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Global Hopf bifurcation for two zooplankton-phytoplankton model with two delays

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Abstract

In this paper, we study the global existence of a bifurcating periodic solution for a two zooplankton-phytoplankton model with two delays. First, we demonstrate that the bifurcating periodic solution exists when one delay increases and the other delay remains unchanged. Second, we give simulation to describe the bifurcating periodic solution when one delay increases. Our work answers the question in Sect. 5 (Shi and Yu in *Chaos Solitons Fractals* 100:62–73, 2017).

Keywords: Zooplankton-phytoplankton model; Delays; Limit cycle; Hopf bifurcation

1 Introduction

The dynamics of a plankton model is important for an aquatic system: phytoplankton produce oxygen by photosynthesis and absorb nearly half carbon dioxide, so phytoplankton exert great influence on our ecosystem. There have been many works about the dynamics of a phytoplankton-zooplankton model in recent years. As we know, delays always exist in nature, which could bring rich dynamics in a large range of biological systems. So the delayed phytoplankton-zooplankton model has attracted much interest [1–10]; the model contains two delays, that is, gestation delay for zooplankton and the maturity delay for toxic phytoplankton, which have been studied by many researchers [1, 4, 6, 8–10] in recent years. In [1, 9, 10], the authors took the gestation delay as a parameter, and the Hopf bifurcation was discussed. In [4, 6, 8], the author took the delay required for the maturation of toxic phytoplankton as a parameter, and the dynamics of system was studied. We know that there always exist more than one delay in the world, so the dynamics of multiple delays should be more real and interesting. The dynamics of more than one delay have turned up in recent years [1, 6, 11–13]. For example, the author [6] discussed Hopf bifurcation of the following phytoplankton-zooplankton system with two delays:

$$\begin{cases} \frac{dP_1}{dt} = r_1P_1\left(1 - \frac{P_1}{K}\right) - \alpha_1P_1P_2 - \rho_1P_1Z, \\ \frac{dP_2}{dt} = r_2P_2\left(1 - \frac{P_2}{K}\right) - \alpha_2P_1P_2 - \rho_2P_2Z, \\ \frac{dZ}{dt} = (r_1P_1 + r_2P_2)Z - dZ - \theta_1P_1(t - \tau_1)Z - \theta_2P_2(t - \tau_2)Z, \end{cases} \quad (1.1)$$

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where the delays τ_1 and τ_2 stand for the time required for maturity of two toxic phytoplankton. Besides we studied the Hopf bifurcation with two delays caused by the gestation of two zooplankton [1]:

$$\begin{cases} \frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{\mu_1 P Z_1}{\alpha_1 + P} - \frac{\mu_2 P Z_2}{\alpha_2 + P}, \\ \frac{dZ_1}{dt} = \frac{\beta_1 P(t-\tau_1) Z_1(t-\tau_1)}{\alpha_1 + P(t-\tau_1)} - \frac{\rho_1 P(t) Z_1}{\alpha_1 + P(t)} - d_1 Z_1 - g_1 Z_1^2, \\ \frac{dZ_2}{dt} = \frac{\beta_2 P(t-\tau_2) Z_2(t-\tau_2)}{\alpha_2 + P(t-\tau_2)} - \frac{\rho_2 P(t) Z_2}{\alpha_2 + P(t)} - d_2 Z_2 - g_2 Z_2^2, \end{cases} \tag{1.2}$$

where $P(t), Z_i(t)$ ($i = 1, 2$) denote the density of phytoplankton and zooplankton respectively, r is the intrinsic growth rate for phytoplankton, K is the environmental carrying capacity of phytoplankton, μ_i denotes the maximum uptake rate for zooplankton. $\frac{P}{\alpha_i + P}$ denotes the Holling Π functional response which describes the grazing for zooplankton. β_i, ρ_i are the ratio of biomass conversion and the rate of toxic substance produced by per unit biomass of phytoplankton, d_i is the death rate, g_i denotes the intrinsic competition coefficients, α_i is the half saturation constant for Holling Π functional response, apparently $\beta_i < \mu_i$ holds. We know that Wu [14] gave the condition about the global Hopf bifurcation for a delay differential system. There exist much work about the global Hopf bifurcation [5, 7–11, 15–17]. Especially there are many works on a phytoplankton-zooplankton system [5, 7–10]. For example, we take the gestation delay as a parameter, and the global Hopf bifurcation of the following system is discussed [10]:

$$\begin{cases} \frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{\mu P Z}{\alpha + P}, \\ \frac{dZ}{dt} = \frac{\rho P(t-\tau) Z(t-\tau)}{\alpha + P(t-\tau)} - \frac{\rho P(t) Z}{\alpha + P(t)} - dZ - gZ^2. \end{cases} \tag{1.3}$$

By now, there have been little work about the global Hopf bifurcation with two delays. In paper [11], the global existence of a bifurcating periodic solution of a neural network model with two delays was studied by limiting $\tau_1 \in (0, \tau_{10})$ and taking τ_2 as a bifurcation parameter.

Based on the above works, we shall study the global Hopf bifurcation of system (1.2) with delays, which is the second question given in Sect. 5 [1]. In this paper we should answer this question: if two delays coexists for system (1.2), limiting $\tau_1 \in (0, \tau_{10})$, when τ_2 increases, does the periodic solution exist for ever? This paper is organized as follows: in Sect. 2, we give some preliminary work about system (1.2). In Sect. 3, based on [14], we demonstrate the global existence of a periodic solution, then we give some simulation of the periodic solution of system (1.2).

2 The mathematical model

Before our discussion, we give the condition for the existence and uniqueness of positive equilibrium.

We know when delays do not exist, $\frac{dz_i}{dt} < 0$ as $\beta_i - \rho_i - d_i < 0$. Hence we assume that $\beta_i - \rho_i - d_i > 0$ holds. From [4], we know that, for system (1.2), a positive equilibrium exists when $h_3\left(\frac{d_1 \alpha_1}{\beta_1 - \rho_1 - d_1}\right) > 0, h_3\left(\frac{d_2 \alpha_2}{\beta_2 - \rho_2 - d_2}\right) > 0$ hold, where

$$h_3(P) = r\left(1 - \frac{P}{K}\right)(\alpha_1 + P)^2(\alpha_2 + P)^2 - \frac{\mu_1}{g_1} \left((\beta_1 - \rho_1 - d_1)P - d_1 \alpha_1 \right) (\alpha_2 + P)^2$$

$$-\frac{\mu_2}{g_2}((\beta_2 - \rho_2 - d_2)P - d_2\alpha_2)(\alpha_1 + P)^2.$$

Furthermore, the positive equilibrium is unique when $h_3(P)$ is monotone on (P_0, P_1) , the definition of P_0, P_1 could be found in [4].

In [1], we discussed the dynamics of system (1.2) under six cases (1) $\tau_1 = \tau_2 = 0$, (2) $\tau_1 > 0, \tau_2 = 0$, (3) $\tau_1 = 0, \tau_2 > 0$, (4) $\tau_1 = \tau_2 = \tau > 0$, (5) $\tau_1 \in (0, \tau_{10}), \tau_2 > 0$, (6) $\tau_2 \in (0, \tau_{20}), \tau_1 > 0$. Now we give two results about case (2) and case (5), which is necessary for our work in this paper.

Lemma 2.1 (Theorem 3.1 [1]) *For system (1.2), when $\tau_1 > 0, \tau_2 = 0$, if H_{21}, H_{22} hold, the equilibrium (P^*, Z_1^*, Z_2^*) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$, Hopf bifurcation occurs when $\tau_1 = \tau_{10}$.*

Lemma 2.2 (Theorem 3.4 [1]) *For system (1.2), when $\tau_1 \in (0, \tau_{10})$, if H_{51}, H_{52} hold, the equilibrium (P^*, Z_1^*, Z_2^*) is asymptotically stable when $\tau_2 \in (0, \tau_{20}^*)$, Hopf bifurcation occurs when $\tau_2 = \tau_{20}^*$.*

From the description of dynamics of system (1.2) under case (5), we could obtain

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\lambda=i\omega_{20}^*, \tau=\tau_{20}^*} > 0, \tag{2.1}$$

where

$$\tau_{2k}^{*(j)} = \frac{1}{\omega_{2k}^*} \arccos\left(\frac{R_{52}R_{53} - R_{51}R_{54}}{R_{51}^2 + R_{52}^2}\right) + \frac{2j\pi}{\omega_{2k}^*}, \tag{2.2}$$

$$\tau_{20}^* = \min \tau_{2k}^{*(0)}, \quad (k = 0, 1, 2, \dots), \tag{2.3}$$

the definition of $H_{21}, H_{22}, H_{51}, H_{52}, R_{51}, R_{52}, R_{53}, R_{54}$, and ω_{20}^* could be found in [1].

3 Global Hopf bifurcation

In appendix [1], assuming $\tau_1 < \tau_{20}^*$, we give the property of Hopf bifurcation for Lemma 2.2 by the center manifold theorem and normal form [18]. Based on the above fact, we study the global Hopf bifurcation under case (5) of system (1.2).

Let $X = C([-\tau_2, 0], R^3)$, $u_t(\theta) = (P(t + \theta), Z_1(t + \theta), Z_2(t + \theta)), t \geq 0, \theta \in [-\tau_2, 0]$, then system (1.2) becomes

$$\dot{u}_t = F(u_t, \tau_2, T), \tag{3.1}$$

$$F(\Phi, \tau_2, T) = \begin{pmatrix} r\phi_1(0)\left(1 - \frac{\phi_1(0)}{K}\right) - \frac{\mu_1\phi_1(0)\phi_2(0)}{\alpha_1 + \phi_1(0)} - \frac{\mu_2\phi_1(0)\phi_3(0)}{\alpha_2 + \phi_1(0)} \\ \frac{\beta_1\phi_1(-\tau_1)\phi_2(-\tau_1)}{\alpha_1 + \phi_1(-\tau_1)} - \frac{\rho_1\phi_1(0)\phi_2(0)}{\alpha_1 + \phi_1(0)} - d_1\phi_2(0) - g_1\phi_2^2(0) \\ \frac{\beta_2\phi_1(-\tau_2)\phi_3(-\tau_2)}{\alpha_2 + \phi_1(-\tau_2)} - \frac{\rho_2\phi_1(0)\phi_3(0)}{\alpha_2 + \phi_1(0)} - d_2\phi_3(0) - g_2\phi_3^2(0) \end{pmatrix} \tag{3.2}$$

with $\Phi = (\phi_1, \phi_2, \phi_3)$ and $F : X \times R_+ \times R_+ \rightarrow R_+^3$ is a mapping, we have a mapping $\hat{F} = F|_{R_+^3 \times R_+ \times R_+}$ by restricting F to the subspace of a constant function in X . If $\hat{F}(\hat{u}, \hat{\tau}_2, \hat{T}) = 0$,

then $(\hat{u}, \hat{\tau}_2, \hat{T})$ is called a stationary point, where

$$\hat{F}(u, \tau_2, T) = \begin{pmatrix} rP(1 - \frac{P}{K}) - \frac{\mu_1 P Z_1}{\alpha_1 + P} - \frac{\mu_2 P Z_2}{\alpha_2 + P} \\ \frac{\beta_1 P Z_1}{\alpha_1 + P} - \frac{\rho_1 P Z_1}{\alpha_1 + P} - d_1 Z_1 - g_1 Z_1^2 \\ \frac{\beta_2 P Z_2}{\alpha_2 + P} - \frac{\rho_2 P Z_2}{\alpha_2 + P} - d_2 Z_2 - g_2 Z_2^2 \end{pmatrix} \tag{3.3}$$

and

$$D_u \hat{F}(u, \tau_2, T) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= r - \frac{2rP}{K} - \frac{\mu_1 Z_1}{\alpha_1 + P} + \frac{\mu_1 P Z_1}{(\alpha_1 + P)^2} - \frac{\mu_2 Z_2}{\alpha_2 + P} + \frac{\mu_2 P Z_2}{(\alpha_2 + P)^2}, \\ a_{12} &= -\frac{\mu_1 P}{\alpha_1 + P}, \quad a_{13} = -\frac{\mu_2 P}{\alpha_2 + P}, \\ a_{21} &= \frac{(\beta_1 - \rho_1) Z_1}{\alpha_1 + P} - \frac{(\beta_1 - \rho_1) P Z_1}{(\alpha_1 + P)^2}, \quad a_{22} = \frac{(\beta_1 - \rho_1) P}{\alpha_1 + P} - d_1 - 2g_1 Z_1, \quad a_{23} = 0, \\ a_{31} &= \frac{(\beta_2 - \rho_2) Z_2}{\alpha_2 + P} - \frac{(\beta_2 - \rho_2) P Z_2}{(\alpha_2 + P)^2}, \quad a_{32} = 0, \quad a_{33} = \frac{(\beta_2 - \rho_2) P}{\alpha_2 + P} - d_2 - 2g_2 Z_2. \end{aligned}$$

F and \hat{F} satisfy the following conditions:

- (A₁) $\hat{F} \in C^2(R_+^3 \times R_+ \times R_+, R_+^3)$,
- (A₂) $D_u \hat{F}(\hat{u}, \tau_2, T)$ is an isomorphism at equilibrium (\hat{u}, τ_2, T) ,
- (A₃) $F(\Phi, \tau_2, T)$ is differentiable with respect to Φ .

For any stationary solution $(\hat{u}, \hat{\tau}_2, \hat{T})$, the characteristic matrix of system (1.2) is $\Delta(\hat{u}, \tau_2, T)(\lambda) = \lambda I - DF(\hat{u}, \tau_2, T)(e^{\lambda \cdot} I)$, that is,

$$\Delta(\hat{u}, \tau_2, T)(\lambda) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where

$$\begin{aligned} b_{11} &= \lambda - r + \frac{2r\hat{P}}{K} + \frac{\mu_1 \hat{Z}_1}{\alpha_1 + \hat{P}} - \frac{\mu_1 \hat{P} \hat{Z}_1}{(\alpha_1 + \hat{P})^2} + \frac{\mu_2 \hat{Z}_2}{\alpha_2 + \hat{P}} - \frac{\mu_2 \hat{P} \hat{Z}_2}{(\alpha_2 + \hat{P})^2}, \\ b_{12} &= \frac{\mu_1 \hat{P}}{\alpha_1 + \hat{P}}, \quad b_{13} = \frac{\mu_2 \hat{P}}{\alpha_2 + \hat{P}}, \\ b_{21} &= \frac{\rho_1 \hat{Z}_1}{\alpha_1 + \hat{P}} - \frac{\rho_1 \hat{P} \hat{Z}_1}{(\alpha_1 + \hat{P})^2} - \left(\frac{\beta_1 \hat{Z}_1}{\alpha_1 + \hat{P}} - \frac{\beta_1 \hat{P} \hat{Z}_1}{(\alpha_1 + \hat{P})^2} \right) e^{-\lambda \tau_1}, \\ b_{22} &= \lambda + d_1 + 2g_1 \hat{Z}_1 + \frac{\rho_1 \hat{P}}{\alpha_1 + \hat{P}} - \frac{\beta_1 \hat{P}}{\alpha_1 + \hat{P}} e^{-\lambda \tau_1}, \quad b_{23} = 0, \\ b_{31} &= \frac{\rho_2 \hat{Z}_2}{\alpha_2 + \hat{P}} - \frac{\rho_2 \hat{P} \hat{Z}_2}{(\alpha_2 + \hat{P})^2} - \left(\frac{\beta_2 \hat{Z}_2}{\alpha_2 + \hat{P}} - \frac{\beta_2 \hat{P} \hat{Z}_2}{(\alpha_2 + \hat{P})^2} \right) e^{-\lambda \tau_2}, \end{aligned}$$

$$b_{32} = 0, \quad b_{33} = \lambda + d_2 + 2g_2\hat{Z}_2 + \frac{\rho_2\hat{P}}{\alpha_2 + \hat{P}} - \frac{\beta_2\hat{P}}{\alpha_2 + \hat{P}}e^{-\lambda\tau_2}.$$

The zeros of $\det(\Delta(\hat{u}, \tau_2, T))(\lambda) = 0$ are called characteristic roots. From (A₂) we know that $\lambda = 0$ is not a characteristic root. $(\hat{u}, \hat{\tau}_2, \hat{T})$ is called a center if $\det(\Delta_{\hat{u}}(\text{im } \frac{2\pi}{T})) = 0$. A center $(\hat{u}, \hat{\tau}_2, \hat{T})$ is said to be isolated if it is the only center in some neighborhood of $(\hat{u}, \hat{\tau}_2, \hat{T})$ and it has a finite characteristic value of form $\text{im } \frac{2\pi}{T}$.

From Lemma 2.2 we conclude that $(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}), j = 0, 1, 2, \dots$, is an isolated center. There exists a smooth curve $\lambda : (\tau_{20}^{*(j)} - \delta, \tau_{20}^{*(j)} + \delta) \rightarrow \mathbb{C}$ such that $\det(\Delta(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}))(\lambda(\tau_2)) = 0, |\lambda(\tau_2) - i\omega_{20}^*| < \varepsilon$ for $\tau_2 \in (\tau_{20}^{*(j)} - \delta, \tau_{20}^{*(j)} + \delta), \lambda(\tau_{20}^{*(j)}) = i\omega_{20}^*$ and

$$\text{Re} \frac{d\lambda}{d\tau_2} \Big|_{\tau_2 = \tau_{20}^{*(j)}} > 0. \tag{3.4}$$

Define $\Omega_{\varepsilon, \frac{2\pi}{\omega_{20}^*}} = \{(v, T) : 0 < v < \varepsilon, |T - \frac{2\pi}{\omega_{20}^*}| < \varepsilon\}$.

(A₄) If and only if $v = 0, \tau_2 = \tau_{20}^{*(j)}, T = \frac{2\pi}{\omega_{20}^*}, j = 0, 1, 2, \dots, \det(\Delta(u^*, \tau_2, T)(v + i\frac{2\pi}{T})) = 0$, hypotheses (A₁)–(A₄) in [14] hold, we put $H^\pm(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})(v, T) = \det(\Delta(u^*, \tau_{20}^{*(j)} \pm \delta, \frac{2\pi}{\omega_{20}^*})(v + \frac{2\pi i}{T}))$.

The crossing number $\gamma(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ of center $(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ can be defined as follows:

$$\begin{aligned} \gamma\left(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}\right) &= \text{deg}_B\left(H^-\left(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}\right), \Omega_{(\varepsilon, \frac{2\pi}{\omega_{20}^*})}\right) \\ &\quad - \text{deg}_B\left(H^+\left(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}\right), \Omega_{(\varepsilon, \frac{2\pi}{\omega_{20}^*})}\right) = -1. \end{aligned}$$

For the periodic solution of (1.2), we define $\Sigma(F) = Cl((u, \tau_2, T) \in X \times R_+ \times R_+ | u \text{ is a } T\text{-periodic solution}) \cap N(F) = ((\hat{u}, \hat{\tau}_2, \hat{T}) \in R_+^5 | F(\hat{u}, \hat{\tau}_2, \hat{T}) = 0)$.

Let $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ denote the connected component of $(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ in $\Sigma(F)$ for $(\hat{u}, \hat{\tau}_2, \hat{T}) = (u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$, we obtain

$$\Sigma_{(\hat{u}, \hat{\tau}_2, \hat{T}) \in l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*}) \cap N(F)} \gamma(\hat{u}, \hat{\tau}_2, \hat{T}) < 0.$$

The connected component $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ through $(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ in $\Sigma(F)$ is not empty, since the first crossing number of each center is always -1 . By Theorem 3.3 [14], we conclude that $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ is unbounded, which is presented as follows.

Lemma 3.1 $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ is unbounded for each center $(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$.

By the fundamental theory [19], system (1.2) admits the existence and uniqueness of the solution with the initial condition

$$\begin{aligned} P(\theta) &= \phi(\theta) \geq 0, \\ Z_i(\theta) &= \psi_i(\theta) \geq 0, \quad \theta \in [-\tau_2, 0], \phi(0) > 0, \psi_i(0) > 0, (i = 1, 2). \end{aligned} \tag{3.5}$$

Lemma 3.2 (Lemma 2.1 [1]) *All the solutions of system (1.2) with initial condition (3.5) are positive and bounded on $[0, +\infty)$.*

In [1], we have demonstrated this theorem, so we omit here.

Lemma 3.3 *If system (1.2) has no nontrivial periodic solution with period τ_2 .*

Proof: For system (1.2), suppose that there is a nontrivial periodic solution with period τ_2 , then the following system has a nontrivial periodic solution:

$$\begin{cases} \frac{dP}{dt} = rP(1 - \frac{P}{K}) - \frac{\mu_1 P Z_1}{\alpha_1 + P} - \frac{\mu_2 P Z_2}{\alpha_2 + P}, \\ \frac{dZ_1}{dt} = \frac{\beta_1 P(t - \tau_1) Z_1(t - \tau_1)}{\alpha_1 + P(t - \tau_1)} - \frac{\rho_1 P Z_1}{\alpha_1 + P} - d_1 Z_1 - g_1 Z_1^2, \\ \frac{dZ_2}{dt} = \frac{\beta_2 P Z_2}{\alpha_2 + P} - \frac{\rho_2 P Z_2}{\alpha_2 + P} - d_2 Z_2 - g_2 Z_2^2. \end{cases} \tag{3.6}$$

We know that the $P - Z_1$ -plane and the $Z_1 - Z_2$ -plane are invariant manifolds for system (3.6), there are no orbits crossing these two planes for the orbits do not intersect each other. Let $t' = \inf\{t : Z_1(t) = 0\}$, combining with initial condition (3.5) and the second equation of (3.6), we get

$$\left. \frac{dZ_1}{dt} \right|_{t=t'} = \frac{\beta_1 P(t' - \tau_1) Z_1(t' - \tau_1)}{\alpha_1 + P(t' - \tau_1)} \geq 0,$$

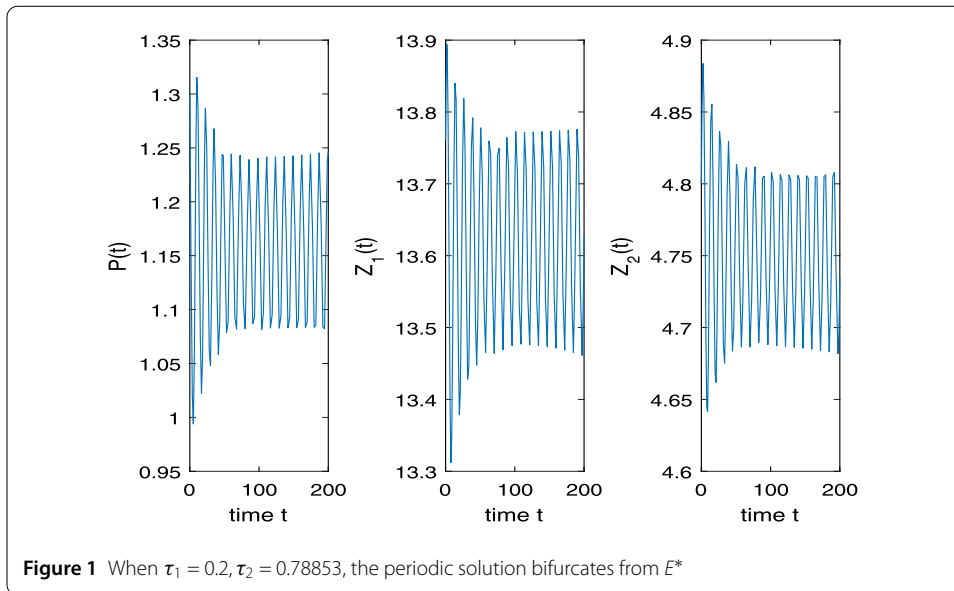
from which the orbits do not cross the $P - Z_2$ -plane, so all the orbits must lie in the first quadrant and equilibrium E^* should be in the interior of a periodic solution. From Lemma 2.1 ($\tau_1 \in (0, \tau_{10}), \tau_2 = 0$), E^* is stable and unique, so it is globally stable in the first quadrant, the periodic solution does not exist. Thus there is no periodic orbit in the first quadrant.

Theorem 3.1 *For each $\tau_2 > \tau_{20}^{*(j)}, j = 0, 1, 2, \dots$, system (1.2) has at least $j + 1$ periodic solutions.*

Proof It is sufficient to prove that the connected component $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ onto the τ_2 space is $(\bar{\tau}, +\infty)$, where $\bar{\tau} \leq \tau_{20}^{*(j)}, j = 0, 1, 2, \dots$. From Lemma 3.3, we know that system (3.6) does not have a nontrivial periodic solution, so the projection of $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ onto the τ_2 space is always from zeros. From expression (2.2), we get

$$\tau_{20}^{*(j)} = \frac{1}{\omega_{20}^*} \arccos\left(\frac{R_{52}R_{53} - R_{51}R_{54}}{R_{51}^2 + R_{52}^2}\right) + \frac{2j\pi}{\omega_{20}^*}, \tag{3.7}$$

so we obtain $\frac{2\pi}{\omega_{20}^*} < \tau_{20}^{*(j)}$ for $j > 0$. Suppose that the projection of $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ onto the τ_2 space is bounded, there exists $\tau^* > 0$, so the projection of $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ onto the τ_2 space is in the interval $(0, \tau^*)$. From $\frac{2\pi}{\omega_{20}^*} < \tau_{20}^{*(j)} (j \geq 1)$ and Lemma 3.3, $0 < T < \tau^*$ for $(u(t), \tau_2, T) \in l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$, which means that the projection of $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ onto the T space is also bounded. By Lemma 3.2, we obtain that the connected com-



ponent $l(u^*, \tau_{20}^{*(j)}, \frac{2\pi}{\omega_{20}^*})$ is bounded, which contradicts Lemma 3.1, so the proof is completed. □

4 Numerical simulation

We give some simulation to support our result. We take the same parameter to [1], system (1.2) becomes

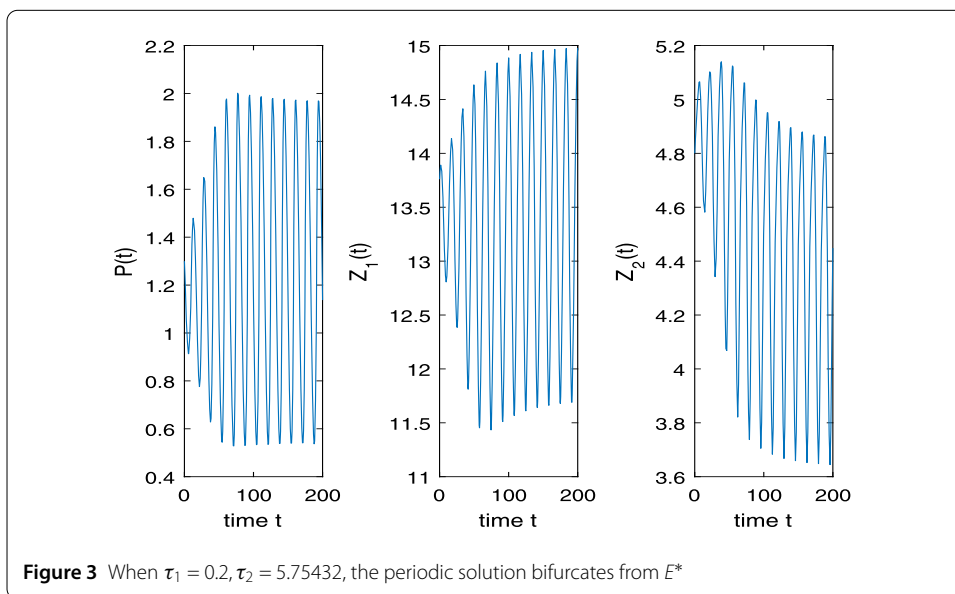
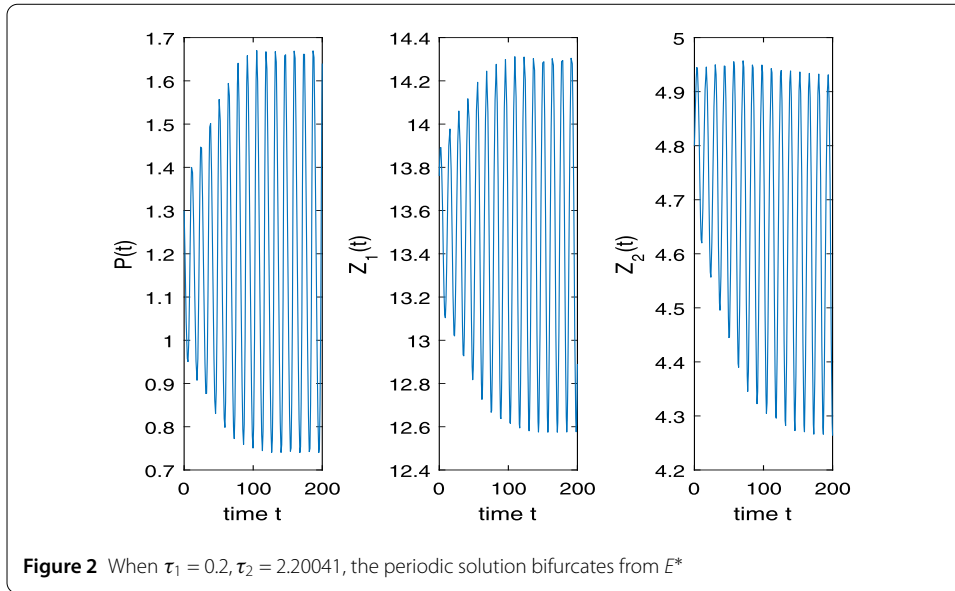
$$\begin{cases} \frac{dP}{dt} = 5P(1 - \frac{P}{4}) - \frac{0.6PZ_1}{2+P} - \frac{0.64PZ_1}{2+P}, \\ \frac{dZ_1}{dt} = \frac{0.43P(t-\tau_1)Z_1(t-\tau_1)}{2+P(t-\tau_1)} - \frac{0.1PZ_1}{2+P} - 0.11Z_1 - 0.0008Z_1^2, \\ \frac{dZ_2}{dt} = \frac{0.53P(t-\tau_2)Z_2(t-\tau_2)}{2+P(t-\tau_2)} - \frac{0.1PZ_2}{2+P} - 0.11Z_2 - 0.01Z_2^2. \end{cases} \tag{4.1}$$

From [1] we know that system (4.1) has a unique equilibrium $E^* = (1.15632, 13.62813, 4.75297)$. For case (2), $\tau_{10} = 0.79432$. For case (5), fixing $\tau_1 = 0.2$, then $\tau_{20}^* = 0.32688$, when $\tau_2 = 0.30115 < \tau_{20}^*$, E^* is asymptotically stable, when $\tau_2 = 0.94733 > \tau_{20}^*$, Hopf bifurcation occurs and a periodic solution bifurcates from E^* , which is illustrated in Figs. 7 and 8 in [1].

Furthermore, we know that system (1.2) has a periodic solution for large $\tau_2, (\tau_1 \in (0, \tau_{10}))$ from global Hopf bifurcation. Figures 1, 2, and 3 show the amplitude and period of a bifurcating periodic solution for different values of $\tau_2 (\tau_2 = 0.78853, 2.20041, 5.75432)$. We conclude that the amplitude and period increase as τ_2 increases.

5 Conclusion

In this paper, we study the global Hopf bifurcation of two zooplankton-phytoplankton system with two delays. Fixing the delay τ_1 in its stable interval and taking the delay τ_2 as a parameter, we demonstrate that the bifurcating periodic solution exists as τ_2 increases, where the delays τ_1, τ_2 are due to gestation of two zooplankton. We conclude the global oscillatory nature for the solution of a delayed system when it exhibits oscillatory behavior. Our work answers the question in Sect. 5 [1].



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Availability of data and materials

The data and material come from papers [1, 4]. We think these data and material satisfy the condition which turns up in the paper. So we think the data and material are available.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by RS and WY. RS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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