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On the Neumann eigenvalues for second-order Sturm–Liouville difference equations

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Abstract
The paper is concerned with the Neumann eigenvalues for second-order Sturm–Liouville difference equations. By analyzing the new discriminant function, we show the interlacing properties between the periodic, antiperiodic, and Neumann eigenvalues. Moreover, when the potential sequence is symmetric and symmetric monotonic, we show the order relation between the first Dirichlet eigenvalue and the second Neumann eigenvalue, and prove that the minimum of the first Neumann eigenvalue gap is attained at the constant potential sequence.

MSC: 39A12; 15A42

Keywords: Second-order difference equations; Eigenvalue gap; Neumann eigenvalues

1 Introduction

Consider the second-order Sturm–Liouville difference equation

$$-\nabla(\triangle y_n) + q_n y_n = \lambda y_n \quad \text{on } [0, N-1], \tag{1}$$

where the potential sequence $q_i \geq 0$ for $i = 0, 1, 2, \dots, N-1$, $\triangle$ is the forward difference operator ($\triangle y_n = y_{n+1} - y_n$), ∇ is the backward difference operator ($\nabla y_n = y_n - y_{n-1}$), and the bracket $[0, N-1]$ means the integers in $[0, N-1]$. Note that equation (1) can be rewritten as the recurrence formula

$$y_{n+1} = (2 + q_n - \lambda) y_n - y_{n-1} \quad \text{on } [0, N-1]$$

or the matrix formula

$$(D + Q)\vec{y} = \lambda \vec{y},$$

where $\vec{y}$ is a vector in $\mathbb{R}^N$, Q is a diagonal matrix whose diagonal elements are $q_0, q_1, \ldots, q_{N-1}$, and D is the $N \times N$ tridiagonal matrix of the form

$$D = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & \cdots & -1 & 2 \end{bmatrix}.$$

It is clear that if $u(x) \in C^2(x_0 - h, x_0 + h)$ for $h > 0$, then there exists $\eta \in [x_0 - h, x_0 + h]$ such that

$$u(x_0 - h) - 2u(x_0) + u(x_0 + h) = u''(\eta)h^2.$$

Equation (1) can be regarded as a discrete analogue of the Sturm–Liouville problem

$$-u''(x) + q(x)u(x) = \lambda u(x) \quad \text{on } (0,1). \tag{2}$$

Sturm–Liouville problem (2) has been widely studied. For the potential function $q \in L^1$, the eigenvalues of Sturm–Liouville problem (2) with separated boundary conditions are real, simple, increasing and tend to infinity [3, 21, 23]. By defining Hill's discriminant of (2) by

$$H(\lambda) = y_1(1,\lambda) + y_2'(1,\lambda),$$

where $y_1(x,\lambda)$ and $y_2(2,\lambda)$ are solutions of (2) and satisfy

$$y_1(0,\lambda) = y_2'(0,\lambda) = 1, \qquad y_1'(0,\lambda) = y_2(0,\lambda) = 0,$$

one can show that [3, 9, 22] the periodic eigenvalues $\{\lambda_n\}_{n\ge 0}$, anti-periodic eigenvalues $\{\lambda_n'\}_{n\ge 1}$, Dirichlet eigenvalues $\{\mu_n\}_{n\ge 1}$, and Neumann eigenvalues $\{\nu_n\}_{n\ge 0}$ satisfy $H(\lambda_n) = 2$, $H(\lambda_n') = -2$, $H(\lambda_n') = -2$, $H(\mu_{2n+1}) < -2$, $H(\mu_{2n}) > 2$, $H(\nu_{2n}) > 2$, and $H(\nu_{2n+1}) < -2$. In particular, we also know $\nu_0 \le \lambda_0$ and

$$\cdots \le \lambda_{2n-2} < \lambda_{2n-1}' \le \frac{\nu_{2n-1}}{\mu_{2n-1}} \le \lambda_{2n}' < \lambda_{2n-1} \le \frac{\nu_{2n}}{\mu_{2n}} \le \lambda_{2n} < \lambda_{2n+1}' \le \cdots. \tag{3}$$

The periodic Sturm–Liouville problem is also called Hill's equation. The intervals $(\lambda_{2n-1}', \lambda_{2n}')$ and $(\lambda_{2n-1}, \lambda_{2n})$ are called the $(2n-1)$th and $2n$th instability intervals. The interval $(-\infty, \lambda_0)$ is called the zero-th instability interval. The name instability interval is used because, for all λ in these intervals, all nontrivial solutions of (2) are unbounded in $(-\infty, \infty)$. In 1946, Borg showed [5] that, for Hill's equation, the potential q is constant if and only if all instability intervals, except the zero-th, are absent. He also showed that all odd instability intervals $(\lambda_{2n-1}', \lambda_{2n}')$ vanish if and only if q has period $1/2$. Later on, Hochstadt [14] generalized Borg's results to show that if q is C^1, then q has period $1/n$ if and only if all those finite instability intervals whose index is not a multiple of n vanish. In particular, Hochstadt also showed that [11] if all but except one instability interval vanish, then the potential function has to be an elliptic function. Furthermore, it was proved that the first instability interval is absent if and only if the potential function is constant when the potential function q is assumed to be symmetric single-well [16]or single-well [6].

Recently, there have been a number of studies on minimum Dirichlet eigenvalue gaps of the Sturm–Liouville equations (2) with convex potentials [19, 20], symmetric single-well potentials [1], or single-well potentials [15], while the symmetric 1-step function is the potential function in $E[h,H,M] \equiv \{q \in \mathrm{PC}(0,1) : h \le q \le H \text{ a.e. and } \int_0^1 q = M\}$ giving the minimal Dirichlet eigenvalue gap [8]. Later on, Cheng et al. [6, 7] showed that if the potential function q is single-well with transition point $a = 1/2$, then $\nu_1 \ge \mu_1$ and $\nu_1 - \nu_0 \ge \pi^2$. Equality holds if and only if q is constant.

In this paper, we study the second-order difference equations (1). So far, there have been results on the second-order difference equations (1) which are analogue to the continuous Sturm–Liouville equation (2). Using the information on more than one set of eigenvalues, the potential sequence can be determined uniquely, for example, two sets of eigenvalues [12, 13], one set of eigenvalues plus a symmetric potential sequence [10], and one set of eigenvalues plus partial information of the potential sequence [25].

In 1990, Ashbaugh and Benguria [2] studied the comparison of the eigenvalues of two discrete Sturm–Liouville equations whose potential sequences satisfy certain relation. We say the sequence $\{x_k\}_{k=0}^{N-1}$ is symmetric if $x_k = x_{N-1-k}$ for $k = 0,1,2,\dots,N-1$, and the sequence $\{x_k\}_{k=0}^{N-1}$ is quasi-symmetric increasing if $x_0 \ge x_{N-1} \ge x_1 \ge \cdots \ge x_{[\frac{N}{2}]}$. In particular, the sequence $\{x_k\}_{k=0}^{N-1}$ is said to be symmetric increasing if it is symmetric and quasi-symmetric increasing. Ashbaugh and Benguria showed that if $\{q_k\}_{k=0}^{N-1}$ is symmetric increasing in (1), then the eigenvalue $\{\mu_k\}_{k=1}^{N-1}$ satisfies

$$\mu_2 - \mu_1 \ge 2\left[\cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{2\pi}{N}\right)\right].$$

Equality holds if and only if $q_k = q_0$ for $k = 0,1,2,\dots,N-1$. Note that if $q_k = 0$ for $k = 0,1,2,\dots,N-1$, then

$$\mu_k = 4\sin^2\left(\frac{k\pi}{2N}\right) = 2\left[1 - \cos\left(\frac{k\pi}{N}\right)\right], \quad k = 1,2,\dots,N-1.$$

Furthermore, the system of (1) with other self-adjoint boundary conditions has also been investigated [3, 18]. Jirari in 1995 showed that problem (1) with the boundary conditions

$$y_{-1} - \alpha y_0 = y_N - \beta y_{N-1} = 0, \tag{4}$$

where $\alpha, \beta \in \mathbb{R}$ has N real and simple eigenvalues. And recently, Ji and Yang [17] studied the eigenvalue comparison of (1) and (4). In particular, they also showed that if $q_i = 0$ for $i = 0,1,2,\dots,N-1$, then the first eigenvalue is simple and associated with the vector whose entries are of all ones.

In 2005, Wang and Shi [26] (see also [24]) were concerned with the eigenvalues for (1) coupled with the periodic boundary conditions

$$y_{-1} = y_{N-1}, \qquad y_0 = y_N, \tag{5}$$

the antiperiodic boundary conditions

$$y_{-1} = -y_{N-1}, \qquad y_0 = -y_N, \tag{6}$$

and the Dirichlet boundary conditions

$$y_0 = y_N = 0. \tag{7}$$

Define the discriminant of (1) by

$$d(\lambda) = \varphi_{N-1}(\lambda) + \psi_N(\lambda),$$

where φ_n and ψ_n are solutions of (1) satisfying the initial conditions

$$\varphi_{-1} = \psi_0 = 1, \qquad \varphi_0 = \psi_{-1} = 0.$$

By the similar argument as the differential equations (see [3, 22]), Wang and Shi showed that the periodic problem (1), (5) and the antiperiodic problem (1), (6) have exactly N real eigenvalues, while the Dirichlet problem (1), (7) has exactly $N-1$ real eigenvalues. Furthermore, they denoted by $\{\lambda_k\}_{k=0}^{N-1}$, $\{\tilde{\lambda}_k\}_{k=1}^{N}$, and $\{\mu_k\}_{k=1}^{N-1}$ the periodic, antiperiodic, and Dirichlet eigenvalues, respectively, and arranged them in the nondecreasing order $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{N-1}$, $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_N$, and $\mu_1 < \mu_1 < \cdots < \mu_{N-1}$. They showed that a set of these three eigenvalues satisfy the following interlacing properties: if N is odd,

$$\lambda_0 < \tilde{\lambda}_1 \le \mu_1 \le \tilde{\lambda}_2 < \lambda_1 \le \mu_2 \le \lambda_2 < \cdots < \lambda_{N-2} \le \mu_{N-1} \le \lambda_{N-1} < \tilde{\lambda}_N,$$

and if N is even,

$$\lambda_0 < \tilde{\lambda}_1 \le \mu_1 \le \tilde{\lambda}_2 < \lambda_1 \le \mu_2 \le \lambda_2 < \cdots < \tilde{\lambda}_{N-1} \le \mu_{N-1} \le \tilde{\lambda}_N < \lambda_{N-1}.$$

In this paper, we consider the second-order difference equations (1) coupled with the Neumann boundary conditions

$$y_{-1} - y_0 = y_N - y_{N-1} = 0. \tag{8}$$

Denote by $\{\nu_k\}_{k=0}^{N-1}$ the Neumann eigenvalues of the second-order difference equations (1). It is known that if $q_k = 0$ for $k = 0, 1, 2, \ldots, N-1$, then

$$\nu_k = 4\sin^2\left(\frac{k\pi}{2N}\right), \quad k = 0, 1, 2, \ldots, N-1.$$

Combined with the result of [26], we will show the interlacing properties of the eigenvalues, which is a discrete analogue result for the continuous Sturm–Liouville problem (see

[9, 22]). We shall remark that, in the continuous case, we analyze Hill's discriminant $H(\lambda)$ to obtain the interlacing property (3). But in the discrete case, we need to define another discriminant

$$f(\lambda) = \varphi_{N-1}(\lambda) + \psi_N(\lambda) - \psi_{N-1}(\lambda),$$

where φ_n and ψ_n are solutions of (1) satisfying the initial conditions

$$\varphi_{-1} = \psi_0 = \varphi_0 = 1, \qquad \psi_{-1} = 0,$$

to show the interlacing property for the Neumann eigenvalues in Theorem 1. By analyzing this new discriminant $f(\lambda)$, we can prove Theorem 1.

Theorem 1 *Consider the second-order difference equations* (1). *The eigenvalues satisfy the following interlacing inequality*: *if N is odd*,

$$\nu_0 \le \lambda_0 < \tilde{\lambda}_1 \le \begin{matrix} \mu_1 \\ \nu_1 \end{matrix} \le \tilde{\lambda}_2 < \lambda_1 \le \begin{matrix} \mu_2 \\ \nu_2 \end{matrix} \le \lambda_2 < \cdots < \lambda_{N-2} \le \begin{matrix} \mu_{N-1} \\ \nu_{N-1} \end{matrix} \le \lambda_{N-1} < \tilde{\lambda}_N,$$

and if N is even,

$$\nu_0 \le \lambda_0 < \tilde{\lambda}_1 \le \begin{matrix} \mu_1 \\ \nu_1 \end{matrix} \le \tilde{\lambda}_2 < \lambda_1 \le \begin{matrix} \mu_2 \\ \nu_2 \end{matrix} \le \lambda_2 < \cdots < \tilde{\lambda}_{N-1} \le \begin{matrix} \mu_{N-1} \\ \nu_{N-1} \end{matrix} \le \tilde{\lambda}_N < \lambda_{N-1}.$$

After obtaining Theorem 1, we will consider the order relation of the first Dirichlet eigenvalue μ_1 and the second Neumann eigenvalue ν_1, and the first Neumann eigenvalue gap $\nu_1 - \nu_0$. Theorems 2 and 3 can be regarded as discrete analogue results of [6] and [7] respectively for the continuous Sturm–Liouville problem (2).

Theorem 2 *Consider the second-order difference equation* (1). *If q_k is symmetric and symmetric decreasing and satisfies* $\max_{k\in[0,N-1]} q_k \le \nu_1$, *then $\mu_1 \le \nu_1$ and the equality holds if and only if $q_k = q_0$ for all $k \in [0, N-1]$.*

Theorem 3 *Consider the second-order difference equation* (1). *If q_n is symmetric and symmetric increasing, then $\nu_1 - \nu_0 \ge 2[1 - \cos(\frac{\pi}{N+1})]$ and the equality holds if and only if $q_k = q_0$ for all $k \in [0, N-1]$.*

The paper is organized as follows. Section 2 gives lemmas about the Wronskian and a variation of constant formula which is used in Sect. 3. In Sect. 3, we study the interlacing properties for the periodic, antiperiodic, and Neumann eigenvalues, and use an argument similar to that in [9, 22] to prove Theorem 1. Finally, the proof of Theorem 2 is given in Sect. 4, while the proof of Theorem 3 is given in Sect. 5.

2 Preliminaries

In this section, we derive some discrete analogous lemmas of the continuous case. One can refer to [4]. Lemmas 1 and 2 have been shown in [18] (see also [26]) by using a similar argument as the continuous case, so we omit the proofs here.

Lemma 1 ([18, Theorem 2.2.3]) *Let y and z be solutions of*

$$-\nabla(\triangle y_n) + q_n y_n = \lambda y_n, \quad n \in [0, N-1]$$

and

$$-\nabla(\triangle z_n) + q_n z_n = \mu z_n, \quad n \in [0, N-1]$$

respectively. Then, for $0 \le n \le N-1$,

$$(\lambda - \mu)\sum_{j=0}^{n} y_j z_j = (y_n z_{n+1} - y_{n+1} z_n) - (y_{-1} z_0 - y_0 z_{-1}).$$

Let $\lambda = \mu$ in Lemma 1, we have the following Wronskian-type identity.

Lemma 2 ([18, Theorem 2.2.8]) *Let y and z be solutions of* (1). *Then the Wronskian*

$$W[y,z](n) \equiv \begin{vmatrix} y_{n+1} & z_{n+1} \\ y_{n+1} - y_n & z_{n+1} - z_n \end{vmatrix} = y_n z_{n+1} - y_{n+1} z_n$$

is a constant on $[-1, N-1]$.

Now, let φ_n and ψ_n be two solutions of (1) satisfying the initial conditions

$$\varphi_{-1} = \psi_0 = \varphi_0 = 1, \qquad \psi_{-1} = 0. \tag{9}$$

Note that $\triangle\varphi_{-1} = \varphi_0 - \varphi_{-1} = 0$ and $\triangle\psi_{-1} = \psi_0 - \psi_{-1} = 1$. In particular, we find that, by Theorem 2,

$$W[\varphi_n, \psi_n](N-1) = \varphi_{N-1}\psi_N - \varphi_N\psi_{N-1} = \varphi_{-1}\psi_0 - \varphi_0\psi_{-1} = 1, \tag{10}$$

and it is known that φ_n, ψ_n are two linear independent solutions of (1). The following theorem is similar to [26, Theorem 2.3], but the initial conditions are different.

Theorem 4 *For any $\{f_n\}_{n=0}^{N-1} \subseteq \mathbb{R}$ and for any $c_0, c_1 \in \mathbb{R}$, the initial value problem*

$$-\nabla(\triangle z_n) + (q_n - \lambda) z_n = f_n, \quad n \in [0, N-1], \tag{11}$$

$$z_{-1} = c_{-1}, \qquad z_0 = c_0 \tag{12}$$

has a unique solution z, which can be expressed as

$$z_n = c_{-1}\varphi_n + (c_0 - c_{-1})\psi_n + \sum_{j=0}^{n-1}(\varphi_n\psi_j - \varphi_j\psi_n) f_j, \quad n \in [-1, N],$$

where $\sum_{j=0}^{-1} \bullet \equiv 0$.

Proof The technique of the proof is based on the variation of parameters on the differential equation. Let

$$z_n = A_n\varphi_n + B_n\psi_n, \quad n \in [-1, N]$$

be a solution of (11). Then

$$\Delta z_n = A_n\Delta\varphi_n + \varphi_{n+1}\Delta A_n + B_n\Delta\psi_n + \psi_{n+1}\Delta B_n, \quad n \in [-1, N-1].$$

Setting

$$\varphi_{n+1}\Delta A_n + \psi_{n+1}\Delta B_n = 0, \quad n \in [-1, N-1], \tag{13}$$

we have

$$\Delta z_n = A_n\Delta\varphi_n + B_n\Delta\psi_n, \quad n \in [-1, N-1].$$

Since φ_n and ψ_n are two solutions of (1), we find

$$\begin{aligned}
&\varphi_{n+1} = (2 + q_n - \lambda)\varphi_n - \varphi_{n-1}, \quad n \in [0, N-1],\\
&\psi_{n+1} = (2 + q_n - \lambda)\psi_n - \psi_{n-1}, \quad n \in [0, N-1],
\end{aligned}$$

and hence

$$\begin{aligned}
-\nabla(\triangle z_{n+1}) &= -A_{n+1}\varphi_{n+2} + A_n\varphi_{n+1} + A_{n+1}\varphi_{n+1} - A_n\varphi_n\\
&\quad - B_{n+1}\psi_{n+2} + B_n\psi_{n+1} + B_{n+1}\psi_{n+1} - B_n\psi_n\\
&= -A_{n+1}\big((2 + q_{n+1} - \lambda)\varphi_{n+1} - \varphi_n\big) + A_n\varphi_{n+1} + A_{n+1}\varphi_{n+1} - A_n\varphi_n\\
&\quad - B_{n+1}\big((2 + q_{n+1} - \lambda)\psi_{n+1} - \psi_n\big) + B_n\psi_{n+1} + B_{n+1}\psi_{n+1} - B_n\psi_n\\
&= -(q_{n+1} - \lambda)(A_{n+1}\varphi_{n+1} + B_{n+1}\psi_{n+1})\\
&\quad - (A_{n+1} - A_n)(\varphi_{n+1} - \varphi_n) - (B_{n+1} - B_n)(\psi_{n+1} - \psi_n)\\
&= -(q_{n+1} - \lambda)z_{n+1} - \Delta A_n\Delta\varphi_n - \Delta B_n\Delta\psi_n
\end{aligned}$$

for $n \in [-1, N-2]$. Combined with (11), we find

$$\Delta\varphi_n\Delta A_n + \Delta\psi_n\Delta B_n = -f_{n+1}, \quad n \in [-1, N-2]. \tag{14}$$

Now, solving system (13) and (14) for $(\Delta A_n, \Delta B_n)$, we find

$$\Delta A_n = \frac{\psi_{n+1}f_{n+1}}{\varphi_{n+1}\Delta\psi_n - \psi_{n+1}\Delta\varphi_n}, \qquad \Delta B_n = \frac{\varphi_{n+1}f_{n+1}}{\psi_{n+1}\Delta\varphi_n - \varphi_{n+1}\Delta\psi_n}, \quad n \in [-1, N-2].$$

By (10), we find $\varphi_{n+1}\Delta\psi_n - \psi_{n+1}\Delta\varphi_n = 1$. Hence

$$\Delta A_n = A_{n+1} - A_n = \psi_{n+1}f_{n+1}, \qquad \Delta B_n = B_{n+1} - B_n = -\varphi_{n+1}f_{n+1}, \quad n \in [-1, N-2]$$

and then, by defining $\sum_{j=0}^{-1}\bullet = 0$, we have

$$A_n = A_{-1} + \sum_{j=0}^{n} \psi_j f_j, \qquad B_n = B_{-1} - \sum_{j=0}^{n} \varphi_j f_j, \quad n \in [-1, N-1].$$

This implies that

$$z_n = A_n \varphi_n + B_n \psi_n = A_{-1}\varphi_n + B_{-1}\psi_n + \sum_{j=0}^{n-1} (\varphi_n \psi_j - \varphi_j \psi_n) f_j, \quad n \in [-1, N-1].$$

By (9) and (12), we find

$$A_{-1} = c_{-1}, \qquad B_{-1} = c_0 - c_{-1}.$$

Finally, for $n = N$, we evaluate

$$\begin{aligned}
z_N &= -f_{N-1} + (2 + q_{N-1} - \lambda) z_{N-1} - z_{N-2} \\
&= -f_{N-1} + (2 + q_{N-1} - \lambda)\left[c_{-1}\varphi_{N-1} + (c_0 - c_{-1})\psi_{N-1} + \sum_{j=0}^{N-2} (\varphi_{N-1}\psi_j - \varphi_j \psi_{N-1}) f_j \right] \\
&\quad - \left[c_{-1}\varphi_{N-2} + (c_0 - c_{-1})\psi_{N-2} + \sum_{j=0}^{N-3} (\varphi_{N-2}\psi_j - \varphi_j \psi_{N-2}) f_j \right] \\
&= -f_{N-1} + c_{-1}\big[(2 + q_{N-1} - \lambda)\varphi_{N-1} - \varphi_{N-2}\big] \\
&\quad + (c_0 - c_{-1})\big[(2 + q_{N-1} - \lambda)\psi_{N-1} - \psi_{N-2}\big] \\
&\quad + \sum_{j=0}^{N-2} \big[(2 + q_{N-1} - \lambda)(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}) - (\varphi_{N-2}\psi_j - \varphi_j \psi_{N-2})\big] f_j \\
&= -f_{N-1} + c_{-1}\varphi_N + (c_0 - c_{-1})\psi_N + \sum_{j=0}^{N-2} [\varphi_N \psi_j - \varphi_j \psi_N] f_j \\
&= c_{-1}\varphi_N + (c_0 - c_{-1})\psi_N + \sum_{j=0}^{N-1} [\varphi_N \psi_j - \varphi_j \psi_N] f_j.
\end{aligned}$$

This proof is complete. □

3 Interlacing properties of eigenvalues

Recall φ_n and ψ_n defined in Sect. 2. It is clear that λ is an eigenvalue of (1) and (5) if and only if $c_1\varphi_n(\lambda) + c_2\psi_n(\lambda)$ satisfies the periodic boundary conditions, i.e.,

$$\begin{aligned}
&c_1\varphi_{-1}(\lambda) + c_2\psi_{-1}(\lambda) = c_1\varphi_{N-1}(\lambda) + c_2\psi_{N-1}(\lambda), \\
&c_1\varphi_0(\lambda) + c_2\psi_0(\lambda) = c_1\varphi_N(\lambda) + c_2\psi_N(\lambda).
\end{aligned}$$

By (9), we find

$$c_1\big[\varphi_{N-1}(\lambda)-1\big]+c_2\psi_{N-1}(\lambda)=0,$$
$$c_1\big[\varphi_N(\lambda)-1\big]+c_2\big[\psi_N(\lambda)-1\big]=0.$$

The above system has a nontrivial solution (c_1,c_2) if and only if

$$\begin{vmatrix}\varphi_{N-1}(\lambda)-1 & \psi_{N-1}(\lambda)\\ \varphi_N(\lambda)-1 & \psi_N(\lambda)-1\end{vmatrix}=0,$$

which implies that, combined with (10),

$$f(\lambda)\equiv\varphi_{N-1}(\lambda)+\psi_N(\lambda)-\psi_{N-1}(\lambda)=2. \tag{15}$$

Hence, we find that $f(\lambda)=2$ if and only if λ is a periodic eigenvalue. Similarly, it can be showed that $f(\lambda)=-2$ if and only if λ is an antiperiodic eigenvalue. In particular, we have the following lemma.

Lemma 3 $f'(\lambda)=0$ *whenever* $f(\lambda)=2$ *if and only if* λ *is a multiple eigenvalue of* (1) *and* (5), *while* $f'(\lambda)=0$ *whenever* $f(\lambda)=-2$ *if and only if* λ *is a multiple eigenvalue of* (1) *and* (6).

Proof

First, we differentiate

$$-\nabla(\triangle\varphi_n)+q_n\varphi_n=\lambda\varphi_n,\qquad -\nabla(\triangle\psi_n)+q_n\psi_n=\lambda\psi_n$$

with respect to λ to obtain

$$-\nabla\big(\triangle\varphi_n'\big)+(q_n-\lambda)\varphi_n'(\lambda)=\varphi_n(\lambda),$$
$$-\nabla\big(\triangle\psi_n'\big)+(q_n-\lambda)\psi_n'(\lambda)=\psi_n(\lambda).$$

In particular, by (9), we also have

$$\varphi_{-1}'=\psi_{-1}'=\varphi_0'=\psi_0'=0.$$

By Theorem 4, we obtain

$$\varphi_n'=\sum_{j=0}^{n-1}\big(\varphi_n(\lambda)\psi_j(\lambda)-\varphi_j(\lambda)\psi_n(\lambda)\big)\varphi_j(\lambda),\quad n\in[-1,N],$$
$$\psi_n'=\sum_{j=0}^{n-1}\big(\varphi_n(\lambda)\psi_j(\lambda)-\varphi_j(\lambda)\psi_n(\lambda)\big)\psi_j(\lambda),\quad n\in[-1,N].$$

Hence, we have

$$\begin{aligned}
f'(\lambda) &= \varphi'_{N-1}(\lambda) + \psi'_N(\lambda) - \psi'_{N-1}(\lambda) \\
&= \sum_{j=0}^{N-2} \big(\varphi_{N-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_{N-1}(\lambda)\big)\varphi_j(\lambda) \\
&\quad + \sum_{j=0}^{N-1} \big(\varphi_N(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_N(\lambda)\big)\psi_j(\lambda) \\
&\quad - \sum_{j=0}^{N-2} \big(\varphi_{N-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_{N-1}(\lambda)\big)\psi_j(\lambda) \\
&= \sum_{j=0}^{N-1} \big(\varphi_{N-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_{N-1}(\lambda)\big)\varphi_j(\lambda) \\
&\quad + \sum_{j=0}^{N-1} \big(\varphi_N(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_N(\lambda)\big)\psi_j(\lambda) \\
&\quad - \sum_{j=0}^{N-1} \big(\varphi_{N-1}(\lambda)\psi_j(\lambda) - \varphi_j(\lambda)\psi_{N-1}(\lambda)\big)\psi_j(\lambda) \\
&= -\sum_{j=0}^{N-1} \big\{\big[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\big]\psi_j^2(\lambda) \\
&\quad + \big[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)\big]\varphi_j(\lambda)\psi_j(\lambda) + \psi_{N-1}(\lambda)\varphi_j^2(\lambda)\big\}.
\end{aligned}$$

Denote

$$I(\lambda) \equiv \begin{bmatrix} \varphi_{N-1}(\lambda) - \varphi_N(\lambda) & \frac{\psi_N(\lambda)-\psi_{N-1}(\lambda)-\varphi_{N-1}(\lambda)}{2} \\ \frac{\psi_N(\lambda)-\psi_{N-1}(\lambda)-\varphi_{N-1}(\lambda)}{2} & \psi_{N-1}(\lambda) \end{bmatrix}, \qquad \vec{\omega} = \begin{bmatrix} \psi_j(\lambda) \\ \varphi_j(\lambda) \end{bmatrix}$$

and

$$\begin{aligned}
\delta_j(\lambda) &\equiv \vec{\omega}^T I \vec{\omega} \\
&= \big[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\big]\psi_j^2(\lambda) + \big[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)\big]\varphi_j(\lambda)\psi_j(\lambda) \\
&\quad + \psi_{N-1}(\lambda)\varphi_j^2(\lambda).
\end{aligned}$$

By (10) and (15), we find

$$\begin{aligned}
\det I &= \psi_{N-1}(\lambda)\big[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\big] - \left(\frac{\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)}{2}\right)^2 \\
&= \psi_{N-1}(\lambda)\big[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\big] - \left(\frac{\psi_N(\lambda) - \psi_{N-1}(\lambda) + \varphi_{N-1}(\lambda)}{2}\right)^2 \\
&\quad + \big[\psi_N(\lambda) - \psi_{N-1}(\lambda)\big]\varphi_{N-1}(\lambda) \\
&= \varphi_{N-1}(\lambda)\psi_N(\lambda) - \varphi_N(\lambda)\psi_{N-1}(\lambda) - 1 \\
&= 0.
\end{aligned}$$

Hence, the matrix $I(\lambda)$ is always positive semi-definite or negative semi-definite in this case. Since $\varphi_j(\lambda)$ and $\psi_j(\lambda)$ are linearly independent, we find that, if $f(\lambda) = 2$, then $f'(\lambda) = 0$ if and only if $\delta_j(\lambda) \equiv 0$ for all $0 \leq j \leq N-1$. In this case, we find I is a zero matrix. Hence, we have $\psi_{N-1}(\lambda) = 0$, $\varphi_{N-1}(\lambda) - \varphi_N(\lambda) = 0$, and $\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda) = 0$. Combined with (10) and (15), we can obtain $\psi_N(\lambda) = \varphi_{N-1}(\lambda) = \varphi_N(\lambda) = 1$ and $\psi_{N-1}(\lambda) = 0$. This implies that $\varphi_j(\lambda)$ and $\psi_j(\lambda)$ are two solutions of (1) and (5), i.e., $f'(\lambda) = 0$ whenever $f(\lambda) = 2$ if and only if λ is not simple. Similarly, it can be showed that $f'(\lambda) = 0$ whenever $f(\lambda) = -2$ if and only if λ is not simple. □

Next, we investigate the Neumann eigenvalues of (1) and (8), and give a proof of Theorem 3. Denote by $\{\nu_k\}_{k=0}^{N-1}$ the set of Neumann eigenvalues. We have the following lemmas.

Lemma 4 *For* $0 \leq k \leq N-1$, $f(\nu_k) \geq 2$ *if kis even, and* $f(\nu_k) \leq -2$ *if k is odd.*

Proof Let $\phi_n(\lambda) \equiv c_1\varphi_n(\lambda) + c_2\psi_n(\lambda)$. Then $\phi_n(\nu_k)$ is an eigenfunction of (1) and (8) if

$$\phi_{-1}(\nu_k) - \phi_0(\nu_k) = 0, \qquad \phi_N(\nu_k) - \phi_{N-1}(\nu_k) = 0,$$

i.e.,

$$\begin{aligned} &c_1\varphi_{-1}(\nu_k) + c_2\psi_{-1}(\nu_k) = c_1\varphi_0(\nu_k) + c_2\psi_0(\nu_k), \\ &c_1\varphi_N(\nu_k) + c_2\psi_N(\nu_k) = c_1\varphi_{N-1}(\nu_k) + c_2\psi_{N-1}(\nu_k). \end{aligned}$$

By (9), we have

$$c_2 = 0, \qquad c_1\bigl[\varphi_N(\nu_i) - \varphi_{N-1}(\nu_i)\bigr] = 0.$$

Since c_1 and c_2 are not both zero, we find $\varphi_N(\nu_k) - \varphi_{N-1}(\nu_k) = 0$. This implies that $\varphi_n(\nu_k)$ is an eigenfunction of (1) and (8) with respect to ν_k.

On the other hand, by (10), we find

$$\begin{aligned} 1 &= \varphi_{N-1}(\nu_k)\psi_N(\nu_k) - \varphi_N(\nu_k)\psi_{N-1}(\nu_k) \\ &= \varphi_{N-1}(\nu_k)\bigl[\psi_N(\nu_k) - \psi_{N-1}(\nu_k)\bigr] + \bigl[\varphi_{N-1}(\nu_k) - \varphi_N(\nu_k)\bigr]\psi_{N-1}(\nu_k) \\ &= \varphi_{N-1}(\nu_k)\bigl[\psi_N(\nu_k) - \psi_{N-1}(\nu_k)\bigr]. \end{aligned}$$

Hence,

$$\psi_N(\nu_k) - \psi_{N-1}(\nu_k) = \frac{1}{\varphi_{N-1}(\nu_k)}.$$

This implies that

$$f(\nu_k) \equiv \varphi_{N-1}(\nu_k) + \psi_N(\nu_k) - \psi_{N-1}(\nu_k) = \varphi_{N-1}(\nu_k) + \frac{1}{\varphi_{N-1}(\nu_k)}.$$

Finally, since $\varphi_n(\nu_k)$, $n = 0,1,2,\dots,N-1$, changes sign k times and $\varphi_0(\nu_k) = 1$, we find $\operatorname{sgn}(\varphi_{N-1}(\nu_k)) = (-1)^k$. This implies that

$$f(\nu_k) = \varphi_{N-1}(\nu_k) + \frac{1}{\varphi_{N-1}(\nu_k)} \begin{cases} \geq 2 & \text{if } k \text{ is even,} \\ \leq -2 & \text{if } k \text{ is odd.} \end{cases} \qquad \square$$

Lemma 5 $f'(\lambda) < 0$ *for all* $\lambda < \nu_0$.

Proof Suppose $f(\lambda) = \pm 2$ for some $\lambda \neq \nu_i$, and $\delta_j(\lambda) \neq 0$. By (10) and (15), we calculate that

$$\begin{aligned} \delta_j(\lambda) &= \left[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\right]\psi_j^2(\lambda) + \left[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)\right]\varphi_j(\lambda)\psi_j(\lambda) \\ &\quad + \psi_{N-1}(\lambda)\varphi_j^2(\lambda) \\ &= \left[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\right]\left\{\psi_j(\lambda) + \frac{[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)]}{2[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)]}\varphi_j(\lambda)\right\}^2 \\ &\quad + \psi_{N-1}(\lambda)\varphi_j^2(\lambda) - \frac{[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)]^2}{4[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)]}\varphi_j^2(\lambda) \\ &= \left[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)\right]\left\{\psi_j(\lambda) + \frac{[\psi_N(\lambda) - \psi_{N-1}(\lambda) - \varphi_{N-1}(\lambda)]}{2[\varphi_{N-1}(\lambda) - \varphi_N(\lambda)]}\varphi_j(\lambda)\right\}^2. \end{aligned}$$

We find

$$f'(\lambda) = [\varphi_N(\lambda) - \varphi_{N-1}(\lambda)]\sum_{j=0}^{N-1}\left\{\psi_j(\lambda) + \frac{\psi_{N+1}(\lambda) - \psi_N(\lambda) - \varphi_N(\lambda)}{2[\varphi_N(\lambda) - \varphi_{N+1}(\lambda)]}\varphi_j(\lambda)\right\}^2,$$

and hence $f'(\lambda)$ and $\varphi_N(\lambda) - \varphi_{N-1}(\lambda)$ have the same sign. Since $f(\lambda)$ is continuous on λ, and ν_0 is the first Neumann eigenvalue and is simple, we find $f'(\lambda) < 0$ if $\lambda < \nu_0$. $\square$

By the above discussion and combining the result of [26, Theorem 3.1], we find that if N is odd,

$$\nu_0 \leq \lambda_0 < \tilde{\lambda}_1 \leq \frac{\mu_1}{\nu_1} \leq \tilde{\lambda}_2 < \lambda_1 \leq \frac{\mu_2}{\nu_2} \leq \lambda_2 < \cdots < \lambda_{N-2} \leq \frac{\mu_{N-1}}{\nu_{N-1}} \leq \lambda_{N-1} < \tilde{\lambda}_N,$$

and if N is even,

$$\nu_0 \leq \lambda_0 < \tilde{\lambda}_1 \leq \frac{\mu_1}{\nu_1} \leq \tilde{\lambda}_2 < \lambda_1 \leq \frac{\mu_2}{\nu_2} \leq \lambda_2 < \cdots < \tilde{\lambda}_{N-1} \leq \frac{\mu_{N-1}}{\nu_{N-1}} \leq \tilde{\lambda}_N < \lambda_{N-1}.$$

4 The order relation of the first Dirichlet eigenvalue and the second Neumann eigenvalue

In this section, we investigate the order relation between the first Dirichlet eigenvalue and the second Neumann eigenvalue. Denote by $(\nu_k, \vec{w}_k)_{k=0}^{N-1}$ the Neumann eigenpairs of (1) and (8) with $\|\vec{w}_k\|_2 = 1$. In particular, $\vec{w}_k$ can be chosen to have exactly k sign changes. We have the following lemma.

Lemma 6 *If* $\max_{k\in[0,N-1]} q_k \leq \nu_1$, *then* $\vec{w}_1$ *is decreasing.*

Proof Denote $\vec{w}_1 = (w_{1,0}, w_{1,1}, w_{1,2}, \dots, w_{1,N-1})$. Since $\vec{w}_1$ satisfies the Neumann boundary conditions, we also assume $w_{1,-1} = w_{1,0}$ and $w_{1,N} = w_{1,N-1}$. Then we have

$$\begin{aligned} &w_{1,j+1} = (2 + q_j - \nu_1)w_{1,j} - w_{1,j-1} \quad \text{for } j = 0, 1, 2, \dots, N-1, \\ &w_{1,-1} = w_{1,0}, \qquad w_{1,N} = w_{1,N-1}. \end{aligned}$$

First, we evaluate that

$$w_{1,1} = (2 + q_0 - \nu_1)w_{1,0} - w_{1,-1} = w_{1,0} + (q_0 - \nu_1)w_{1,0} \le w_{1,0}.$$

Now, assume $w_{1,j} \le w_{1,j-1}$. Then

$$w_{1,j+1} = (2 + q_j - \nu_1)w_{1,j} - w_{1,j-1} = w_{1,j} + (q_j - \nu_1)w_{1,j} + w_{1,j} - w_{1,j-1} \le w_{1,j}.$$

By induction, we find $\vec{w}_1$ is decreasing. □

Proof of Theorem 2 Define $\vec{z} = (z_j)_{j=0}^N$ where $z_j = w_{1,j} - w_{1,j-1}$ for $j = 0, 1, 2, \dots, N$. Then $z_0 = z_N = 0$. In particular, by Lemma 6, $z_j > 0$. Hence, $\vec{z}$ is the first Dirichlet eigenfunction. By the variational principle, we have

$$\mu_1 \le \frac{\langle \vec{z}, H\vec{z} \rangle}{\langle \vec{z}, \vec{z} \rangle},$$

where $(Hz)_j = -z_{j+1} + (2 + q_j)z_j - z_{j-1}$. Since $\vec{w}_1$ changes sign only once and q_k is symmetric, we find that if $N-1$ is even, then $w_{1,j} > 0$ for $j \in [0, \frac{N-1}{2})$, $w_{1,j} < 0$ for $j \in (\frac{N-1}{2}, N-1]$, and $w_{1,\frac{N-1}{2}} = 0$; while if $N-1$ is odd, then $w_{1,j} > 0$ for $j \in [0, \frac{N-2}{2}]$ and $w_{1,j} < 0$ for $j \in [\frac{N}{2}, N-1]$. Furthermore, since q_k is symmetric and symmetric decreasing, we have, for $j = 0, 1, \dots, N-1$,

$$\begin{aligned} &-z_{j+1} + (2 + q_j)z_j - z_{j-1} \\ &\quad = -(w_{1,j+1} - w_{1,j}) + (2 + q_j)(w_{1,j} - w_{1,j-1}) - (w_{1,j-1} - w_{1,j}) \\ &\quad = \bigl(-w_{1,j+1} + (2 + q_j)w_{1,j} - w_{1,j-1}\bigr) - \bigl(-w_{1,j} + (2 + q_j)w_{1,j-1} - w_{1,j-1}\bigr) \\ &\quad \le \bigl(-w_{1,j+1} + (2 + q_j)w_{1,j} - w_{1,j-1}\bigr) - \bigl(-w_{1,j} + (2 + q_{j-1})w_{1,j-1} - w_{1,j-1}\bigr) \\ &\quad = \nu_1 w_{1,j} - \nu_1 w_{1,j-1} \\ &\quad = \nu_1 z_j. \end{aligned}$$

This implies that

$$\mu_1 \le \frac{\langle \vec{z}, H\vec{z} \rangle}{\langle \vec{z}, \vec{z} \rangle} = \frac{\nu_1 \sum_{i=0}^{N-1} z_i^2}{\sum_{i=0}^{N-1} z_i^2} = \nu_1.$$

The equality holds if and only if $q_j = q_0$ for $j = 1, 2, \dots, N-1$. □

5 The lower bound of the first Neumann eigenvalue gap

In this section, we give an optimal lower bound of the first Neumann eigenvalue gap.

Lemma 7 *Consider* (1) *and* (8), *and let $q_n(t)$ be a one-parameter family of potential sequences such that $q_n'(t)$ exists. Then*

$$\nu_k'(t)=\sum_{j=0}^{N-1}q_j'(t)w_{k,j}^2(t).$$

Proof Consider (1) and (8) with the potential sequence $q_j(t)$:

$$w_{k,j+1}(t)=\bigl(2+q_j(t)-\nu_k(t)\bigr)w_{k,j}(t)-w_{k,j-1}(t).$$

Differentiating the above equation with respect to t, we have

$$w_{k,j+1}'(t)=\bigl(q_j'(t)-\nu_k'(t)\bigr)w_{k,j}(t)+\bigl(2+q_j(t)-\nu_k(t)\bigr)w_{k,j}'(t)-w_{k,j-1}'(t).$$

Furthermore, we can obtain

$$\begin{aligned}&w_{k,j+1}'(t)w_{k,j}(t)-w_{k,j+1}(t)w_{k,j}'(t)\\&\quad=\bigl(q_j'(t)-\nu_k'(t)\bigr)w_{k,j}^2(t)+w_{k,j-1}(t)w_{k,j}'(t)-w_{k,j}(t)w_{k,j-1}'(t).\end{aligned}$$

Summing up the above equation from $j=0$ to $j=N-1$, we have

$$\nu_k'(t)\sum_{j=0}^{N-1}w_{k,j}^2(t)=\sum_{j=0}^{N-1}q_j'(t)w_{k,j}^2(t).$$

Since $\|\vec{w}_k\|_2=1$, we find

$$\nu_k'(t)=\sum_{j=0}^{N-1}q_j'(t)w_{k,j}^2(t).$$

□

Lemma 8 *There exist $j_1,j_2\in(0,N-1)$ with $j_1<j_2$ such that*

$$w_{1,j}^2(t)-w_{0,j}^2(t)\begin{cases}\geq 0 & \text{if } j\in[0,j_1]\cup[j_2,N-1],\\ \leq 0 & \text{if } j\in[j_1,j_2].\end{cases}$$

Proof First, since $\vec{w}_1$ changes sign only once, we may assume $w_{1,j}\geq 0$ for $j\in[0,\bar{k}]$ and $w_{1,j}\leq 0$ for $j\in[\bar{k},N-1]$. Since

$$\begin{aligned}w_{0,1}&=(2+q_0-\nu_0)w_{0,0}-w_{0,-1}=(1+q_0-\nu_0)w_{0,0},\\w_{1,1}&=(2+q_0-\nu_1)w_{1,0}-w_{1,-1}=(1+q_0-\nu_1)w_{1,0},\end{aligned}$$

we have

$$w_{0,1}w_{1,0}-w_{0,0}w_{1,1}=(\nu_1-\nu_0)w_{0,0}w_{1,0}>0,$$

and hence

$$\frac{w_{1,0}}{w_{0,0}} > \frac{w_{1,1}}{w_{0,1}}.$$

Now, for $j \in [0, \bar{k} - 1]$, assume $\frac{w_{1,j}}{w_{0,j}} > \frac{w_{1,j+1}}{w_{0,j+1}}$ holds. Then

$$w_{0,j+2}w_{1,j+1} - w_{0,j+1}w_{1,j+2} = (\nu_1 - \nu_0)w_{0,j+1}w_{1,j+1} + w_{0,j+1}w_{1,j} - w_{0,j}w_{1,j+1} > 0,$$

and hence

$$\frac{w_{1,j+1}}{w_{0,j+1}} > \frac{w_{1,j+2}}{w_{0,j+2}}.$$

By induction, we find

$$\frac{w_{1,0}}{w_{0,0}} > \frac{w_{1,1}}{w_{0,1}} > \cdots > \frac{w_{1,\bar{k}-1}}{w_{0,\bar{k}-1}} > \frac{w_{1,\bar{k}}}{w_{0,\bar{k}}}.$$

This implies that there exists $j_1 \in (0, \bar{k})$ such that

$$\frac{w_{1,j_1}}{w_{0,j_1}} \geq 1 > \frac{w_{1,j_1+1}}{w_{0,j_1+1}}.$$

Hence,

$$w_{1,j}^2(t) - w_{0,j}^2(t) \begin{cases} \geq 0 & \text{if } j \in [0, j_1], \\ \leq 0 & \text{if } j \in [j_1, \bar{k}]. \end{cases}$$

Similarly, it can be showed that there exists $j_2 \in (\bar{k}, N-1)$ such that

$$w_{1,j}^2(t) - w_{0,j}^2(t) \begin{cases} \geq 0 & \text{if } j \in [j_2, N-1], \\ \leq 0 & \text{if } j \in [\bar{k}, j_2]. \end{cases}$$

The proof is complete. □

Now, we are prepared to prove Theorem 3.

Proof of Theorem 3

First, according to Lemma 8 and since q_j is symmetric, we find that there exists $\bar{j} \in (0, N - 1)$ such that

$$w_{1,j}^2(t) - w_{0,j}^2(t) \begin{cases} \geq 0 & \text{if } j \in [0, \bar{j}] \cup [N-1-\bar{j}, N-1], \\ \leq 0 & \text{if } j \in [\bar{j}, j_2]. \end{cases}$$

Now, consider

$$S = \{q_n : q_n \text{ is symmetric and symmetric increasing}\}.$$

Then $\nu_1 - \nu_0$ attains its minimum at some $\hat{q}_n \in S$. Let $q_j(t) = (1-t)\hat{q}_j + t\hat{q}_{\bar{j}}$. By Lemma 7, we find

$$\nu_1'(t) - \nu_0'(t) = \sum_{j=0}^{N-1} q_j'(t)\bigl(w_{1,j}^2(t) - w_{0,j}^2(t)\bigr) = \sum_{j=0}^{N-1} (\hat{q}_{\bar{j}} - \hat{q}_j)\bigl(w_{1,j}^2(t) - w_{0,j}^2(t)\bigr) < 0.$$

This implies that

$$\nu_1(t) - \nu_0(t) > \nu_1(1) - \nu_0(1) = (\nu_1 - \nu_0)[\hat{q}_{\bar{j}}] = 2\left[1 - \cos\left(\frac{\pi}{N+1}\right)\right]. \qquad \square$$

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