \\ \left|\hat{\mathcal{I}}_1\left(w\left(\frac{1}{3}\right)\right) - \hat{\mathcal{I}}_1\left(\bar{w}\left(\frac{1}{3}\right)\right)\right| &= \left|\frac{|w(\frac{1}{3})|}{25 + |w(\frac{1}{3})|} - \frac{|w(\frac{1}{3})|}{25 + |\bar{w}(\frac{1}{3})|}\right| \le \frac{1}{25}\left|w\left(\frac{1}{3}\right) - \bar{w}\left(\frac{1}{3}\right)\right| \end{aligned}$$

and

$$\left|\bar{\mathcal{I}}_1\left(w\left(\frac{1}{3}\right)\right) - \bar{\mathcal{I}}_1\left(\bar{w}\left(\frac{1}{3}\right)\right)\right| = \left|\frac{|w(\frac{1}{3})|}{20 + |w(\frac{1}{3})|} - \frac{|w(\frac{1}{3})|}{20 + |\bar{w}(\frac{1}{3})|}\right| \le \frac{1}{20}\left|w\left(\frac{1}{3}\right) - \bar{w}\left(\frac{1}{3}\right)\right|,$$

which satisfy (H_3) with $C_1 = \frac{1}{26}$, $C_2 = \frac{1}{25}$, $C_3 = \frac{1}{20}$. So we have

$$\begin{aligned} &\left[\frac{L_{\mathrm{g}}}{(1 - N_{\mathrm{g}})}\left(\frac{3(n+1)}{\Gamma(\varsigma + 1)} + \frac{13n - 3}{2\Gamma(\varsigma)} + \frac{15n - 8}{2\Gamma(\varsigma - 1)}\right) + \frac{13nC_1 + 5(3n - 2)C_2 + 6(n - 1)C_3}{4}\right] \\ &\quad = \frac{1771}{25{,}800\sqrt{\pi}} < 1. \end{aligned}$$

Therefore, by Theorem 1, problem (21) has a unique solution. By Theorem 3, problem (21) is Hyers–Ulam stable. For any $t \in [0, 1]$, we set $\phi(t) = t$, $\varphi = 1$. Then

$$\begin{aligned} {}_0\mathrm{I}_t^{\frac{5}{2}}\phi(t) &= \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t - s)^{\frac{5}{2} - 1} s\, ds \\ &\le \frac{8}{15\sqrt{\pi}} t. \end{aligned}$$

We see assumption (H_6) holds with $\lambda_\phi = \frac{8}{15\sqrt{\pi}}$. Also since

$$\begin{aligned} &\left[\frac{L_{\mathrm{g}}}{(1 - N_{\mathrm{g}})}\left(\frac{3(n+1)}{\Gamma(\varsigma + 1)} + \frac{13n - 3}{2\Gamma(\varsigma)} + \frac{15n - 8}{2\Gamma(\varsigma - 1)}\right) + \frac{13nC_1 + 5(3n - 2)C_2 + 6(n - 1)C_3}{4}\right] \\ &\quad = \frac{1771}{25{,}800\sqrt{\pi}} < 1. \end{aligned}$$

Therefore, the numerical problem (21) is Hyers–Ulam–Rassias stable with respect to (φ, ϕ).

6 Conclusion

In this paper, using Schaefer's fixed point theorem, we derived a result of at least one solution to system (2). By the application of Banach contraction theorem, we obtained conditions for unique solution of problem (2). Further, by the applications of qualitative theory and nonlinear functional analysis, we investigated Ulam–Hyers stability to the considered system. We applied our obtained results to a numerical problem.

Acknowledgements
We are really thankful to the reviewers for their useful suggestions.

Funding
We have no funds for this research work.

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally contributed to this manuscript and approved the final version.

Author details
[1]Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan. [2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. [3]Department of Medical Research, China Medical University, Taichung, 40402, Taiwan. [4]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. [5]Department of Mathematics, Shaheed BB University, Dir Uppr, Khyber Pakhtunkhwa, Pakistan.

Publisher's Note

Received: 23 July 2020 Accepted: 5 October 2020 Published online: 12 October 2020

References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Butkovskii, A.G., Postnov, S.S., Postnova, E.A.: Fractional integrodifferential calculus and its control theoretical applications. I. Mathematical fundamentals and the problem of interpretation. Autom. Remote Control **74**(4), 543–574 (2013)
3. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
4. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Elwy, O., Abdelaty, A.M., Said, L.A., Radwan, A.G.: Fractional calculus definitions, approximations, and engineering applications. J. Eng. Appl. Sci. **67**(1), 1–30 (2020)
6. Torvik, P.J., Bagley, R.L.: On the appearance of fractional derivatives in the behaviour of real materials. J. Appl. Mech. **51**(2), 294–298 (1984)
7. Wang, G., Ren, X.: Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy–Schrödinger systems. Appl. Math. Lett. **2020**, 106560 (2020)
8. Zhang, L., Hou, W.: Standing waves of nonlinear fractional *p*-Laplacian Schrödinger equation involving logarithmic nonlinearity. Appl. Math. Lett. **102**, 106149 (2020)
9. Wang, G., Ren, X., Bai, Z., Hou, W.: Radial symmetry of standing waves for nonlinear fractional Hardy–Schrödinger equation. Appl. Math. Lett. **96**, 131–137 (2019)
10. Wang, F., Yang, Y., Hu, M.: Asymptotic stability of delayed fractional-order neural networks with impulsive effects. Neurocomputing **154**, 239–244 (2015)
11. Wang, F., et al.: Global asymptotic stability of impulsive fractional-order BAM neural networks with time delay. Neural Comput. Appl. **28**(2), 345–352 (2017)
12. Ali, A., Rabiei, F., Shah, K.: On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions. J. Nonlinear Sci. Appl. **10**(9), 4760–4775 (2017)
13. Ali, A., Shah, K., Jarad, F., Gupta, V., Abdeljawad, T.: Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations. Adv. Differ. Equ. **2019**, 101 (2019)
14. Ahmad, B., Alghanmi, M., Alsaedi, A., Agarwal, R.P.: On an impulsive hybrid system of conformable fractional differential equations with boundary conditions. Int. J. Syst. Sci. **51**(5), 958–970 (2020)
15. Balachandran, K., Kiruthika, S., Trujillo, J.J.: Existence of solution of nonlinear fractional pantograph equations. Acta Math. Sci. **33**(3), 712–720 (2013)
16. Yu, Z.H.: Variational iteration method for solving the multi-pantograph delay equation. Phys. Lett. A **372**(43), 6475–6479 (2008)

17. Tohidi, E., Bhrawy, A.H., Erfani, K.A.: Collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Appl. Math. Model. **37**(6), 4283–4294 (2012)
18. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1940)
19. Hyers, D.H.: On the stability of the linear functional equations. Proc. Natl. Acad. Sci. **27**, 222–224 (1941)
20. Rassias, T.M.: On the stability of linear mappings in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
21. Jung, S.M.: Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011)
22. Wang, Z., Yang, D., Ma, T., Sun, N.: Stability analysis for nonlinear fractional-order systems based on comparison principle. Nonlinear Dyn. **75**(1–2), 387–402 (2014)
23. Stamova, I.: Global Mittag-Leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays. Nonlinear Dyn. **77**(4), 1251–1260 (2014)
24. Agarwal, R.P., Benchohra, M., Hamani, S.: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973–1033 (2010)
25. Cabada, A., Dimitrijevic, S., Tomovic, T., Aleksic, S.: The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions. Math. Methods Appl. Sci. **40**(6), 1880–1891 (2017)
26. Cabada, A., Wang, G.: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**(1), 403–411 (2013)
27. Ali, A., Shah, K.: Ulam–Hyers stability analysis of a three-point boundary-value problem for fractional differential equations. Ukr. Mat. Ž. **72**(2), 147–160 (2020)
28. Khan, A., Abdeljawad, T., Gomez-Aguilar, J.F., Khan, H.: Dynamical study of fractional order mutualism parasitism food web module. Chaos Solitons Fractals **134**, 109685 (2020)
29. Khan, A., Gomez-Aguilar, J.F., Abdeljawad, T., Khan, H.: Stability and numerical simulation of a fractional order plant nectar pollinator model. Alex. Eng. J. **59**, 49–59 (2020)
30. Khan, H., Tunc, C., Khan, A.: Green function's properties and existence theorems for nonlinear singular-delay-fractional differential equations. Discrete Contin. Dyn. Syst., Ser. S **13**(9), 2475–2487 (2020)
31. Khan, H., Gomez-Aguilar, J.F., Alkhazzan, A., Khan, A.A.: Fractional order HIV-TB coinfection model with nonsingular Mittag-Leffler law. Math. Methods Appl. Sci. **43**(6), 3786–3806 (2020)
32. Khan, H., Khan, A., Jarad, F., Shah, A.: Existence and data dependence theorems for solutions of an ABC-fractional order impulsive system. Chaos Solitons Fractals **131**, 109477 (2019)
33. Khan, A., Khan, H., Gomez-Aguilar, J.F., Abdeljawad, T.: Existence and Hyers–Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. Chaos Solitons Fractals **127**, 422–427 (2019)
34. Khan, H., Jarad, F., Abdeljawad, T., Khan, A.: A singular ABC-fractional differential equation with p-Laplacian operator. Chaos Solitons Fractals **129**, 56–61 (2019)
35. Khan, A., Gomez-Aguilar, J.F., Khan, T.S., Khan, H.: Stability analysis and numerical solutions of fractional order HIV/AIDS model. Chaos Solitons Fractals **122**, 119–128 (2019)
36. Ahmad, B., Alruwaily, Y., Alsaedi, A., Nieto, J.J.: Fractional integro-differential equations with dual anti-periodic boundary conditions. Differ. Integral Equ. **33**(3/4), 181–206 (2020)
37. Agarwal, R.P., Ahmad, B., Alsaedi, A.: Fractional-order differential equations with anti-periodic boundary conditions: a survey. Bound. Value Probl. **2017**(1), 1 (2017)
38. Wang, G., Ahmad, B., Zhang, L.: Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. Nonlinear Anal., Theory Methods Appl. **74**(3), 792–804 (2011)
39. Rus, I.A.: Ulam stabilities of ordinary differential equations in a Banach space. Carpath. J. Math. **26**(1), 103–107 (2010)