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A new fourth-order explicit group method in the solution of two-dimensional fractional Rayleigh–Stokes problem for a heated generalized second-grade fluid

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Abstract

In this article, a new explicit group iterative scheme is developed for the solution of two-dimensional fractional Rayleigh–Stokes problem for a heated generalized second-grade fluid. The proposed scheme is based on the high-order compact Crank–Nicolson finite difference method. The resulting scheme consists of three-level finite difference approximations. The stability and convergence of the proposed method are studied using the matrix energy method. Finally, some numerical examples are provided to show the accuracy of the proposed method.

Keywords: Two-dimensional fractional Rayleigh–Stokes problem; Finite difference method; Explicit group method; Crank–Nicolson high-order; Stability and convergence

1 Introduction

The fractional calculus has gained attention because of its application in engineering, physics, and chemistry [1–5]. Fractional differential equations represent more complex models, but mostly it is difficult to solve them analytically. Therefore different researchers are looking for numerical methods, e.g., finite element method, spectral method, and finite difference method, to find the solution to these fractional differential equations [6–22]. The finite difference method is relatively simple and easy; that is why it has been seen more in the literature for the solution of fractional differential equations.

In this paper, we consider two dimensional (2D) Rayleigh–Stokes problem for a heated generalized second-grade fluid with fractional derivative and a nonhomogeneous term of the form:

$$\begin{aligned}\frac{\partial w(x,y,t)}{\partial t} &= {}_0\mathrm{D}_t^{1-\gamma}\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right)\\ &\quad + \frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} + f(x,y,t) \qquad (1)\end{aligned}$$

with initial and boundary conditions

$$\begin{aligned} &w(x,y,t) = g(x,y,t), \quad (x,y) \in \partial\Omega, \\ &w(x,y,0) = h(x,y), \quad (x,y) \in \Omega, \end{aligned} \tag{2}$$

where $0 < \gamma < 1$, $\Omega = \{(x,y)|0 \leq x \leq L, 0 \leq y \leq L\}$.

The Rayleigh–Stokes problem has gained attention in recent years. This problem plays a vital role to show the dynamic behavior of some non-Newtonian fluids, and the fractional derivative in this model is used to capture the viscoelastic behavior of the flow [23, 24].

Several numerical methods are presented in the literature for the solution of fractional Rayleigh–Stokes problem, for example, Chen et al. [25] have solved the problem using explicit and implicit finite difference methods, they have also presented its stability and convergence using Fourier analysis. The convergence order for both schemes is $O(\tau + \Delta x^2 + \Delta y^2)$. Ramy et al. [26] solved Rayleigh–Stokes problem using Jacobi spectral Galerkin method. The method they derived is efficient and easily generalizes to multiple dimensions. The advantages of this method are reasonable accuracy and relatively fewer degrees of freedom. Mohebbi et al. [27] used a higher-order implicit finite difference scheme for two-dimensional Rayleigh–Stokes problem and discussed its convergence and stability by Fourier analysis. The convergence order of their scheme is shown to be $O(\tau + \Delta x^4 + \Delta y^4)$.

High-order schemes produce more accurate results, but suffer from slow convergence due to the increase of complexity in the algorithm. Since explicit group methods reduce algorithm complexity [28–31], we propose the use of explicit group method for the solution of two-dimensional Rayleigh–Stokes problem for a heated generalized second-grade fluid. The main purpose of this article is to solve two-dimensional Rayleigh–Stokes problem with the high-order explicit group method (HEGM).

The paper is arranged as follows; in Sect. 2, we give the formulation of the high-order compact explicit group scheme, and its stability is discussed in Sect. 3. In Sect. 4, the convergence of the proposed scheme is discussed. In Sect. 5, some numerical examples are presented with discussion, and finally, the conclusion is presented in Sect. 6.

2 The group explicit scheme

First, let us define the following notations:

$$\begin{aligned} &\delta_x^2 w_{i,j}^k = w_{i+1,j}^k - 2w_{i,j}^k + w_{i-1,j}^k, \qquad \delta_y^2 w_{i,j}^k = w_{i,j+1}^k - 2w_{i,j}^k + w_{i,j-1}^k, \\ &w_{i,j}^{k+\frac{1}{2}} = \frac{w_{i,j}^{k+1} + w_{i,j}^k}{2}, \qquad x_i = i\Delta x, y_j = j\Delta y, \quad \{i,j = 0,1,2,3,\ldots,M\}, \\ &t_k = k\tau, \quad \{k = 0,1,2,3,\ldots,N\}, \end{aligned}$$

where $\Delta x = \Delta y = h = \frac{L}{M}$ which represent the space step and $\Delta t = \frac{T}{N}$ represents the time step. The operators δ_x^2 and δ_y^2, which consist of the three-point stencil [32], satisfy

$$\frac{\delta_x^2}{h^2(1 + \frac{1}{12}\delta_x^2)} w_{i,j}^k = \frac{\partial^2 w}{\partial x^2}\bigg|_{i,j}^k - \frac{1}{240}\frac{\partial^4 w}{\partial x^4}\bigg|_{i,j}^k + O(h^6) \tag{3}$$

and

$$\frac{\delta_y^2}{h^2(1+\frac{1}{12}\delta_y^2)}w_{i,j}^k = \frac{\partial^2 w}{\partial y^2}\bigg|_{i,j}^k - \frac{1}{240}\frac{\partial^4 w}{\partial y^4}\bigg|_{i,j}^k + O(h^6). \tag{4}$$

The relationship between the Grunwald–Letnikov and Riemann–Liouville fractional derivatives is defined as [27, 33]

$${}_0D_l^{1-\gamma}f(t) = \frac{1}{\tau^{1-\gamma}}\sum_{k=0}^{[\frac{t}{\tau}]}\omega_k^{1-\gamma}f(t-k\tau) + O(\tau^p), \tag{5}$$

where $\omega_k^{1-\gamma}$ are the coefficients of the generating function, that is, $\omega(z,\gamma) = \sum_{k=0}^{\infty}\omega_k^{\gamma}z^k$. We consider $\omega(z,\gamma) = (1-z)^{\gamma}$ for $p = 1$, so the coefficients are $\omega_0^{\gamma} = 1$ and

$$\begin{aligned}\omega_k^{\gamma} &= (-1)^k\binom{\gamma}{k} = (-1)^k\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!} \\ &= \left(1-\frac{2-\gamma}{k}\right)\omega_{k-1}^{\gamma}, \quad k \geq 1.\end{aligned} \tag{6}$$

Let $\eta_l = \omega_l^{1-\gamma}$, then

$$\eta_0 = 1 \quad \text{and} \quad \eta_l = (-1)^l\binom{1-\gamma}{l} = \left(1-\frac{2-\gamma}{k}\right)\eta_{l-1}, \quad k \geq 1.$$

From (5) we can obtain the following:

$${}_0D_t^{1-\gamma}\frac{\partial^2 w(x,y,t)}{\partial x^2} = \tau^{\gamma-1}\sum_{l=0}^{[\frac{t}{\tau}]}\eta_l\frac{\partial^2 w(x,y,t-l\tau)}{\partial x^2} + O(\tau^p), \tag{7}$$

$${}_0D_t^{1-\gamma}\frac{\partial^2 w(x,y,t)}{\partial y^2} = \tau^{\gamma-1}\sum_{l=0}^{[\frac{t}{\tau}]}\eta_l\frac{\partial^2 w(x,y,t-l\tau)}{\partial y^2} + O(\tau^p). \tag{8}$$

Using (3), (4), (7), (8), and (1), we have

$$\begin{aligned}\frac{w_{i,j}^{k+1}-w_{i,j}^k}{\tau} &= \tau^{1-\gamma}\left(\sum_{l=0}^{k}\eta_l\frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2)} + \sum_{l=0}^{k}\eta_l\frac{\delta_y^2}{h^2(1+\frac{1}{12}\delta_y^2)}\right)w_{i,j}^{k-l+\frac{1}{2}} \\ &\quad + \frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2)}w_{i,j}^{k+\frac{1}{2}} + \frac{\delta_y^2}{h^2(1+\frac{1}{12}\delta_y^2)}w_{i,j}^{k+\frac{1}{2}} + f_{i,j}^{k+\frac{1}{2}}.\end{aligned} \tag{9}$$

Multiplying both sides by $\tau(1+\frac{1}{12}\delta_x^2)(1+\frac{1}{12}\delta_y^2)$, we have

$$\begin{aligned}\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)\left(w_{i,j}^{k+1}-w_{i,j}^k\right) &= \frac{\tau^{2-\gamma}}{2h^2}\sum_{l=0}^{k+1}\eta_l\left(\delta_x^2+\delta_y^2+\frac{\delta_x^2\delta_y^2}{6}\right)w_{i,j}^{k+1-l} \\ &\quad + \frac{\tau^{2-\gamma}}{2h^2}\sum_{l=0}^{k}\eta_l\left(\delta_x^2+\delta_y^2+\frac{\delta_x^2\delta_y^2}{6}\right)w_{i,j}^{k-l}\end{aligned}$$

$$+\frac{\tau}{2h^2}\left(\delta_x^2+\delta_y^2+\frac{\delta_x^2\delta_y^2}{6}\right)\left(w_{i,j}^{k+1}+w_{i,j}^{k}\right)$$
$$+\tau\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)f_{i,j}^{k+\frac{1}{2}}.$$

After simplifying and rearranging, we get Crank–Nicoslon (C–N) high-order compact scheme

$$\begin{aligned}
a_1w_{i,j}^{k+1} &= a_2\left(w_{i+1,j}^{k+1}+w_{i-1,j}^{k+1}+w_{i,j+1}^{k+1}+w_{i,j-1}^{k+1}\right)+a_3\left(w_{i+1,j+1}^{k+1}+w_{i-1,j+1}^{k+1}\right.\\
&\quad\left.+w_{i+1,j-1}^{k+1}+w_{i-1,j-1}^{k+1}\right)+a_4w_{i,j}^{k}+a_5\left(w_{i+1,j}^{k}+w_{i-1,j}^{k}+w_{i,j+1}^{k}+w_{i,j-1}^{k}\right)\\
&\quad+a_6\left(w_{i+1,j+1}^{k}+w_{i-1,j+1}^{k}+w_{i+1,j-1}^{k}+w_{i-1,j-1}^{k}\right)+\frac{25\tau}{36}f_{i,j}^{k+\frac{1}{2}}+\frac{5\tau}{72}\left(f_{i+1,j}^{k+\frac{1}{2}}\right.\\
&\quad\left.+f_{i-1,j}^{k+\frac{1}{2}}+f_{i,j+1}^{k+\frac{1}{2}}+f_{i,j-1}^{k+\frac{1}{2}}\right)+\frac{\tau}{144}\left(f_{i+1,j+1}^{k+\frac{1}{2}}+f_{i-1,j+1}^{k+\frac{1}{2}}+f_{i+1,j-1}^{k+\frac{1}{2}}+f_{i-1,j-1}^{k+\frac{1}{2}}\right)\\
&\quad+S_1\left[\sum_{l=2}^{k+1}\eta_l\left(\frac{-10}{3}w_{i,j}^{k+1-l}+\frac{2}{3}\left(w_{i+1,j}^{k+1-l}+w_{i-1,j}^{k+1-l}+w_{i,j+1}^{k+1-l}+w_{i,j-1}^{k+1-l}\right)\right)\right]\\
&\quad+\frac{S_1}{6}\left(w_{i+1,j+1}^{k+1-l}+w_{i-1,j+1}^{k+1-l}+w_{i+1,j-1}^{k+1-l}+w_{i-1,j-1}^{k+1-l}\right)+S_1\left[\sum_{l=1}^{k}\eta_l\left(\frac{-10}{3}w_{i,j}^{k-l}\right.\right.\\
&\quad\left.\left.+\frac{2}{3}\left(w_{i+1,j}^{k-l}+w_{i-1,j}^{k-l}+w_{i,j+1}^{k-l}+w_{i,j-1}^{k-l}\right)\right)\right]+\frac{S_1}{6}\left(w_{i+1,j+1}^{k-l}+w_{i-1,j+1}^{k-l}\right.\\
&\quad\left.+w_{i+1,j-1}^{k-l}+w_{i-1,j-1}^{k-l}\right)+O\left(\tau+h^4\right),
\end{aligned}\tag{10}$$

where

$$\begin{aligned}
&S_1=\frac{\tau^{\gamma}}{2h^2},\qquad S_2=\frac{\tau}{2h^2},\qquad H=S_1+S_2,\\
&a_1=\frac{5}{36}(5+24H),\qquad a_2=\frac{1}{144}(-10+96H),\qquad a_3=\frac{1}{144}(-1+24H),\\
&a_4=\frac{1}{144}\left(100-480(H+S_1\eta_1)\right),\qquad a_5=\frac{1}{144}\left(10+96(H+S_1\eta_1)\right),\\
&a_6=\frac{1}{144}\left(1+24(H+S_1\eta_1)\right).
\end{aligned}$$

Applying (8) to the group of four points (as shown in Fig. 1) will result in the following 4×4 system:

$$\begin{bmatrix} a_1 & -a_2 & -a_3 & -a_2\\ -a_2 & a_1 & -a_2 & -a_3\\ -a_3 & -a_2 & a_1 & -a_2\\ -a_2 & -a_3 & -a_2 & a_1\end{bmatrix}\begin{bmatrix} w_{i,j}\\ w_{i+1,j}\\ w_{i+1,j+1}\\ w_{i,j+1}\end{bmatrix}=\begin{bmatrix} rhs_{i,j}\\ rhs_{i+1,j}\\ rhs_{i+1,j+1}\\ rhs_{i,j+1}\end{bmatrix},\tag{11}$$

where

$$\begin{aligned}
rhs_{i,j} &= a_2\left(w_{i-1,j}^{k+1}+w_{i,j-1}^{k+1}\right)+a_3\left(w_{i-1,j+1}^{k+1}+w_{i+1,j-1}^{k+1}+w_{i-1,j-1}^{k+1}\right)+a_4w_{i,j}^{k}\\
&\quad+a_5\left(w_{i+1,j}^{k}+w_{i-1,j}^{k}+w_{i,j+1}^{k}+w_{i,j-1}^{k}\right)+a_6\left(w_{i+1,j+1}^{k}+w_{i-1,j+1}^{k}\right.
\end{aligned}$$

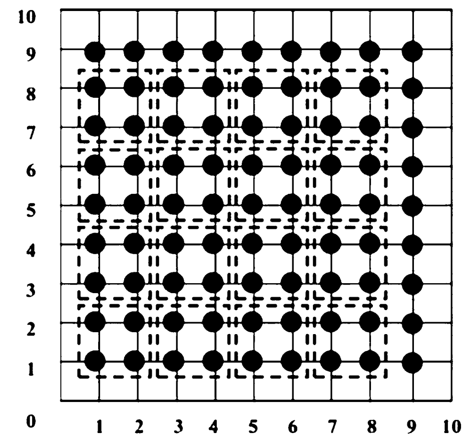


Figure 1 Groups of four points for HEGM

$$
\begin{aligned}
&+ w^{k}_{i+1,j-1} + w^{k}_{i-1,j-1}\big) + \frac{25}{36}\tau f^{k+\frac{1}{2}}_{i,j} + \frac{5}{72}\tau\big(f^{k+\frac{1}{2}}_{i+1,j} + f^{k+\frac{1}{2}}_{i-1,j} + f^{k+\frac{1}{2}}_{i,j+1} \\
&+ f^{k+\frac{1}{2}}_{i,j-1}\big) + \frac{\tau}{144}\big(f^{k+\frac{1}{2}}_{i+1,j+1} + f^{k+\frac{1}{2}}_{i-1,j+1} + f^{k+\frac{1}{2}}_{i+1,j-1} + f^{k+\frac{1}{2}}_{i-1,j-1}\big) \\
&+ s_1 \sum_{l=2}^{k+1} \lambda_l \left(\frac{-10}{3} w^{k+1-l}_{i,j} + \frac{2}{3}\big(w^{k+1-l}_{i+1,j} + w^{k+1-l}_{i-1,j} + w^{k+1-l}_{i,j+1}\right. \\
&\left. + w^{k+1-l}_{i,j-1}\big) + \frac{1}{6}\big(w^{k+1-l}_{i+1,j+1} + w^{k+1-l}_{i-1,j+1} + w^{k+1-l}_{i+1,j-1} + w^{k+1-l}_{i-1,j-1}\big)\right) \\
&+ + s_1 \sum_{l=1}^{k} \lambda_l \left(\frac{-10}{3} w^{k-l}_{i,j} + \frac{2}{3}\big(w^{k-l}_{i+1,j} + w^{k-l}_{i-1,j} + w^{k-l}_{i,j+1} + w^{k-l}_{i,j-1}\big)\right. \\
&\left. + \frac{1}{6}\big(w^{k-l}_{i+1,j+1} + w^{k-l}_{i-1,j+1} + w^{k-l}_{i+1,j-1} + w^{k-l}_{i-1,j-1}\big)\right), \\
rhs_{i+1,j} &= a_2\big(w^{k+1}_{i+2,j} + w^{k+1}_{i+1,j-1}\big) + a_3\big(w^{k+1}_{i+2,j+1} + w^{k+1}_{i+2,j-1} + w^{k+1}_{i,j-1}\big) + a_4 w^{k}_{i+1,j} \\
&+ a_5\big(w^{k}_{i+2,j} + w^{k}_{i,j} + w^{k}_{i+1,j+1} + w^{k}_{i+1,j-1}\big) + a_6\big(w^{k}_{i+2,j+1} + w^{k}_{i,j+1} \\
&+ w^{k}_{i+2,j-1} + w^{k}_{i,j-1}\big) + \frac{25}{36}\tau f^{k+\frac{1}{2}}_{i+1,j} + \frac{5}{72}\tau\big(f^{k+\frac{1}{2}}_{i+2,j} + f^{k+\frac{1}{2}}_{i,j} + f^{k+\frac{1}{2}}_{i+1,j+1} \\
&+ f^{k+\frac{1}{2}}_{i+1,j-1}\big) + \frac{\tau}{144}\big(f^{k+\frac{1}{2}}_{i+2,j+1} + f^{k+\frac{1}{2}}_{i,j+1} + f^{k+\frac{1}{2}}_{i+2,j-1} + f^{k+\frac{1}{2}}_{i,j-1}\big) \\
&+ s_1 \sum_{l=2}^{k+1} \lambda_l \left(\frac{-10}{3} w^{k+1-l}_{i+1,j} + \frac{2}{3}\big(w^{k+1-l}_{i+2,j} + w^{k+1-l}_{i,j} + w^{k+1-l}_{i+1,j+1}\right. \\
&\left. + w^{k+1-l}_{i+1,j-1}\big) + \frac{1}{6}\big(w^{k+1-l}_{i+2,j+1} + w^{k+1-l}_{i,j+1} + w^{k+1-l}_{i+2,j-1} + w^{k+1-l}_{i,j-1}\big)\right) \\
&+ s_1 \sum_{l=1}^{k} \lambda_l \left(\frac{-10}{3} w^{k-l}_{i+1,j} + \frac{2}{3}\big(w^{k-l}_{i+2,j} + w^{k-l}_{i,j} + w^{k-l}_{i+1,j+1} + w^{k-l}_{i+1,j-1}\big)\right. \\
&\left. + \frac{1}{6}\big(w^{k-l}_{i+2,j+1} + w^{k-l}_{i,j+1} + w^{k-l}_{i+2,j-1} + w^{k-l}_{i,j-1}\big)\right), \\
rhs_{i+1,j+1} &= a_2\big(w^{k+1}_{i+2,j+1} + w^{k+1}_{i+1,j+2}\big) + a_3\big(w^{k+1}_{i+2,j+2} + w^{k+1}_{i,j+2} + w^{k+1}_{i+2,j}\big) + a_4 w^{k}_{i+1,j+1} \\
&+ a_5\big(w^{k}_{i+2,j+1} + w^{k}_{i,j+1} + w^{k}_{i+1,j+2} + w^{k}_{i+1,j}\big) + a_6\big(w^{k}_{i+2,j+2} + w^{k}_{i,j+2}
\end{aligned}
$$

$$
\begin{aligned}
&+ w^{k}_{i+2,j} + w^{k}_{i,j}\big) + \frac{25}{36}\tau f^{k+\frac{1}{2}}_{i+1,j+1} + \frac{5}{72}\tau\big(f^{k+\frac{1}{2}}_{i+2,j+1} + f^{k+\frac{1}{2}}_{i,j+1} + f^{k+\frac{1}{2}}_{i+1,j+2} \\
&+ f^{k+\frac{1}{2}}_{i+1,j}\big) + \frac{\tau}{144}\big(f^{k+\frac{1}{2}}_{i+2,j+2} + f^{k+\frac{1}{2}}_{i,j+2} + f^{k+\frac{1}{2}}_{i+2,j} + f^{k+\frac{1}{2}}_{i,j}\big) \\
&+ s_1\sum_{l=2}^{k+1}\lambda_l\bigg(\frac{-10}{3}w^{k+1-l}_{i+1,j+1} + \frac{2}{3}\big(w^{k+1-l}_{i+2,j+1} + w^{k+1-l}_{i,j+1} + w^{k+1-l}_{i+1,j+2} \\
&+ w^{k+1-l}_{i+1,j}\big) + \frac{1}{6}\big(w^{k+1-l}_{i+2,j+2} + w^{k+1-l}_{i,j+2} + w^{k+1-l}_{i+2,j} + w^{k+1-l}_{i,j}\big)\bigg) \\
&+ + s_1\sum_{l=1}^{k}\lambda_l\bigg(\frac{-10}{3}w^{k-l}_{i+1,j+1} + \frac{2}{3}\big(w^{k-l}_{i+2,j+1} + w^{k-l}_{i,j+1} + w^{k-l}_{i+1,j+2} + w^{k-l}_{i+1,j}\big) \\
&+ \frac{1}{6}\big(w^{k-l}_{i+2,j+2} + w^{k-l}_{i,j+2} + w^{k-l}_{i+2,j} + w^{k-l}_{i,j}\big)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
rhs_{i,j+1} = {}& a_2\big(w^{k+1}_{i-1,j+1} + w^{k+1}_{i,j+2}\big) + a_3\big(w^{k+1}_{i-1,j+2} + w^{k+1}_{i+1,j+2} + w^{k+1}_{i-1,j}\big) + a_4 w^{k}_{i,j+1} \\
&+ a_5\big(w^{k}_{i+1,j+1} + w^{k}_{i-1,j+1} + w^{k}_{i,j+2} + w^{k}_{i,j}\big) + a_6\big(w^{k}_{i+1,j+2} + w^{k}_{i-1,j+2} \\
&+ w^{k}_{i+1,j} + w^{k}_{i-1,j}\big) + \frac{25}{36}\tau f^{k+\frac{1}{2}}_{i,j+1} + \frac{5}{72}\tau\big(f^{k+\frac{1}{2}}_{i+1,j+1} + f^{k+\frac{1}{2}}_{i-1,j+1} + f^{k+\frac{1}{2}}_{i,j+2} \\
&+ f^{k+\frac{1}{2}}_{i,j}\big) + \frac{\tau}{144}\big(f^{k+\frac{1}{2}}_{i+1,j+2} + f^{k+\frac{1}{2}}_{i-1,j+2} + f^{k+\frac{1}{2}}_{i+1,j} + f^{k+\frac{1}{2}}_{i-1,j}\big) \\
&+ s_1\sum_{l=2}^{k+1}\lambda_l\bigg(\frac{-10}{3}w^{k+1-l}_{i,j+1} + \frac{2}{3}\big(w^{k+1-l}_{i+1,j+1} + w^{k+1-l}_{i-1,j+1} + w^{k+1-l}_{i,j+2} \\
&+ w^{k+1-l}_{i,j}\big) + \frac{1}{6}\big(w^{k+1-l}_{i+1,j+2} + w^{k+1-l}_{i-1,j+2} + w^{k+1-l}_{i+1,j} + w^{k+1-l}_{i-1,j}\big)\bigg) \\
&+ s_1\sum_{l=1}^{k}\lambda_l\bigg(\frac{-10}{3}w^{k-l}_{i,j+1} + \frac{2}{3}\big(w^{k-l}_{i+1,j+1} + w^{k-l}_{i-1,j+1} + w^{k-l}_{i,j+2} + w^{k-l}_{i,j}\big) \\
&+ \frac{1}{6}\big(w^{k-l}_{i+1,j+2} + w^{k-l}_{i-1,j+2} + w^{k-l}_{i+1,j} + w^{k-l}_{i-1,j}\big)\bigg).
\end{aligned}
$$

The matrix (9) is inverted to get the high-order compact explicit group equation

$$
\begin{bmatrix} w_{i,j} \\ w_{i+1,j} \\ w_{i+1,j+1} \\ w_{i,j+1} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_2 \\ \phi_2 & \phi_1 & \phi_2 & \phi_3 \\ \phi_3 & \phi_2 & \phi_1 & \phi_2 \\ \phi_2 & \phi_3 & \phi_2 & \phi_1 \end{bmatrix}\begin{bmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{bmatrix}, \tag{12}
$$

where

$$
\begin{aligned}
&\phi_1 = a_1^3 - 2a_1a_2^2 - 2a_2^2a_3 - a_1a_3^2, \qquad \phi_2 = a_1^2a_2 + 2a_1a_2a_3 + a_2a_3^2, \\
&\phi_3 = 2a_1a_2^2 + a_1^2a_3 + 2a_2^2a_3 - a_3^3, \qquad d = \big(-4a_2^2 + (a_1-a_3)^2\big)(a_1+a_3)^2.
\end{aligned}
$$

Figure 1 shows grid points on the x–y plane with mesh size $m = 10$, where the groups of four points are computed using (10) and the remaining points are computed using (8).

3 Stability of the proposed method

First we recall the following lemma.

Lemma 1 ([34]) *The coefficients η_l satisfy the following relations*:

$$
\begin{aligned}
&(1)\quad \eta_0 = 1, \qquad \eta_1 = \gamma - 1, \qquad \eta_l < 0, \quad l = 1, 2, \ldots, \\
&(2)\quad \sum_{l=0}^{\infty} \eta_l = 0, \quad \forall n \in N, \quad -\sum_{l=1}^{n} \eta_l < 1.
\end{aligned}
$$

The stability of the proposed method is analyzed using the matrix analysis method. Form (9), *we obtain*

$$
\begin{aligned}
&M_1 w^1 = N_1 w^0 + \tau P_1\bigl(F^{\frac{1}{2}}\bigr), \quad k = 0, \\
&M_1 w^{k+1} = N_1 w^k + s_1 \sum_{l=2}^{k+1} \lambda_l P_1\bigl(w^{k+1-l}\bigr) \\
&\qquad + s_1 \sum_{l=1}^{k} \lambda_l P_1\bigl(w^{k-l}\bigr) + \tau P_1\bigl(F^{k+\frac{1}{2}}\bigr), \quad k > 0,
\end{aligned}
\tag{13}
$$

$$
M_1 = \begin{bmatrix} R_1 & R_3 & & & 0 \\ R_2 & R_1 & R_3 & & \\ & R_2 & R_1 & & \\ & & & \ddots & R_3 \\ 0 & & & R_2 & R_1 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} P_1 & P_3 & & & 0 \\ P_2 & P_1 & P_3 & & \\ & P_2 & P_1 & & \\ & & & \ddots & P_3 \\ 0 & & & P_2 & P_1 \end{bmatrix},
$$

$$
P_1 = \begin{bmatrix} Q_1 & Q_3 & & & 0 \\ Q_2 & Q_1 & Q_3 & & \\ & Q_2 & Q_1 & & \\ & & & \ddots & Q_3 \\ 0 & & & Q_2 & Q_1 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} G_1 & G_3 & & & \\ G_2 & G_1 & G_3 & & \\ & G_2 & G_1 & & \\ & & & \ddots & G_3 \\ & & & G_2 & G_1 \end{bmatrix},
$$

$$
R_2 = \begin{bmatrix} G_6 & G_4 & & & \\ G_8 & G_6 & G_4 & & \\ & G_8 & G_6 & & \\ & & & \ddots & G_4 \\ & & & G_8 & G_6 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} G_7 & G_9 & & & \\ G_5 & G_7 & G_9 & & \\ & G_5 & G_7 & & \\ & & & \ddots & G_9 \\ & & & G_5 & G_7 \end{bmatrix},
$$

$$
P_1 = \begin{bmatrix} H_1 & H_3 & & & \\ H_2 & H_1 & H_3 & & \\ & H_2 & H_1 & & \\ & & & \ddots & H_3 \\ & & & H_2 & H_1 \end{bmatrix}, \qquad p_2 = \begin{bmatrix} H_6 & H_4 & & & \\ H_8 & H_6 & H_4 & & \\ & H_8 & H_6 & & \\ & & & \ddots & H_4 \\ & & & H_8 & H_6 \end{bmatrix},
$$

$$P_3 = \begin{bmatrix} H_7 & H_9 & & & \\ H_5 & H_7 & H_9 & & \\ & H_5 & H_7 & & \\ & & & \ddots & H_9 \\ & & & H_5 & H_7 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} L_1 & L_3 & & & \\ L_2 & L_1 & L_3 & & \\ & L_2 & L_1 & & \\ & & & \ddots & L_3 \\ & & & L_2 & L_1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} L_6 & L_4 & & & \\ L_8 & L_6 & L_4 & & \\ & L_8 & L_6 & & \\ & & & \ddots & L_4 \\ & & & L_8 & L_6 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} L_7 & L_9 & & & \\ L_5 & L_7 & L_9 & & \\ & L_5 & L_7 & & \\ & & & \ddots & L_9 \\ & & & L_5 & L_7 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_2 \\ -a_2 & a_1 & -a_2 & -a_3 \\ -a_3 & -a_2 & a_1 & -a_2 \\ -a_2 & -a_3 & -a_2 & a_1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0 & 0 & -a_3 & -a_2 \\ 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_3 & -a_2 & 0 & 0 \\ -a_2 & -a_3 & 0 & 0 \end{bmatrix}, \qquad G_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & 0 \end{bmatrix},$$

$$G_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_6 = \begin{bmatrix} 0 & -a_2 & -a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -a_3 & -a_2 & 0 \end{bmatrix},$$

$$G_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & -a_3 \\ -a_3 & 0 & 0 & -a_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_8 = \begin{bmatrix} 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_9 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} a_4 & a_5 & a_6 & a_5 \\ a_5 & a_4 & a_5 & a_6 \\ a_6 & a_5 & a_4 & a_5 \\ a_5 & a_6 & a_5 & a_4 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 & a_6 & a_5 \\ 0 & 0 & a_5 & a_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_6 & a_5 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \end{bmatrix},$$

$$H_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_6 = \begin{bmatrix} 0 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & 0 \end{bmatrix},$$

$$H_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & a_6 \\ a_6 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_8 = \begin{bmatrix} 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_9 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_1 = \frac{1}{3}\begin{bmatrix} -10 & 2 & \frac{1}{2} & 2 \\ 2 & -10 & 2 & \frac{1}{2} \\ \frac{1}{2} & 2 & -10 & 2 \\ 2 & \frac{1}{2} & 2 & -10 \end{bmatrix}, \qquad L_2 = \frac{1}{3}\begin{bmatrix} 0 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_3 = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 2 & 0 & 0 \\ 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad L_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \end{bmatrix},$$

$$L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L_6 = \frac{1}{3}\begin{bmatrix} 0 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & 0 \end{bmatrix},$$

$$L_7 = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L_8 = \begin{bmatrix} 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L_9 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 1 *The high-order explicit group scheme* (12) *is unconditionally stable.*

Proof Let $w_{i,j}^k$ and $W_{i,j}^k$ be the approximate and exact solutions, respectively, for (1), and let $\epsilon_{i,j}^k = W_{i,j}^k - w_{i,j}^k$ denote the error at time level k. Then from (11),

$$\begin{aligned} ME^1 &= NE^0 + \tau P_1\bigl(F^{\frac{1}{2}}\bigr), \quad k = 0, \\ ME^{k+1} &= NE^k + s_1 \sum_{l=2}^{k+1} \lambda_l P_1\bigl(E^{k+1-l}\bigr) + s_1 \sum_{l=1}^{k} \lambda_l P_1\bigl(E^{k+1-l}\bigr) + \tau P_1\bigl(F^{k+\frac{1}{2}}\bigr), \quad k > 0, \end{aligned} \tag{14}$$

where

$$E^{k+1} = \begin{bmatrix} E_1^{k+1} \\ E_1^{k+1} \\ \vdots \\ E_1^{k+1} \\ E_1^{k+1} \end{bmatrix}, \qquad E_1^{k+1} = \begin{bmatrix} \epsilon_1^{k+1} \\ \epsilon_2^{k+1} \\ \vdots \\ \epsilon_{m-2}^{k+1} \\ \epsilon_{m-1}^{k+1} \end{bmatrix}, \qquad \epsilon_i^{k+1} = \begin{bmatrix} \epsilon_{i,j}^{k+1} \\ \epsilon_{i+1,j}^{k+1} \\ \epsilon_{i+1,j+1}^{k+1} \\ \epsilon_{i,j+1}^{k+1} \end{bmatrix}, \quad i = 1, 2, \ldots, m-1,$$

$$j = 1, 2, \ldots, m-1.$$

From (11) we know

$$\begin{aligned}
M_1 &= G_1 I + (G_2 + G_3)E + G_6 I + (G_4 + G_8)E + G_7 I + (G_5 + G_9)E, \\
N_1 &= H_1 I + (H_2 + H_3)E + H_6 I + (H_4 + H_8)E + H_7 I + (H_5 + H_9)E, \\
P_1 &= L_1 I + (L_2 + L_3)E + L_6 I + (L_4 + L_8)E + L_7 I + (L_5 + L_9)E,
\end{aligned} \tag{15}$$

where I is the identity matrix and E is the matrix with unity values along each diagonal immediately below and above the main diagonal. Let ρ_1, ρ_2, and ρ_3 represent the maximum eigenvalues for M_1, N_1, and P_1, respectively, then

$$\rho_1 = a_1 - a_3 - 2a_2, \qquad \rho_2 = a_4 - 2a_6 + a_5, \qquad \rho_3 = \frac{9}{2}. \tag{16}$$

From (12), when $k = 0$,

$$\begin{aligned}
&E^1 = M_1^{-1} N_1 E^0, \\
&\|E^1\| \le \|M_1^{-1} N_1\| \|E^0\| \le \frac{|a_4 + 2a_6 + a_5|}{|a_1 - a_3 - 2a_2|} \|E^0\|, \\
&\|E^1\| \le \frac{|121h^2 - 132(\tau + \gamma\tau^\gamma)|}{|81(h^2 + 4(\tau + \tau^\gamma))|} \|E^0\|, \\
&\|E^1\| \le \|E^0\| \quad \because \text{ denominator} > \text{numerator}.
\end{aligned}$$

Supposing

$$\|E^s\| \le \|E^0\|, \quad s = 2, 3, \dots, k, \tag{17}$$

we will prove it for $s = k + 1$. Indeed, from (12)

$$\begin{aligned}
\|E^{k+1}\| &= \left\| M_1^{-1} \left(N_1 E^k + s_1 \sum_{l=2}^{k+1} \lambda_l P_1 E^{k+1-l} + s_1 \sum_{l=1}^{k} \lambda_l P_1 E^{k-l} \right)) \right\| \\
&\le \|M_1^{-1} N_1\| \|E^k\| + s_1 \sum_{l=2}^{k+1} \lambda_l P_1 E^{k+1-l} \|M_1^{-1} P_1\| \|E^{k+1-l}\| \\
&\quad + s_1 \sum_{l=1}^{k} \lambda_l \|M_1^{-1} P_1\| \|E^{k-l}\| \\
&\le \|M_1^{-1} N_1\| \|E^k\| + s_1 \|M_1^{-1} P_1\| \left(\sum_{l=2}^{k+1} \lambda_l \|E^{k+1-l}\| + \sum_{l=1}^{k} \lambda_l \|E^{k-l}\| \right) \\
&\le \left(\|M_1^{-1} N_1\| + s_1 \|M_1^{-1} P_1\| \left(\sum_{l=2}^{k+1} \lambda_l + \sum_{l=1}^{k} \lambda_l \right) \right) \|E^0\| \quad \because \text{ using (17)} \\
&= \left(\frac{a_4 - 2a_6 + a_5}{a_1 - a_3 - 2a_2} + \frac{9s_1}{2(a_1 - a_3 - 2a_2)} \left(\sum_{l=2}^{k+1} \lambda_l + \sum_{l=1}^{k} \lambda_l \right) \right) \|E^0\| \\
&\le \left(\frac{a_4 - 2a_6 + a_5}{a_1 - a_3 - 2a_2} + \frac{9s_1}{2(a_1 - a_3 - 2a_2)} \left(\sum_{l=2}^{k+1} \lambda_l - 1 \right) \right) \|E^0\| \quad \because \text{ using Lemma 1}
\end{aligned}$$

$$= \left(\frac{a_4 - 2a_6 + a_5}{a_1 - a_3 - 2a_2} + \frac{9s_1}{2(a_1 - a_3 - 2a_2)}\left(\sum_{l=1}^{k+1}\lambda_l - 1 - \lambda_1\right)\right)\|E^0\|$$
$$\leq \left(\frac{a_4 - 2a_6 + a_5}{a_1 - a_3 - 2a_2} + \frac{9s_1}{2(a_1 - a_3 - 2a_2)}(-1 - 1 - \lambda_1)\right)\|E^0\| \quad \because \text{ using Lemma 1}$$
$$= \left(\frac{81h^2 - 324(\gamma\tau^\gamma + \tau) - 4.5s_1(\gamma + 1)}{79h^2 + 348(\tau^\gamma + \tau)}\right)\|E^0\|$$
$$= \left(\frac{81h^2 - (324d_0 + 4.5d_1)}{79h^2 + 348d_0}\right)\|E^0\| \quad \text{where } d_0 = (\gamma\tau^\gamma + \tau) \text{ and } d_1 = s_1(\gamma + 1)$$
$$\|E^{k+1}\| \leq \|E^0\|, \quad \because \text{ denominator > numerator, because } d_0, d_1 > 0 \text{ and } h \in (0,1).$$

So, using matrix analysis via mathematical induction, we proved that the proposed method is unconditionally stable. □

4 Convergence of the proposed method

Let us denote the truncation errors for the group of four points $w_{i,j}^{k+\frac{1}{2}}, w_{i+1,j}^{k+\frac{1}{2}}, w_{i+1,j+1}^{k+\frac{1}{2}}, w_{i,j+1}^{k+\frac{1}{2}}$ by $e_{i,j}^{k+\frac{1}{2}}, e_{i+1,j}^{k+\frac{1}{2}}, e_{i+1,j+1}^{k+\frac{1}{2}}, e_{i,j+1}^{k+\frac{1}{2}}$ then let $R^{k+\frac{1}{2}} = \{R_{1,1}^{k+\frac{1}{2}}, R_{1,2}^{k+\frac{1}{2}}, \ldots, R_{1,\frac{m-1}{4}}^{k+\frac{1}{2}}, R_{2,1}^{k+\frac{1}{2}}, R_{2,2}^{k+\frac{1}{2}}, \ldots, R_{\frac{m-1}{4},\frac{m-1}{4}}^{k+\frac{1}{2}}\}$ where $R_{i,j}^{k+\frac{1}{2}} = \{e_{i,j}^{k+\frac{1}{2}}, e_{i+1,j}^{k+\frac{1}{2}}, e_{i+1,j+1}^{k+\frac{1}{2}}, e_{i,j+1}^{k+\frac{1}{2}}\}$ and $i,j = 1,2,\ldots,\frac{m-1}{4}$. Then from (8) we have

$$\|R^{k+\frac{1}{2}}\| \leq \varphi(\tau + h^4), \quad k = 0,1,2,\ldots,N-1, \tag{18}$$

where φ is a constant.

Define the error as

$$E^{k+1} = \begin{bmatrix} E_1^{k+1} \\ E_1^{k+1} \\ \vdots \\ E_1^{k+1} \\ E_1^{k+1} \end{bmatrix}, \qquad E_1^{k+1} = \begin{bmatrix} \epsilon_1^{k+1} \\ \epsilon_2^{k+1} \\ \vdots \\ \epsilon_{m-2}^{k+1} \\ \epsilon_{m-1}^{k+1} \end{bmatrix}, \qquad \epsilon_i^{k+1} = \begin{bmatrix} \epsilon_{i,j}^{k+1} \\ \epsilon_{i+1,j}^{k+1} \\ \epsilon_{i+1,j+1}^{k+1} \\ \epsilon_{i,j+1}^{k+1} \end{bmatrix}, \quad i = 1,2,\ldots,m-1,$$
$$j = 1,2,\ldots,m-1. \tag{19}$$

By substituting (19) into (11) and using $E^0 = 0$, we get

$$\begin{aligned} &M_1E^1 = R^{\frac{1}{2}}, \quad k = 0, \\ &M_1E^{k+1} = N_1E^k + s_1\sum_{l=2}^{k+1}\lambda_l P_1(E^{k+1-l}) + s_1\sum_{l=1}^{k}\lambda_l P_1(E^{k-l}) + (R^{k+\frac{1}{2}}), \quad k > 0. \end{aligned} \tag{20}$$

Proposition 2 *Suppose* E^{k+1} $(k = 0,1,\ldots,N)$ *satisfy* (20), *then*

$$\|E^{k+1}\| \leq \|R^{k+\frac{1}{2}}\|.$$

Proof We will use mathematical induction. When $k = 0$,

$$M_1E^1 = R^{\frac{1}{2}},$$

$$
\begin{aligned}
&\|E^1\| \le \|M_1^{-1}\|\|R^{\frac{1}{2}}\| = \frac{1}{|a_1-a_3-2a_2|}\|R^{\frac{1}{2}}\| \le \frac{1}{81(h^2+4(\tau+\tau^\gamma))}\|R^{\frac{1}{2}}\|,\\
&\|E^1\| \le \mu_0\|R^{\frac{1}{2}}\|, \quad \text{where } \mu_0 = \frac{1}{81(h^2+4(\tau+\tau^\gamma))},\\
&\|E^1\| \le \|R^{\frac{1}{2}}\|.
\end{aligned}
$$

Assume that

$$
\|E^s\| \le \|R^{(s-1)+\frac{1}{2}}\|, \quad s = 2,3,\dots,k, \tag{21}
$$

then for $s = k+1$,

$$
\begin{aligned}
&M_1E^{k+1} = N_1E^k + s_1\sum_{l=2}^{k+1}P_1(E^{k+1-s}) + s_1\sum_{l=1}^{k}P_1(E^{k-s}) + R^{k+\frac{1}{2}},\\
&\|E^{k+1}\| = \left\|M_1^{-1}\left(N_1E^k + s_1\sum_{l=2}^{k+1}\lambda_lP_1(E^{k+1-s}) + s_1\sum_{l=1}^{k}\lambda_lP_1(E^{k-s}) + R^{k+\frac{1}{2}}\right)\right\|,\\
&\|E^{k+1}\| \le \|M_1^{-1}N_1\|\|E^k\| + \|M_1^{-1}P_1\|s_1\left(\sum_{l=2}^{k+1}\lambda_l\|E^{k+1-s}\| + \sum_{l=1}^{k}\lambda_l\|E^{k-s}\|\right)\\
&\qquad + \|M_1^{-1}R^{k+\frac{1}{2}}\|,\\
&\|E^{k+1}\| \le \|M_1^{-1}N_1\|\|E^k\| + \|M_1^{-1}P_1\|s_1\left(\sum_{l=2}^{k+1}\lambda_l\|E^{k+1-s}\| + \sum_{l=1}^{k}\lambda_l\|E^{k-s}\|\right)\\
&\qquad + \|M_1^{-1}R^{k+\frac{1}{2}}\|,\\
&\|E^{k+1}\| \le \|M_1^{-1}N_1\|\|R^{(k-1)+\frac{1}{2}}\| + \|M_1^{-1}P_1\|s_1\left(\sum_{l=2}^{k+1}\lambda_l + \sum_{l=1}^{k}\lambda_l\right)\|R^{(k-1)+\frac{1}{2}}\|\\
&\qquad + \|M_1^{-1}\|\|R^{k+\frac{1}{2}}\|,\\
&\|E^{k+1}\| = \left(\|M_1^{-1}N_1\| + \|M_1^{-1}P_1\|s_1\left(\sum_{l=2}^{k+1}\lambda_l + \sum_{l=1}^{k}\lambda_l\right) + \|M_1^{-1}\|\right)R^{k+\frac{1}{2}}\|,\\
&\|E^{k+1}\| \le \left(\|M_1^{-1}N_1\| + s_1\|M^{-1}P_1\|(-2+\lambda_1) + \|M_1^{-1}\|\right)\|R^{k+\frac{1}{2}}\|\\
&\qquad = \left(\frac{a_4-2a_6+a_5}{a_1-a_3-2a_2} + \frac{9s_1(-2-\lambda_1)}{2(a_1-a_3-2a_2)} + \frac{1}{a_1-a_3-2a_2}\right)\|R^{k+\frac{1}{2}}\|\\
&\qquad = \left(\frac{a_4-2a_6+a_5-9s_1(2+\lambda_1)+2}{2(a_1-a_3-2a_2)}\right)\|R^{k+\frac{1}{2}}\|\\
&\qquad = \left(\frac{81h^2-324(\gamma\tau^\gamma+\tau)-4.5s_1(\gamma+1)+2}{79h^2+348(\tau^\gamma+\tau)}\right)\|R^{k+\frac{1}{2}}\|,\\
&\|E^{k+1}\| \le \phi\|R^{k+\frac{1}{2}}\|,
\end{aligned}
$$

where $\phi = \frac{81h^2-324(\gamma\tau^\gamma+\tau)-4.5s_1(\gamma+1)+2}{79h^2+348(\tau^\gamma+\tau)}$ and $\phi \in (0,1)$, so

$$
\|E^{k+1}\| \le \|R^{k+\frac{1}{2}}\|. \qquad \square
$$

Theorem 1 *The high-order explicit group scheme* (10) *is convergent with the order of convergence* $O(\tau + h^4)$.

Proof From (18), we have

$$\begin{aligned} &\|E^{k+1}\|_2 \le \|R^{k+\frac{1}{2}}\| \le \varphi(\tau + h^4), \\ &\|E^{k+1}\|_2 \le \varphi(\tau + h^4), \quad \forall k = 0, 1, 2, \dots, N-1. \end{aligned}$$

Hence, we proved that the high-order explicit group scheme (10) is convergent with the order of convergence $O(\tau + h^4)$. □

5 Numerical experiments and discussion

In this section, three numerical experiments were simulated using Core i7 Duo 3.40 GHz, 4 GB RAM and Windows 7 using Mathematica software. The acceleration technique "Successive over-relaxation (SOR)" is used with relaxation factor $\omega = 1.8$ and convergence tolerance $\zeta = 10^{-5}$ for the maximum error (L_∞); C_1- and C_2-order of convergence are used for space and time variables and calculated using [34]

$$C_1\text{-order} = \log_2\left(\frac{\|L_\infty(2\tau, h)\|}{\|L_\infty(\tau, h)\|}\right), \tag{22}$$

$$C_2\text{-order} = \log_2\left(\frac{\|L_\infty(16\tau, 2h)\|}{\|L_\infty(\tau, h)\|}\right), \tag{23}$$

where h, τ and L_∞ represent the space-step, the time-step, and the infinity norm, respectively.

The following three numerical experiments are considered:

Example 1 ([27])

$$\begin{aligned} \frac{\partial w(x,y,t)}{\partial t} = {}_0D_1^{1-\gamma}\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 w(x,y,t)}{\partial x^2} \\ + \frac{\partial^2 w(x,y,t)}{\partial y^2} + e^{x+y}\left((1+\gamma)t^\gamma - 2\frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}t^{2\gamma} - 2t^{1+\gamma}\right), \end{aligned}$$

where $0 < x, y < 1$, with initial and boundary conditions

$$\begin{aligned} &w(x,y,0) = 0, \\ &w(0,y,t) = e^y t^{1+\gamma}, \qquad w(x,0,t) = e^x t^{1+\gamma}, \\ &w(n,y,t) = e^{1+y} t^{1+\gamma}, \qquad w(x,n,t) = e^{1+x} t^{1+\gamma}, \end{aligned}$$

and with the exact solution

$$w(x,y,t) = e^{x+y} t^{1+\gamma}.$$

Example 2 ([27])

$$\frac{\partial w(x,y,t)}{\partial t} = {}_0D_1^{1-\gamma}\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 w(x,y,t)}{\partial x^2}$$

Table 1 Comparison between Crank–Nicolson (C–N) high-order finite difference method and HEGM for Example 1 when $\gamma = 0.75$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h = \tau = \frac{1}{10}$	46	53	16.59	19.23	3.9320×10^{-3}	3.9473×10^{-3}	2.2799×10^{-3}	2.2815×10^{-3}
$h = \tau = \frac{1}{18}$	49	52	208.44	220.02	2.4058×10^{-3}	2.4059×10^{-3}	1.2545×10^{-3}	1.2567×10^{-3}
$h = \tau = \frac{1}{22}$	48	57	514.81	603.56	2.0054×10^{-3}	2.0087×10^{-3}	1.0222×10^{-3}	1.0258×10^{-3}
$h = \tau = \frac{1}{30}$	47	65	1902.32	2571.45	1.5041×10^{-3}	1.5053×10^{-3}	7.4966×10^{-4}	7.5067×10^{-4}

Table 2 Comparison between Crank–Nicolson (C–N) high-order finite difference method and HEGM for Example 2 when $\gamma = 0.5$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h = \tau = \frac{1}{10}$	46	50	16.22	18.37	7.5870×10^{-3}	7.5733×10^{-3}	4.3269×10^{-3}	4.3301×10^{-3}
$h = \tau = \frac{1}{18}$	48	52	207.69	224.26	4.3170×10^{-3}	4.3041×10^{-3}	2.2571×10^{-3}	2.2547×10^{-3}
$h = \tau = \frac{1}{22}$	47	57	483.15	594.33	3.5419×10^{-3}	3.5520×10^{-3}	1.8133×10^{-3}	1.8190×10^{-3}
$h = \tau = \frac{1}{30}$	47	65	2018.26	2727.72	2.6249×10^{-3}	2.6329×10^{-3}	1.3123×10^{-3}	1.3130×10^{-3}

$$+ \frac{\partial^2 w(x,y,t)}{\partial y^2} + \exp\left(-\frac{(x-0.5)^2}{\nu} - \frac{(t-0.5)^2}{\nu}\right)(1+\gamma)t^{\gamma} + \left(\frac{(\Gamma(2+\gamma))}{\Gamma(1+2\gamma)}t^{2\gamma} + t^{1+\gamma}\right)\left(\frac{4}{\nu} - \frac{4(x-0.5)^2}{\nu^2} - \frac{4(y-0.5)^2}{\nu^2}\right),$$

where $0 < x, y < 1$, with initial and boundary conditions

$$\begin{aligned}
&w(x,y,0) = 0, \\
&w(0,y,t) = \exp\left(-\left(\frac{0.25}{\nu} + \frac{(y-0.5)^2}{\nu}\right)\right)t^{1+\gamma}, \\
&w(x,0,t) = \exp\left(-\left(\frac{(x-0.5)^2}{\nu} + \frac{0.25}{\nu}\right)\right)t^{1+\gamma}, \\
&w(n,y,t) = \exp\left(-\left(\frac{0.25}{\nu} + \frac{(y-0.5)^2}{\nu}\right)\right)t^{1+\gamma}, \\
&w(x,n,t) = \exp\left(-\left(\frac{(x-0.5)^2}{\nu} + \frac{0.25}{\nu}\right)\right)t^{1+\gamma},
\end{aligned}$$

and with the exact solution

$$w(x,y,t) = \exp\left(-\left(\frac{(x-0.5)^2}{\nu} + \frac{(y-0.5)^2}{\nu}\right)\right)t^{1+\gamma}.$$

Example 3

$$\begin{aligned}
\frac{\partial w(x,y,t)}{\partial t} = {}_0D_1^{1-\gamma}\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) + \frac{\partial^2 w(x,y,t)}{\partial x^2} \\
+ \frac{\partial^2 w(x,y,t)}{\partial y^2} + e^t \sin(x+y) + \frac{3t^{\gamma-1}\sin(x+y)}{\Gamma(\gamma)} \\
+ \frac{3e^t\Gamma(\gamma) - \Gamma(\gamma)\sin(x+y)}{\Gamma(\gamma)},
\end{aligned}$$

Table 3 Comparison between Crank–Nicolson (C–N) method and HEGM for Example 2 when $\gamma = 0.5$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h=\tau=\frac{1}{10}$	42	38	16.72	15.72	1.7912×10^{-2}	1.7921×10^{-2}	5.3164×10^{-3}	5.3176×10^{-3}
$h=\tau=\frac{1}{18}$	34	44	160.91	201.84	9.6466×10^{-3}	9.6753×10^{-3}	2.7624×10^{-3}	2.7719×10^{-3}
$h=\tau=\frac{1}{22}$	34	38	377.19	416.85	7.9298×10^{-3}	7.9215×10^{-3}	2.2387×10^{-3}	2.2376×10^{-3}
$h=\tau=\frac{1}{30}$	33	37	1446.76	1766.20	5.8326×10^{-3}	5.8344×10^{-3}	1.6201×10^{-3}	1.6210×10^{-3}

Table 4 Comparison between C–N high-order finite difference method and HEGM for Example 2 when $\gamma = 0.75$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h=\tau=\frac{1}{10}$	41	39	17.39	15.67	1.0025×10^{-2}	1.0025×10^{-2}	2.7431×10^{-3}	2.7423×10^{-3}
$h=\tau=\frac{1}{18}$	36	45	164.84	196.07	5.4258×10^{-3}	5.4142×10^{-3}	1.5410×10^{-3}	1.5363×10^{-3}
$h=\tau=\frac{1}{22}$	36	41	387.83	421.71	4.4612×10^{-3}	4.4652×10^{-3}	1.2528×10^{-3}	1.2530×10^{-3}
$h=\tau=\frac{1}{30}$	37	37	1602.14	1810.40	3.3128×10^{-3}	3.3150×10^{-3}	9.1139×10^{-4}	9.1040×10^{-4}

Table 5 Comparison between C–N high-order finite difference method and HEGM for Example 3 when $\gamma = 0.75$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h=\tau=\frac{1}{10}$	44	48	46.75	53.44	1.1762×10^{-2}	1.1780×10^{-2}	2.0762×10^{-3}	2.0850×10^{-3}
$h=\tau=\frac{1}{14}$	44	52	152.45	188.46	8.5326×10^{-3}	8.7058×10^{-4}	9.1589×10^{-4}	9.1579×10^{-4}
$h=\tau=\frac{1}{18}$	45	58	462.44	552.32	4.9486×10^{-3}	4.9496×10^{-3}	5.9148×10^{-4}	5.9035×10^{-4}
$h=\tau=\frac{1}{22}$	48	56	828.46	1158.37	1.8846×10^{-3}	1.8875×10^{-4}	2.6648×10^{-4}	2.6678×10^{-4}

Table 6 Comparison between C–N high-order finite difference method and HEGM for Example 3 when $\gamma = 0.1$

h/τ	Iteration		Time		Maximum error		Average error	
	HEGM	C–N	HEGM	C–N	HEGM	C–N	HEGM	C–N
$h=\tau=\frac{1}{10}$	43	50	14.71	25.47	1.0779×10^{-1}	1.3229×10^{-1}	6.5071×10^{-2}	6.7078×10^{-2}
$h=\tau=\frac{1}{14}$	45	53	56.87	97.87	9.72047×10^{-2}	9.6072×10^{-2}	5.5989×10^{-2}	5.8080×10^{-2}
$h=\tau=\frac{1}{18}$	46	56	151.07	221.53	9.1148×10^{-2}	9.4512×10^{-2}	5.0646×10^{-2}	5.1572×10^{-2}
$h=\tau=\frac{1}{22}$	44	57	342.85	435.62	8.74869×10^{-2}	8.9345×10^{-2}	4.74195×10^{-2}	4.6185×10^{-2}

Table 7 Errors and CPU time with $\tau = \frac{1}{20}$ for Example 1

h	$\gamma = 0.1$		$\gamma = 0.5$	
	Max error	CPU time	Max error	CPU time
$h=\frac{1}{2}$	5.20503×10^{-1}	0.56	3.97534×10^{-1}	0.56
$h=\frac{1}{4}$	4.26844×10^{-2}	5.34	3.75884×10^{-2}	5.68
$h=\frac{1}{8}$	8.27965×10^{-3}	28.28	7.84629×10^{-2}	28.15
$h=\frac{1}{16}$	4.32245×10^{-3}	133.54	3.89731×10^{-3}	137.01

Table 8 Errors and CPU time with $\tau = \frac{1}{20}$ for Example 2

h	$\gamma = 0.1$		$\gamma = 0.5$	
	Max error	CPU time	Max error	CPU time
$h=\frac{1}{2}$	1.0305	0.60	1.0241	0.60
$h=\frac{1}{4}$	4.9756×10^{-2}	5.23	5.1451×10^{-2}	5.29
$h=\frac{1}{8}$	9.1542×10^{-3}	25.81	1.07962×10^{-2}	23.81
$h=\frac{1}{16}$	7.1005×10^{-3}	111.87	8.7834×10^{-3}	113.76

Table 9 Errors and relaxation factor (ω) with $\tau = \frac{1}{20}$ and $\gamma = 0.1$ for Example 2

h	$\omega = 0.5$	$\omega = 0.9$	$\omega = 1.5$	$\omega = 1.8$
$h = \frac{1}{2}$	1.0305	1.0305	1.0241	1.0305
$h = \frac{1}{4}$	4.9842×10^{-2}	4.9812×10^{-2}	4.9844×10^{-2}	4.9756×10^{-2}
$h = \frac{1}{8}$	9.4342×10^{-3}	9.92685×10^{-3}	9.44614×10^{-2}	9.1542×10^{-3}
$h = \frac{1}{16}$	7.4882×10^{-3}	8.36823×10^{-3}	8.5763×10^{-3}	7.1005×10^{-3}

Table 10 C_2-order of convergence for Example 1 and different γ's

$\gamma = 0.4$			$\gamma = 0.5$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h = \tau = \frac{1}{2}$	3.7570×10^{-2}	–	$h = \tau = \frac{1}{2}$	4.2072×10^{-2}	–
$h = \frac{1}{4}, \tau = \frac{1}{32}$	2.6621×10^{-3}	3.81	$h = \frac{1}{4}, \tau = \frac{1}{32}$	2.3925×10^{-3}	4.13
$h = \tau = \frac{1}{4}$	1.9645×10^{-2}	–	$h = \tau = \frac{1}{4}$	1.9308×10^{-2}	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	1.3235×10^{-3}	3.89	$h = \frac{1}{8}, \tau = \frac{1}{64}$	1.2333×10^{-3}	3.96
$\gamma = 0.6$			$\gamma = 0.7$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h = \tau = \frac{1}{2}$	4.0658×10^{-2}	–	$h = \tau = \frac{1}{2}$	3.3869×10^{-2}	–
$h = \frac{1}{4}, \tau = \frac{1}{32}$	2.0573×10^{-3}	4.30	$h = \frac{1}{4}, \tau = \frac{1}{32}$	1.6180×10^{-3}	4.38
$h = \tau = \frac{1}{4}$	1.6658×10^{-2}	–	$h = \tau = \frac{1}{4}$	1.2312×10^{-2}	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	1.0452×10^{-3}	3.99	$h = \frac{1}{8}, \tau = \frac{1}{64}$	8.3252×10^{-3}	3.88

Table 11 C_2-order of convergence of HEGM for Example 2 and different γ's

$\gamma = 0.4$			$\gamma = 0.5$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h = \tau = \frac{1}{2}$	1.1717	–	$h = \tau = \frac{1}{2}$	1.1891	–
$h = \frac{1}{4}, \tau = \frac{1}{32}$	6.8679×10^{-2}	4.09	$h = \frac{1}{4}, \tau = \frac{1}{32}$	6.8093×10^{-2}	4.12
$h = \tau = \frac{1}{4}$	8.5608×10^{-2}	–	$h = \tau = \frac{1}{4}$	8.7109×10^{-2}	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	5.1955×10^{-3}	4.04	$h = \frac{1}{8}, \tau = \frac{1}{64}$	4.8825×10^{-3}	4.15
$\gamma = 0.6$			$\gamma = 0.7$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h = \tau = \frac{1}{2}$	1.1802	–	$h = \tau = \frac{1}{2}$	1.1474	–
$h = \frac{1}{4}, \tau = \frac{1}{32}$	6.7241×10^{-2}	4.13	$h = \frac{1}{4}, \tau = \frac{1}{32}$	6.6197×10^{-2}	4.11
$h = \tau = \frac{1}{4}$	8.2701×10^{-2}	–	$h = \tau = \frac{1}{4}$	7.3781×10^{-2}	–
$h = \frac{1}{8}, \tau = \frac{1}{64}$	4.4968×10^{-3}	4.20	$h = \frac{1}{8}, \tau = \frac{1}{64}$	4.0021×10^{-3}	4.20

Table 12 C_1-order of convergence for Example 1, when $h = \frac{1}{8}$

τ	$\gamma = 0.5$		$\gamma = 0.75$	
	L_∞	C_1-order	L_∞	C_1-order
$\tau = \frac{1}{10}$	7.5050×10^{-3}	–	3.9019×10^{-3}	–
$\tau = \frac{1}{20}$	3.8462×10^{-3}	0.96	2.1660×10^{-3}	0.86
$\tau = \frac{1}{40}$	1.9623×10^{-3}	0.97	1.1278×10^{-3}	0.94
$\tau = \frac{1}{80}$	9.9769×10^{-4}	0.98	5.8404×10^{-4}	0.95

Table 13 C_1-order of convergence for Example 2, when $h = \frac{1}{8}$

τ	$\gamma = 0.5$		$\gamma = 0.1$	
	L_∞	C_1-order	L_∞	C_1-order
$\tau = \frac{1}{10}$	1.9223×10^{-2}	–	9.4036×10^{-3}	–
$\tau = \frac{1}{20}$	9.7960×10^{-3}	0.97	5.1542×10^{-3}	0.86
$\tau = \frac{1}{40}$	5.1042×10^{-3}	0.94	2.5442×10^{-3}	1.01
$\tau = \frac{1}{80}$	2.5146×10^{-3}	1.02	1.2420×10^{-3}	1.03

Table 14 C_1-order of convergence for Example 3, when $h = \frac{1}{8}$

τ	$\gamma = 0.5$		$\gamma = 0.1$	
	L_∞	C_1-order	L_∞	C_1-order
$\tau = \frac{1}{10}$	1.56836×10^{-1}	–	1.06481×10^{-1}	–
$\tau = \frac{1}{20}$	7.9683×10^{-2}	0.97	5.3363×10^{-2}	0.99
$\tau = \frac{1}{40}$	3.7875×10^{-2}	1.07	2.6884×10^{-2}	0.98
$\tau = \frac{1}{80}$	2.00105×10^{-2}	0.92	1.2946×10^{-2}	1.05

Table 15 Computational complexity for the HEGM and C–N high-order finite difference method method

Methods	Per iteration	
	Addition/subtraction	Multiplication/division
C–N	$(35 + 8(k-1))m^2$	$(13 + 4(k-1))m^2$
HEGM	$(34 + 8(k-1))(m-1)^2 + (35 + 8(k-1))(2m-1)$	$(10 + 4(k-1))(m-1)^2 + (13 + 4(k-1))(2m-1)$

Figure 2 Exact solution for Example 1

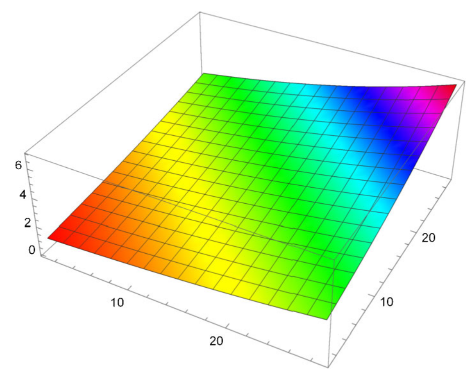
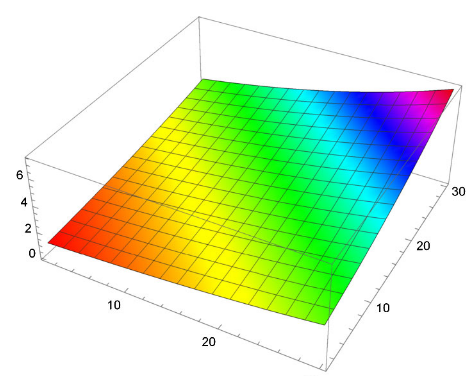


Figure 3 Approximate solution for Example 1 when $h = \tau = \frac{1}{30}$



where $0 < x, y < 1$, with initial and boundary conditions

$$w(x,y,0) = \sin(x+y),$$

$$w(0,y,t) = e^t \sin(y), \qquad w(x,0,t) = e^t \sin(x),$$

$$w(n,y,t) = e^t \sin(1+y), \qquad w(x,n,t) = e^t \sin(1+x),$$

Figure 4 Exact solution for Example 2

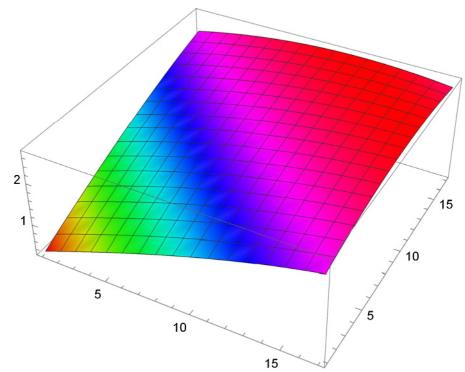


Figure 5 Approximate solution for Example 2 when $h = \tau = \frac{1}{30}$

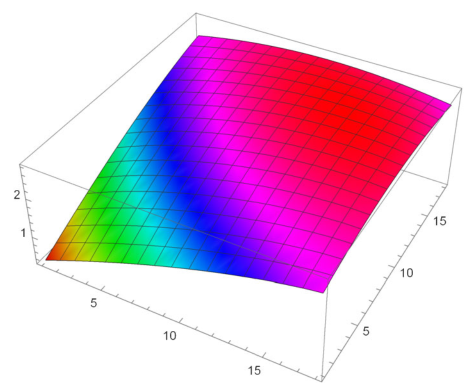
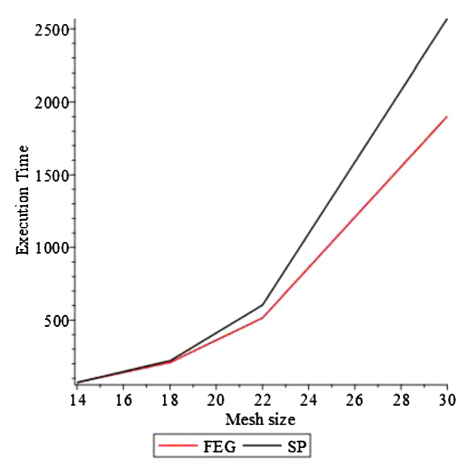


Figure 6 Execution time (in s) for different mesh sizes when $\gamma = 0.75$ for Example 1

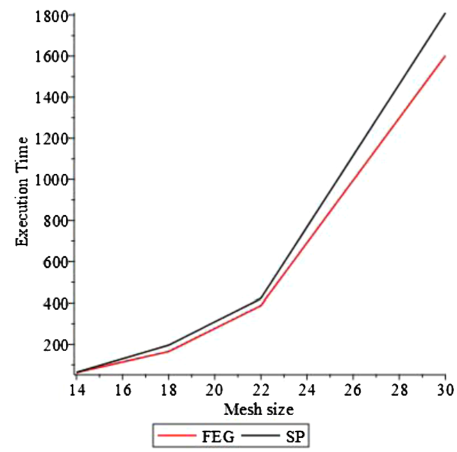


and with the exact solution

$$w(x, y, t) = e^t \sin(x + y).$$

The execution time, error, and number of iteration are shown for the comparison between standard point and HEGM from Table 1 to Table 6. The execution time in HEGM is reduced by (5–35)%, (7–35)%, (10–25)%, (8–18)%, (12.5–28.48)%, and (21.29–42.24)% as compared to C–N point method in Tables 1 to 6, respectively, and it can also be seen

Figure 7 Execution time (in s) for different mesh sizes when $\gamma = 0.75$ for Example 2



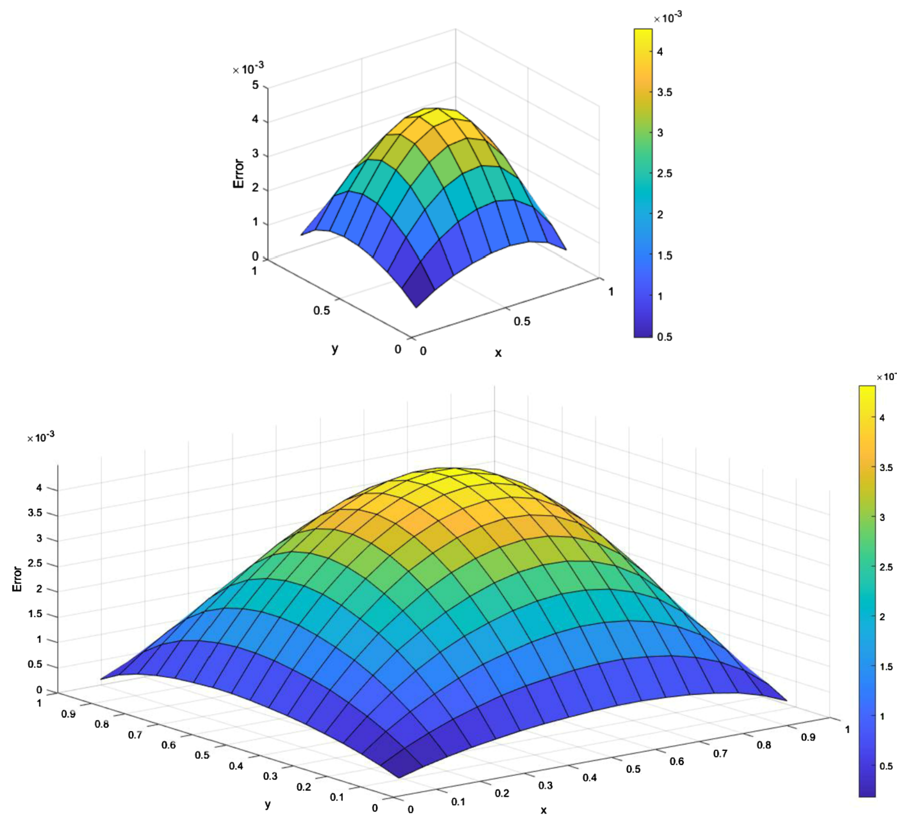


Figure 8 Graphs of maximum errors using HEGM with $\gamma = 0.5$, $\tau = 1/20$ and $h = \frac{1}{10}$ (a), $h = \frac{1}{20}$ (b), for Test problem 1

in Figs. 4 and 5. Table 7 and Table 8 show the maximum errors and CPU timing at different values of γ's for Example 1 and Example 2 respectively. Table 9 shows the maximum error at different values of the relaxation factor (ω's). Tables 10 to 14 represent the space and time variables' order of convergence for the HEGM, which show that the theoretical order of convergence is in agreement with the experimental order of convergence. Figures 2 to 5 represent 3D graphs for the exact and approximate solutions of Examples 1

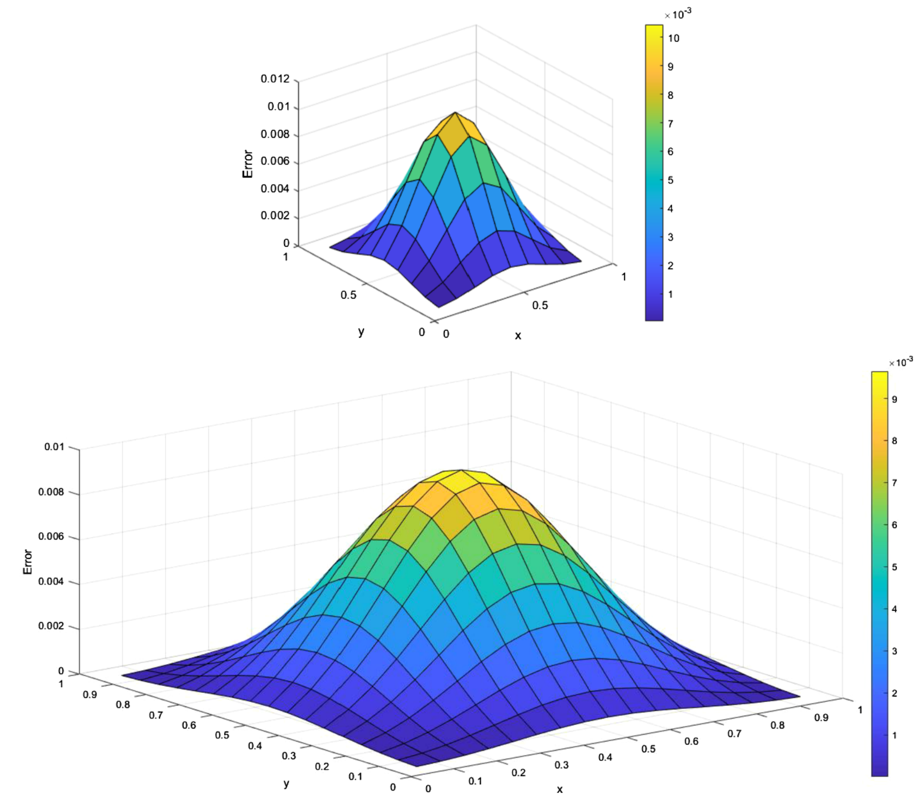


Figure 9 Graphs of maximum errors using HEGM with $\gamma = 0.5$, $\tau = 1/20$ and $h = \frac{1}{10}$ (a), $h = \frac{1}{20}$ (b), for Test problem 2

Table 16 The total computation effort for different mesh size for Example 1 when $\alpha = 0.75$

k/m	High-order finite difference method		HEGM	
	No. of iter.	Total operations	No. of iter.	Total operations
$k = m = 10$	46	702,696	53	826,800
$k = m = 18$	49	1,926,092	52	2,079,168
$k = m = 22$	48	3,863,616	57	4,653,936
$k = m = 30$	47	16,592,692	65	23,166,000

Table 17 The total computation effort for different mesh size for Example 2 when $\alpha = 0.5$

k/m	High-order finite difference method		HEGM	
	No. of iter.	Total operations	No. of iter.	Total operations
$k = m = 10$	39	1,559,376	38	1,493,704
$k = m = 18$	44	3,592,512	34	2,736,728
$k = m = 22$	38	5,517,600	34	4,876,824
$k = m = 30$	37	13,186,800	33	11,650,188

and 2, which show that the proposed method is effective and reliable. The comparison of execution timing between FEG (HEGM) and SP (C-N) for Example 1 and Example 2 are shown in Figure 6 and Figure 7 respectively, which depicted that HEGM method required less execution time as compared to the C-N. Figures 8 and 9 show the graphs of the maximum error using HEGM when $\gamma = 0.5$ and $\tau = \frac{1}{20}$ for Examples 1 and 2, respectively. The computational effort is shown in Tables 16 and 17; it can be seen that the HEGM requires fewer operations as compared to the high-order Crank–Nicolson finite difference method.

6 Conclusion

In this paper, we have solved two-dimensional fractional Rayleigh–Stokes problem for a heated generalized second-grade fluid using the HEGM. The C_2-order of convergence shows that the theoretical order of convergence agrees with the experimental order of convergence. The proposed method reduces execution time and computational complexity as compared to the high-order compact Crank–Nicolson finite difference scheme. We proved the unconditional stability using the matrix analysis method; moreover, the proposed method is convergent.

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Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this article was proposed by MAK, NHMA and NNAH. MAK prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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References

1. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations (1993)
2. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering., vol. 198. Academic Press, San Diego (1999)
3. Khan, M.A., Ali, N.H.M.: High-order compact scheme for the two-dimensional fractional Rayleigh–Stokes problem for a heated generalized second-grade fluid. Adv. Differ. Equ. **2020**(1), 1 (2020)
4. Saadatmandi, A., Dehghan, M.: A new operational matrix for solving fractional-order differential equations. Comput. Math. Appl. **59**(3), 1326–1336 (2010)
5. Dehghan, M., Manafian, J., Saadatmandi, A.: Solving nonlinear fractional partial differential equations using the homotopy analysis method. Numer. Methods Partial Differ. Equ. **26**(2), 448–479 (2010)
6. Liu, Q., Liu, F., Turner, I., Anh, V.: Finite element approximation for a modified anomalous subdiffusion equation. Appl. Math. Model. **35**(8), 4103–4116 (2011)
7. Li, X., Xu, C.: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. **47**(3), 2108–2131 (2009)
8. Abbaszadeh, M., Dehghan, M.: A finite-difference procedure to solve weakly singular integro partial differential equation with space-time fractional derivatives. Eng. Comput., 1–10 (2020)
9. Abbaszadeh, M., Dehghan, M.: A reduced order finite difference method for solving space-fractional reaction-diffusion systems: the Gray–Scott model. Eur. Phys. J. Plus **134**(12), 620 (2019)
10. Dehghan, M., Abbaszadeh, M.: A finite element method for the numerical solution of Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivatives. Eng. Comput. **33**(3), 587–605 (2017)
11. Hendy, A.S., Zaky, M.A.: Global consistency analysis of L_1-Galerkin spectral schemes for coupled nonlinear space-time fractional Schrödinger equations. Appl. Numer. Math. (2020)
12. Zaky, M.A., Hendy, A.S., Macías-Díaz, J.E.: Semi-implicit Galerkin–Legendre spectral schemes for nonlinear time-space fractional diffusion–reaction equations with smooth and nonsmooth solutions. J. Sci. Comput. **82**(1), 1–27 (2020)
13. Zaky, M.A.: Recovery of high order accuracy in Jacobi spectral collocation methods for fractional terminal value problems with non-smooth solutions. J. Comput. Appl. Math. **357**, 103–122 (2019)

14. Zaky, M.A.: An accurate spectral collocation method for nonlinear systems of fractional differential equations and related integral equations with nonsmooth solutions. Appl. Numer. Math. (2020)
15. Zaky, M.A.: An improved tau method for the multi-dimensional fractional Rayleigh–Stokes problem for a heated generalized second grade fluid. Comput. Math. Appl. **75**(7), 2243–2258 (2018)
16. Yanbing, Y., Ahmed, M.S., Lanlan, Q., Runzhang, X.: Global well-posedness of a class of fourth-order strongly damped nonlinear wave equations. Opusc. Math. **39**(2), 297 (2019)
17. Ali, U., Abdullah, F.A., Mohyud-Din, S.T.: Modified implicit fractional difference scheme for 2D modified anomalous fractional sub-diffusion equation. Adv. Differ. Equ. **2017**(1), 1 (2017)
18. Khan, M.A., Ali, N.H.M.: Fourth-order compact iterative scheme for the two-dimensional time fractional sub-diffusion equations. Math. Stat. **8**(2A), 52–57 (2020)
19. Salama, F.M., Ali, N.H.M., Abd Hamid, N.N.: Efficient hybrid group iterative methods in the solution of two-dimensional time fractional cable equation. Adv. Differ. Equ. **2020**(1), 1 (2020)
20. Abbaszadeh, M., Dehghan, M.: Investigation of the Oldroyd model as a generalized incompressible Navier–Stokes equation via the interpolating stabilized element free Galerkin technique. Appl. Numer. Math. **150**, 274–294 (2020)
21. Mirzaei, D., Dehghan, M.: New implementation of MLBIE method for heat conduction analysis in functionally graded materials. Eng. Anal. Bound. Elem. **36**(4), 511–519 (2012)
22. Dehghan, M.: Three-level techniques for one-dimensional parabolic equation with nonlinear initial condition. Appl. Math. Comput. **151**(2), 567–579 (2004)
23. Tan, W.-C., Xu, M.-Y.: The impulsive motion of flat plate in a generalized second grade fluid. Mech. Res. Commun. **29**(1), 3–9 (2002)
24. Fetecau, C., Jamil, M., Fetecau, C., Vieru, D.: The Rayleigh–Stokes problem for an edge in a generalized Oldroyd-b fluid. Z. Angew. Math. Phys. **60**(5), 921–933 (2009)
25. Chen, C.-M., Liu, F., Burrage, K., Chen, Y.: Numerical methods of the variable-order Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivative. IMA J. Appl. Math. **78**(5), 924–944 (2013)
26. Hafez, R.M., Zaky, M.A., Abdelkawy, M.A.: Jacobi spectral Galerkin method for distributed-order fractional Rayleigh–Stokes problem for a generalized second grade fluid. Front. Phys. **7**, 240 (2020). https://doi.org/10.3389/fphy
27. Mohebbi, A., Abbaszadeh, M., Dehghan, M.: Compact finite difference scheme and RBF meshless approach for solving 2D Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivatives. Comput. Methods Appl. Mech. Eng. **264**, 163–177 (2013)
28. Kew, L.M., Ali, N.H.M.: New explicit group iterative methods in the solution of three dimensional hyperbolic telegraph equations. J. Comput. Phys. **294**, 382–404 (2015)
29. Ali, N.H.M., Saeed, A.M.: Preconditioned modified explicit decoupled group for the solution of steady state Navier–Stokes equation. Appl. Math. Inf. Sci. **7**(5), 1837 (2013)
30. Ali, N.H.M., Kew, L.M.: New explicit group iterative methods in the solution of two dimensional hyperbolic equations. J. Comput. Phys. **231**(20), 6953–6968 (2012)
31. Balasim, A.T., Ali, N.H.M.: Group Iterative Methods for the Solution of Two-Dimensional Time-Fractional Diffusion Equation. AIP Conference Proceedings, vol. 1750. AIP, New York (2016)
32. Cui, M.: Compact finite difference method for the fractional diffusion equation. J. Comput. Phys. **228**(20), 7792–7804 (2009)
33. Yuste, S.B.: Weighted average finite difference methods for fractional diffusion equations. J. Comput. Phys. **216**(1), 264–274 (2006)
34. Abbaszadeh, M., Mohebbi, A.: A fourth-order compact solution of the two-dimensional modified anomalous fractional sub-diffusion equation with a nonlinear source term. Comput. Math. Appl. **66**(8), 1345–1359 (2013)