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Analysis of a coupled system of fractional differential equations with non-separated boundary conditions

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Abstract

Solutions to fractional differential equations is an emerging part of current research, since such equations appear in different applied fields. A study of existence, uniqueness, and stability of solutions to a coupled system of fractional differential equations with non-separated boundary conditions is the main target of this paper. The existence and uniqueness results are obtained by employing the Leray–Schauder fixed point theorem and the Banach contraction principle. Additionally, we examine different types of stabilities in the sense of Ulam–Hyers such as Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. To prove the effectiveness of our main results, we study a few interesting examples.

MSC: 26A33; 34A08; 34B27

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1 Introduction

In the last few decades, the theory of fractional differential equations (FDEs) has performed a significant role in a new branch of applied mathematics. Many researchers addressed FDEs for various models because of the fact that FDEs are considered to be more applicable and realistic as compared to integer order or classical differential equations. Fractional order differential and integral equations, which constitute a coupled system, became an important field of research in view of their nonlocal nature and applications in many real-world problems like anomalous diffusion [30], disease models [8, 9, 22], synchronization of chaotic system, etc. [10, 45]. We refer the reader to a series of papers [1, 5, 12, 15, 16, 19, 28, 31, 33–35, 40] for the theoretical works on coupled systems of FDEs and classical differential equations.

In the area of mathematical analysis Ulam [32] stability of functional equations is one of the essential subjects. The verdict of this type of stability plays a key role concerning this subject. When Hyers [13] and Rassias [24] generalized this stability to Banach spaces, then a number of mathematicians spread the idea of stability to different classes of differential equations. Obloza [20, 21] proved for the first time Ulam–Hyers–Rassias stability of first order linear differential equations, after that many researchers furnished this idea with different results [7, 17, 25, 26, 37, 39, 41–44].

Nowadays, the investigation of initial and boundary value problems has attracted the attention of many mathematicians. Particularly, the existence, uniqueness, and stability properties of coupled systems supplemented with boundary conditions have grown to be one of the central interest areas in mathematical analysis, see [2–4, 36, 38].

Li et al. [18] investigated the existence and uniqueness of solutions to the following FDEs system with non-separated boundary conditions:

$$\begin{aligned} &{}^cD^{\alpha}x(t) = f\bigl(t,x(t)\bigr), \quad t \in [0,T], 1 < \alpha \leq 2, T > 0, \\ &a_1x(0) + b_1x(T) = c_1, \qquad a_2\bigl({}^cD^{\gamma}x(0)\bigr) + b_2\bigl({}^cD^{\gamma}x(T)\bigr) = c_2, \quad 0 < \gamma < 1, \end{aligned}$$

where $f \in ([0,T] \times \mathbb{R})$, ${}^cD^{\alpha}$ represents the Caputo fractional derivative of order α and a_i, b_i, c_i, for $i = 1,2$, are real constants with $a_1 + b_1 \neq 0$ and $b_2 \neq 0$. Alsulami et al. [6] studied fractional order coupled systems with non-separated coupled boundary conditions:

$$\begin{cases} {}^cD^{a}_{0^+}u(t) = f(t,u(t),v(t)), \\ {}^cD^{b}_{0^+}v(t) = g(t,u(t),v(t)), \\ u(0) = \lambda_1 v(T), \qquad u'(0) = \lambda_2 v'(T), \\ v(0) = \mu_1 u(T), \qquad v'(0) = \mu_2 u'(T), \end{cases}$$

where $t \in [0,T]$, $a,b \in (1,2]$, ${}^cD^a$, ${}^cD^b$ denote the Caputo fractional derivatives of order a and b, respectively, $u,v : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\lambda_j, \mu_j, j = 1,2$, are real constants.

Recently, Rao and Alesemi [23] investigated the existence and uniqueness of solutions for a coupled system of fractional differential equations with fractional non-separated coupled boundary conditions. As far as we know, the Ulam–Hyers stability analysis for the solutions of nonlinear coupled FDEs with non-separated coupled boundary conditions has been rarely investigated.

Motivated by the mentioned work, in this paper we study the existence, uniqueness, and stability results to the following nonlinear coupled FDEs:

$$\begin{cases} {}^cD^{m}_{0^+}\varpi(t) = k(t,w(t),{}^cD^{m}_{0^+}\varpi(t)), \quad 0 < t < 1, \\ {}^cD^{n}_{0^+}w(t) = l(t,\varpi(t),{}^cD^{n}_{0^+}w(t)), \quad 0 < t < 1, \end{cases} \tag{1.1}$$

with non-separated coupled boundary conditions

$$\begin{cases} \varpi(0) = \alpha_1 w(1), \qquad \varpi'(0) = \alpha_2 w'(\zeta), \qquad \varpi''(0) = \alpha_3 w''(\eta), \\ w(0) = \beta_1 \varpi(1), \qquad w'(0) = \beta_2 \varpi'(\zeta), \qquad w''(0) = \beta_3 \varpi''(\eta), \end{cases} \tag{1.2}$$

where the symbols ${}^cD^{m}_{0^+}$ and ${}^cD^{n}_{0^+}$ stand for Caputo fractional derivatives, $m,n \in (2,3]$, α_i, β_i, $i = 1,2,3$, are real constants, and $k,l \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$.

We can say that model (1.1)–(1.2) is of quite general and flexible nature, because the parameters involved in the problem cover a wide range of important cases. Some of them are explained here. If we choose $\alpha_1, \beta_1 = 0$ and $t \in [0, T]$, then the results correspond to a problem with coupled flux type boundary conditions. Next, if we set $t \in [0, T]$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\beta_1 = \beta_2 = \beta_3 = -1$ or vice versa, then the results correspond to nonlinear coupled fractional differential equations with coupled periodic and anti-periodic boundary conditions of the form: $\varpi(0) = w(T)$, $\varpi'(0) = w'(T)$, $\varpi''(0) = w''(T)$, $w(0) = -\varpi(T)$, $w'(0) = -\varpi'(T)$, $w''(0) = -\varpi''(T)$ or $\varpi(0) = -w(T)$, $\varpi'(0) = -w'(T)$, $\varpi''(0) = -w''(T)$, $w(0) = \varpi(T)$, $w'(0) = \varpi'(T)$, $w''(0) = \varpi''(T)$.

The remaining part of the paper is designed in the following way: In Section 2, we recall a few important definitions and lemmas from fractional calculus used throughout this article. In Section 3, we present the application of some standard fixed point approaches already mentioned in the abstract, through which the existence and uniqueness results for (1.1)–(1.2) are obtained. Ulam–Hyers stability results are established in Section 4. In Section 5, applications of the main results are provided, while in the final section, we present the conclusion of the paper.

2 Preliminaries

In this part we state some important lemmas and definitions about fractional derivatives and fractional integrals taken from [46].

Definition 2.1 (Riemann–Liouville fractional integral) The Riemann–Liouville fractional integral of a function $f : [0,\infty) \to \mathbb{R}$ of order $\alpha > 0$ is defined by

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0,$$

if the integral on the right-hand side exists, where Γ denotes the Euler gamma function.

Definition 2.2 (Riemann–Liouville fractional derivative) The Riemann–Liouville fractional derivative of a function $f : [0,\infty) \to \mathbb{R}$ of order $\alpha > 0$, $n-1 < \alpha < n$ is defined by

$$D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1+\alpha-n}}\, ds, \quad t > 0.$$

Definition 2.3 (Caputo fractional derivative) The Caputo fractional derivative of a function $f : [0,\infty) \to \mathbb{R}$ of order $\alpha > 0$, $n-1 < \alpha < n$ is given by

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{n}(s)}{(t-s)^{1+\alpha-n}}\, ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ is the integer part of α.

Theorem 2.4 ([29], Banach contraction theorem) *Consider a Banach space $Y \neq \emptyset$, and let a map $\Omega : Y \to Y$ be a contraction on Y. Then Ω has precisely one fixed point.*

Lemma 2.5 ([14], Method of undetermined coefficients) *Let $\alpha > 0$, $h(t) \in C([0,1],\mathbb{R})$, then the homogeneous fractional differential equation*

$$^{c}D^{\alpha} h(t) = 0$$

has a solution of the form

$$h(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, *and* $n = [\alpha] + 1$.

Lemma 2.6 *Let* $\theta, \tau \in C([0,1], \mathbb{R})$. *Then the solution of the following boundary value problem*

$$\begin{cases} {}^cD^m_{0^+}\varpi(t) = \theta(t), \quad m \in (2,3], \\ {}^cD^n_{0^+}w(t) = \tau(t), \quad n \in (2,3], \\ \varpi(0) = \alpha_1 w(1), \qquad \varpi'(0) = \alpha_2 w'(\zeta), \qquad \varpi''(0) = \alpha_3 w''(\eta), \\ w(0) = \beta_1 \varpi(1), \qquad w'(0) = \beta_2 \varpi'(\zeta), \qquad w''(0) = \beta_3 \varpi''(\eta), \end{cases} \tag{2.1}$$

enriched with the boundary conditions (1.2) *is given by*

$$\varpi(t) = \begin{cases} [\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*} \\ \quad + \frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_2\beta_3\zeta t}{\varrho} \\ \quad + \frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)t}{\varrho} + \frac{\alpha_3\beta_3 t^2}{2(1-\alpha_3\beta_3)}]U_3 + [\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} \\ \quad + \frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} + \frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)t}{\varrho} \\ \quad + \frac{\alpha_3 t^2}{2(1-\alpha_3\beta_3)}]V_3 + [\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_1\alpha_2\beta_2^2}{z} + \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2}]U_2 \\ \quad + [\frac{\alpha_1\alpha_2\beta_1}{z} + \frac{\alpha_1\alpha_2\beta_2}{z} + \frac{\alpha_2 t}{(1-\alpha_2\beta_2)}]V_2 + \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}U_1 + \frac{\alpha_1}{1-\alpha_1\beta_1}V_1 \\ \quad + \frac{1}{\Gamma(m)}\int_0^t (t-s)^{m-1}\theta(s)\,ds \end{cases} \tag{2.2}$$

and

$$w(t) = \begin{cases} [\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3^2}{2\tilde{z}} \\ \quad + \frac{\alpha_1\beta_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_3\beta_1\beta_3}{2\tilde{z}} \\ \quad + \frac{\alpha_2\alpha_3\beta_2^2\beta_3\zeta t}{\varrho} + \frac{\alpha_2\beta_2\beta_3\zeta t(\alpha_3\beta_3+1)}{\varrho} + \frac{\alpha_3\beta_2\beta_3\zeta t}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3^2 t^2}{2(1-\alpha_3\beta_3)} + \frac{\beta_3 t^2}{2}]U_3 \\ \quad + [\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3}{2\tilde{z}} \\ \quad + \frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_3\beta_1}{2\tilde{z}} + \frac{\alpha_2\alpha_3\beta_2\zeta t(\beta_2+\beta_3)}{\varrho} + \frac{\alpha_3\beta_2\zeta t}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3 t^2}{2(1-\alpha_3\beta_3)}]V_3 \\ \quad + [\frac{\alpha_1\alpha_2\beta_1\beta_2^2}{z} + \frac{\alpha_1\beta_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2^2 t}{1-\alpha_2\beta_2} + \beta_2 t]U_2 + [\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} \\ \quad + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2} + \frac{\alpha_1\alpha_2\beta_1}{z}]V_2 + \frac{\alpha_1}{1-\alpha_1\beta_1}U_1 + \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}V_1 \\ \quad + \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\tau(s)\,ds, \end{cases} \tag{2.3}$$

where

$$U_1 = \frac{1}{\Gamma(m)}\int_0^1 (1-s)^{m-1}\theta(s)\,ds, \qquad V_1 = \frac{1}{\Gamma(n)}\int_0^1 (1-s)^{n-1}\tau(s)\,ds,$$

$$U_2 = \frac{1}{\Gamma(m-1)}\int_0^\zeta (\zeta-s)^{m-2}\theta(s)\,ds, \qquad V_2 = \frac{1}{\Gamma(n-1)}\int_0^\zeta (\zeta-s)^{n-2}\tau(s)\,ds,$$

$$U_3 = \frac{1}{\Gamma(m-2)}\int_0^\eta (\eta-s)^{m-3}\theta(s)\,ds, \qquad V_3 = \frac{1}{\Gamma(n-2)}\int_0^\eta (\eta-s)^{n-3}\tau(s)\,ds,$$

$\varrho = (1-\alpha_2\beta_2)(1-\alpha_3\beta_3)$, $\varrho^* = \varrho(1-\alpha_1\beta_1)$, $z = (1-\alpha_1\beta_1)(1-\alpha_2\beta_2)$, $\tilde{z} = (1-\alpha_1\beta_1)(1-\alpha_3\beta_3)$.

Proof We know by the method of undetermined coefficients that the general solution of (2.1) can be drafted as follows:

$$\varpi(t) = a_0 + a_1 t + a_2 t^2 + \frac{1}{\Gamma(m)}\int_0^t (t-s)^{m-1}\theta(s)\,ds, \tag{2.4}$$

$$w(t) = b_0 + b_1 t + b_2 t^2 + \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\tau(s)\,ds, \tag{2.5}$$

where $a_i, b_i \in \mathbb{R}$, $i = 0,1,2$.

Using the boundary conditions $\varpi(0) = \alpha_1 w(1)$ and $w(0) = \beta_1\varpi(1)$, we get

$$\begin{aligned} a_0 &= \alpha_1\left[b_0 + b_1 + b_2 + \frac{1}{\Gamma(n)}\int_0^1 (1-s)^{n-1}\tau(s)\,ds\right], \\ b_0 &= \beta_1\left[a_0 + a_1 + a_2 + \frac{1}{\Gamma(m)}\int_0^1 (1-s)^{m-1}\theta(s)\,ds\right]. \end{aligned} \tag{2.6}$$

By applying the conditions $\varpi'(0) = \alpha_2 w'(\zeta)$, $w'(0) = \beta_2\varpi'(\zeta)$ and using (2.4) and (2.5), we obtain

$$a_1 = \alpha_2\left[b_1 + 2b_2\zeta + \frac{1}{\Gamma(n-1)}\int_0^\zeta (\zeta-s)^{n-2}\tau(s)\,ds\right] \tag{2.7}$$

and

$$b_1 = \beta_2\left[a_1 + 2a_2\zeta + \frac{1}{\Gamma(m-1)}\int_0^\zeta (\zeta-s)^{m-2}\theta(s)\,ds\right]. \tag{2.8}$$

In view of $\varpi''(0) = \alpha_3 w''(\eta)$ and $w''(0) = \beta_3\varpi''(\eta)$ along with (2.4) and (2.5), we have

$$\begin{aligned} a_2 &= \frac{\alpha_3}{2}\left[2b_2 + \frac{1}{\Gamma(n-2)}\int_0^\eta (\eta-s)^{n-3}\tau(s)\,ds\right], \\ b_2 &= \frac{\beta_3}{2}\left[2a_2 + \frac{1}{\Gamma(m-2)}\int_0^\eta (\eta-s)^{m-3}\theta(s)\,ds\right]. \end{aligned}$$

By solving the last two equations, we get

$$\begin{aligned} a_2 &= \frac{1}{1-\alpha_3\beta_3}\left[\frac{\alpha_3\beta_3}{2\Gamma(m-2)}\int_0^\eta (\eta-s)^{m-3}\theta(s)\,ds\right] \\ &\quad + \frac{\alpha_3}{2\Gamma(n-2)}\int_0^\eta (\eta-s)^{n-3}\tau(s)\,ds, \\ b_2 &= \frac{\alpha_3\beta_3^2}{2(1-\alpha_3\beta_3)}\frac{1}{\Gamma(m-2)}\int_0^\eta (\eta-s)^{m-3}\theta(s)\,ds \\ &\quad + \frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\frac{1}{\Gamma(n-2)}\int_0^\eta (\eta-s)^{n-3}\tau(s)\,ds \\ &\quad + \frac{\beta_3}{2\Gamma(m-2)}\int_0^\eta (\eta-s)^{m-3}\theta(s)\,ds. \end{aligned}$$

Substituting the values of a_2 and b_2 in (2.7) and (2.8), we get

$$\begin{aligned} a_1 ={}& \frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_2\alpha_3(\beta_2+\beta_3)\zeta}{\varrho}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \\ &+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\frac{1}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds \\ &+\frac{\alpha_2}{1-\alpha_2\beta_2}\frac{1}{\Gamma(n-1)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \end{aligned}$$

and

$$\begin{aligned} b_1 ={}& \frac{\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_2\alpha_3\beta_2\zeta(\beta_2+\beta_3)}{\varrho}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \\ &+\frac{\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_2\beta_2^2}{1-\alpha_2\beta_2}\frac{1}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds \\ &+\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\frac{1}{\Gamma(n-1)}\int_0^\eta(\eta-s)^{n-2}\tau(s)\,ds \\ &+\frac{\alpha_3\beta_2\beta_3\zeta}{1-\alpha_3\beta_3}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_3\beta_2\zeta}{1-\alpha_3\beta_3}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \\ &+\frac{\beta_2}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds. \end{aligned}$$

By substituting the values of b_0, b_1, and b_2 in (2.6), we get

$$\begin{aligned} a_0 ={}& \frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \\ &+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \\ &+\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}\frac{1}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds \\ &+\frac{\alpha_1\alpha_2\beta_1}{z}\frac{1}{\Gamma(n-1)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds \\ &+\frac{\alpha_1\alpha_3\beta_1\beta_3}{2\tilde{z}}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha_1\alpha_3\beta_1}{2\tilde{z}} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1} \frac{1}{\Gamma(m)} \int_0^1 (1-s)^{m-1}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_2^2}{z} \frac{1}{\Gamma(m-1)} \int_0^\zeta (\zeta - s)^{m-2}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_2}{z} \frac{1}{\Gamma(n-1)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} \frac{1}{\Gamma(m-1)} \int_0^\zeta (\zeta - s)^{m-2}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_3^2}{2\tilde{z}} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_3}{2\tilde{z}} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1}{1-\alpha_1\beta_1} \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-1}\tau(s)\,ds.
\end{aligned}
$$

After some calculations, we get (2.2). By following the same steps, we obtain

$$
\begin{aligned}
b_0 = {}& \frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2\beta_3\zeta}{\varrho^*} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\zeta(\beta_2+\beta_3)}{\varrho^*} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_1\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-2}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_1\beta_2^2}{z} \frac{1}{\Gamma(m-1)} \int_0^\zeta (\zeta - s)^{m-2}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_2\beta_1\beta_2}{z} \frac{1}{\Gamma(n-1)} \int_0^\zeta (\zeta - s)^{n-2}\tau(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_1\beta_2\beta_3\zeta}{\tilde{z}} \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta - s)^{m-3}\theta(s)\,ds \\
&+ \frac{\alpha_1\alpha_3\beta_1\beta_2\zeta}{\tilde{z}} \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta - s)^{n-3}\tau(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_1\beta_1\beta_2}{1-\alpha_1\beta_1}\frac{1}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds\\
&+\frac{\alpha_1\alpha_3\beta_1\beta_3^2}{2\tilde{z}}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds\\
&+\frac{\alpha_1\alpha_3\beta_1\beta_3}{2\tilde{z}}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds\\
&+\frac{\alpha_1\beta_1\beta_3}{2(1-\alpha_1\beta_1)}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds\\
&+\frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-1}\tau(s)\,ds\\
&+\frac{\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds\\
&+\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds\\
&+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds\\
&+\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}\frac{1}{\Gamma(m-1)}\int_0^\zeta(\zeta-s)^{m-2}\theta(s)\,ds\\
&+\frac{\alpha_1\alpha_2\beta_1}{z}\frac{1}{\Gamma(n-1)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds\\
&+\frac{\alpha_3\beta_1\beta_3}{2\tilde{z}}\frac{1}{\Gamma(m-2)}\int_0^\eta(\eta-s)^{m-3}\theta(s)\,ds\\
&+\frac{\alpha_3\beta_1}{2\tilde{z}}\frac{1}{\Gamma(n-2)}\int_0^\eta(\eta-s)^{n-3}\tau(s)\,ds+\frac{\alpha_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(m)}\int_0^1(1-s)^{m-1}\theta(s)\,ds.
\end{aligned}
$$

With the help of b_0, b_1, and b_2, we get (2.3). □

3 Main results

Let $\mathbb{Y}=\{\varpi(t):\varpi\in C([0,1],\mathbb{R})\}$ denote the Banach space of all continuous functions from the interval $[0,1]$ into $\mathbb{R}$ supplied by the norm $\|\varpi\|=\sup\{|\varpi(t)|:t\in[0,1]\}$. Definitely, the product space $\mathbb{X}=\mathbb{Y}\times\mathbb{Y},\|(.,.)\|$ is a Banach space equipped with the norm $\|(\varpi,w)\|=\|\varpi\|+\|w\|$.

Next, in view of Lemma 3.2, we define an operator $\Upsilon:\mathbb{X}\to\mathbb{X}$ by

$$
\Upsilon(\varpi,w)(t)=\bigl(\Upsilon_1(\varpi,w)(t),\Upsilon_2(\varpi,w)(t)\bigr),
$$

where

$$
\begin{aligned}
&\Upsilon_1(\varpi,w)(t)\\
&\quad=\biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*}\\
&\qquad+\frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}}+\frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)}+\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta t}{\varrho}\\
&\qquad+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)t}{\varrho}+\frac{\alpha_3\beta_3t^2}{2(1-\alpha_3\beta_3)}\biggr]U_{3k}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*} \right. \\
&\left. + \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} + \frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)t}{\varrho} + \frac{\alpha_3 t^2}{2(1-\alpha_3\beta_3)} \right] V_{3l} \\
&+ \left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_1\alpha_2\beta_2^2}{z} + \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2} \right] U_{2k} \\
&+ \left[\frac{\alpha_1\alpha_2\beta_1}{z} + \frac{\alpha_1\alpha_2\beta_2}{z} + \frac{\alpha_2 t}{(1-\alpha_2\beta_2)} \right] V_{2l} + \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1} U_{1k} + \frac{\alpha_1}{1-\alpha_1\beta_1} V_{1l} \\
&+ \frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} k\bigl(s, w(s), {}^c D_{0^+}^m \varpi(s)\bigr)\, ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_2(\varpi, w)(t) \\
&= \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3^2}{2\tilde{z}} \right. \\
&+ \frac{\alpha_1\beta_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_3\beta_1\beta_3}{2\tilde{z}} \\
&\left. + \frac{\alpha_2\alpha_3\beta_2^2\beta_3\zeta t}{\varrho} + \frac{\alpha_2\beta_2\beta_3\zeta t(\alpha_3\beta_3+1)}{\varrho} + \frac{\alpha_3\beta_2\beta_3\zeta t}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3^2 t^2}{2(1-\alpha_3\beta_3)} + \frac{\beta_3 t^2}{2} \right] U_{3k} \\
&+ \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} \right. \\
&\left. + \frac{\alpha_3\beta_1}{2\tilde{z}} + \frac{\alpha_2\alpha_3\beta_2\zeta t(\beta_2+\beta_3)}{\varrho} + \frac{\alpha_3\beta_2\zeta t}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3 t^2}{2(1-\alpha_3\beta_3)} \right] V_{3l} \\
&+ \left[\frac{\alpha_1\alpha_2\beta_1\beta_2^2}{z} + \frac{\alpha_1\beta_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2^2 t}{1-\alpha_2\beta_2} + \beta_2 t \right] U_{2k} \\
&+ \left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2} + \frac{\alpha_1\alpha_2\beta_1}{z} \right] V_{2l} + \frac{\alpha_1}{1-\alpha_1\beta_1} U_{1k} + \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1} V_{1l} \\
&+ \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} l\bigl(s, \varpi(s), {}^c D_{0^+}^n w(s)\bigr)\, ds.
\end{aligned}
$$

Here,

$$
\begin{aligned}
U_{1k} &= \frac{1}{\Gamma(m)} \int_0^1 (1-s)^{m-1} k\bigl(s, w(s), {}^c D_{0^+}^m \varpi(s)\bigr)\, ds, \\
V_{1l} &= \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-1} l\bigl(s, \varpi(s), {}^c D_{0^+}^n w(s)\bigr)\, ds, \\
U_{2k} &= \frac{1}{\Gamma(m-1)} \int_0^\zeta (\zeta-s)^{m-2} k\bigl(s, w(s), {}^c D_{0^+}^m \varpi(s)\bigr)\, ds, \\
V_{2l} &= \frac{1}{\Gamma(n-1)} \int_0^\zeta (\zeta-s)^{n-2} l\bigl(s, \varpi(s), {}^c D_{0^+}^n w(s)\bigr)\, ds, \\
U_{3k} &= \frac{1}{\Gamma(m-2)} \int_0^\eta (\eta-s)^{m-3} k\bigl(s, w(s), {}^c D_{0^+}^m \varpi(s)\bigr)\, ds, \\
V_{3l} &= \frac{1}{\Gamma(n-2)} \int_0^\eta (\eta-s)^{n-3} l\bigl(s, \varpi(s), {}^c D_{0^+}^n w(s)\bigr)\, ds.
\end{aligned}
$$

For convenience, we use the subsequent symbols:

$$\begin{aligned}
\omega_1 = {}& \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*} \\
&+ \frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho} \\
&+ \frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho} + \frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{(m-1)\eta^{m-2}}{\Gamma(m)} \\
&+ \biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_1\alpha_2\beta_2^2}{z} + \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\biggr]\frac{\zeta^{m-1}}{\Gamma(m)} \\
&+ \frac{\alpha_1}{(1-\alpha_1\beta_1)\Gamma(m+1)} + \frac{1}{\Gamma(m+1)}, \\
\omega_2 = {}& \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*} \\
&+ \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} + \frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho} + \frac{\alpha_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{(n-1)\eta^{n-2}}{\Gamma(n)} \\
&+ \biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2}{1-\alpha_2\beta_2} + \frac{\alpha_1\alpha_2\beta_1}{z}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)} + \frac{\alpha_1\beta_1}{(1-\alpha_1\beta_1)\Gamma(n+1)}, \\
\omega_3 = {}& \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3^2}{2\tilde{z}} \\
&+ \frac{\alpha_1\beta_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_3\beta_1\beta_3}{2\tilde{z}} + \frac{\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho} \\
&+ \frac{\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho} + \frac{\alpha_3\beta_2\beta_3\zeta}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3^2}{2(1-\alpha_3\beta_3)} + \frac{\beta_3}{2}\biggr]\frac{(m-1)\eta^{m-2}}{\Gamma(m)} \\
&+ \biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2^2}{z} + \frac{\alpha_1\beta_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2^2}{1-\alpha_2\beta_2} + \beta_2\biggr]\frac{\zeta^{m-1}}{\Gamma(m)} \\
&+ \frac{\alpha_1}{(1-\alpha_1\beta_1)\Gamma(m+1)},
\end{aligned}$$

and

$$\begin{aligned}
\omega_4 = {}& \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_1\beta_2\zeta}{\tilde{z}} + \frac{\alpha_1\alpha_3\beta_1\beta_3}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} \\
&+ \frac{\alpha_3\beta_1}{2\tilde{z}} + \frac{\alpha_2\alpha_3\beta_2\zeta(\beta_2+\beta_3)}{\varrho} + \frac{\alpha_3\beta_2\zeta}{1-\alpha_3\beta_3} + \frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{(n-1)\eta^{n-2}}{\Gamma(n)} \\
&+ \biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2}{1-\alpha_2\beta_2} + \frac{\alpha_1\alpha_2\beta_1}{z}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)} + \frac{\alpha_1\beta_1}{(1-\alpha_1\beta_1)\Gamma(n+1)} + \frac{1}{\Gamma(n+1)}.
\end{aligned}$$

Now we present our main results.

Theorem 3.1 *Suppose that*:

(E_1) $k, l : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous such that, for any* $y_1, y_2, z_1, z_2 \in \mathbb{R}$, *there exist constants* $\wp_1, \wp_2 > 0$ *satisfying*

$$\bigl|k(t,y_1,z_1) - k(t,y_2,z_2)\bigr| \le \wp_1\bigl(|y_1-y_2| + |z_1-z_2|\bigr) \le \wp_1\bigl|(y_1,z_1) - (y_2,z_2)\bigr|,$$

$$\bigl|l(t,y_1,z_1)-l(t,y_2,z_2)\bigr|\le\wp_2\bigl(|y_1-y_2|+|z_1-z_2|\bigr)\le\wp_2\bigl|(y_1,z_1)-(y_2,z_2)\bigr|;$$

(E$_2$) *The linear operator $D:\mathbb{R}\to\mathbb{R}$ is continuous such that, for each $\varpi,w,u,v\in\mathbb{R}$, we can find constants $0<l_1,l_2<1$ satisfying*

$$\begin{aligned}&\bigl|D\varpi(t)-D(u)\bigr|\le l_1\bigl|\varpi(t)-u(t)\bigr|,\\&\bigl|Dw(t)-D(v)\bigr|\le l_2\bigl|w(t)-v(t)\bigr|.\end{aligned}$$

If $(\omega_1+\omega_3)\wp_1+(\omega_2+\omega_4)\wp_2<1$, then system (1.1)–(1.2) has a unique solution on the interval $[0,1]$.

Proof Define $\sup_{t\in[0,1]}k(t,0,0)=\delta_1<\infty$ and $\sup_{t\in[0,1]}l(t,0,0)=\delta_2<\infty$ and $q>0$, where

$$q>\frac{(\omega_1+\omega_3)\delta_1+(\omega_2+\omega_4)\delta_2}{1-(\omega_1+\omega_3)\wp_1-(\omega_2+\omega_4)\wp_2}.$$

Define $T_q=\{(\varpi,w)\in\mathbb{X}:\|(\varpi,w)\le q\|\}$, we will show that $\Upsilon(T_q)\subset T_q$.

By assumptions E$_1$ and E$_2$ for each $(\varpi,w)\in T_q$ and every $t\in[0,1]$, we have

$$\begin{aligned}\bigl|k\bigl(t,w(t),D^m\varpi(t)\bigr)\bigr|&\le\bigl|k\bigl(t,w(t),D^m\varpi(t)\bigr)-k(t,0,0)\bigr|+\bigl|k(t,0,0)\bigr|\\&\le\wp_1\bigl(\bigl|w(t)\bigr|+l_1\bigl|\varpi(t)\bigr|\bigr)+\delta_1\\&\le\wp_1\bigl(\|s\|+\|w\|\bigr)+\delta_1\\&\le\wp_1q+\delta_1.\end{aligned}$$

Following the same procedure, we obtain

$$\bigl|l\bigl(t,\varpi(t),D^nw(t)\bigr)\bigr|\le\wp_2q+\delta_2,$$

which further yields

$$\begin{aligned}&\bigl|\Upsilon_1(\varpi,w)(t)\bigr|\\&\quad\le\biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*}\\&\qquad+\frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}}+\frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)}+\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}\\&\qquad+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}+\frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(m-1)}\eta^{m-2}(\wp_1q+\delta_1)\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*}+\frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*}\\&\qquad+\frac{\alpha_1\alpha_3\beta_2}{\tilde{z}}+\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho}+\frac{\alpha_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(n-1)}\eta^{n-2}(\wp_2q+\delta_2)\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}+\frac{\alpha_1\alpha_2\beta_2^2}{z}+\frac{\alpha_1\beta_2}{1-\alpha_1\beta_1}+\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\biggr]\frac{\zeta^{m-1}}{\Gamma(m)}(\wp_1q+\delta_1)\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\beta_1}{z}+\frac{\alpha_1\alpha_2\beta_2}{z}+\frac{\alpha_2}{(1-\alpha_2\beta_2)}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)}(\wp_2q+\delta_2)\end{aligned}$$

$$+ \frac{\alpha_1\beta_1}{(1-\alpha_1\beta_1)\Gamma(m+1)}(\wp_1 q + \delta_1) + \frac{\alpha_1}{(1-\alpha_1\beta_1)\Gamma(n+1)}(\wp_2 q + \delta_2)$$
$$+ \frac{1}{\Gamma(m+1)}(\wp_1 q + \delta_1).$$

Thus,

$$\left\|\Upsilon_1(\varpi, w)\right\| \le (\omega_1\wp_1 + \omega_2\wp_2)q + \omega_1\delta_1 + \omega_2\delta_2. \tag{3.1}$$

Likewise,

$$\left\|\Upsilon_2(\varpi, w)\right\| \le (\omega_3\wp_1 + \omega_4\wp_2)q + \omega_3\delta_1 + \omega_4\delta_2. \tag{3.2}$$

Since

$$\left\|\Upsilon(\varpi, w)\right\| \le \left\|\Upsilon_1(\varpi, w)\right\| + \left\|\Upsilon_1(\varpi, w)\right\|. \tag{3.3}$$

Therefore, combining inequalities (3.1) and (3.2) into (3.3), we obtain

$$\left\|\Upsilon(\varpi, w)\right\| \le \left[(\omega_1 + \omega_3)\wp_1 + (\omega_2 + \omega_4)\wp_2\right]q + (\omega_1 + \omega_3)\delta_1 + (\omega_2 + \omega_4)\delta_2 \le q.$$

Hence, Υ maps a bounded subset of T_q into a bounded subset of T_q.

Next, for any $(\varpi_2, w_2), (\varpi_1, w_1) \in \mathbb{X}$ and each $t \in [0,1]$, we have

$$\begin{aligned}
&\left|\Upsilon_1(\varpi_2, w_2)(t) - \Upsilon_1(\varpi_1, w_1)(t)\right| \\
&\quad \le \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}}\right. \\
&\qquad + \frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)} \\
&\qquad \left. + \frac{\alpha_2\alpha_3\beta_2\beta_3\zeta t}{\varrho} + \frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)t}{\varrho} + \frac{\alpha_3\beta_3 t^2}{2(1-\alpha_3\beta_3)}\right] \\
&\qquad \times \frac{(m-1)\eta^{m-2}}{\Gamma(m)}\left|k\bigl(t, w_2(t), D^m\varpi_2(t)\bigr) - k\bigl(t, w_1(t), D^m\varpi_1(t)\bigr)\right| \\
&\qquad + \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*}\right. \\
&\qquad \left. + \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} + \frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)t}{\varrho} + \frac{\alpha_3 t^2}{2(1-\alpha_3\beta_3)}\right] \\
&\qquad \times \frac{(n-1)\eta^{n-2}}{\Gamma(n)}\left|l\bigl(t, \varpi_2(t), D^n w_2(t)\bigr) - l\bigl(t, \varpi_1(t), D^n w_1(t)\bigr)\right| \\
&\qquad + \left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_1\alpha_2\beta_2^2}{z} + \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2}\right] \\
&\qquad \times \frac{\zeta^{m-1}}{\Gamma(m)}\left|k\bigl(t, w_2(t), D^m\varpi_2(t)\bigr) - k\bigl(t, w_1(t), D^m\varpi_1(t)\bigr)\right| + \left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2}\right. \\
&\qquad \left. + \frac{\alpha_1\alpha_2\beta_1}{z}\right]\frac{\zeta^{n-1}}{\Gamma(n)}\left|l\bigl(t, \varpi_2(t), D^n w_2(t)\bigr) - l\bigl(t, \varpi_1(t), D^n w_1(t)\bigr)\right| + \frac{\alpha_1}{(1-\alpha_1\beta_1)\Gamma(m+1)}
\end{aligned}$$

$$\times \left|k\big(t,w_2(t),D^m\varpi_2(t)\big)-k\big(t,w_1(t),D^m\varpi_1(t)\big)\right| + \frac{\alpha_1\beta_1}{(1-\alpha_1\beta_1)\Gamma(n+1)}\left|l\big(t,\varpi_2(t),D^n w_2(t)\big)-l\big(t,\varpi_1(t),D^n w_1(t)\big)\right| + \frac{1}{\Gamma(m+1)}t^m\left|k\big(t,w_2(t),D^m\varpi_2(t)\big)-k\big(t,w_1(t),D^m\varpi_1(t)\big)\right|.$$

Employing assumptions E_1 and E_2, we obtain

$$\begin{aligned}
&\left|\Upsilon_1(\varpi_2,w_2)(t)-\Upsilon_1(\varpi_1,w_1)(t)\right| \\
&\le \left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}}\right. \\
&\quad+\frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*}+\frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} \\
&\quad+\frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}}+\frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)}+\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho} \\
&\quad\left.+\frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\right]\frac{(m-1)\eta^{m-2}}{\Gamma(m)}\wp_1\big(l_1\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big) \\
&\quad+\left[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*}+\frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*}\right. \\
&\quad\left.+\frac{\alpha_1\alpha_3\beta_2}{\tilde{z}}+\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho}+\frac{\alpha_3}{2(1-\alpha_3\beta_3)}\right] \\
&\quad\times\frac{(n-1)\eta^{n-2}}{\Gamma(n)}\wp_2\big(\|\varpi_2-\varpi_1\|+l_2\|w_2-w_1\|\big)+\left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}+\frac{\alpha_1\alpha_2\beta_2^2}{z}\right. \\
&\quad\left.+\frac{\alpha_1\beta_2}{1-\alpha_1\beta_1}+\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\right]\frac{\zeta^{m-1}}{\Gamma(m)}\wp_1\big(l_1\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big) \\
&\quad+\left[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}+\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}+\frac{\alpha_1\alpha_2\beta_1}{z}\right]\frac{\zeta^{n-1}}{\Gamma(n)}\wp_2\big(\|\varpi_2-\varpi_1\|+l_2\|w_2-w_1\|\big) \\
&\quad+\frac{\alpha_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(m+1)}\wp_1\big(l_1\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big) \\
&\quad+\frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(n+1)}\wp_2\big(\|\varpi_2-\varpi_1\|+l_2\|w_2-w_1\|\big) \\
&\quad+\frac{1}{\Gamma(m+1)}\wp_1\big(l_1\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big).
\end{aligned}$$

The condition $0 < l_1, l_2 < 1$ will lead us to

$$\left\|\Upsilon_1(\varpi_2,w_2)(t)-\Upsilon_1(\varpi_1,w_1)(t)\right\| \le (\omega_1\wp_1+\omega_2\wp_2)\big(\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big). \tag{3.4}$$

Similarly,

$$\left\|\Upsilon_2(\varpi_2,w_2)(t)-\Upsilon_2(\varpi_1,w_1)(t)\right\| \le (\omega_3\wp_1+\omega_4\wp_2)\big(\|\varpi_2-\varpi_1\|+\|w_2-w_1\|\big). \tag{3.5}$$

From inequalities (3.4) and (3.5), it follows that

$$\left\|\Upsilon(\varpi_2,w_2)(t)-\Upsilon(\varpi_1,w_1)(t)\right\| \le \big[(\omega_1+\omega_3)\wp_1+(\omega_2+\omega_4)\wp_2\big]\big(\left\|(w_1,w_2)-(\varpi_1,\varpi_2)\right\|\big).$$

As $[(\omega_1+\omega_3)\wp_1+(\omega_2+\omega_4)\wp_2]<1$. Therefore, Υ is a contractive operator. By the Banach contraction principle, we deduce that the operator Υ has a unique fixed point which is the unique solution of problem (1.1)–(1.2). □

The coming result is established on the basis of Leray–Schauder fixed point alternative.

Lemma 3.2 (Leray–Schauder alternative [11]) *Let $A: S\to S$ be a completely continuous operator (i.e., a map restricted to any bounded set in S is compact), and let*

$$Z(F)=\{x\in S: x=\lambda F(x) \quad \text{for some } 0<\lambda<1\}.$$

Then either the set $Z(F)$ is unbounded or F has at least one fixed point.

Theorem 3.3 *Assume that:*

(E_3) *$k,l:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ such that, for all $x,\bar{x},y,\bar{y}$, we can find positive real constants d_0,d_1,d_2,e_0,e_1,e_2 satisfying*

$$\begin{aligned}&\bigl|k(t,x,\bar{x})\bigr|\le d_0+d_1|x|+d_2|\bar{x}|,\\&\bigl|l(t,y,\bar{y})\bigr|\le e_0+e_1|y|+e_2|\bar{y}|.\end{aligned}$$

If $(\omega_1+\omega_3)d_2+(\omega_2+\omega_4)e_1<1$ and $(\omega_1+\omega_3)d_1+(\omega_2+\omega_4)e_2<1$, then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.

Proof The proof will be finished in the subsequent steps.

Step 1. First we show that the operator $\Upsilon:\mathbb{X}\to\mathbb{X}$ is completely continuous. It is clear that Υ is continuous due to the continuity of functions k and l.

Let Δ be any bounded subset of $\mathbb{X}$. Then, for all $(\varpi,w)\in\Delta$, there exist some positive constants A_1 and A_2 such that $|k(t,w(t),D^m\varpi(t))|\le A_1$, $|l(t,\varpi(t),D^nw(t))|\le A_2$. Therefore, for any $(\varpi,w)\in\Delta$, we have

$$\begin{aligned}&\bigl|\Upsilon_1(\varpi,w)(t)\bigr|\\&\quad=\biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*}+\frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*}\\&\qquad+\frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}}+\frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)}+\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta t}{\varrho}\\&\qquad+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)t}{\varrho}+\frac{\alpha_3\beta_3t^2}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(m-1)}\eta^{m-2}\bigl|k\bigl(t,w(t),D^m\varpi(t)\bigr)\bigr|\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*}+\frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}}+\frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*}+\frac{\alpha_1\alpha_3\beta_2}{\tilde{z}}\\&\qquad+\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)t}{\varrho}+\frac{\alpha_3t^2}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(n-1)}\eta^{n-2}\bigl|l\bigl(t,\varpi(t),D^nw(t)\bigr)\bigr|\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z}+\frac{\alpha_1\alpha_2\beta_2^2}{z}+\frac{\alpha_1\beta_2}{1-\alpha_1\beta_1}+\frac{\alpha_2\beta_2t}{1-\alpha_2\beta_2}\biggr]\frac{\zeta^{m-1}}{\Gamma(m)}\bigl|k\bigl(t,w(t),D^m\varpi(t)\bigr)\bigr|\\&\qquad+\biggl[\frac{\alpha_1\alpha_2\beta_1}{z}+\frac{\alpha_1\alpha_2\beta_2}{z}+\frac{\alpha_2t}{(1-\alpha_2\beta_2)}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)}\bigl|l\bigl(t,\varpi(t),D^nw(t)\bigr)\bigr|\end{aligned}$$

$$+\frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(m+1)}\big|k\big(t,w(t),D^m\varpi(t)\big)\big| + \frac{\alpha_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(n+1)}\big|l\big(t,\varpi(t),D^n w(t)\big)\big| + \frac{1}{\Gamma(m+1)}t^m\big|k\big(t,w(t),D^m\varpi(t)\big)\big|.$$

By the boundedness of k and l, we get

$$\begin{aligned}
&\big|\Upsilon_1(\varpi,w)(t)\big| \\
&\leq \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\zeta}{\varrho^*} + \frac{\alpha_1\alpha_2\beta_1\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho^*} \\
&\quad + \frac{\alpha_1\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho^*} + \frac{\alpha_1\alpha_3\beta_2\beta_3\zeta}{\tilde{z}} + \frac{\alpha_1\beta_3}{2(1-\alpha_1\beta_1)} + \frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho} \\
&\quad + \frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho} + \frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(m-1)}\eta^{m-2}A_1 \\
&\quad + \biggl[\frac{\alpha_1\alpha_2\alpha_3\beta_1\zeta(\beta_2+\beta_3)}{\varrho^*} + \frac{\alpha_1\alpha_3(\beta_1+\beta_3)}{2\tilde{z}} + \frac{\alpha_1\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho^*} \\
&\quad + \frac{\alpha_1\alpha_3\beta_2}{\tilde{z}} + \frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho} + \frac{\alpha_3}{2(1-\alpha_3\beta_3)}\biggr]\frac{1}{\Gamma(n-1)}\eta^{n-2}A_2 \\
&\quad + \biggl[\frac{\alpha_1\alpha_2\beta_1\beta_2}{z} + \frac{\alpha_1\alpha_2\beta_2^2}{z} + \frac{\alpha_1\beta_2}{1-\alpha_1\beta_1} + \frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\biggr]\frac{\zeta^{m-1}}{\Gamma(m)}A_1 \\
&\quad + \biggl[\frac{\alpha_1\alpha_2\beta_1}{z} + \frac{\alpha_1\alpha_2\beta_2}{z} + \frac{\alpha_2}{(1-\alpha_2\beta_2)}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)}A_2 \\
&\quad + \frac{\alpha_1\beta_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(m+1)}A_1 + \frac{\alpha_1}{1-\alpha_1\beta_1}\frac{1}{\Gamma(n+1)}A_2 + \frac{1}{\Gamma(m+1)}A_1.
\end{aligned}$$

Consequently,

$$\big\|\Upsilon_1(\varpi,w)\big\| \leq \omega_1A_1 + \omega_2A_2. \tag{3.6}$$

Similarly,

$$\big\|\Upsilon_2(\varpi,w)\big\| \leq \omega_3A_1 + \omega_4A_2. \tag{3.7}$$

Thus, it follows from inequalities (3.6) and (3.7) that Υ is uniformly bounded since $\|\Upsilon(\varpi,w)\| \leq (\omega_1+\omega_3)A_1 + (\omega_2+\omega_4)A_2$.

Step 2. Next, we show that Υ is equicontinuous. Let $t_1,t_2 \in [0,1]$ with $t_1 < t_2$. Then

$$\begin{aligned}
&\big|\Upsilon_1\big(\varpi(t_2),w(t_2)\big) - \Upsilon_1\big(\varpi\big(t_1,w(t_1)\big)\big)\big| \\
&\leq A_1\biggl[\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}(t_2-t_1) + \frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}(t_2-t_1) + \frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\biggr] \\
&\quad \times \frac{1}{\Gamma(m-1)}\eta^{m-2} + A_2\biggl[\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho}(t_2-t_1) + \frac{\alpha_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\biggr] \\
&\quad \times \frac{1}{\Gamma(n-1)}\eta^{n-2} + A_1\biggl[\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\biggr]\frac{\zeta^{m-1}}{\Gamma(m)}(t_2-t_1) + A_2\biggl[\frac{\alpha_2}{(1-\alpha_2\beta_2)}\biggr]\frac{\zeta^{n-1}}{\Gamma(n)}(t_2-t_1) \\
&\quad + A_1\biggl|\frac{1}{\Gamma(m)}\int_0^{t_2}(t_2-s)^{m-1}\,ds - \frac{1}{\Gamma(m)}\int_0^{t_1}(t_1-s)^{m-1}\,ds\biggr|
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}(t_2-t_1)+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}(t_2-t_1)+\frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\right]\\
&\quad\times\frac{1}{\Gamma(m-1)}\eta^{m-2}+A_2\left[\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho}(t_2-t_1)+\frac{\alpha_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\right]\\
&\quad\times\frac{1}{\Gamma(n-1)}\eta^{n-2}+A_1\left[\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\right]\frac{\zeta^{m-1}}{\Gamma(m)}(t_2-t_1)+A_2\left[\frac{\alpha_2}{(1-\alpha_2\beta_2)}\right]\frac{\zeta^{n-1}}{\Gamma(n)}(t_2-t_1)\\
&\quad+\frac{A_1}{\Gamma(m)}\left|\int_{t_1}^{t_2}(t_2-s)^{m-1}\,ds\right|+A_1\left|\int_0^{t_1}\frac{(t_2-s)^{m-1}-(t_1-s)^{m-1}}{\Gamma(m)}\,ds\right|
\end{aligned}$$

or

$$\begin{aligned}
&\bigl|\Upsilon_1\bigl(\varpi(t_2),w(t_2)\bigr)-\Upsilon_1\bigl(\varpi\bigl(t_1,w(t_1)\bigr)\bigr|\\
&\leq A_1\left[\frac{\alpha_2\alpha_3\beta_2\beta_3\zeta}{\varrho}(t_2-t_1)+\frac{\alpha_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}(t_2-t_1)\right.\\
&\quad\left.+\frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\right]\frac{1}{\Gamma(m-1)}\eta^{m-2}+A_2\left[\frac{\alpha_2\alpha_3\zeta(\beta_2+\beta_3)}{\varrho}(t_2-t_1)\right.\\
&\quad\left.+\frac{\alpha_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\right]\frac{1}{\Gamma(n-1)}\eta^{n-2}+A_1\left[\frac{\alpha_2\beta_2}{1-\alpha_2\beta_2}\right]\frac{\zeta^{m-1}}{\Gamma(m)}(t_2-t_1)\\
&\quad+A_2\left[\frac{\alpha_2}{(1-\alpha_2\beta_2)}\right]\frac{\zeta^{n-1}}{\Gamma(n)}(t_2-t_1)+A_1\frac{(t_2^m-t_1^m)}{\Gamma(m+1)}.
\end{aligned}\tag{3.8}$$

By following the same procedure, we get

$$\begin{aligned}
&\bigl|\Upsilon_2\bigl(\varpi(t_2),w(t_2)\bigr)-\Upsilon_2\bigl(\varpi\bigl(t_1,w(t_1)\bigr)\bigr|\\
&\leq A_1\left[\frac{\alpha_2\alpha_3\beta_2^2\beta_3\zeta}{\varrho}(t_2-t_1)+\frac{\alpha_2\beta_2\beta_3\zeta(\alpha_3\beta_3+1)}{\varrho}(t_2-t_1)\right.\\
&\quad\left.+\frac{\alpha_3\beta_2\beta_3\zeta}{1-\alpha_3\beta_3}(t_2-t_1)+\frac{\alpha_3\beta_3^2}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2+\frac{\beta_3}{2}(t_2-t_1)^2\right]\frac{1}{\Gamma(m-1)}\eta^{m-2}\\
&\quad+A_2\left[\frac{\alpha_2\alpha_3\beta_2\zeta(\beta_2+\beta_3)}{\varrho}(t_2-t_1)+\frac{\alpha_3\beta_2\zeta}{1-\alpha_3\beta_3}(t_2-t_1)\right.\\
&\quad\left.+\frac{\alpha_3\beta_3}{2(1-\alpha_3\beta_3)}(t_2-t_1)^2\right]\frac{1}{\Gamma(n-1)}\eta^{n-2}+A_1\left[\frac{\alpha_2\beta_2^2}{1-\alpha_2\beta_2}(t_2-t_1)\right.\\
&\quad\left.+\beta_2(t_2-t_1)\right]\frac{1}{\Gamma(m)}\zeta^{m-1}+A_2\left[\frac{\alpha_2\beta_2 t}{1-\alpha_2\beta_2}\right]\frac{1}{\Gamma(n)}\zeta^{n-1}+A_2\frac{(t_2^n-t_1^n)}{\Gamma(n+1)}.
\end{aligned}\tag{3.9}$$

From inequalities (3.8) and (3.9), we conclude that $|\Upsilon_1(\varpi(t_2),w(t_2))-\Upsilon_1(\varpi(t_1,w(t_1))|\to 0$ as $t_1\to t_2$ and $|\Upsilon_2(\varpi(t_2),w(t_2))-\Upsilon_2(\varpi(t_1,w(t_1))|\to 0$ as $t_1\to t_2$. Therefore, $\Upsilon(\varpi,w)$ is equicontinuous. Hence, by Arzela–Ascoli theorem the operator $\Upsilon(\varpi,w)$ is completely continuous.

Step 3. It remains to show that the set

$$Z=\bigl\{(\varpi,w)\in\mathbb{X}:(\varpi,w)=\mu\Upsilon(\varpi,w)\text{ for some }\mu\in(0,1)\bigr\}$$

is bounded. Let $(\varpi, w) \in Z$ along with $(\varpi, w) = \mu\Upsilon(\varpi, w)$, then for any $t \in [0,1]$, we have $\varpi(t) = \mu\Upsilon_1(\varpi, w)(t)$, $w(t) = \mu\Upsilon_2(\varpi, w)(t)$.

$$\begin{aligned}\left|\varpi(t)\right| &\le \mu\left[\omega_1\left|k\left(t, w(t), D^m\varpi(t)\right)\right| + \omega_2\left|l\left(t, \varpi(t), D^n w(t)\right)\right|\right] \\ &\le \omega_1\left(d_0 + d_1\left|w(t)\right| + d_2k\left|\varpi(t)\right|\right) + \omega_2\left(e_0 + e_1\left|\varpi(t)\right| + e_2k\left|w(t)\right|\right) \\ &= \omega_1 d_0 + \omega_2 e_0 + (\omega_1 d_2 + \omega_2 e_1)\left|\varpi(t)\right| + (\omega_1 d_1 + \omega_2 e_2)\left|w(t)\right|.\end{aligned}$$

Also

$$\begin{aligned}\left|w(t)\right| &\le \mu\left[\omega_3\left(d_0 + d_1\left|w(t)\right| + d_2k\left|\varpi(t)\right|\right) + \omega_4\left(e_0 + e_1\left|\varpi(t)\right| + e_2k\left|w(t)\right|\right)\right] \\ &= \omega_3 d_0 + \omega_4 e_0 + (\omega_3 d_2 + \omega_4 e_1)\left|\varpi(t)\right| + (\omega_3 d_1 + \omega_4 e_2)\left|w(t)\right|,\end{aligned}$$

which further gives

$$\begin{aligned}\|\varpi\| + \|w\| \le{}& (\omega_1 + \omega_3)d_0 + (\omega_2 + \omega_4)e_0 + \left[(\omega_1 + \omega_3)d_2 + (\omega_2 + \omega_4)e_1\right]\|\varpi\| \\ &+ \left[(\omega_1 + \omega_3)d_1 + (\omega_2 + \omega_4)e_2\right]\|w\|.\end{aligned}$$

Consequently,

$$\left\|(\varpi, w)\right\| \le \frac{(\omega_1 + \omega_3)d_0 + (\omega_2 + \omega_4)e_0}{\gamma_0},$$

where

$$\gamma_0 = \min\left\{1 - \left[(\omega_1 + \omega_3)d_2 + (\omega_2 + \omega_4)e_1\right], 1 - \left[(\omega_1 + \omega_3)d_1 + (\omega_2 + \omega_4)e_2\right]\right\},$$

which implies that Z is bounded. Thus, the operator Υ has at least one fixed point which is the solution of (1.1)–(1.2), thanks to Lemma 3.2. □

4 Ulam–Hyers stability

This section is dedicated to the investigation of Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability results corresponding to the solutions of (1.1)–(1.2). We will only establish the most general stability result, that is, generalized Ulam–Hyers–Rassias stability result. The following definitions are adopted from [27].

Let $\varepsilon_m, \varepsilon_n > 0$ and $\Theta_m, \Theta_n : [0,1] \to \mathbb{R}_+$ be nondecreasing continuous functions. We will focus on the following inequalities for $t \in [0,1]$:

$$\begin{aligned}&\left|D^m\varpi(t) - k\left(t, w(t), D^m\varpi(t)\right)\right| \le \varepsilon_m, \\ &\left|D^n w(t) - l\left(t, \varpi(t), {}^cD^n w(t)\right)\right| \le \varepsilon_n,\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\left|D^m\varpi(t) - k\left(t, w(t), D^m\varpi(t)\right)\right| \le \varepsilon_m, \\ &\left|D^n w(t) - l\left(t, \varpi(t), {}^cD^n w(t)\right)\right| \le \varepsilon_n,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\left|D^m\varpi(t) - k\left(t, w(t), D^m\varpi(t)\right)\right| \le \Theta_m(t)\varepsilon_m, \\ &\left|D^n w(t) - l\left(t, \varpi(t), {}^cD^n w(t)\right)\right| \le \Theta_n(t)\varepsilon_n,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\left|D^m \varpi(t) - k\left(t, w(t), D^m \varpi(t)\right)\right| \le \Theta_m(t),\\&\left|D^n w(t) - l\left(t, \varpi(t), {}^c D^n w(t)\right)\right| \le \Theta_n(t).\end{aligned} \tag{4.4}$$

Definition 4.1 Problem (1.1)–(1.2) is called Ulam–Hyers stable if we can find a constant $\mathcal{P}_{m,n} > 0$ ($\mathcal{P}_{m,n} = \max\{\mathcal{P}_m, \mathcal{P}_n\}$) such that, for each $\varepsilon > 0$ ($\varepsilon = \max\{\varepsilon_m, \varepsilon_n\}$) and every solution $(\varpi, w) \in \mathbb{X}$ of (4.1), there exists a solution $(s^*, w^*) \in \mathbb{X}$ of (1.1)–(1.2) with

$$\left|(\varpi, w)(t) - \left(s^*, w^*\right)(t)\right| \le \mathcal{P}_{m,n}\varepsilon, \quad t \in [0,1].$$

Definition 4.2 Problem (1.1)–(1.2) is said to be generalized Ulam–Hyers stable if there is $\Lambda \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\Lambda(0) = 0$ such that, for every solution $(\varpi, w) \in \mathbb{X}$ of (4.2), there is a solution $(s^*, w^*) \in \mathbb{X}$ of (1.1)–(1.2) with

$$\left|(\varpi, w)(t) - \left(s^*, w^*\right)(t)\right| \le \Lambda(\varepsilon), \quad t \in [0,1].$$

Definition 4.3 Problem (1.1)–(1.2) is Ulam–Hyers–Rassias stable on the interval $[0,1]$ with respect to $\Theta_{m,n} \in C([0,1], \mathbb{R})$, ($\Theta_{m,n} = \max\{\Theta_m, \Theta_n\}$) if there exists a positive real number C such that, for each solution $(\varpi^*, w^*) \in \mathbb{X}$ of (4.3), there exists a solution $(\varpi, w) \in \mathbb{X}$ of (1.1)–(1.2) with

$$\left|(\varpi, w)(t) - \left(\varpi^*, w^*\right)(t)\right| \le C\Theta_{m,n}(t)(\varepsilon), \quad t \in [0,1].$$

Definition 4.4 Problem (1.1)–(1.2) is generalized Ulam–Hyers–Rassias stable on $[0,1]$ with respect to $\Theta_{m,n} \in C([0,1], \mathbb{R})$ if there exists a real number $C_\Theta > 0$ such that, for each solution $(\varpi, w) \in \mathbb{X}$ of (4.4), we have a solution $(s^*, w^*) \in \mathbb{X}$ of (1.1)–(1.2) with

$$\left|(\varpi, w)(t) - \left(s^*, w^*\right)(t)\right| \le C_\Theta \Theta_{m,n}(t), \quad t \in [0,1].$$

Remark 4.5 We say that the functions $\varpi, w \in \mathbb{X}$ are the solutions of (4.4) if there exist functions $\hbar_1, \hbar_1 \in \mathbb{X}$ which depend upon ϖ, w, respectively, such that

(I)

$$\left|\hbar_1(t)\right| \le \Theta_m, \qquad \left|\hbar_2(t)\right| \le \Theta_n$$

(II)

$$\begin{aligned}&{}^c D^m \varpi(t) = k\left(t, w(t), {}^c D^m \varpi(t)\right) + \hbar_1(t), \quad 0 < t < 1,\\&{}^c D^n w(t) = l\left(t, \varpi(t), {}^c D^n w(t)\right) + \hbar_2(t), \quad 0 < t < 1.\end{aligned}$$

Before going to the main result, we need the following assumption:

(E_4) $\hbar_1, \hbar_2 : J \to \mathbb{R}^+$ are nondecreasing functions such that

$$\int_0^t \hbar_1(s)\,ds \le \lambda_m \hbar_1(t), \qquad \int_0^t \hbar_2(s)\,ds \le \lambda_n \hbar_2(t)$$

for all $t \in [0,1]$ and $\lambda_m, \lambda_n > 0$.

Lemma 4.6 *Let ϖ, w be the solutions of inequality (4.4), then*

$$\begin{cases} |\varpi(t) - u(t)| \leq C_{\Theta_m}\Theta_m, \\ |w(t) - v(t)| \leq C_{\Theta_n}\Theta_n. \end{cases}$$

Proof From Remark 4.5(II), we have

$$\begin{cases} {}^cD^m\varpi(t) = k(t, w(t), {}^cD^m\varpi(t)) + \hbar_1(t), \quad 0 < t < 1, \\ {}^cD^n w(t) = l(t, \varpi(t), {}^cD^n w(t)) + \hbar_2(t), \quad 0 < t < 1, \\ \varpi(0) = \alpha_1 w(1), \qquad \varpi'(0) = \alpha_2 w'(\zeta), \qquad \varpi''(0) = \alpha_3 w''(\eta), \\ w(0) = \beta_1\varpi(1), \qquad w'(0) = \beta_2\varpi'(\zeta), \qquad w''(0) = \beta_3\varpi''(\eta). \end{cases} \tag{4.5}$$

In view of Lemma 3.2, the solution of (4.5) will be equivalent to the subsequent integral equations:

$$\varpi(t) = \omega_1 k\bigl(t, w(t), {}^cD^m\varpi(t)\bigr) + \omega_2 l\bigl(t, \varpi(t), {}^cD^n w(t)\bigr) + \frac{1}{\Gamma(m)}\int_0^t (t-s)^{m-1}\hbar_1(s)\,ds, \tag{4.6}$$

$$w(t) = \omega_3 k\bigl(t, w(t), {}^cD^m\varpi(t)\bigr) + \omega_4 l\bigl(t, \varpi(t), {}^cD^n w(t)\bigr) + \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\hbar_2(s)\,ds. \tag{4.7}$$

It follows from (4.6) that

$$\bigl|\varpi(t) - u(t)\bigr| \leq \frac{1}{\Gamma(m)}\int_0^t (t-s)^{m-1}\hbar_1(s)\,ds, \tag{4.8}$$

where

$$u(t) = \omega_1 k\bigl(t, w(t), {}^cD^m\varpi(t)\bigr) + \omega_2 l\bigl(t, \varpi(t), {}^cD^n w(t)\bigr).$$

By using (I) of Remark 4.5 and assumption E_4, (4.8) leads to

$$\bigl|\varpi(t) - u(t)\bigr| \leq \lambda_m\Theta_m.$$

Also from (4.7) we have

$$\bigl|w(t) - v(t)\bigr| \leq \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\hbar_2(s)\,ds, \tag{4.9}$$

where

$$v(t) = \omega_3 k\bigl(t, w(t), {}^cD^m\varpi(t)\bigr) + \omega_4 l\bigl(t, \varpi(t), {}^cD^n w(t)\bigr).$$

Following the same procedures, we can write equation (4.9) as

$$\bigl|w(t) - v(t)\bigr| \leq \lambda_n\Theta_n.$$

□

Theorem 4.7 *Under assumption* E_4, *system* (1.1)–(1.2) *has generalized Ulam–Hyers–Rassias stability provided that*

$$1-\aleph_m\aleph_n>0.$$

Proof Let $(\varpi,w)\in\mathbb{X}$ be a solution of inequality (4.4) and $(\varpi^*,w^*)\in\mathbb{X}$ be the unique solution of the following problem:

$$\begin{cases} {}^cD^m_{0^+}\varpi^*(t)=k(t,w^*(t),{}^cD^m_{0^+}\varpi^*(t)), \quad 0<t<1\\ {}^cD^n_{0^+}w^*(t)=l(t,\varpi^*(t),{}^cD^n_{0^+}w^*(t)), \quad 0<t<1\\ \varpi^*(0)=\alpha_1w^*(1), \qquad \varpi^{*'}(0)=\alpha_2w^{*'}(\zeta), \qquad \varpi^{*''}(0)=\alpha_3w^{*''}(\eta),\\ w^*(0)=\beta_1\varpi^*(1), \qquad w^{*'}(0)=\beta_2\varpi^{*'}(\zeta), \qquad w^{*''}(0)=\beta_3\varpi^{*''}(\eta). \end{cases} \tag{4.10}$$

Thus, we write the solution of (4.10) as follows:

$$\begin{cases} \varpi^*(t)=\omega_1k(t,w^*(t),{}^cD^m_{0^+}\varpi^*(t))+\omega_2l(t,\varpi^*(t),{}^cD^n_{0^+}w^*(t)),\\ w^*(t)=\omega_3k(t,w^*(t),{}^cD^m_{0^+}\varpi^*(t))+\omega_4l(t,\varpi^*(t),{}^cD^n_{0^+}w^*(t)). \end{cases}$$

Now

$$\begin{aligned} \bigl|\varpi(t)-\varpi^*(t)\bigr| &\le \bigl|\varpi(t)-u(t)\bigr|+\bigl|u(t)-\varpi^*(t)\bigr|\\ &\le \lambda_m\Theta_m+\bigl|\omega_1k\bigl(t,w(t),{}^cD^m_{0^+}\varpi(t)\bigr)+\omega_2l\bigl(t,\varpi(t),{}^cD^n_{0^+}(t)\bigr)\\ &\quad -\omega_1k\bigl(t,w^*(t),{}^cD^m_{0^+}s^*(t)\bigr)-\omega_2l\bigl(t,s^*(t),{}^cD^n_{0^+}w^*(t)\bigr)\bigr|\\ &\le \lambda_m\Theta_m+(\omega_1\wp_1+\omega_2\wp_2l_2)\bigl|w(t)-w^*(t)\bigr|\\ &\quad +(\omega_1\wp_1l_1+\omega_2\wp_2)\bigl|\varpi(t)-\varpi^*(t)\bigr|, \end{aligned}$$

which further gives

$$\bigl|\varpi(t)-\varpi^*(t)\bigr|\le\frac{\lambda_m\Theta_m}{1-(\omega_1\wp_1+\omega_2\wp_2)}+\frac{\omega_1\wp_1+\omega_2\wp_2}{1-(\omega_1\wp_1+\omega_2\wp_2)}\bigl|w(t)-w^*(t)\bigr|. \tag{4.11}$$

Similarly,

$$\bigl|w(t)-w^*(t)\bigr|\le\frac{\lambda_n\Theta_n}{1-(\omega_3\wp_1+\omega_4\wp_2)}+\frac{\omega_3\wp_1+\omega_4\wp_2}{1-(\omega_3\wp_1+\omega_4\wp_2)}\bigl|\varpi(t)-\varpi^*(t)\bigr|. \tag{4.12}$$

Inequality (4.11) can be written as

$$\bigl\|\varpi-\varpi^*\bigr\|\le C_m\Theta_m+\aleph_m\bigl\|w-w^*\bigr\|$$

or

$$\bigl\|\varpi-\varpi^*\bigr\|-\aleph_m\bigl\|w-w^*\bigr\|\le C_m\Theta_m, \tag{4.13}$$

where

$$C_m = \frac{\lambda_m}{1-(\omega_1\wp_1+\omega_2\wp_2)}, \qquad \aleph_m = \frac{\omega_1\wp_1+\omega_2\wp_2}{1-(\omega_1\wp_1+\omega_2\wp_2)}.$$

Also from (4.12) we gain

$$\|w-w^*\| - \aleph_n\|\varpi-\varpi^*\| \le C_n\Theta_n, \tag{4.14}$$

where

$$C_n = \frac{\lambda_n}{1-(\omega_3\wp_1+\omega_4\wp_2)}, \qquad \aleph_n = \frac{\omega_3\wp_1+\omega_4\wp_2}{1-(\omega_3\wp_1+\omega_4\wp_2)}.$$

From (4.13) and (4.14), we have

$$\begin{bmatrix} 1 & -\aleph_m \\ -\aleph_n & 1 \end{bmatrix} \begin{bmatrix} \|\varpi-\varpi^*\| \\ \|w-w^*\| \end{bmatrix} \le \begin{bmatrix} C_m\theta_m \\ C_n\theta_n \end{bmatrix}.$$

Set

$$\Delta = 1-\aleph_m\aleph_n > 0.$$

Simplification yields

$$\begin{aligned} \|\varpi-\varpi^*\| &\le \frac{C_m\Theta_m}{\Delta} + \frac{\aleph_m C_n\Theta_n}{\Delta}, \\ \|w-w^*\| &\le \frac{C_n\Theta_n}{\Delta} + \frac{\aleph_n C_m\Theta_m}{\Delta}, \\ \|\varpi-\varpi^*\| + \|w-w^*\| &\le \frac{C_m\Theta_m}{\Delta} + \frac{C_n\Theta_n}{\Delta} + \frac{\aleph_m C_n\Theta_n}{\Delta} + \frac{\aleph_n C_m\Theta_m}{\Delta}. \end{aligned} \tag{4.15}$$

Inequality (4.15) becomes

$$\|(\varpi,w)-(\varpi^*,w^*)\| \le C_{m,n}\Theta_{m,n},$$

where

$$C_{m,n} = \frac{C_m}{\Delta} + \frac{C_n}{\Delta} + \frac{\aleph_m C_n}{\Delta} + \frac{\aleph_n C_m}{\Delta}.$$

Thus, by Definition 4.4, the proposed system is generalized Ulam–Hyers–Rassias stable. □

Remark 4.8 From the last theorem, the Ulam–Hyers stability, generalized Ulam–Hyers stability, and Ulam–Hyers–Rassias stability of system (1.1)–(1.2) can be obtained as corollaries.

5 Example

In this section, we present some examples to achieve the existence, uniqueness, and stability of the proposed system.

Example 5.1 Consider the following system of fractional differential equation with non-separated coupled boundary conditions:

$$\begin{cases} {}^cD^{\frac{5}{2}}\varpi(t) = \frac{e^t w(t)}{16} + \frac{\cos|{}^cD^{\frac{5}{2}}\varpi(t)|}{16+t^2}, & t \in (0,1), \\ {}^cD^{\frac{5}{2}}w(t) = \frac{1+\sin^2\varpi(t)}{\sqrt{t(1-t)^2}} + \frac{|{}^cD^{\frac{5}{2}}w(t)|}{16(1+|{}^cD^{\frac{5}{2}}w(t)|)}, & t \in (0,1), \\ \varpi(0) = \frac{1}{2}w(1), \quad \varpi'(0) = \frac{1}{3}w'(\frac{1}{4}), \quad \varpi''(0) = \frac{1}{4}w''(\frac{1}{4}), \\ w(0) = \frac{3}{7}\varpi(1), \quad w'(0) = \frac{2}{3}\varpi'(\frac{1}{4}), \quad w''(0) = \frac{1}{5}\varpi''(\frac{1}{6}). \end{cases} \tag{5.1}$$

Here, $m = n = \frac{5}{2}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{4}$, $\beta_1 = \frac{3}{7}$, $\beta_2 = \frac{2}{3}$, $\beta_3 = \frac{1}{5}$, $\zeta = \frac{1}{4}$, $\eta = \frac{1}{6}$. We can easily find that $\omega_1 = 0.648937084$, $\omega_2 = 0.5431954534$, $\omega_3 = 0.8409326373$, and $\omega_4 = 0.5537849552$.

Also, we have

$$\begin{aligned} \bigl|f(t,\varpi_1,\varpi_2) - f(t,w_1,w_2)\bigr| &\le \frac{1}{16}|\varpi_1 - \varpi_2| + \frac{1}{16}|w_1 - w_2|, \\ \bigl|g(t,\varpi_1,\varpi_2) - g(t,w_1,w_2)\bigr| &\le \frac{1}{16}|\varpi_1 - \varpi_2| + \frac{1}{16}|w_1 - w_2|. \end{aligned}$$

Also $(\omega_1 + \omega_3)\wp_1 + (\omega_2 + \omega_4)\wp_2 \approx 0.1616781 < 1$. Since all the requirements of Theorem 3.1 are satisfied, hence problem (5.1) has a unique solution.

Example 5.2 Consider the following system of fractional differential equations:

$$\begin{cases} {}^cD^{\frac{5}{2}}\varpi(t) = 1 + \frac{1}{2(t+1)^2}\frac{|{}^cD^{\frac{5}{2}}\varpi(t)|}{1+|D\varpi(t)|} + \frac{\sin w(t)}{9}, & t \in (0,1), \\ {}^cD^{\frac{5}{2}}w(t) = \frac{1}{3} + \frac{\sin 2\pi\varpi(t)}{27} + \frac{{}^cD^{\frac{5}{2}}w(t)}{9}, & t \in (0,1), \\ \varpi(0) = \frac{1}{2}w(1), \quad \varpi'(0) = \frac{1}{3}w'(\frac{1}{4}), \quad \varpi''(0) = \frac{1}{4}w''(\frac{1}{4}), \\ w(0) = \frac{3}{7}\varpi(1), \quad w'(0) = \frac{2}{3}\varpi'(\frac{1}{4}), \quad w''(0) = \frac{1}{5}\varpi''(\frac{1}{6}). \end{cases} \tag{5.2}$$

Comparing (5.2) with system (1.1)–(1.2) yields $m = n = \frac{5}{2}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{4}$, $\beta_1 = \frac{3}{7}$, $\beta_2 = \frac{2}{3}$, $\beta_3 = \frac{1}{5}$, $\zeta = \frac{1}{4}$, $\eta = \frac{1}{6}$. Furthermore,

$$\bigl|k(t,x,\bar{x})\bigr| \le 1 + \frac{1}{9}|x| + \frac{1}{9}|\bar{x}|, \qquad \bigl|l(t,y,\bar{y})\bigr| \le \frac{1}{3} + \frac{1}{9}|y| + \frac{1}{9}|\bar{y}|.$$

As $d_0 = 1$, $d_1 = \frac{1}{9}$, $d_2 = \frac{1}{9}$, $e_0 = \frac{1}{3}$, $e_1 = \frac{1}{9}$, $e_2 = \frac{1}{9}$, we can easily find that $\omega_1 = 0.648937084$, $\omega_2 = 0.5431954534$, $\omega_3 = 0.8409326373$, and $\omega_4 = 0.5537849552$. Note that $(\omega_1 + \omega_3)d_2 + (\omega_2 + \omega_4)e_1 \approx 0.2874278 < 1$ and $(\omega_1 + \omega_3)d_1 + (\omega_2 + \omega_4)e_2 < 1$. As a conclusion, we can say that a coupled system has at least one solution. Furthermore,

$$1 - \aleph_m \aleph_n \approx 0.6639308221 > 0$$

and condition E_4 is also satisfied. Consequently, problem (5.2) is generalized Ulam–Hyers–Rassias stable.

6 Conclusion

In this manuscript, we have successfully derived the existence and uniqueness results for nonlinear coupled FODEs with non-separated boundary conditions. To guarantee the existence and uniqueness of solutions, some sufficient criteria have been established by the Banach contraction principle and the Leray–Schauder alternative. Moreover, we presented the generalized Ulam–Hyers–Rassias stability for model (1.1)–(1.2) by classical functional analysis. At the end, for the justification of our results, we stated some examples. The results obtained in this article are of quite general nature, because by changing the parameters and the interval from $(0,1)$ to $[0,T]$ in the proposed system, one can get different types of boundary conditions like coupled flux type conditions, periodic and anti periodic boundary conditions, etc.

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