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# Positive solutions of a discrete second-order boundary value problems with fully nonlinear term

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## Abstract

In this paper, we mainly consider a kind of discrete second-order boundary value problem with fully nonlinear term. By using the fixed-point index theory, we obtain some existence results of positive solutions of this kind of problems. Instead of the upper and lower limits condition on  $f$ , we may only impose some weaker conditions on  $f$ .

**MSC:** 39A12

**Keywords:** Second-order discrete boundary value problems; Positive solutions; Fixed-point index theory

## 1 Introduction

Let  $a$  and  $b$  are two integers with  $a < b$  and  $[a, b]_{\mathbb{Z}} = \{a, a + 1, \dots, b\}$ . In this paper, we consider the existence of positive solutions of the following discrete problem:

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t), \Delta u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0, \end{cases} \quad (1.1)$$

where  $T > 1$  is a positive integer,  $\Delta$  is the forward difference operator with  $\Delta u(t) = u(t+1) - u(t)$ ,  $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$  is a continuous function and  $\mathbb{R}^+ = [0, \infty)$ .

In the past few years, boundary value problems for difference equations have been deduced from different disciplines, such as the computer sciences, economics, mechanical engineering and control systems and so on; see, for instance, [1, 6, 23, 24]. Therefore, many scholars studied the discrete boundary value problems, including the linear discrete problems and nonlinear discrete problems [2–5, 8, 10–17, 19–21, 26, 28–30, 33–35]. In 1999, by using the upper and lower solution method, Agarwal and O'Regan [2] studied the existence of solutions and nonnegative solutions for the following discrete problem:

$$\begin{cases} \Delta^2 u(t-1) + \mu f(t, u(t)) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases} \quad (1.2)$$

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Thereafter, many authors focused on the existence of solutions and positive solutions of (1.2). In particular, since Zhou et al. [26] introduced the variation method to solve the discrete boundary condition, several excellent existence results of discrete boundary value problems have been obtained by using this method; see, for instance, [8, 11, 26, 33, 35] and the references therein. For example, by using the variation method, Bonanno et al. [8] studied the existence of multiple positive solutions of (1.2). Meanwhile, as a very important method, the bifurcation technique has also been introduced to discuss the discrete problem as (1.2). For example, by using the bifurcation technique, Gao et al. [12] studied the continuum of the positive and negative solutions of the boundary value problem (1.2) and they also obtained the existence of positive solutions and negative solutions of (1.2). Meanwhile, Ma et al. [27–29] and Gao [10] also used the same method to consider different discrete boundary value problems. Finally, another important method used to discuss the positive solutions of the discrete boundary value problems should be noted: fixed-point theory in cones. In fact, since Merdivenci [31] introduced the fixed-point theory in cones to consider the positive solutions of the two-point discrete boundary value problems as (1.2), lots of interesting and excellent results have been obtained. For example, by using the fixed-point theory in cones, Wong and Agarwal [34] considered the existence results of positive solutions for a boundary value problems of a higher-order difference equation, Ma and Raffoul [30] considered the existence of positive solutions of the discrete three-point boundary value problems in 2004. Later, Henderson and Luca [19–21], Agarwal and Luca [2] considered the existence of positive solutions of the discrete multi-point systems.

However, it is noted that most of the above results focus on the problems as (1.2) which does not contain the damping term  $\Delta u$  in the nonlinear term  $f$ . As we know, the damping phenomenon exists widely in the real world. Therefore, it is interesting to consider such a problem which has the damping term in the nonlinear term; see, for instance, [7, 22, 32]. In [7], Anderson et al. considered the existence of the solutions of this kind of problems by using Schaefer's theorem. In [22, 32], the method of lower and upper solutions are used to consider the existence of solutions a kind of discrete problems with the fully nonlinear term. Therefore, inspired by the above the results, we try our best to consider the existence of positive solutions of the discrete boundary value problem (1.1), which has a damping term  $\Delta u$  in the nonlinear term. Our main tools here are also some fixed-point theories in a cone, called the fixed-point index theories, we only briefly list them in Sect. 3 and we can find them in the references [9, 18] for more details. Furthermore, in the present paper, the superlinear and the sublinear conditions on the nonlinear term  $f$  at 0 and  $\infty$  do not hold as the limitation form, but some weaker conditions hold at 0 and  $\infty$ ; see Remarks 3.1 and 3.2. Finally, it is noted that the continuous problems with fully nonlinear terms have been studied by [25].

The rest of the present paper is organized as follows: In Sect. 2, we give some preliminaries, including the work space, the properties of the Green's function and the spectral results of the linear eigenvalue problems. In Sect. 3, we give our main results and prove them.

## 2 Preliminaries

At first, let us introduce our work space. Let

$$E = \{u | u : [0, T + 1]_{\mathbb{Z}}, u(0) = u(T + 1) = 0\}$$

with the maximum norm  $\|u\|_E = \max_{t \in [0, T+1]_{\mathbb{Z}}} |u(t)|$  and

$$Y = \{u|u : [0, T]_{\mathbb{Z}} \rightarrow \mathbb{R}\}$$

with the maximum norm  $\|u\|_Y = \max_{t \in [0, T]_{\mathbb{Z}}} |u(t)|$ .

Let  $j : E \rightarrow Y$  by

$$j(0, u(1), u(2), \dots, u(T), 0) = (u(1), u(2), \dots, u(T)).$$

Then  $j$  is an isomorphism from  $E$  to  $Y$ . Furthermore, define

$$P = \{u \in E | u(t) \geq 0\},$$

then  $P$  is a cone in  $E$ .

Now, let us consider the following linear boundary value problems:

$$\Delta^2 u(t - 1) + h(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}, \tag{2.1}$$

$$u(0) = u(T + 1) = 0. \tag{2.2}$$

Then the following results hold.

**Lemma 2.1** *Let  $h \in P$ . Then the problem (2.1), (2.2) has a unique nonnegative solution*

$$u(t) = \sum_{s=1}^T G(t, s)h(s), \tag{2.3}$$

where  $G(t, s)$  is the Green's function defined as

$$G(t, s) = \frac{1}{T + 1} \begin{cases} (T + 1 - t)s, & s \leq t, \\ (T + 1 - s)t, & t \leq s. \end{cases}$$

*Proof* Summing Eq. (2.1) from  $s = 1$  to  $s = t - 1$ , we get

$$\Delta u(t - 1) = \Delta u(0) - \sum_{s=1}^{t-1} h(s).$$

Then continuing to sum the above equation from  $s = 1$  to  $s = t - 1$ , we obtain

$$u(t) = t\Delta u(0) - \sum_{s=1}^t \sum_{\tau=1}^{s-1} h(\tau) = t\Delta u(0) - \sum_{s=1}^{t-1} (t - s)h(s).$$

Combining this with the boundary condition  $u(T + 1) = 0$ , we get

$$u(t) = \frac{t}{T + 1} \sum_{s=1}^T (T + 1 - s)h(s) - \sum_{s=1}^{t-1} (t - s)h(s) = \sum_{s=1}^T G(t, s)h(s). \quad \square$$

**Lemma 2.2** *The Green’s function  $G(t, s)$  satisfies the following properties:*

- (i)  $G(t, s) = G(s, t)$ , for  $t, s \in [0, T + 1]_{\mathbb{Z}} \times [0, T + 1]_{\mathbb{Z}}$ ;
- (ii)  $G(0, s) = G(T + 1, s) = 0$  for  $s \in [0, T + 1]_{\mathbb{Z}}$ ;
- (iii)  $G(t, s) > 0$ , for  $t, s \in [1, T]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$ ;
- (iv)  $G(t, s) \leq G(s, s)$ , for  $t, s \in [0, T + 1]_{\mathbb{Z}} \times [0, T + 1]_{\mathbb{Z}}$ ;
- (v)  $G(t, s) \geq \frac{1}{T+1}G(t, t)G(s, s)$ .

*Proof* The properties (i)–(iv) are obvious. We only prove (v) here. In fact,

$$\frac{G(t, s)}{G(t, t)G(s, s)} = \begin{cases} \frac{(T+1)(T+1-t)s}{(T+1-t)t(T+1-s)s}, & 1 \leq s \leq t \leq T, \\ \frac{(T+1)(T+1-s)t}{(T+1-t)t(T+1-s)s}, & 1 \leq s \leq t \leq T. \end{cases}$$

Therefore, (v) holds. □

**Lemma 2.3** *Let  $u \in P$  be a solution of (2.1), (2.2). Then  $u$  satisfies the following properties:*

- (i)  $u(t) \geq \frac{1}{T+1}G(t, t)\|u\|_E$  for  $t \in [0, T + 1]_{\mathbb{Z}}$ ;
- (ii)  $\|u\|_E \leq T \max_{t \in [0, T-1]_{\mathbb{Z}}} |\Delta u(t)|$ ;
- (iii)  $\max_{t \in [0, T]_{\mathbb{Z}}} |\Delta u(t)| \leq \Delta u(0) - \Delta u(T)$ .

*Proof* (i) For  $t \in [1, T]_{\mathbb{Z}}$ , by the properties of  $G(t, s)$ , we have

$$u(t) = \sum_{s=1}^T G(t, s)h(s) \leq \sum_{s=1}^T G(s, s)h(s).$$

Therefore,

$$\|u\|_E \leq \sum_{s=1}^T G(s, s)h(s).$$

Furthermore, by the property (v) of  $G(t, s)$ , we know that

$$u(t) = \sum_{s=1}^T G(t, s)h(s) \geq \frac{1}{T+1}G(t, t) \sum_{s=1}^T G(s, s)h(s) \geq \frac{1}{T+1}G(t, t)\|u\|_E.$$

(ii) By direct calculation, we know that

$$u(t) = \sum_{s=1}^t \Delta u(s - 1).$$

Then, for  $t \in [1, T]_{\mathbb{Z}}$ , we get

$$|u(t)| \leq \sum_{s=1}^t |\Delta u(s - 1)| \leq \sum_{s=1}^T |\Delta u(s - 1)| \leq T \max_{t \in [0, T-1]_{\mathbb{Z}}} |\Delta u(t)|.$$

Combining this with the fact that  $u(0) = u(T + 1) = 0$ , we see that the assertion (ii) holds.

(iii) By Lemma 2.1 and the fact  $h \in P$ , it is not difficult to see that  $\Delta u(0) \geq 0$  and  $\Delta u(T) \leq 0$ . Moreover,  $-\Delta^2 u(t - 1) = h(t) \geq 0$ , we know that  $\Delta u(t)$  is an increasing function on  $[0, T]_{\mathbb{Z}}$ . Therefore,

$$\max_{t \in [0, T]_{\mathbb{Z}}} |\Delta u(t)| = \max\{\Delta u(0), -\Delta u(T)\} \leq \Delta u(0) - \Delta u(T). \quad \square$$

**Lemma 2.4** *The linear eigenvalue problem*

$$\begin{cases} \Delta^2 u(t - 1) + \lambda u(t) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T + 1) = 0, \end{cases} \quad (2.4)$$

has  $T$  real and simple eigenvalues  $2 - 2 \cos \frac{k\pi}{T+1}, k = 1, 2, \dots, T$ , and the corresponding eigenfunction is  $\varphi_k = \sin \frac{k\pi t}{T+1}, k = 1, 2, \dots, T$ .

*Proof* This result is the well-known discrete Sturm–Liouville theory, we can find it in several classical book, like Kelly and Peterson [23]. To be complete, we give a brief proof here.

The characteristic equation of the equation in (2.4) is  $\mu^2 + (\lambda - 2)\mu + 1 = 0$ . Then

$$m_{1,2} = \frac{(2 - \lambda) \pm \sqrt{(\lambda - 2)^2 - 4}}{2}.$$

If  $|\lambda - 2| \geq 2$ , then the general solution of the equation in (2.4) is

$$u(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

Combining the boundary condition  $u(0) = u(T + 1) = 0$ , we know that  $c_1 = c_2 = 0$ . Therefore, the problem (2.4) has only a trivial solution in this case.

If  $|\lambda - 2| < 2$ , we could set  $2 - \lambda = 2 \cos \theta$ . Then

$$m_{1,2} = \cos \theta \pm i \sin \theta = e^{\pm i\theta}.$$

Therefore, the general solution of the equation in (2.4) is

$$u(t) = c_1 \cos \theta t + c_2 \sin \theta t.$$

Combining the boundary condition  $u(0) = u(T + 1) = 0$ , we know that

$$y(0) = c_1 = 0, \quad y(T + 1) = c_2 \sin(T + 1)\theta = 0.$$

Let

$$\theta_k = \frac{k\pi}{T + 1}, \quad k = 1, 2, \dots, T.$$

Then we get the eigenvalue of the problem (2.4) is

$$\lambda_k = 2 - 2 \cos \frac{k\pi}{T + 1}, \quad k = 1, 2, \dots, T,$$

and the corresponding eigenfunction is  $\varphi_k = \sin \frac{k\pi t}{T+1}$ ,  $k = 1, 2, \dots, T$ . The proof is complete.  $\square$

### 3 Main results

In this section, we try our best to find the nontrivial positive solution of the problem (1.1). Let

$$K = \left\{ u \in P \mid u(t) \geq \frac{1}{T+1} G(t, t) \|u\|_E, t \in [1, T]_{\mathbb{Z}} \right\}.$$

Then  $K$  is a positive cone in  $E$ . Define an operator  $A : K \rightarrow E$  by

$$Au(t) = \sum_{s=1}^T G(t, s) f(s, u(s), \Delta u(s)).$$

Since  $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$  is a continuous function, it is not difficult to see that  $A : K \rightarrow K$  is a completely continuous mapping. Now, it suffices to find the nontrivial positive fixed-point of  $A$ . To get it, let us recall some basic concepts and lemmas on the fixed-point theory in a cone; see [9, 18].

Let  $E$  be a Banach space,  $K \subset E$  is a closed convex cone. Suppose that  $D$  is a bounded open subset of  $E$  with boundary  $\partial D$ , and  $K \cap D \neq \emptyset$ . Then the following lemmas hold.

**Lemma 3.1** *Let  $D$  be a bounded open subset of  $E$  with  $\theta \in D$ , and  $A : K \cap \bar{D} \rightarrow K$  a completely continuous mapping. If  $\mu Au \neq u$  for every  $u \in K \cap \partial D$  and  $0 < \mu < 1$ , then  $i(A, K \cap D, K) = 1$ .*

**Lemma 3.2** *Let  $D$  be a bounded open subset of  $E$  and  $A : K \cap \bar{D} \rightarrow K$  a completely continuous mapping. If there exists  $v_0 \in K \setminus \{\theta\}$  such that  $u - Au \neq \tau v_0$  for every  $u \in K \cap \partial D$  and  $\tau \geq 0$ , then  $i(A, K \cap D, K) = 0$ .*

**Lemma 3.3** *Let  $D$  be a bounded open subset of  $E$ , and  $A, A_1 : K \cap \bar{D} \rightarrow K$  be two completely continuous mappings. If  $(1 - t)Au + tA_1u \neq u$  for every  $u \in K \cap \partial D$  and  $0 \leq t \leq 1$ , then  $i(A, K \cap D, K) = i(A_1, K \cap D, K)$ .*

Now, let us introduce two notations. For  $r > 0$ , let

$$\Omega_r = \{u \in E \mid \|u\|_E < r\}, \quad \partial\Omega_r = \{u \in E \mid \|u\|_E = r\}.$$

The first main result is as follows.

**Theorem 3.1** *Let  $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$  be a continuous function. Suppose that the conditions*

(H1) *there exist three positive constants  $a > 0, b > 0, \delta > 0$  with  $a + 2b < \frac{1}{T^2}$  such that*

$$f(t, u, v) \leq au + b|v|, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times [0, \delta] \times [-2\delta, 2\delta],$$

and

(H2) there exist constants  $c > \lambda_1 = 2 - 2 \cos \frac{\pi}{T+1}$  and  $H > 0$  such that

$$f(t, u, v) \geq cu, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}, |u| + |v| > H,$$

hold.

Then the boundary value problem (1.1) has at least one positive solution in  $K$ .

*Proof* Let  $r_1 \in (0, \delta)$  small enough, where  $\delta$  is the positive constant introduced by (H1). Then, by Lemma 3.1, we try to prove that, for any  $u \in K \cap \partial\Omega_{r_1}$  and  $0 < \mu \leq 1$ ,

$$\mu Au \neq u. \tag{3.1}$$

Suppose to the contrary that there exists  $u_0 \in K \cap \partial\Omega_{r_1}$  and  $0 < \mu_0 \leq 1$  such that  $\mu_0 Au_0 = u_0$ . This implies that  $u_0$  is a positive solution of the problem

$$\begin{cases} \Delta^2 u(t-1) + \mu_0 f(t, u(t), \Delta u(t)) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Now, by (H1), we have

$$\begin{aligned} f(t, u_0(t), \Delta u_0(t)) &\leq a u_0(t) + b |\Delta u_0(t)| \\ &\leq a \|u_0\|_E + b \max_{t \in [0, T]_{\mathbb{Z}}} |\Delta u_0(t)| \\ &\leq (a + 2b) \|u_0\|_E. \end{aligned}$$

Therefore, combining this with the fact

$$-\Delta^2 u_0(t-1) = \mu_0 f(t, u_0(t), \Delta u_0(t)),$$

we get

$$\Delta u_0(0) - \Delta u_0(T) \leq (a + 2b) T \|u_0\|_E.$$

Combining this with Lemma 2.3 (ii) and (iii), we obtain

$$\frac{1}{T} \|u_0\|_E \leq (a + 2b) T \|u_0\|_E.$$

This contradicts the assumption  $a + 2b < \frac{1}{T^2}$ . Therefore, (3.1) holds. By Lemma 3.1, we get

$$i(A, K \cap \Omega_{r_1}, K) = 1. \tag{3.2}$$

Now, let  $L_0 = \max\{|f(t, u, v) - cu| : (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}, |u| + |v| \leq H\} + 1$ . Then, the condition (H2) implies that

$$f(t, u, v) \geq cu - L_0, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}.$$

Define a operator  $A_1 : K \rightarrow E$  by

$$A_1 u(t) = \sum_{s=1}^T G(t,s) (f(s, u(s), \Delta u(s)) + L_0).$$

Then  $A_1 : K \rightarrow K$  is a completely continuous operator. Now, let  $r_2 > \delta$ , we show that

$$i(A_1, K \cap \Omega_{r_2}, K) = 0. \tag{3.3}$$

To get it, by Lemma 3.2, we only need to show that

$$u - A_1 u \neq \tau \varphi_1, \quad u \in K \cap \partial \Omega_{r_2}, \tau \geq 0, \tag{3.4}$$

where  $\varphi_1(t) = \sin \frac{\pi t}{T+1} / \|\sin \frac{\pi t}{T+1}\|_E$  is the eigenfunction of the linear eigenvalue problem (2.4), which corresponds to the first eigenvalue  $\lambda_1 = 2 - 2 \cos \frac{\pi}{T+1}$ . Then  $\|\varphi_1\|_E = 1$  and  $\varphi_1(t) > 0$  on  $[1, T]_{\mathbb{Z}}$ . Suppose to the contrary that there exist  $u_1 \in K \cap \partial \Omega_{r_2}$  and  $\tau_1 \geq 0$  such that  $u_1 - A_1 u_1 = \tau_1 \varphi_1$ . Combining this with the definition of  $A_1$ , we know that  $u_1$  is a solution of the problem

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t), \Delta u(t)) + L_0 + \tau_1 \lambda_1 \varphi_1, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Therefore, by (H2), we get

$$\begin{aligned} -\Delta^2 u_1(t-1) &= f(t, u_1(t), \Delta u_1(t)) + L_0 + \tau_1 \lambda_1 \varphi_1 \\ &\geq c u_1(t) + \tau_1 \lambda_1 \varphi_1 \\ &\geq c u_1(t). \end{aligned}$$

Multiplying this inequality by  $\varphi_1(t)$  and summing from  $s = 1$  to  $s = t$ , we get

$$\lambda_1 \sum_{t=1}^T u_1(t) \varphi_1(t) \geq c \sum_{t=1}^T u_1(t) \varphi_1(t).$$

Now, if  $\sum_{t=1}^T u_1(t) \varphi_1(t) \neq 0$ , then we get  $\lambda_1 \geq c$ . In fact, by Lemma 2.3 (i), for  $t \in [0, T+1]_{\mathbb{Z}}$ , we get

$$u_1(t) \geq \frac{1}{T+1} G(t,t) \|u_1\|_E, \quad \varphi_1(t) \geq \frac{1}{T+1} G(t,t) \|\varphi_1\|_E.$$

This implies that

$$\sum_{t=1}^T u_1(t) \varphi_1(t) \geq \frac{1}{(T+1)^2} \|u_1\|_E \sum_{s=1}^T G^2(s,s) > 0.$$

Therefore,  $\lambda_1 \geq c$ . However, this contradicts the condition (H2). So, we see that (3.4) holds and then (3.3) holds too.



Now, let us show that the operator  $A$  and  $A_1$  satisfy the condition of Lemma 3.3 for  $r_2 > 0$  large enough, i.e.,

$$(1 - t)Au + tA_1u \neq u, \quad u \in K \cap \partial\Omega_{r_2}, 0 \leq t \leq 1. \tag{3.5}$$

Suppose to the contrary that there exist  $u_2 \in K \cap \partial\Omega_{r_2}$  and  $0 \leq t_0 \leq 1$  such that

$$(1 - t_0)Au_2 + t_0A_1u_2 = u_2.$$

Therefore, by the definition of  $A$  and  $A_2$ , we know that  $u_2$  is a solution of the problem

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t), \Delta u(t)) + t_0L_0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Therefore,

$$\begin{aligned} -\Delta^2 u_2(t-1) &= f(t, u_2(t), \Delta u_2(t)) + t_0L_0 \\ &\geq cu_2 - (1 - t_0)L_0 \\ &\geq cu_2 - L_0. \end{aligned}$$

Multiplying both sides of this inequality by  $\varphi_1(t)$  and summing from  $s = 1$  to  $s = T$ , we get

$$\lambda_1 \sum_{s=1}^T u_2(t)\varphi_1(t) \geq c \sum_{s=1}^T u_2(t)\varphi_1(t) - L_0T.$$

This implies that

$$\sum_{s=1}^T u_2(t)\varphi_1(t) \leq \frac{L_0T}{c - \lambda_1}. \tag{3.6}$$

Furthermore, by Lemma 2.3 (i), we know that

$$u_1(t) \geq \frac{1}{T+1}G(t, t)\|u_1\|_E, \quad \varphi_1(t) \geq \frac{1}{T+1}G(t, t)\|\varphi_1\|_E.$$

So,

$$\sum_{s=1}^T u_2(t)\varphi_1(t) \geq \frac{1}{(T+1)^2}\|u_2\|_E \sum_{s=1}^T G^2(s, s) > 0.$$

Combining this with (3.6), we obtain

$$\|u_2\|_E \leq \frac{L_0T(T+1)^2}{(c - \lambda_1) \sum_{s=1}^T G^2(s, s)}. \tag{3.7}$$

Let

$$M := \frac{L_0T(T+1)^2}{(c - \lambda_1) \sum_{s=1}^T G^2(s, s)}.$$

Now, let  $r_2 = \max\{M, \delta\}$ . Then, by the definition of  $\Omega_{r_2}$ , we get  $\|u_2\|_E = r_2 > M$  if  $u \in K \cap \partial\Omega_{r_2}$ . However, this contradicts (3.7). Therefore, (3.5) holds, which implies that the operator  $A$  and  $A_1$  satisfy the condition in Lemma 3.2. Therefore, by Lemma 3.2, we get

$$i(A, K \cap \Omega_{r_2}, K) = i(A_1, K \cap \Omega_{r_2}, K).$$

Combining this with (3.3), we get

$$i(A, K \cap \Omega_{r_2}, K) = 0.$$

Hence,

$$i(A, K \cap (\Omega_{r_2} \setminus \overline{\Omega_{r_1}}), K) = i(A, K \cap \Omega_{r_2}, K) - i(A, K \cap \Omega_{r_1}, K) = -1.$$

Therefore,  $A$  has a fixed-point  $u \in K \cap (\Omega_{r_2} \setminus \overline{\Omega_{r_1}})$ . Furthermore, it is a positive solution of (1.1). □

*Remark 3.1* In this remark, we try to show that our condition (H1) and (H2) are weaker than the usual limitation conditions. In fact, Let  $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$  is continuous and

$$\begin{aligned} f^0 &= \limsup_{|u|+|v| \rightarrow 0^+} \max_{t \in [1, T]_{\mathbb{Z}}} \frac{f(t, u, v)}{|u| + |v|}, & f_0 &= \liminf_{|u|+|v| \rightarrow 0^+} \min_{t \in [1, T]_{\mathbb{Z}}} \frac{f(t, u, v)}{|u| + |v|}, \\ f^\infty &= \limsup_{|u|+|v| \rightarrow +\infty} \max_{t \in [1, T]_{\mathbb{Z}}} \frac{f(t, u, v)}{|u| + |v|}, & f_\infty &= \liminf_{|u|+|v| \rightarrow +\infty} \min_{t \in [1, T]_{\mathbb{Z}}} \frac{f(t, u, v)}{|u| + |v|}. \end{aligned}$$

Then it is not difficult to see that

$$f^0 < \frac{1}{2T^2}, \quad f_\infty > \lambda_1 = 2 - 2 \cos \frac{\pi}{T+1}$$

imply the condition (H1) and the condition (H2) hold, respectively. In fact, if  $f^0 < \frac{1}{2T^2}$ , then there exist two positive constants  $\varepsilon_1 > 0$  and  $\delta > 0$  small enough such that  $f^0 + \varepsilon_1 < \frac{1}{2T^2}$  and

$$f(t, u, v) \leq (f^0 + \varepsilon_1)(u + |v|), \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times [0, \delta] \times [-2\delta, 2\delta].$$

Now, if we choose  $a = f^0 + \varepsilon_1$  and  $b = \frac{f^0 + \varepsilon_1}{2}$ , then  $a + b < \frac{1}{T^2}$  and

$$f(t, u, v) \leq au + b|v|, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times [0, \delta] \times [-2\delta, 2\delta].$$

This means that the condition (H1) holds. If  $f_\infty > \lambda_1$ , then there exist a constant  $\varepsilon_2 > 0$  small enough and a positive constant  $H > 0$  big enough such that  $f_\infty - \varepsilon_2 > \lambda_1$  and

$$f(t, u, v) \geq (f_\infty - \varepsilon_2)(u + |v|), \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}, |u| + |v| > H.$$

Now, let  $c = f_\infty - \varepsilon_2$ . Then (H2) holds.

**Theorem 3.2**  $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$  be a continuous function. Suppose that the conditions

(H3) there exist constants  $c > \lambda_1$  and  $\eta > 0$  such that

$$f(t, u, v) \geq cu, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times [0, \eta] \times [-2\eta, 2\eta];$$

(H4) there exist three positive constants  $a > 0, b > 0$  and  $H > 0$  with  $a + 2b < \frac{1}{T^2}$ , such that

$$f(t, u, v) \leq au + b|v|, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}, |u| + |v| > H,$$

hold.

Then the boundary value problem (1.1) has at least one positive solution in  $K$ .

*Proof* Let  $r_3 \in (0, \eta)$ , where  $\eta$  is the constant in (H3). Then we will show that

$$i(A, K \cap \Omega_{r_3}, K) = 0.$$

To get it, we choose  $v_0 = \varphi_1(t)$  and verify the condition of Lemma 3.2 holds, that is,

$$u - Au \neq \tau \varphi_1, \quad u \in K \cap \partial \Omega_{r_3}, \tau \geq 0. \tag{3.8}$$

Suppose to the contrary that there exist  $u_3 \in K \cap \partial \Omega_{r_3}$  and  $\tau_3 \geq 0$  such that

$$u_3 - Au_3 = \tau_3 \varphi_1.$$

Then, by the definition of  $A$ , we know that  $u_3$  is a solution of the problem

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t), \Delta u(t)) + L_0 + \tau_3 \lambda_1 \varphi_1, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Furthermore, since  $u_3 \in K \cap \partial \Omega_{r_3}$ , we know that  $0 \leq |u_3(t)| \leq \|u_3\|_E = r_3 < \eta$  and  $0 \leq |\Delta u_3(t)| \leq 2\|u_3\|_E = 2r_3 < 2\eta$ . Therefore, by (H4), we get

$$\begin{aligned} -\Delta^2 u_3(t-1) &= f(t, u_3(t), \Delta u_3(t)) + L_0 + \tau_3 \lambda_1 \varphi_1 \\ &\geq cu_3(t) + \tau_3 \lambda_1 \varphi_1 \\ &\geq cu_3(t). \end{aligned}$$

Now, similar to the proof of (3.4), we get a contradiction. Therefore, (3.8) holds and then, by Lemma 3.2,

$$i(A, K \cap \Omega_{r_3}, K) = 0. \tag{3.9}$$

Next, let  $r_4 > \delta$  large enough. Then, by Lemma 3.1, we only need to show that

$$\mu Au \neq u, \quad u \in K \cap \partial \Omega_{r_4}, 0 < \mu \leq 1. \tag{3.10}$$

Suppose to the contrary that there exist  $u_4 \in K \cap \partial\Omega_{r_4}$  and  $\mu_4 \in (0, 1]$  such that

$$\mu_4 Au_4 = u_4.$$

Then, by the definition of  $A$ , we know that  $u_4$  is a solution of the problem

$$\begin{cases} \Delta^2 u(t-1) + \mu_4 f(t, u(t), \Delta u(t)) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Now, choose  $L_1 = \max\{|f(t, u, v) - (au + b|v)| : (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}, |u| + |v| \leq H\} + 1$ . Then the condition (H4) implies that

$$f(t, u, v) \leq au + b|v| + L_1, \quad (t, u, v) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}.$$

Then, by the facts that  $u_4 \in K \cap \partial\Omega_{r_4}$  and  $\mu_4 \in (0, 1]$ , we obtain

$$\begin{aligned} -\Delta^2 u_4(t-1) &= \mu_4 f(t, u_4(t), \Delta u_4(t)) \\ &\leq au_4(t) + b|\Delta u_4(t)| + L_1 \\ &\leq a\|u_4\|_E + b \max_{t \in [0, T]_{\mathbb{Z}}} |\Delta u_4(t)| + L_1 \\ &\leq (a + 2b)\|u_4\|_E + L_1. \end{aligned}$$

Summing both sides of the above inequality from  $s = 1$  to  $s = T$ , then we get

$$\Delta u_4(0) - \Delta u_4(T) \leq T[(a + 2b)\|u_4\|_E + L_1]. \tag{3.11}$$

Furthermore, by Lemma 2.3 (ii) and (iii),

$$\|u_4\|_E \leq T \max_{t \in [0, T]_{\mathbb{Z}}} |\Delta u_4(t)| \leq T(\Delta u_4(0) - \Delta u_4(T)).$$

Combining this with (3.11), we obtain

$$\|u_4\|_E \leq \frac{L_1 T}{1 - T^2(a + 2b)} := M_1. \tag{3.12}$$

Let  $r_4 > \max\{M_1, \delta\}$ . Since  $u_4 \in K \cap \partial\Omega_{r_4}$ , we know that  $\|u_4\|_E = r_4 > M_1$ . However, this contradicts (3.12). Therefore, (3.10) holds. Now, by Lemma 3.1, we get

$$i(A, K \cap \Omega_{r_4}, K) = 1. \tag{3.13}$$

Combining (3.9) with (3.13), we obtain

$$i(A, K \cap (\Omega_{r_4} \setminus \overline{\Omega_{r_3}}), K) = i(A, K \cap \Omega_{r_4}, K) - i(A, K \cap \Omega_{r_3}, K) = 1.$$

Therefore,  $A$  has a fixed-point  $u \in K \cap (\Omega_{r_4} \setminus \overline{\Omega_{r_3}})$ . Furthermore, it is a positive solution of (1.1). □

**Remark 3.2** Similar to Remark 3.1, it is not difficult to see that the condition (H3) and (H4) are also weaker than the usual limitation conditions.

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#### Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally in writing this paper. All authors read and approved the final manuscript.

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