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Traveling waves in nonlocal dispersal SIR epidemic model with nonlinear incidence and distributed latent delay

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Abstract

This paper studies the traveling waves in a nonlocal dispersal SIR epidemic model with nonlinear incidence and distributed latent delay. It is found that the traveling waves connecting the disease-free equilibrium with endemic equilibrium are determined by the basic reproduction number $\mathcal{R}_0$ and the minimal wave speed c^*. When $\mathcal{R}_0 > 1$ and $c > c^*$, the existence of traveling waves is established by using the upper-lower solutions, auxiliary system, constructing the solution map, and then the fixed point theorem, limiting argument, diagonal extraction method, and Lyapunov functions. When $\mathcal{R}_0 > 1$ and $0 < c < c^*$, the nonexistence result is also obtained by using the reduction to absurdity and the theory of asymptotic spreading.

Keywords: Nonlocal dispersal epidemic model; Nonlinear incidence; Distributed latent delay; Traveling waves; Upper-lower solutions; Limiting argument

1 Introduction

Mathematical models can be a powerful tool for designing strategies to control the spread of diseases. Over the past decades, great attention has been paid to describing the spread of an epidemic by mathematical methods, especially ordinary differential equations. However, because individuals (humans, birds, mosquitoes) always move frequently between regions, so when we study the spread of infectious diseases, the factors about the spatial diffusion of individuals cannot be ignored. Therefore, the reaction-diffusion model has attracted a lot of attention of researchers [1–9]. The equilibria, basic reproduction number, asymptotic and global stability, uniform persistence, bifurcation, and traveling waves are the focus of reaction-diffusion models [2, 3, 5–7, 10–20].

As is well known, the incidence function plays a very important role in modeling infectious diseases. Some factors, such as media coverage, density of population, and life style, may affect the incidence rate directly or indirectly. In 1978, Capasso et al. [21] introduced a saturated incidence rate $g(I)S$ by research of the cholera epidemic spread in Bari, which includes the behavioral change and crowding effect and avoids the unboundedness of the effective contact. If the function $g(I)$ is decreasing on $I > 0$, it can be interpreted as the "psychological" effect. In 1995, this effect was also observed, when Brown et al.

[22] studied infection of the two-spotted spider mites, Tetranychus urticae, with the entomopathogenic fungus, Neozygites floridana. Thus, it can be seen that many realistic epidemic systems can be accurately modeled only by using the nonlinear incidence. Recently, Zhou et al. [23] proposed the following nonlocal dispersal susceptible-infected-removed (SIR) epidemic model with nonlinear incidence $f(S)g(I)$:

$$\begin{cases} \partial_t S(x,t) = d_1(J * S(x,t) - S(x,t)) - f(S(x,t))g(I(x,t)), \\ \partial_t I(x,t) = d_2(J * I(x,t) - I(x,t)) + f(S(x,t))g(I(x,t)) - \gamma I(x,t), \\ \partial_t R(x,t) = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t), \end{cases} \tag{1}$$

where $J * u(x,t)$ with $u(x,t) = S(x,t)$, $I(x,t)$ and $R(x,t)$ denote nonlocal diffusion with the following form:

$$J * u(x,\cdot) = \int_{\mathbb{R}} J(x-y)u(y,\cdot)\,dy = \int_{\mathbb{R}} J(y)u(x-y,\cdot)\,dy. \tag{2}$$

The kernel function $J(x-y)$ is the probability of dispersal from location y to x, and $\int_{\mathbb{R}} J(x-y)u(y,\cdot)\,dy$ stands for the rate at which individuals arrive at location x from all other locations. Zhou et al. [23] studied the existence and nonexistence of traveling waves. It is shown that the traveling wave solutions are completely dependent on critical wave speed c^* and basic reproduction number $\mathcal{R}_0$. When $\mathcal{R}_0 > 1$ and wave speed $c > c^*$, then the existence theorem is established for model (1) (see Theorem 2.3 in [23]); otherwise, when $\mathcal{R}_0 > 1$ and $0 < c < c^*$ or $\mathcal{R}_0 < 1$, then the nonexistence theorems are obtained (see Theorems 3.1 and 3.2 in [23]).

On the one hand, we also know that the time delay has a great influence on dynamic behavior in many infectious diseases, since some disease may take time to reach the infection stage from the point of being infected [24–33]. Zhang et al. [34] established the following SIR epidemic model with nonlocal dispersal and nonlinear incidence:

$$\begin{cases} \partial_t S(x,t) = d_1(J * S(x,t) - S(x,t)) - f(S(x,t))g(I(x,t-\tau)), \\ \partial_t I(x,t) = d_2(J * I(x,t) - I(x,t)) + f(S(x,t))g(I(x,t-\tau)) - \gamma I(x,t), \\ \partial_t R(x,t) = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t). \end{cases} \tag{3}$$

The existence and nonexistence of traveling waves of the system are established. It is shown that the spread speed c is dependent on the dispersal rate of the infected individuals and the time delay. In model (3), the time delay is considered to be a constant.

However, for some diseases, such as rabies, the incubation period fluctuates in a wide range. According to a study of dog diseases, the incubation period for many rabies cases is within two months, while for about 5% of rabies cases incubation period is more than two months (2–5 months). In humans, the incubation period of most cases is 1 to 2 months. There are still more than 15% cases where incubation period is more than 3 months or even years [35]. Therefore, the latent period from the point of being infected to infection is often a variable [36]. The above facts make us ponder what would be the conclusion when considering the distributed latent delay and the nonlinear incidence.

On the other hand, we also notice that in models (1) and (3), the authors do not consider the supplement of the susceptible, the natural death of the susceptible, infected, and removed, and the disease-related death of the infected. We know that if these factors are considered, then we will get a completely different situation from models (1) and (3). Particularly, the endemic equilibrium point will appear in this case.

Therefore, in this paper, we propose the following nonlocal dispersal SIR epidemic model with the general nonlinear incidence $f(S,I)$ and the distributed latent delay:

$$\begin{cases} \partial_t S(x,t) = d_1(J * S(x,t) - S(x,t)) + \Lambda - \mu S(x,t) - \beta f(S(x,t), I(x,t)), \\ \partial_t I(x,t) = d_2(J * I(x,t) - I(x,t)) + \beta \int_0^\tau h(s) f(S(x,t-s), I(x,t-s))\,ds \\ \qquad\qquad - (\mu + \gamma + \alpha) I(x,t), \\ \partial_t R(x,t) = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t) - \mu R(x,t). \end{cases} \tag{4}$$

Here, S, I, and R denote the amount of susceptible, infected, and removed individuals at location x and time t, respectively. Parameter $\Lambda > 0$ is the total recruitment rate; $\beta > 0$ stands for the per-capita effective transmission rate; $\mu > 0$ and $\alpha > 0$ are the natural death rate and the disease-related death rate, respectively; $d_i \geq 0$ ($i = 1, 2, 3$) describes the diffusion rates for the three groups, respectively; γ represents the recovery rate of infected individuals and $\tau > 0$ is a constant. Function $f(S,I)$ denotes the nonlinear incidence rate; the distributed latent delay term $\int_0^\tau h(s) f(S(x,t-s), I(x,t-s))\,ds$ shows that the disease transmission has an incubation period, and the period of incubation is not constant.

In addition, for model (1) and model (3), the authors investigated the traveling wave solution $(S(\xi), I(\xi), R(\xi))$ with $\xi = x + ct$ satisfying $(S(-\infty), I(-\infty), R(-\infty)) = (S_0, 0, 0)$ and $(S(\infty), I(\infty), R(\infty)) = (S_\infty, 0, 0)$, where $S_0 > 0$ is interpreted as the number of the susceptible individuals before being infected, and $S_\infty < S_0$. However, in this paper, we study the traveling wave solution $(S(\xi), I(\xi), R(\xi))$ connecting the disease-free equilibrium $(S_0, 0, 0)$ with endemic equilibrium (S^*, I^*, R^*), namely $(S(-\infty), I(-\infty), R(-\infty)) = (S_0, 0, 0)$ and $(S(\infty), I(\infty), R(\infty)) = (S^*, I^*, R^*)$.

The paper is organized as follows. In the next section, we study the existence of equilibria and critical wave speed c^* of model (4). In Sect. 3, the upper-lower solutions of the auxiliary system are defined. In Sect. 4, a convex set and a solution map defined on this set are constructed for the auxiliary system. In Sect. 5, the existence of traveling waves is established firstly for the auxiliary system, and then for model (4) by using Schauder's fixed point theorem, the limiting argument, and the diagonal extraction method. Furthermore, asymptotic boundary properties are obtained by means of the Lyapunov functions technique. In Sect. 6, the nonexistence of traveling waves is discussed by the asymptotic spreading theory. In Sect. 7, we derive some number simulations to verify our results. In Sect. 8, a brief conclusion is given.

2 Preliminary

For convenience, let $p(x)$ with $x = (x_1, x_2, \ldots, x_n)$ be a quadratic continuously differentiable function, we denote $p_{x_i}(x) = \frac{\partial p(x)}{\partial x_i}$ and $p_{x_i x_j}(x) = \frac{\partial^2 p(x)}{\partial x_i \partial x_j}$.

We always assume that functions $f(S,I)$, $J(x)$, and $h(s)$ in model (4) satisfy the following assumptions:

(A1) $f(S,I)$ is quadratic continuously differentiable, nondecreasing and $f(0,I) = f(S,0) = 0$ for all $S > 0$ and $I > 0$; $\frac{f(S,I)}{I}$ is nonincreasing for all $I > 0$.

(A2) $J(x)$ is local Lipschitz continuous and compactly supported on $\mathbb{R}$, $\int_{\mathbb{R}} J(x)\,dx = 1$, $J(x) = J(-x) \geq 0$ for any $x \in \mathbb{R}$, $J(0) > 0$, $\lim_{\lambda\to+\infty} \frac{1}{\lambda}\int_{\mathbb{R}} J(y)\,e^{-\lambda y}\,dy = +\infty$ and $\lim_{\lambda\to 0^+} \frac{1}{\lambda}\int_{\mathbb{R}} J(y)(e^{-\lambda y}-1)\,dy = 0$.

(A3) $h(s)$ is nonnegative and integrable on $(0,\tau)$ and $\int_0^\tau h(s)\,ds = 1$.

Remark 1 It is clear that the nonlinear incidence $f(S,I)$ satisfying assumption (A1) includes many common incidence functions, such as $f(S,I) = SI$, which was studied by Zhang et al. in [37]; $f(S,I) = \frac{SI}{S+I}$, which was studied by Xu in [38]; $f(S,I) = \frac{SI^2}{1+aI^2}$ with $a > 0$, which was studied by Ruan et al. in [39].

Define the basic reproduction number of model (4) as follows:

$$\mathcal{R}_0 = \frac{\beta f_I(S_0,0)}{\mu+\gamma+\alpha},$$

where $S_0 = \frac{\Lambda}{\mu}$. Model (4) always has a disease-free equilibrium $E_0 = (S_0,0,0)$. When $\mathcal{R}_0 > 1$, by (A1) we easily prove that model (4) has a unique endemic equilibrium $E^* = (S^*,I^*,R^*)$.

The traveling wave in model (4) is defined as a special solution $(S(x+ct), I(x+ct), R(x+ct))$, where $c > 0$ is the wave speed. Let $\xi = x+ct$, then from model (4) we obtain the following system:

$$\begin{cases} cS'(\xi) = d_1(J*S(\xi)-S(\xi)) + \Lambda - \mu S(\xi) - \beta f(S(\xi),I(\xi)), \\ cI'(\xi) = d_2(J*I(\xi)-I(\xi)) + \beta\int_0^\tau h(s) f(S(\xi-cs),I(\xi-cs))\,ds - (\mu+\gamma+\alpha)I(\xi), \\ cR'(\xi) = d_3(J*R(\xi)-R(\xi)) + \gamma I(\xi) - \mu R(\xi), \end{cases} \tag{5}$$

where $J*u(\xi) = \int_{\mathbb{R}} J(\xi-y)u(y)\,dy$ with $u(\xi) = S(\xi), I(\xi)$ and $R(\xi)$.

In this paper, we investigate the existence of traveling wave $(S(\xi),I(\xi),R(\xi))$ of model (5) for $\xi \in \mathbb{R}$ with the asymptotic boundary conditions:

$$\lim_{\xi\to-\infty}\bigl(S(\xi),I(\xi),R(\xi)\bigr) = (S_0,0,0), \qquad \lim_{\xi\to\infty}\bigl(S(\xi),I(\xi),R(\xi)\bigr) = \bigl(S^*,I^*,R^*\bigr). \tag{6}$$

Since the third equation in system (5) is fully decoupled with the first two equations, in the following we first consider the subsystem

$$\begin{cases} cS'(\xi) = d_1(J*S(\xi)-S(\xi)) + \Lambda - \mu S(\xi) - \beta f(S(\xi),I(\xi)), \\ cI'(\xi) = d_2(J*I(\xi)-I(\xi)) + \beta\int_0^\tau h(s) f(S(\xi-cs),I(\xi-cs))\,ds - (\mu+\gamma+\alpha)I(\xi). \end{cases} \tag{7}$$

Linearizing the second equation of system (7) at equilibrium E_0, we have

$$cI'(\xi) = d_2\int_{\mathbb{R}} J(y)\bigl(I(\xi-y)-I(\xi)\bigr)\,dy + \beta f_I(S_0,0)\int_0^\tau h(s)I(\xi-cs)\,ds - (\mu+\gamma+\alpha)I(\xi). \tag{8}$$

Substituting $I(\xi) = e^{\lambda\xi}$ into (8), it follows that

$$\begin{aligned} \Delta(\lambda,c) := {}& d_2\int_R J(y)\bigl(e^{-\lambda y}-1\bigr)\,dy - c\lambda + \beta f_I(S_0,0)\int_0^\tau h(s)e^{-\lambda cs}\,ds \\ & - (\mu+\gamma+\alpha) = 0. \end{aligned} \tag{9}$$

Lemma 1 *Assume $\mathcal{R}_0 > 1$, then there exist unique $c^* > 0$ and $\lambda^* > 0$ satisfying $\Delta(\lambda^*, c^*) = 0$ and $\frac{\partial \Delta(\lambda^*, c^*)}{\partial \lambda} = 0$. Furthermore,*

(i) *if $c > c^*$, then there are two positive constants $\lambda_{1(c)} < \lambda_{2(c)}$ such that $\Delta(\lambda_{i(c)}, c) = 0$ $(i = 1, 2)$, $\Delta(\lambda, c) > 0$ for $\lambda \in (0, \lambda_{1(c)}) \cup (\lambda_{2(c)}, +\infty)$ and $\Delta(\lambda, c) < 0$ for $\lambda \in (\lambda_{1(c)}, \lambda_{2(c)})$;*

(ii) *if $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda > 0$.*

Proof We have $\Delta(0, c) = \beta f_I(S_0, 0) - (\mu + \gamma + \alpha) > 0$ since $\mathcal{R}_0 > 1$, $\Delta(+\infty, c) = +\infty$ by (A2) and

$$\begin{aligned} \frac{\partial \Delta(\lambda, c)}{\partial \lambda}\bigg|_{\lambda=0} &= -c - c\beta f_I(S_0, 0) \int_0^\tau sh(s)\,dy\,ds < 0, \\ \frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} &= d_2 \int_{\mathbb{R}} y^2 J(y) e^{-\lambda y}\,dy + \beta f_I(S_0, 0) \int_0^\tau h(s)(cs)^2 e^{-\lambda cs}\,ds > 0. \end{aligned}$$

Besides, for any $\lambda > 0$, we have $\Delta(\lambda, +\infty) = -\infty$ and

$$\begin{aligned} \Delta(\lambda, 0) &\geq d_2 \int_{\mathbb{R}} J(y)(-\lambda y)\,dy + \beta f_I(S_0, 0) \int_0^\tau h(s)\,ds - (\mu + \gamma + \alpha) \\ &= \beta f_I(S_0, 0) - (\mu + \gamma + \alpha) > 0, \\ \frac{\partial \Delta(\lambda, c)}{\partial c} &= -\lambda - \lambda \beta f_I(S_0, 0) \int_0^\tau sh(s) e^{-\lambda cs}\,ds < 0. \end{aligned}$$

Therefore, there exist unique $c^* > 0$ and $\lambda^* > 0$ satisfying $\Delta(\lambda^*, c^*) = 0$ and $\frac{\partial \Delta(\lambda^*, c^*)}{\partial \lambda} = 0$.

When $c > c^*$, we easily get that there are two positive numbers $\lambda_{1(c)} < \lambda_{2(c)}$ such that $\Delta(\lambda_{i(c)}, c) = 0$ $(i = 1, 2)$, $\Delta(\lambda, c) > 0$ for $\lambda \in (0, \lambda_{1(c)}) \cup (\lambda_{2(c)}, +\infty)$ and $\Delta(\lambda, c) < 0$ for $\lambda \in (\lambda_{1(c)}, \lambda_{2(c)})$. If $0 < c < c^*$, then it is clear that $\Delta(\lambda, c) > 0$ for all $\lambda > 0$. This completes the proof. □

3 Upper-lower solutions

In this section, we always assume $\mathcal{R}_0 > 1$ and $c > c^*$. We introduce the auxiliary system

$$\begin{cases} cS'(\xi) = d_1(J * S(\xi) - S(\xi)) + \Lambda - \mu S(\xi) - \beta f(S(\xi), I(\xi)), \\ cI'(\xi) = d_2(J * I(\xi) - I(\xi)) + \beta \int_0^\tau h(s) f(S(\xi - cs), I(\xi - cs))\,ds \\ \qquad - (\mu + \gamma + \alpha) I(\xi) - \varepsilon I^2(\xi), \end{cases} \tag{10}$$

where $\varepsilon > 0$ is a constant. We define four functions: $\overline{S}(\xi) = S_0$, $\underline{S}(\xi) = \max\{S_0(1 - M_1 e^{\varepsilon_1 \xi}), 0\}$, $\overline{I}(\xi) = \min\{e^{\lambda_1 \xi}, K_\varepsilon\}$, and $\underline{I}(\xi) = \max\{e^{\lambda_1 \xi}(1 - M_2 e^{\varepsilon_2 \xi}), 0\}$ for $\xi \in \mathbb{R}$, where $\lambda_1 = \lambda_{1(c)}$ is given in Lemma 1, $K_\varepsilon = \frac{\beta f_I(S_0, 0) - (\mu + \gamma + \alpha)}{\varepsilon}$ and M_i, ε_i $(i = 1, 2)$ are positive constants to be determined in the following lemmas. Now, we prove that $(\overline{S}(\xi), \overline{I}(\xi))$ and $(\underline{S}(\xi), \underline{I}(\xi))$ are the upper and lower solutions respectively for system (10).

Lemma 2 *Function $\overline{S}(\xi)$ satisfies*

$$c\overline{S}'(\xi) \geq d_1\big(J * \overline{S}(\xi) - \overline{S}(\xi)\big) + \Lambda - \mu \overline{S}(\xi) - \beta f\big(\overline{S}(\xi), \underline{I}(\xi)\big), \quad \xi \in \mathbb{R}.$$

Lemma 3 *Function* $\overline{I}(\xi)$ *satisfies*

$$\begin{aligned} c\overline{I}'(\xi) \geq {} & d_2\big(J * \overline{I}(\xi) - \overline{I}(\xi)\big) + \beta \int_0^\tau h(s) f\big(\overline{S}(\xi - cs), \overline{I}(\xi - cs)\big)\, ds \\ & - (\mu + \gamma + \alpha)\overline{I}(\xi) - \varepsilon \overline{I}^2(\xi) \end{aligned} \tag{11}$$

for any $\xi \neq \xi_1 := \frac{1}{\lambda_1} \ln K_\varepsilon$.

The proofs of Lemmas 2 and 3 are simple, we here omit them.

Lemma 4 *There exists a constant* $\varepsilon_1 \in (0, \lambda_1)$ *small enough such that function* $\underline{S}(\xi)$ *satisfies*

$$c\underline{S}'(\xi) \leq d_1\big(J * \underline{S}(\xi) - \underline{S}(\xi)\big) + \Lambda - \mu \underline{S}(\xi) - \beta f\big(\underline{S}(\xi), \overline{I}(\xi)\big) \tag{12}$$

for any $\xi \neq \xi_2 := \frac{1}{\varepsilon_1} \ln \varepsilon_1$.

Proof Choose $M_1 = \frac{1}{\varepsilon_1}$. When $\xi > \xi_2$, then $\underline{S}(\xi) = 0$. Since $d_1 J * \underline{S}(\xi) + \Lambda > 0$, we obtain that (12) holds. When $\xi < \xi_2$, then $\underline{S}(\xi) = S_0(1 - M_1 e^{\varepsilon_1 \xi})$ and $\overline{I}(\xi) \leq e^{\lambda_1 \xi}$. To obtain (12), it is sufficient to prove the following inequality:

$$c\underline{S}'(\xi) \leq d_1\big(J * \underline{S}(\xi) - \underline{S}(\xi)\big) + \Lambda - \mu \underline{S}(\xi) - \beta f_I(S_0, 0)\overline{I}(\xi)$$

for $\xi < \xi_2$, which is equivalent to proving

$$cM_1 S_0 \varepsilon_1 e^{\varepsilon_1 \xi} \geq d_1 S_0 M_1 e^{\varepsilon_1 \xi} \int_{\mathbb{R}} J(y)\big(e^{-\varepsilon_1 y} - 1\big)\, dy - \mu M_1 e^{\varepsilon_1 \xi} S_0 + \beta f_I(S_0, 0) e^{\lambda_1 \xi},$$

that is,

$$d_1 S_0 \frac{1}{\varepsilon_1} \int_{\mathbb{R}} J(y)\big(e^{-\varepsilon_1 y} - 1\big)\, dy - \mu \frac{1}{\varepsilon_1} S_0 + \beta f_I(S_0, 0) e^{(\lambda_1 - \varepsilon_1)\xi} \leq cS_0 \tag{13}$$

for $\xi < \xi_2$. By (A2), it is clear that inequality (13) holds for $0 < \varepsilon_1 < \lambda_1$ small enough. This completes the proof. □

Lemma 5 *There exist the constants* $0 < \varepsilon_2 < \min\{\varepsilon_1, \lambda_2 - \lambda_1\}$ *and* $M_2 > 1$ *large enough with* $-\frac{1}{\varepsilon_2} \ln M_2 < \xi_2$ *such that function* $\underline{I}(\xi)$ *satisfies*

$$\begin{aligned} c\underline{I}'(\xi) \leq {} & d_2\big(J * \underline{I}(\xi) - \underline{I}(\xi)\big) + \beta \int_0^\tau h(s) f\big(\underline{S}(\xi - cs), \underline{I}(\xi - cs)\big)\, ds \\ & - (\mu + \gamma + \alpha)\underline{I}(\xi) - \varepsilon \underline{I}^2(\xi) \end{aligned} \tag{14}$$

for any $\xi \neq \xi_3 := -\frac{1}{\varepsilon_2} \ln M_2$.

Proof Obviously, (14) can be rewritten as follows:

$$
\begin{aligned}
c\underline{I}'(\xi) \le{}& d_2\big(J * \underline{I}(\xi) - \underline{I}(\xi)\big) + \beta f_I(S_0,0)\int_0^\tau h(s)\underline{I}(\xi - cs)\,ds - (\mu + \gamma + \alpha)\underline{I}(\xi) \\
&+ \beta \int_0^\tau h(s) f\big(\underline{S}(\xi - cs), \underline{I}(\xi - cs)\big)\,ds \\
&- \beta f_I(S_0,0)\int_0^\tau h(s)\underline{I}(\xi - cs)\,ds - \varepsilon \underline{I}^2(\xi).
\end{aligned}
\tag{15}
$$

When $\xi > \xi_3$, then $\underline{I}(\xi) = 0$. Hence, (15) clearly holds. When $\xi < \xi_3$, since $\xi_3 < \xi_2$, we have $\overline{S}(\xi) = S_0$, $\underline{S}(\xi) = S_0(1 - M_1 e^{\varepsilon_1 \xi})$ and $\underline{I}(\xi) = e^{\lambda_1 \xi}(1 - M_2 e^{\varepsilon_2 \xi})$. Hence, for $\xi < \xi_3$, we obtain

$$
\begin{aligned}
&c\underline{I}'(\xi) - d_2\big(J * \underline{I}(\xi) - \underline{I}(\xi)\big) - \beta f_I(S_0,0)\int_0^\tau h(s)\underline{I}(\xi - cs)\,ds + (\mu + \gamma + \alpha)\underline{I}(\xi) \\
&\quad= -e^{\lambda_1 \xi}\Delta(\lambda_1, c) + M_2 e^{(\lambda_1 + \varepsilon_2)\xi}\Delta(\lambda_1 + \varepsilon_2, c) \\
&\quad= M_2 e^{(\lambda_1 + \varepsilon_2)\xi}\Delta(\lambda_1 + \varepsilon_2, c).
\end{aligned}
\tag{16}
$$

From (A1), we can obtain $f(\underline{S}(\xi - cs), \underline{I}(\xi - cs)) \le f_I(S_0,0)\underline{I}(\xi - cs)$, which implies

$$
\beta \int_0^\tau h(s) f\big(\underline{S}(\xi - cs), \underline{I}(\xi - cs)\big)\,ds - \beta f_I(S_0,0)\int_0^\tau h(s)\underline{I}(\xi - cs)\,ds - \varepsilon \underline{I}^2(\xi) < 0
$$

and

$$
\begin{aligned}
&\beta \int_0^\tau h(s) f\big(\underline{S}(\xi - cs), \underline{I}(\xi - cs)\big)\,ds - \beta f_I(S_0,0)\int_0^\tau h(s)\underline{I}(\xi - cs)\,ds \\
&\quad= \beta \int_0^\tau h(s) f_S\big(\delta, \underline{I}(\xi - cs)\big)\big(\underline{S}(\xi - cs) - S_0\big)\,ds \\
&\qquad+ \beta \int_0^\tau h(s)\left[\frac{f(S_0, \underline{I}(\xi - cs))}{\underline{I}(\xi - cs)} - f_I(S_0,0)\right]\underline{I}(\xi - cs)\,ds \\
&\quad= \beta \int_0^\tau h(s)\frac{f_S(\delta, \underline{I}(\xi - cs))}{\underline{I}(\xi - cs)}\underline{I}(\xi - cs)\big(\underline{S}(\xi - cs) - S_0\big)\,ds \\
&\qquad+ \beta \int_0^\tau h(s)\left[\frac{f_I(S_0,\theta)\theta - f(S_0,\theta)}{\theta^2}\right]\big(\underline{I}(\xi - cs)\big)^2\,ds,
\end{aligned}
\tag{17}
$$

where $\delta = \delta(\xi, s) \in [\underline{S}(\xi - cs), S_0]$ and $\theta = \theta(\xi, s) \in [0, \underline{I}(\xi - cs)]$.

Noting $\underline{S}(\xi - cs) \ge S_0(1 - M_1 e^{\varepsilon\xi})$ for $\xi \le \xi_3$ and $s \in [0,\infty)$, we have $\delta(\xi, s) \in [S_0(1 - M_1 e^{\varepsilon\xi}), S_0]$ for $\xi \le \xi_3$ and $s \in [0,\infty)$. Hence, $\lim_{\xi\to-\infty}\delta(\xi, s) = S_0$ uniformly for $s \in [0,\infty)$. It follows from (A1) that $f_I(S,I)I - f(S,I) \le 0$ and $\frac{f(S,I)}{I} \le f_I(S,0)$ for $S \ge 0$ and $I > 0$. Hence, $\frac{f_S(\delta, \underline{I}(\xi - cs))}{\underline{I}(\xi - cs)} \le f_{SI}(\delta(\xi, s), 0)$ for $\xi \le \xi_3$ and $s \in [0,\infty)$. Then we have

$$
\lim_{\xi\to-\infty} f_{SI}\big(\delta(\xi, s), 0\big) = f_{SI}(S_0, 0) \quad \text{uniformly for } s \in [0,\infty).
\tag{18}
$$

Noting that $\underline{I}(\xi - cs) \le e^{\lambda_1 \xi}$ for $\xi \le \xi_3$ and $s \in [0,\infty)$, we have $\theta(\xi, s) \in [0, e^{\lambda_1 \xi}]$ for $\xi \le \xi_3$ and $s \in [0,\infty)$. Hence, $\lim_{\xi\to-\infty}\theta(\xi, s) = 0$ uniformly for $s \in [0,\infty)$. Then we obtain

$$
\lim_{\xi\to-\infty}\frac{f_I(S_0, \theta(\xi, s))\theta(\xi, s) - f(S_0, \theta(\xi, s))}{\theta(\xi, s)^2} = f_{II}(S_0, 0),
\tag{19}
$$

uniformly for $s \in [0,\infty)$.

Since $0 \geq \underline{S}(\xi - cs) - S_0 \geq -S_0 M_1 e^{\varepsilon_1 \xi}$ for $\xi \leq \xi_3$ and $s \in [0, \infty)$, we also have

$$\begin{aligned}&\int_0^\tau h(s) \frac{f_S(\delta(\xi,s), \underline{I}(\xi - cs))}{\underline{I}(\xi - cs)} \underline{I}(\xi - cs)\bigl(\underline{S}(\xi - cs) - S_0\bigr)\, ds \\ &\quad \geq -S_0 M_1 \int_0^\tau h(s) f_{SI}\bigl(\delta(\xi,s), 0\bigr)\, ds\, e^{(\lambda_1+\varepsilon_1)\xi}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\int_0^\tau h(s)\biggl[\frac{f_I(S_0,\theta(\xi,s))\theta(\xi,s) - f(S_0,\theta(\xi,s))}{\theta(\xi,s)^2}\biggr]\bigl(\underline{I}(\xi - cs)\bigr)^2\, ds \\ &\quad \geq \int_0^\tau h(s)\biggl[\frac{f_I(S_0,\theta(\xi,s))\theta(\xi,s) - f(S_0,\theta(\xi,s))}{\theta(\xi,s)^2}\biggr]\, ds\, e^{2\lambda_1 \xi}.\end{aligned}$$

Then, from (17), for $\xi \leq \xi_3$ we have

$$\begin{aligned}&\beta \int_0^\tau h(s) f\bigl(\underline{S}(\xi - cs), \underline{I}(\xi - cs)\bigr)\, ds - \beta f_I(S_0, 0) \int_0^\tau h(s) \underline{I}(\xi - cs)\, ds - \varepsilon\bigl(\underline{I}(\xi)\bigr)^2 \\ &\quad \geq \biggl(-S_0 M_1 \beta \int_0^\tau h(s) f_{SI}\bigl(\delta(\xi,s),0\bigr)\, ds\, e^{(\varepsilon_1+\lambda_1)\xi} \\ &\qquad + \beta \int_0^\tau h(s)\biggl[\frac{f_I(S_0,\theta(\xi,s))\theta(\xi,s) - f(S_0,\theta(\xi,s))}{\theta(\xi,s)^2}\biggr]\, ds - \varepsilon\biggr) e^{2\lambda_1 \xi} =: P(\xi).\end{aligned}$$

To obtain (15), from (16) we only need to show that there is a constant $M_2 > 1$ such that $M_2 e^{(\lambda_1+\varepsilon_2)\xi} \Delta(\lambda_1 + \varepsilon_2, c) \leq P(\xi)$ for any $\xi \leq \xi_3$, which is equivalent to

$$\begin{aligned}&M_2 \Delta(\lambda_1 + \varepsilon_3, c) \leq P(\xi) e^{-(\varepsilon_1+\lambda_1)\xi} \\ &\quad = -S_0 M_1 \int_0^\tau h(s) f_{SI}\bigl(\delta(\xi,s),0\bigr)\, ds\, e^{(\varepsilon_1-\varepsilon_2)\xi} \\ &\qquad + \int_0^\tau h(s)\biggl[\frac{f_I(S_0,\theta(\xi))\theta(\xi) - f(S_0,\theta(\xi))}{\theta(\xi)^2}\biggr]\, ds\, e^{(\lambda_1-\varepsilon_2)\xi} - \varepsilon e^{(\lambda_1-\varepsilon_2)\xi}.\end{aligned} \tag{20}$$

From Lemma 1, we know $\Delta(\lambda_1 + \varepsilon_2, c) < 0$ as $\lambda_1 < \lambda_1 + \varepsilon_2 < \lambda_2$. Since $\lim_{\xi \to -\infty} e^{(\varepsilon_1-\varepsilon_2)\xi} = 0$, $\lim_{\xi \to -\infty} e^{(\lambda_1-\varepsilon_2)\xi} = 0$, from (18) and (19) we can obtain $\lim_{\xi \to -\infty} P(\xi) e^{-(\varepsilon_1+\lambda_1)\xi} = 0$. Therefore, there is $\xi_3 < 0$ with $\xi_3 < \xi_2$ such that (20) holds for $\xi < \xi_3$. Choose $M_2 > 1$ such that $\xi_3 = -\frac{1}{\xi_2} \ln M_2$. Then we have that (15) holds for $\xi < \xi_3$. This completes the proof. □

4 Solution map on a convex set

For any given $X > \max\{|\xi_1|, |\xi_3|, r\}$, we construct a set of functions as follows:

$$\Gamma_X = \left\{ \bigl(\phi(\xi), \varphi(-X)\bigr) \in C\bigl([-X, X], \mathbb{R}^2\bigr) : \begin{array}{l} \phi(-X) = \underline{S}(-X) \\ \varphi(-X) = \underline{I}(-X) \\ \underline{S}(\xi) \leq \phi(\xi) \leq S_0 \\ \underline{I}(\xi) \leq \varphi(\xi) \leq \overline{I}(\xi) \\ \xi \in [-X, X] \end{array} \right\}. \tag{21}$$

For any $(\phi(\xi),\varphi(\xi)) \in \Gamma_X$, we define

$$\hat{\phi}(\xi) = \begin{cases} \phi(X), & \xi > X, \\ \phi(\xi), & |\xi| \le X, \\ \underline{S}(\xi), & \xi < -X, \end{cases} \qquad \hat{\varphi}(\xi) = \begin{cases} \varphi(X), & \xi > X, \\ \varphi(\xi), & |\xi| \le X, \\ \underline{I}(\xi), & \xi < -X. \end{cases} \tag{22}$$

Obviously, Γ_X is a closed and convex set. $(\hat{\phi}(\xi),\hat{\varphi}(\xi))$ satisfies

$$\underline{S}(\xi) \le \hat{\phi}(\xi) \le S_0, \qquad \underline{I}(\xi) \le \hat{\varphi}(\xi) \le \bar{I}(\xi), \quad \xi \in \mathbb{R}. \tag{23}$$

Consider the initial value problem

$$\begin{cases} cS'(\xi) = d_1(J * \hat{\phi}(\xi) - S(\xi)) + \Lambda - \mu S(\xi) - \beta f(S(\xi),\varphi(\xi)), \\ cI'(\xi) = d_2(J * \hat{\varphi}(\xi) - I(\xi)) + \beta \int_0^\tau h(s) f(\hat{\phi}(\xi - cs),\hat{\varphi}(\xi - cs))\,ds \\ \qquad\qquad - (\mu + \gamma + \alpha)I(\xi) - \varepsilon I^2(\xi) \end{cases} \tag{24}$$

with

$$S(-X) = \underline{S}(-X), \qquad I(-X) = \underline{I}(-X). \tag{25}$$

The ODE theory ensures that initial value problem (24) and (25) admits a unique solution $(S_X(\xi),I_X(\xi))$ defined for $\xi \in [-X,X]$. Thus, we define a map $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$ on Γ_X by

$$\mathcal{F}_1(\phi,\varphi) = S_X, \qquad \mathcal{F}_2(\phi,\varphi) = I_X. \tag{26}$$

Lemma 6 *For any given $X > \max\{|\xi_1|,|\xi_3|,r\}$, map $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$ is $\Gamma_X \to \Gamma_X$.*

Lemma 6 can be easily proved by using (A1), (A2), Lemmas 2–5, and the comparison principle, we hence omit it here.

Lemma 7 *Map $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2) : \Gamma_X \to \Gamma_X$ is completely continuous.*

Proof For any $(\phi,\varphi) \in \Gamma_X$, we easily obtain from (24) that $(S_X(\xi),I_X(\xi)) \in C^1([-X,X],\mathbb{R}^2)$. Thus, the compactness of map $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$ can be obtained by the Arzelà–Ascoli theorem.

Now, we show the continuity of $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$. Let $S_{X,i}(\xi) = \mathcal{F}_1(\phi_i,\varphi_i)(\xi)$ and $I_{X,i}(\xi) = \mathcal{F}_2(\phi_i,\varphi_i)(\xi)$, where $(\phi_i(\xi),\varphi_i(\xi)) \in \Gamma_X$ $(i = 1,2)$ for $\xi \in [-X,X]$. We first consider the continuity of $\mathcal{F}_1$. It follows from the first equation of (24) that

$$\begin{aligned} &c\big(S'_{X,1}(\xi) - S'_{X,2}(\xi)\big) + (d_1 + \mu)\big(S_{X,1}(\xi) - S_{X,2}(\xi)\big) \\ &\quad = d_1 \int_{\mathbb{R}} J(y)\big(\hat{\phi}_1(\xi - y) - \hat{\phi}_2(\xi - y)\big)\,dy + \beta\big(f\big(S_{X,2}(\xi),\varphi_2(\xi)\big) - f\big(S_{X,1}(\xi),\varphi_1(\xi)\big)\big). \end{aligned} \tag{27}$$

Since

$$\int_{\mathbb{R}} J(y)\hat{\phi}(\xi - y)\,dy = \int_{-\infty}^{-X} J(\xi - y)\underline{S}(y)\,dy + \int_{-X}^{X} J(\xi - y)\phi(y)\,dy + \int_{X}^{+\infty} J(\xi - y)\phi(X)\,dy,$$

we have

$$\left|\int_{\mathbb{R}} J(y)\big(\hat{\phi}_1(\xi-y)\,dy-\hat{\phi}_2(\xi-y)\big)\,dy\right| \le 2\max_{y\in[-X,X]}\big|\phi_1(y)-\phi_2(y)\big|. \tag{28}$$

From (A1), for any $(\phi_1,\varphi_1),(\phi_2,\varphi_2)\in\Gamma_X$, since $\bar{I}(\xi)\le K_\varepsilon$ for $\xi\in[-X,X]$, then

$$\big|f\big(\phi_1(\xi),\varphi_1(\xi)\big)-f\big(\phi_2(\xi),\varphi_2(\xi)\big)\big| \le M_4\big(\big|\phi_1(\xi)-\phi_2(\xi)\big|+\big|\varphi_1(\xi)-\varphi_2(\xi)\big|\big), \tag{29}$$

where $M_4=\sup\{f_I(S_0,0),f_S(\vartheta,K_\varepsilon):0\le\vartheta\le S_0\}$.

Let $u(\xi)=c|S_{X,1}(\xi)-S_{X,2}(\xi)|$. Then from (27)–(29) we obtain

$$\begin{aligned} u'(\xi) &= c\,\mathrm{sign}\big(S_{X,1}(\xi)-S_{X,2}(\xi)\big)\big(S'_{X,1}(\xi)-S'_{X,2}(\xi)\big)\\ &\le 2d_1\max_{y\in[-X,X]}\big|\phi_1(y)-\phi_2(y)\big|-(d_1+\mu-\beta M_4)\big|S_{X,1}(\xi)-S_{X,2}(\xi)\big|+\beta M_4|\varphi_2-\varphi_1|\\ &= \left(-\frac{d_1+\mu}{c}+\frac{\beta M_4}{c}\right)u(\xi)+2d_1\max_{y\in[-X,X]}\big|\phi_1(y)-\phi_2(y)\big|\\ &\quad+\beta M_4\max_{y\in[-X,X]}\big|\varphi_2(y)-\varphi_1(y)\big|. \end{aligned}$$

Thus, for all $\xi\in[-X,X]$, we obtain

$$\begin{aligned} u(\xi) &\le u(-X)e^{(-\frac{d_1+\mu}{c}+\frac{\beta M_4}{c})(\xi+X)}+\int_{-X}^{\xi}\Big(2d_1\max_{y\in[-X,X]}\big|\phi_1(y)-\phi_2(y)\big|\\ &\quad+\beta M_4\max_{y\in[-X,X]}\big|\varphi_2(y)-\varphi_1(y)\big|\Big)e^{(-\frac{d_1+\mu}{c}+\frac{\beta M_4}{c})(\xi-\tau)}\,d\tau. \end{aligned} \tag{30}$$

Since $u(-X)=0$, from (30) we finally have $\|u(\xi)\|_{\Gamma_X}\to 0$ as $\|(\phi_2,\varphi_2)-(\phi_1,\varphi_1)\|_{\Gamma_X}\to 0$. Therefore, $\mathcal{F}_1$ is continuous on Γ_X. Similarly, we can obtain the continuity of $\mathcal{F}_2$. □

Since Γ_X is closed and convex, combining Lemmas 6 and 7, applying Schauder's fixed point theorem, we obtain the following theorem.

Theorem 1 *Map $\mathcal{F}$ has at least one fixed point $(S^*_X(\xi),I^*_X(\xi))\in\Gamma_X$.*

Now, we give some estimates for the fixed point $(S^*_X(\xi),I^*_X(\xi))$ of map $\mathcal{F}$ in the space $C^{1,1}([-X,X])$, where

$$C^{1,1}\big([-X,X]\big)=\big\{u\in C^1\big([-X,X]\big):u \text{ and } u' \text{ are Lipschitz continuous}\big\},$$

with the norm

$$\|u\|_{C^{1,1}([-X,X])}=\max_{x\in[-X,X]}\big|u(x)\big|+\max_{x\in[-X,X]}\big|u'(x)\big|+\sup_{x,y\in[-X,X],x\neq y}\frac{|u'(x)-u'(y)|}{|x-y|}. \tag{31}$$

We have the following result.

Lemma 8 *Let $(S^*_X(\xi),I^*_X(\xi))$ be the fixed point of map $\mathcal{F}$, then there exists a constant $C>0$ independent of X satisfying $\|S^*_X(\xi)\|_{C^{1,1}([-X,X])}\le C$ and $\|I^*_X(\xi)\|_{C^{1,1}([-X,X])}\le C$ for any $X>$* $\max\{|\xi_1|,|\xi_3|,r\}$.

Proof Obviously, we have

$$\begin{cases} cS_X^{*\prime}(\xi) = d_1(J * \hat{S}_X(\xi) - S_X^*(\xi)) + \Lambda - \mu S_X^*(\xi) - \beta f(S_X^*(\xi), I_X^*(\xi)), \\ cI_X^{*\prime}(\xi) = d_2 J * \hat{I}_X(\xi) + \beta \int_0^\tau h(s) f(\hat{S}_X(\xi - cs), \hat{I}_X(\xi - cs))\, ds \\ \qquad - (d_2 + \mu + \gamma + \alpha) I_X^*(\xi) - \varepsilon I_X^{*2}(\xi) \end{cases} \tag{32}$$

for $\xi \in [-X, X]$, where

$$\hat{S}_X(\xi) = \begin{cases} S_X^*(X), & \xi > X, \\ S_X^*(\xi), & |\xi| \le X, \\ \underline{S}(\xi), & \xi < -X, \end{cases} \qquad \hat{I}_X(\xi) = \begin{cases} I_X^*(X), & \xi > X, \\ I_X^*(\xi), & |\xi| \le X, \\ \underline{I}(\xi), & \xi < -X. \end{cases}$$

Since $S_X^*(\xi) \le S_0$ and $I_X^*(\xi) \le K_\varepsilon$ for $\xi \in [-X, X]$, from (32) we can obtain

$$\begin{aligned} \left|S_X^{*\prime}(\xi)\right| &\le \frac{1}{c}\left(2d_1 S_0 + \Lambda + \mu S_0 + \beta f_I(S_0, 0) K_\varepsilon\right) := L_1, \\ \left|I_X^{*\prime}(\xi)\right| &\le \frac{1}{c}\left(2d_2 K_\varepsilon + (\mu + \gamma + \alpha) K_\varepsilon + \beta f_I(S_0, 0) K_\varepsilon + \varepsilon K_\varepsilon^2\right) := L_2. \end{aligned} \tag{33}$$

It follows that

$$\left|S_X^*(\xi) - S_X^*(\eta)\right| \le L_1 |\xi - \eta|, \qquad \left|I_X^*(\xi) - I_X^*(\eta)\right| \le L_2 |\xi - \eta|. \tag{34}$$

Combining (32) and (33), we further have

$$\begin{aligned} c\left|S_X^{*\prime}(\xi) - S_X^{*\prime}(\eta)\right| \le{} & d_1 \left| \int_{\mathbb{R}} J(y)\left(\hat{S}_X(\xi - y) - \hat{S}_X(\eta - y)\right) dy \right| \\ & + (d_1 + \mu)\left|S_X^*(\xi) - S_X^*(\eta)\right| \\ & + \beta \left| f\left(S_X^*(\xi), I_X^*(\xi)\right) - f\left(S_X^*(\eta), I_X^*(\eta)\right)\right|, \end{aligned} \tag{35}$$

and

$$\begin{aligned} c\left|I_X^{*\prime}(\xi) - I_X^{*\prime}(\eta)\right| \le{} & d_2 \left| \int_{\mathbb{R}} J(y)\left(\hat{I}_X(\xi - y) - \hat{I}_X(\eta - y)\right) dy \right| \\ & + (d_2 + \mu + \gamma + \alpha)\left|I_X^*(\xi) - I_X^*(\eta)\right| \\ & + \beta \int_0^\tau h(s) \left| f\left(\hat{S}_X(\xi - cs), \hat{I}_X(\xi - cs)\right) \right. \\ & \left. - f\left(\hat{S}_X(\eta - cs), \hat{I}_X(\eta - cs)\right)\right| ds \\ & + \varepsilon \left| {I^*}_X^2(\xi) - {I^*}_X^2(\eta)\right|. \end{aligned} \tag{36}$$

Let $[-r, r]$ be the compact support of $J(x)$. Since $J(x)$ is Lipschitz continuous, there is a constant $L_J > 0$ satisfying $J(x) \le L_J r$ and $|J(x) - J(y)| \le L_J |x - y|$ for all $x, y \in [-r, r]$. Then we infer that

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} J(y)\hat{S}_X(\xi-y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{S}_X(\eta-y)\,dy\right| \\
&\quad= \left|\int_{\eta+r}^{\xi+r} J(\xi-y)\hat{S}_X(y)\,dy + \int_{\xi-r}^{\eta-r} J(\xi-y)\hat{S}_X(y)\,dy \right. \\
&\qquad\left. + \int_{\eta-r}^{\eta+r} \bigl(J(\xi-y)-J(\eta-y)\bigr)\hat{S}_X(y)\,dy\right| \\
&\quad\le 4L_J r S_0|\xi-\eta|.
\end{aligned} \tag{37}$$

Similarly, we have

$$\left|\int_{-\infty}^{+\infty} J(y)\hat{I}_X(\xi-y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{I}_X(\eta-y)\,dy\right| \le 4L_J r K_\varepsilon|\xi-\eta|. \tag{38}$$

Then it follows from (29) and (34) that

$$\int_0^\infty h(s)\bigl|f\bigl(\hat{S}_X(\xi-cs),\hat{I}_X(\xi-cs)\bigr) - f\bigl(\hat{S}_X(\eta-cs),\hat{I}_X(\eta-cs)\bigr)\bigr|\,ds \le M_4(L_1+L_2)|\xi-\eta|.$$

Meanwhile,

$$\bigl|{I^*}^2_X(\xi) - {I^*}^2_X(\eta)\bigr| = \bigl|{I^*}_X(\xi)+{I^*}_X(\eta)\bigr|\bigl|{I^*}_X(\xi)-{I^*}_X(\eta)\bigr| \le 2K_\varepsilon\bigl|{I^*}_X(\xi)-{I^*}_X(\eta)\bigr|. \tag{39}$$

Combining (35)–(39), we know $|{S_X^*}'(\xi)-{S_X^*}'(\eta)| \le C_S|\xi-\eta|$ and $|{I_X^*}'(\xi)-{I_X^*}'(\eta)| \le C_I|\xi-\eta|$, where

$$\begin{aligned}
C_S &= \frac{1}{c}\bigl(4d_1L_J r S_0 + (d_1+\mu)L_1 + \beta M_4(L_1+L_2)\bigr), \\
C_I &= \frac{1}{c}\bigl(4d_2L_J r K_\varepsilon + (d_2+\mu+\gamma+\alpha)L_2 + 2\varepsilon K_\varepsilon + \beta M_4(L_1+L_2)\bigr).
\end{aligned}$$

From the above discussions, we finally obtain $\|S_X^*(\xi)\|_{C^{1,1}([-X,X])} \le C$ and $\|I_X^*(\xi)\|_{C^{1,1}([-X,X])} \le C$ with $C = \max\{S_0+L_1+C_S, K_\varepsilon+L_2+C_I\}$. This completes the proof. □

5 Existence of traveling waves

In this section, we investigate the existence of traveling waves of system (5). Firstly, for auxiliary system (10), we have the following result.

Theorem 2 *Assume $\mathcal{R}_0 > 1$ and $c > c^*$, then system* (10) *admits a solution* $(S^*(\xi), I^*(\xi))$ *defined for $\xi \in \mathbb{R}$ satisfying $\underline{S}(\xi) \le S^*(\xi) < S_0$, $\underline{I}(\xi) \le I^*(\xi) \le \overline{I}(\xi)$, $S^*(\xi) > 0$, and $I^*(\xi) > 0$ for $\xi \in \mathbb{R}$.*

Proof Let the sequence $\{X_n\}_{n=1}^\infty$ satisfy $X_n > \max\{|\xi_1|, |\xi_3|, r\}$ and $\lim_{n\to\infty} X_n = +\infty$. Schauder's fixed point theorem ensures that there exists the fixed point $(S_{X_n}^*(\xi), I_{X_n}^*(\xi)) \in \Gamma_{X_n}$ of map $\mathcal{F}$ for every X_n. It follows from Lemma 8 that $\|S_{X_n}^*(\xi)\|_{C^{1,1}([-X_n,X_n])} \le C$ and $\|I_{X_n}^*(\xi)\|_{C^{1,1}([-X_n,X_n])} \le C$ for $n = 1, 2, \ldots$. Therefore, for any integer k, sequences $\{(S_{X_n}^*(\xi), I_{X_n}^*(\xi))\}$ and $\{(S_{X_n}^{*\prime}(\xi), I_{X_n}^{*\prime}(\xi))\}$ for $n \ge k$ are uniformly bounded and equicontinuous on $[-X_k, X_k]$. Thus, the Arzelà–Ascoli theorem and the diagonal extraction method ensure that there exists a subsequence $\{(S^*_{X_m}(\xi), I^*_{X_m}(\xi))\}$ such that $(S^*_{X_m}(\xi), I^*_{X_m}(\xi))$ and $(S^{*\prime}_{X_m}(\xi), I^{*\prime}_{X_m}(\xi))$ uniformly converge in each interval $[-X_k, X_k]$ $(k = 1, 2, \ldots)$ as $m \to \infty$.

Let $\lim_{m\to\infty}(S^*_{X_m}(\xi), I^*_{X_m}(\xi)) = (S^*(\xi), I^*(\xi))$, then we have $\lim_{m\to\infty}(S^{*\prime}_{X_m}(\xi), I^{*\prime}_{X_m}(\xi)) = (S^{*\prime}(\xi), I^{*\prime}(\xi))$. Let r be the supported radius of $J(x)$. Since $(S^*_{X_m}(\xi), I^*_{X_m}(\xi)) \le (\overline{S}(\xi), \overline{I}(\xi))$ for $\xi \in \mathbb{R}$ and $m = 1, 2, \ldots$, using the Lebesgue dominated convergence theorem, it follows that

$$\lim_{m\to\infty} \int_{\mathbb{R}} J(y) S_{X_m}(\xi - y)\, dy = \lim_{m\to\infty} \int_{-r}^{r} J(y) S_{X_m}(\xi - y)\, dy = J * S^*(\xi).$$

Similarly, we can obtain $\lim_{m\to\infty} J * I_{X_m}(\xi) = J * I^*(\xi)$. Therefore, $(S^*(\xi), I^*(\xi))$ satisfies (10) and $\underline{S}(\xi) \le S^*(\xi) \le S_0$ and $\underline{I}(\xi) \le I^*(\xi) \le \overline{I}(\xi)$ for $\xi \in \mathbb{R}$.

Now, we prove $S_0 > S^*(\xi) > 0$ and $I^*(\xi) > 0$. Since $S(-\infty) = S_0 > 0$, suppose there exists $\xi_0 \in \mathbb{R}$ such that $S(\xi_0) = 0$ and $S(\xi) > 0$ for all $\xi \in (-\infty, \xi_0)$, then we have $S'(\xi_0) \le 0$. From the first equation of system (10), we have $d_1 \int_{\mathbb{R}} J(y) S(\xi_0 - y)\, dy + \Lambda \le 0$. This leads to a contradiction. Hence, $S^*(\xi) > 0$ for all $\xi \in \mathbb{R}$. Similarly, we have $I^*(\xi) > 0$ for $\xi \in \mathbb{R}$. Next, we prove $S^*(\xi) < S_0$. Suppose that there exists $\xi_0 \in \mathbb{R}$ such that $S^*(\xi_0) = S_0$, then we have $S^{*\prime}(\xi_0) \ge 0$. Combining the first equation of system (10), we know

$$d_1 \int_{\mathbb{R}} J(y) \bigl(S^*(\xi_0 - y) - S_0\bigr)\, dy + \Lambda - \mu S_0 - \beta f\bigl(S^*(\xi_0), I(\xi_0)\bigr) \ge 0,$$

that is, $d_1 \int_{\mathbb{R}} J(y)(S^*(\xi_0 - y) - S_0)\, dy - \beta f(S_0, I(\xi_0)) \ge 0$, which reduces to a contradiction since $S^*(\xi_0 - y) - S_0 \le 0$ and $f(S_0, I(\xi_0)) > 0$. Thus, $S^*(\xi) < S_0$ for $\xi \in \mathbb{R}$. This completes the proof. □

Next, for subsystem (7), we further have the following result.

Theorem 3 *Assume* $\mathcal{R}_0 > 1$ *and* $c > c^*$, *then system* (7) *admits a solution* $(S^*(\xi), I^*(\xi))$ *defined for* $\xi \in \mathbb{R}$ *satisfying* $\lim_{\xi\to-\infty}(S^*(\xi), I^*(\xi)) = (S_0, 0)$, $0 < S^*(\xi) < S_0$, *and* $I^*(\xi) > 0$ *for* $\xi \in \mathbb{R}$.

Proof Let the sequence $\{\varepsilon_n\}$ satisfy $0 < \varepsilon_{n+1} < \varepsilon_n < 1$ for $n = 1, 2, \ldots$ and $\lim_{n\to+\infty} \varepsilon_n = 0$. According to Theorem 2, there exists a solution sequence $\Phi_n(\xi) = (S^*_n(\xi), I^*_n(\xi))$ with $\varepsilon = \varepsilon_n$ for each $n \in \mathbb{N}^*$ and $\xi \in \mathbb{R}$, satisfying

$$\begin{cases} cS^{*\prime}_n(\xi) = d_1 J * S^*_n(\xi) - d_1 S^*_n(\xi) + \Lambda - \mu S^*_n(\xi) - \beta f(S^*_n(\xi), I^*_n(\xi)), \\ cI^{*\prime}_n(\xi) = d_2 J * I^*_n(\xi) + \beta \int_0^\tau h(s) f(S^*_n(\xi - cs), I^*_n(\xi - cs))\, ds \\ \qquad\qquad - (d_2 + \mu + \gamma + \alpha) I^*_n(\xi) - \varepsilon_n I^{*2}_n(\xi), \end{cases} \tag{40}$$

and

$$\underline{S}(\xi) < S^*_n(\xi) < S_0, \qquad \underline{I}(\xi) \le I^*_n(\xi) \le \overline{I}(\xi), \qquad S^*_n(\xi) > 0, \qquad I^*_n(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{41}$$

In the interval $[-1, 1]$, since $K_{\varepsilon_n} = \frac{1}{\varepsilon_n}(\beta f_I(S_0, 0) - (\mu + \gamma + \alpha)) \to +\infty$ as $n \to +\infty$, there exists $n_1 \in \mathbb{N}^*$ such that $e^{\lambda_1 \xi} < K_{\varepsilon_n}$, that is, $\overline{I}(\xi) = e^{\lambda_1 \xi}$ for any $n > n_1$ and $\xi \in [-1, 1]$. Therefore, when $n > n_1$, $\{\Phi_n(\xi)\}$ is uniformly bounded on $[-1, 1]$. From (40), we further obtain that both $\{\Phi_n(\xi)\}$ and $\{\Phi'_n(\xi)\}$ for $n > n_1$ are equicontinuous and uniformly bounded on $[-1,1]$. Therefore, there exists a subsequence $\{\Phi_{1,m}(\xi)\}$ of $\{\Phi_n(\xi)\}$ such that $\{\Phi_{1,m}(\xi)\}$ and $\{\Phi'_{1,m}(\xi)\}$ uniformly converge on $[-1,1]$ as $m\to\infty$ by using the Arzelà–Ascoli theorem. Furthermore, we obtain $I^*_{1,m}(\xi)\le e^{\lambda_1\xi}$ for all $\xi\in[-1,1]$.

Assume that in the interval $[-(k-1),k-1]$ we have selected a subsequence $\{\Phi_{k-1,m}(\xi)\}$ of $\{\Phi_{k-2,m}(\xi)\}$ such that $\{\Phi_{k-1,m}(\xi)\}$ and $\{\Phi'_{k-1,m}(\xi)\}$ uniformly converge on $[-(k-1),k-1]$ as $m\to\infty$. We also have $I^*_{k-1,m}(\xi)\le e^{\lambda_1\xi}$ for all $\xi\in[-(k-1),k-1]$. Then in the interval $[-k,k]$, since $K_{\varepsilon_{k-1,m}}\to+\infty$ as $m\to+\infty$, there exists $m_k\in\mathbb{N}^*$ such that $e^{\lambda_1\xi}<K_{\varepsilon_{k-1,m}}$ for any $m>m_k$ and $\xi\in[-k,k]$. Hence $\overline{I}(\xi)=e^{\lambda_1\xi}$ for $\xi\in[-k,k]$. Thus, when $m>m_k$, $\{\Phi_{k-1,m}(\xi)\}$ is uniformly bounded on $[-k,k]$. Deduced by (36), we further obtain that both $\{\Phi_{k-1,m}(\xi)\}$ and $\{\Phi'_{k-1,m}(\xi)\}$ are equicontinuous and uniformly bounded on $[-k,k]$. Therefore, there exists a subsequence $\{\Phi_{k,m}(\xi)\}$ of $\{\Phi_{k-1,m}(\xi)\}$ such that $\{\Phi_{k,m}(\xi)\}$ and $\{\Phi'_{k,m}(\xi)\}$ uniformly converge on $[-k,k]$ as $m\to\infty$. We also have $I^*_{k,m}(\xi)\le e^{\lambda_1\xi}$ for all $\xi\in[-k,k]$.

Thus, by using the diagonal extraction method, we can select the subsequences $\{\Phi_{m,m}(\xi)\}$ and $\{\Phi'_{m,m}(\xi)\}$ which uniformly converge on each interval $[-k,k]$ $(k=1,2,3,\ldots)$. Let $\{\Phi_{m,m}(\xi)\}\to(S^*(\xi),I^*(\xi))$ as $m\to+\infty$. Then we further have $\{\Phi'_{m,m}(\xi)\}\to(S^{*\prime}(\xi),I^{*\prime}(\xi))$ as $m\to+\infty$. Since for every $m\in\mathbb{N}^*$ we have

$$\begin{cases} cS^{*\prime}_{m,m}(\xi)=d_1J*S^*_{m,m}(\xi)-d_1S^*_{m,m}(\xi)+\Lambda-\mu S^*_{m,m}(\xi)-\beta f(S^*_{m,m}(\xi),I^*_{m,m}(\xi)),\\ cI^{*\prime}_{m,m}(\xi)=d_2J*I^*_{m,m}(\xi)-(d_2+\mu+\gamma+\alpha)I^*_{m,m}(\xi)\\ \qquad\qquad +\beta\int_0^\tau h(s)f(S^*_{m,m}(\xi-cs),I^*_{m,m}(\xi-cs))\,ds-\varepsilon_{m,m}I^{*2}_{m,m}(\xi). \end{cases} \tag{42}$$

Taking $m\to+\infty$, combining (A1), using the continuity of $f(S,I)$, $\lim_{m\to\infty}\varepsilon_{m,m}=0$, and the dominated convergence theorem ensures that we finally obtain

$$\begin{cases} cS^{*\prime}(\xi)=d_1(J*S^*(\xi)-S^*(\xi))+\Lambda-\mu S^*(\xi)-\beta f(S^*(\xi),I^*(\xi)),\\ cI^{*\prime}(\xi)=d_2(J*I^*(\xi)-I^*(\xi))+\beta\int_0^\tau h(s)f(S^*(\xi-cs),I^*(\xi-cs))\,ds\\ \qquad\qquad -(\mu+\gamma+\alpha)I^*(\xi) \end{cases} \tag{43}$$

for all $\xi\in\mathbb{R}$. That is, $(S^*(\xi),I^*(\xi))$ is the solution of system (7) defined for $\xi\in\mathbb{R}$.

From (41), we obtain $\underline{S}(\xi)<S^*(\xi)\le S_0$ and $\underline{I}(\xi)\le I^*(\xi)$ for all $\xi\in\mathbb{R}$. Since for any integer $k>0$, when $m\ge k$, $I^*_{m,m}(\xi)\le e^{\lambda_1\xi}$ for all $\xi\in[-k,k]$, we further obtain $I^*(\xi)\le e^{\lambda_1\xi}$ for $\xi\in\mathbb{R}$. Combining the upper-lower solutions, it follows that $(S^*(\xi),I^*(\xi))$ satisfies $\lim_{\xi\to-\infty}(S^*(\xi),I^*(\xi))=(S_0,0)$.

Using similar arguments as in Theorem 2, we easily prove that $0<S^*(\xi)<S_0$ for all $\xi\in\mathbb{R}$. Similarly, suppose there exists $\hat{\xi}\in\mathbb{R}$ such that $I^*(\hat{\xi})=0$; moreover, $I^*(\xi)>0$ for all $\xi\in(-\infty,\hat{\xi})$. It is clear that $\hat{\xi}>\xi_3$ and $I^{*\prime}(\hat{\xi})\le0$. Then the second equation of (43) yields

$$cI^{*\prime}(\hat{\xi})=d_2J*I^*(\hat{\xi})+\beta\int_0^\tau h(s)f\big(S^*(\hat{\xi}-cs),I^*(\hat{\xi}-cs)\big)\,ds>0.$$

This is a contradiction. Thus, $I^*(\xi)>0$ for all $\xi\in\mathbb{R}$. This completes the proof. □

Let the solution $(S^*(\xi),I^*(\xi))$ be determined in Theorem 3. To obtain the asymptotic boundary condition $(S^*(\xi),I^*(\xi))\to(S^*,I^*)$ as $\xi\to+\infty$, we need to introduce the following assumption.

(A4) For any $S > 0$ and $I > 0$,

$$\frac{f(S,I)}{f(S,I^*)} - \frac{f(S^*,I^*)}{f(S,I^*)} - \frac{I}{I^*} + \frac{If(S^*,I^*)}{I^*f(S^*,I)} + \frac{f(S^*,I)}{f(S,I)} - 1 \le 0.$$

Theorem 4 *Assume that* $\mathcal{R}_0 > 1$, $c > c^*$ *and* (A4) *holds. Then system* (7) *admits a positive traveling wave* $(S^*(\xi), I^*(\xi))$ *which satisfies* $\lim_{\xi\to-\infty}(S^*(\xi), I^*(\xi)) = (S_0, 0)$ *and* $\lim_{\xi\to\infty}(S^*(\xi), I^*(\xi)) = (S^*, I^*)$.

Proof Let $H(x) = x - 1 - \ln x$, $\alpha_1(y) = \int_y^{+\infty} J(x)\,dx$, and $\alpha_2(y) = \int_{-\infty}^y J(x)\,dx$. From (A2), let the compact support of $J(x)$ be $[-r, r]$, we have $\alpha_1(y) = 0$ for $y \ge r$ and $\alpha_2(y) = 0$ for $y \le -r$. Consider the Lyapunov function

$$L = c_1 V_1 + c_2 V_2 + c_3 V_S + c_4 V_I,$$

where the constants c_1, c_2, c_3, and c_4 will be determined later, and

$$V_1 = \left(S(\xi) - S^* - \int_{S^*}^{S(\xi)} \frac{f(S^*,I^*)}{f(\eta,I^*)}\,d\eta\right) + I(\xi) - I^* - \int_{I^*}^{I(\xi)} \frac{f(S^*,I^*)}{f(S^*,\eta)}\,d\eta,$$

$$V_2 = \int_0^\tau h(s) \int_{\xi-cs}^{\xi} \left(\frac{f(S(u),I(u))}{f(S^*,I^*)} - 1 - \ln\frac{f(S(u),I(u))}{f(S^*,I^*)}\right) du\,ds,$$

$$V_S = \int_0^{+\infty} \alpha_1(y)\left[H\left(\frac{S(\xi-y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi-y)}{f(S(\xi),I^*)S^*}\right)\right] dy$$
$$- \int_{-\infty}^0 \alpha_2(y)\left[H\left(\frac{S(\xi-y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi-y)}{f(S(\xi),I^*)S^*}\right)\right] dy,$$

$$V_I = \int_0^{+\infty} \alpha_1(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I(\xi))I^*}\right)\right] dy$$
$$- \int_{-\infty}^0 \alpha_2(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I(\xi))I^*}\right)\right] dy.$$

Calculating the derivative of V_1, V_2, V_S, and V_I with respect to system (7), we have

$$\frac{dV_1}{d\xi} = \left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\frac{1}{c}\big(d_1(J*S - S) + \Lambda - \mu S - \beta f(S,I)\big) + \left(1 - \frac{f(S^*,I^*)}{f(S^*,I)}\right)$$
$$\times \frac{1}{c}\left(d_2(J*I - I) + \beta\int_0^\tau h(s) f\big(S(\xi-cs), I(\xi-cs)\big)\,ds - (\mu+\gamma+\alpha)I\right),$$

$$\frac{dV_2}{d\xi} = \int_0^\tau h(s)\left[\frac{f(S,I)}{f(S^*,I^*)} - \frac{f(S(\xi-cs),I(\xi-cs))}{f(S^*,I^*)} + \ln\frac{f(S(\xi-cs),I(\xi-cs))}{f(S,I)}\right] ds.$$

Since $\alpha_i(0) = \frac{1}{2}$, $\frac{d\alpha_1(y)}{dy} = -J(y)$ and $\frac{d\alpha_2(y)}{dy} = J(y)$, we have

$$\frac{dV_S}{d\xi} = -\int_0^{+\infty} \alpha_1(y)\frac{d}{dy}\left[H\left(\frac{S(\xi-y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi-y)}{f(S,I^*)S^*}\right)\right] dy$$
$$+ \int_{-\infty}^0 \alpha_2(y)\frac{d}{dy}\left[H\left(\frac{S(\xi-y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi-y)}{f(S,I^*)S^*}\right)\right] dy$$
$$= H\left(\frac{S}{S^*}\right) - H\left(\frac{f(S^*,I^*)S}{f(S,I^*)S^*}\right) - \int_{\mathbb{R}} J(y)\left[H\left(\frac{S(\xi-y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi-y)}{f(S,I^*)S^*}\right)\right] dy.$$

Similarly,

$$\frac{dV_I}{d\xi} = H\left(\frac{I}{I^*}\right) - H\left(\frac{f(S^*,I^*)I}{f(S^*,I)I^*}\right) - \int_{\mathbb{R}} J(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I)I^*}\right)\right] dy.$$

Choose $c_1 = c$, $c_2 = \beta f(S^*,I^*)$, $c_3 = d_1 S^*$, and $c_4 = d_2 I^*$, then we obtain

$$\frac{dL}{d\xi} = B_1 + d_1 B_2 + d_2 B_3,$$

where

$$\begin{aligned}
B_1 &= \left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\left(\Lambda - \mu S - \beta f(S,I)\right) \\
&\quad + \left(1 - \frac{f(S^*,I^*)}{f(S^*,I)}\right)\left(\beta \int_0^\tau h(s) f\left(S(\xi - cs), I(\xi - cs)\right) ds - (\mu + \gamma + \alpha) I\right) \\
&\quad + \beta f\left(S^*,I^*\right) \int_0^\tau h(s)\left[\frac{f(S,I)}{f(S^*,I^*)} - \frac{f(S(\xi - cs), I(\xi - cs))}{f(S^*,I^*)}\right. \\
&\quad \left. + \ln \frac{f(S(\xi - cs), I(\xi - cs))}{f(S,I)}\right] ds, \\
B_2 &= \left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)(J * S - S) + S^* H\left(\frac{S}{S^*}\right) - S^* H\left(\frac{f(S^*,I^*)S}{f(S,I^*)S^*}\right) \\
&\quad - S^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{S(\xi - y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi - y)}{f(S,I^*)S^*}\right)\right] dy, \\
B_3 &= \left(1 - \frac{f(S^*,I^*)}{f(S^*,I)}\right)(J * I - I) + I^* H\left(\frac{I}{I^*}\right) - I^* H\left(\frac{f(S^*,I^*)I}{f(S^*,I)I^*}\right) \\
&\quad - I^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{I(\xi - y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi - y)}{f(S^*,I)I^*}\right)\right] dy.
\end{aligned}$$

By computation, we further obtain

$$\begin{aligned}
B_1 &= \mu\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\left(S^* - S\right) + \beta f\left(S^*,I^*\right) \int_0^\tau h(s)\left[1 - \frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S,I)}{f(S,I^*)}\right. \\
&\quad \left. - \frac{I}{I^*} + \frac{If(S^*,I^*)}{I^* f(S^*,I)} - \frac{f(S(\xi - cs), I(\xi - cs))}{f(S^*,I)} + \ln \frac{f(S(\xi - cs), I(\xi - cs))}{f(S,I)}\right] ds \\
&= \mu\left(1 - \frac{f(S^*,I^*)}{f(S,I^*)}\right)\left(S^* - S\right) + \beta f\left(S^*,I^*\right) \int_0^\tau h(s)\left[-\frac{f(S^*,I^*)}{f(S,I^*)} + \frac{f(S,I)}{f(S,I^*)}\right. \\
&\quad \left. - \frac{I}{I^*} + \frac{If(S^*,I^*)}{I^* f(S^*,I)} + \frac{f(S^*,I)}{f(S,I)} - 1 - H\left(\frac{f(S(\xi - cs), I(\xi - cs))}{f(S^*,I)}\right) - H\left(\frac{f(S^*,I)}{f(S,I)}\right)\right] ds, \\
B_2 &= S^* \int_{\mathbb{R}} J(y)\left[\frac{S(\xi - y)}{S^*} - \frac{f(S^*,I^*)S(\xi - y)}{f(S,I^*)S^*} - \ln \frac{S}{S^*} + \ln \frac{f(S^*,I^*)S}{f(S,I^*)S^*}\right] dy \\
&\quad - S^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{S(\xi - y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi - y)}{f(S,I^*)S^*}\right)\right] dy \\
&= S^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{S(\xi - y)}{S^*}\right) - \frac{f(S^*,I^*)S(\xi - y)}{f(S,I^*)S^*} + 1 + \ln \frac{f(S^*,I^*)}{f(S,I^*)} \frac{S(\xi - y)}{S^*}\right] dy \\
&\quad - S^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{S(\xi - y)}{S^*}\right) - H\left(\frac{f(S^*,I^*)S(\xi - y)}{f(S,I^*)S^*}\right)\right] dy = 0,
\end{aligned}$$

$$\begin{aligned} B_3 &= I^* \int_{\mathbb{R}} J(y)\left[\frac{I(\xi-y)}{I^*} - \frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I)I^*} - \ln\frac{I}{I^*} + \ln\frac{f(S^*,I^*)I}{f(S^*,I)I^*}\right] dy \\ &\quad - I^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I)I^*}\right)\right] dy \\ &= I^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - \frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I)I^*} + 1 + \ln\frac{f(S^*,I^*)}{f(S^*,I)}\frac{I(\xi-y)}{I^*}\right] dy \\ &\quad - I^* \int_{\mathbb{R}} J(y)\left[H\left(\frac{I(\xi-y)}{I^*}\right) - H\left(\frac{f(S^*,I^*)I(\xi-y)}{f(S^*,I)I^*}\right)\right] dy = 0. \end{aligned}$$

It follows from (A1) that $(1-\frac{f(S^*,I^*)}{f(S,I^*)})(S^*-S) \leq 0$. (A4) implies $\frac{dL}{d\xi} \leq 0$. Furthermore, $\frac{dL}{d\xi} = 0$ if and only if $S = S^*$ and $I = I^*$. Using LaSalle's invariance principle, we know that $\lim_{\xi\to\infty} S(\xi) = S^*$ and $\lim_{\xi\to\infty} I(\xi) = I^*$. From Theorem 3, we know that $\lim_{\xi\to-\infty}(S^*(\xi), I^*(\xi)) = (S_0, 0)$. This completes the proof. □

Lastly, on the existence of traveling waves for system (5), we establish the following result.

Theorem 5 *Assume that* $\mathcal{R}_0 > 1$, $c > c^*$ *and* (A4) *holds. Then system* (5) *admits a positive traveling wave* $(S^*(\xi), I^*(\xi), R^*(\xi))$ *defined for* $\xi \in \mathbb{R}$ *which satisfies asymptotic boundary conditions* (6).

Proof From Theorems 3 and 4, it follows that there exists a positive traveling wave $(S^*(\xi), I^*(\xi))$ defined for $\xi \in \mathbb{R}$ satisfying the first two equations of system (5) and $I^*(\xi) \leq e^{\lambda_1\xi}$ for $\xi \in \mathbb{R}$.

From the third equation of system (5) we have

$$cR'(\xi) = d_3\big(J * R(\xi) - R(\xi)\big) + \gamma I^*(\xi) - \mu R(\xi). \tag{44}$$

It is sufficient to prove that equation (44) has a solution $R^*(\xi)$ defined for $\xi \in \mathbb{R}$ satisfying the asymptotic boundary conditions $\lim_{\xi\to-\infty} R^*(\xi) = 0$ and $\lim_{\xi\to\infty} R^*(\xi) = R^*$.

Let $\underline{R}(\xi) \equiv 0$ and $\overline{R}(\xi) = \min\{Ae^{\alpha\xi}, A\}$ for $\xi \in \mathbb{R}$, where $\alpha > 0$ is a constant, $A = \frac{\gamma}{\mu}(I_M^* + 1)$ and $I_M^* = \sup_{\xi\in\mathbb{R}} I^*(\xi) < \infty$. It is clear that $\underline{R}(\xi)$ is the lower solution of equation (44). When $\xi \geq 0$, since $\overline{R}(\xi) = A$, we obtain

$$d_3\big(J * \overline{R}(\xi) - \overline{R}(\xi)\big) + \gamma I^*(\xi) - \mu\overline{R}(\xi) - c\overline{R}'(\xi) \leq \gamma I_\mu^* - \mu A < 0.$$

Since $\lim_{\alpha\to 0^+}\int_{-r}^{r} J(y)(e^{-\alpha y}-1)\,dy = 0$, there exists $\alpha \in (0, \lambda_1)$ such that $d_3A\int_{-r}^{r} J(y)(e^{-\alpha y} - 1)\,dy + \gamma e^{(\lambda_1-\alpha)\xi} - \mu A < 0$ for all $\xi < 0$. When $\xi < 0$, since $\overline{R}(\xi) = Ae^{\alpha\xi}$, we obtain

$$\begin{aligned} &d_3\big(J * \overline{R}(\xi) - \overline{R}(\xi)\big) + \gamma I^*(\xi) - \mu\overline{R}(\xi) - c\overline{R}'(\xi) \\ &\quad= e^{\alpha\xi}\left(d_3A\int_{\mathbb{R}} J(y)\big(e^{-\alpha y}-1\big)\,dy + \gamma I^*(\xi)e^{-\alpha\xi} - \mu A\right) - cA\alpha e^{\alpha\xi} \\ &\quad\leq e^{\alpha\xi}\left(d_3A\int_{-r}^{r} J(y)\big(e^{-\alpha y}-1\big)\,dy + \gamma e^{(\lambda_1-\alpha)\xi} - \mu A\right) < 0. \end{aligned}$$

This shows that $\overline{R}(\xi)$ is the upper solution of equation (44).

Define a function set

$$\Gamma_H = \left\{ \omega(\xi) \in C\big([-H,H],R\big) : \begin{array}{l} \omega(-H) = \underline{R}(-H) \\ \underline{R}(\xi) \le \omega(\xi) \le \overline{R}(\xi), \end{array} \xi \in [-H,H], H > 0 \right\}.$$

For any $\omega(\xi) \in \Gamma_H$, we define a function $\hat{\omega}(\xi)$ for $\xi \in \mathbb{R}$ as follows: $\hat{\omega}(\xi) = \omega(H)$ if $\xi > H$, $\hat{\omega}(\xi) = \omega(\xi)$ if $\xi \in [-H,H]$ and $\hat{\omega}(\xi) = \underline{R}(\xi)$ if $\xi < -H$. Obviously, Γ_H is closed and convex, and $\underline{R}(\xi) \le \hat{\omega}(\xi) \le \overline{R}(\xi)$ for all $\xi \in \mathbb{R}$.

Consider the initial value problem

$$\begin{cases} cR'(\xi) = d_3(J * \hat{\omega}(\xi) - R(\xi)) + \gamma I^*(\xi) - \mu R(\xi), \\ R(-H) = \underline{R}(-H). \end{cases} \tag{45}$$

The ODE theory implies that equation (45) has a unique solution $R_H(\xi)$ defined for $[-H,H]$. Thus, we can define the map G as follows:

$$G(\omega)(\xi) = R_H(\xi), \quad \omega(\xi) \in \Gamma_H.$$

Similar to Lemma 6, Lemma 7, and Theorem 1, we can prove that operator G maps Γ_H to Γ_H and is completely continuous. Schauder's fixed point theorem ensures that map G admits a fixed point $R_H^*(\xi) \in \Gamma_H$ such that $cR_H^{*\prime}(\xi) = d_3(J * \hat{R}_H^*(\xi) - R_H^*(\xi)) + \gamma I^*(\xi) - \mu R_H^*(\xi)$ and $\underline{R}(\xi) \le R_H^*(\xi) \le \overline{R}(\xi)$ for all $\xi \in [-H,H]$.

Choose $H = H_k$ for $k = 1,2,\dots$ such that the sequence $\{H_k\}$ is strictly increasing and $\lim_{k\to\infty} H_k = +\infty$, then we can obtain a solution sequence $\{R_{H_k}^*(\xi)\}$. By a similar argument as in Lemma 8, we know that there is a constant C which is independent of k such that $\|R_{H_k}^*(\xi)\|_{C^{1,1}([-H_k,H_k])} \le C$ for each $k = 1,2,\dots$. Furthermore, using a similar argument as in Theorem 3, we can obtain that there exists a solution $R^*(\xi)$ of equation (44) defined for $\xi \in \mathbb{R}$ satisfying $\underline{R}(\xi) < R^*(\xi) \le \overline{R}(\xi)$. Obviously, $\lim_{\xi\to-\infty} R^*(\xi) = R^*(-\infty) = 0$. Next, we prove $\lim_{\xi\to+\infty} R^*(\xi) = R^*(+\infty) = R^*$. In fact, define the Lyapunov function

$$\begin{aligned} L(\xi) &= cR^* H\left(\frac{R(\xi)}{R^*}\right) \\ &\quad + d_3R^*\left(\int_0^{+\infty} \alpha_1(y) H\left(\frac{R(\xi-y)}{R^*}\right) dy - \int_{-\infty}^0 \alpha_2(y) H\left(\frac{R(\xi-y)}{R^*}\right) dy\right), \end{aligned}$$

where $H(x)$ and $\alpha_i(y)$ $(i = 1,2)$ are defined in Theorem 4. By a similar calculation as in Theorem 4, we have

$$\begin{aligned} \frac{dL(\xi)}{d\xi} &= \left(1 - \frac{R^*}{R}\right)\big(d_3(J * R - R) + \gamma I^* - \mu R\big) \\ &\quad + d_3R^*\left(H\left(\frac{R(\xi)}{R^*}\right) - \int_{\mathbb{R}} J(y) H\left(\frac{R(\xi-y)}{R^*}\right) dy\right) \\ &= \mu\left(1 - \frac{R^*}{R}\right)\big(R^* - R\big) - d_3R^* \int_{\mathbb{R}} J(y) H\left(\frac{R(\xi-y)}{R(\xi)}\right) dy \le 0, \end{aligned}$$

and $\frac{dL}{d\xi} = 0$ if and only if $R(\xi) = R^*$. Therefore, by LaSalle's invariance principle, we have $R(+\infty) = R^*$. This completes the proof. □

As a special case, we consider the nonlinear incidence function $f(S,I) = p(S)g(I)$ and introduce the following assumption.

(A5) Functions $g(I)$ and $p(S)$ are quadric continuously differentiable and nondecreasing for $S \geq 0$ and $I \geq 0$, $\frac{g(I)}{I}$ is nonincreasing for $I > 0$, and $p(0) = g(0) = 0$.

It is easy to verify that when (A5) holds, then (A4) also holds. Therefore, as a consequence of Theorem 5, we have the following corollary.

Corollary 1 *Assume that* $f(S,I) = p(S)g(I)$, $\mathcal{R}_0 > 1$, $c > c^*$ *and* (A5) *holds. Then system* (5) *admits a positive traveling wave* $(S^*(\xi), I^*(\xi), R^*(\xi))$ *defined for* $\xi \in \mathbb{R}$ *satisfying asymptotic boundary conditions* (6).

6 Nonexistence of traveling waves

In this section we investigate the nonexistence of a traveling wave $(S^*(\xi), I^*(\xi), R^*(\xi))$ of system (5). We have the following result.

Theorem 6 *Assume that* $\mathcal{R}_0 > 1$ *and* $0 < c < c^*$, *then there does not exist a traveling wave* $(S^*(\xi), I^*(\xi), R^*(\xi))$ *of system* (5) *defined for* $\xi \in \mathbb{R}$ *satisfying asymptotic boundary conditions* (6).

Proof Suppose that there exists a traveling wave $(S^*(\xi), I^*(\xi), R^*(\xi))$ of system (5) satisfying conditions (6) for some $0 < c_1 < c^*$. From (6) and $\mathcal{R}_0 > 1$, for any given $\epsilon > 0$, there exists some $M_\epsilon > 0$ large enough such that $S_0 - \epsilon \leq S^*(\xi) < S_0$ for all $\xi < -M_\epsilon$. Combining the second equation of system (5), we have

$$\begin{aligned} c_1 {I^*}'(\xi) &= d_2\bigl(J * I^*(\xi) - I^*(\xi)\bigr) \\ &\quad + \beta \int_0^\tau h(s) f\bigl(S^*(\xi - c_1 s), I^*(\xi - c_1 s)\bigr)\, ds - (\mu + \gamma + \alpha) I^*(\xi) \\ &\geq d_2\bigl(J * I^*(\xi) - I^*(\xi)\bigr) \\ &\quad + \beta \int_0^\tau h(s) f\bigl(S_0 - \epsilon, I^*(\xi - c_1 s)\bigr)\, ds - (\mu + \gamma + \alpha) I^*(\xi) \end{aligned} \tag{46}$$

for $\xi < -M_\epsilon$. Noting the continuity and asymptotic boundary conditions (6) of traveling waves, there exist positive constants δ and M_0 such that $S^*(\xi) \geq \delta$ and $I^*(\xi) \leq M_0$ for all $\xi \in \mathbb{R}$. Using assumption (A1), we obtain that

$$\begin{aligned} \frac{f((S_0 - \epsilon, I^*(\xi - c_1 s))}{f(S^*(\xi - c_1 s), I^*(\xi - c_1 s))} &\leq \frac{f(S_0 - \epsilon, I^*(\xi - c_1 s))}{f(\delta, I^*(\xi - c_1 s))} \\ &= \frac{f(S_0 - \epsilon, I^*(\xi - c_1 s))}{I^*(\xi - c_1 s)} \frac{I^*(\xi - c_1 s)}{f(\delta, I^*(\xi - c_1 s))} \\ &\leq \frac{M_0}{f(\delta, M_0)} f_I(S_0, 0) < \infty, \quad \xi > -M_\epsilon. \end{aligned}$$

Noting that $I^*(\xi) > 0$ for $\xi \in \mathbb{R}$ and $I^*(+\infty) = I^* > 0$, there exists a positive constant $\underline{I} > 0$ such that $I^*(\xi) \geq \underline{I}$ for all $\xi > -M_\epsilon$. Therefore, we can choose a constant $h > 1$ such that $\frac{f(S_0 - \epsilon, I^*(\xi - c_1 s))}{(1 + I^*(\xi - c_1 s))^h} \leq f(S^*(\xi - c_1 s), I^*(\xi - c_1 s))$ for $\xi > -M_\epsilon$. Then, for $\xi > -M_\epsilon$, the following inequality holds:

$$\begin{aligned} c_1 I^{*\prime}(\xi) \geq{}& d_2\big(J * I^*(\xi) - I^*(\xi)\big) + \beta \int_0^\tau h(s) \frac{f(S_0 - \epsilon, I^*(\xi - c_1 s))}{(1 + I^*(\xi - c_1 s))^h}\, ds \\ &- (\mu + \gamma + \alpha) I^*(\xi). \end{aligned} \tag{47}$$

Combining (46) and (47), we finally obtain

$$\begin{aligned} c_1 I^{*\prime}(\xi) \geq{}& d_2\big(J * I^*(\xi) - I^*(\xi)\big) + \beta \int_0^\tau h(s) \frac{f(S_0 - \epsilon, I^*(\xi - c_1 s))}{(1 + I^*(\xi - c_1 s))^h}\, ds \\ &- (\mu + \gamma + \alpha) I^*(\xi), \quad \xi \in \mathbb{R}. \end{aligned} \tag{48}$$

Let $b(u) = \inf_{u \leq v \leq M_0} \{\frac{\beta f(S_0 - \epsilon, v)}{(1+v)^h}\}$ and $u(x,t) = I^*(x + c_1 t)$. It follows from (48) that

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} \geq d_2(J * u(x,t) - u(x,t)) + \beta \int_0^\tau h(s) b(u(x,t-s))\, ds - (\mu + \gamma + \alpha) u(x,t), \\ u(x,s) = I^*(x + c_1 s), \quad x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases}$$

By the comparison principle [40], we have

$$u(x,t) \geq v(x,t), \quad x \in \mathbb{R}, t \geq 0, \tag{49}$$

where $v(x,t)$ is the solution of the following equation:

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = d_2(J * v(x,t) - v(x,t)) + \beta \int_0^\tau h(s) b(v(x,t-s))\, ds - (\mu + \gamma + \alpha) v(x,t), \\ v(x,s) = I^*(x + c_1 s), \quad x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases} \tag{50}$$

Now, we prove that for any $\hat{c} \in (0, c^*)$

$$\lim_{t \to \infty} \inf_{|x| \leq \hat{c} t} v(x,t) > 0 \tag{51}$$

by using the asymptotic spreading theory [41]. We know that the operator $J * \cdot - \cdot$ can generate a C_0-semigroup [42, 43]. It is clear that system (50) is Fisher–KPP type equation and admits only two equilibria: $v \equiv 0$ and a positive equilibrium v^* satisfying $\beta b(v^*) - (\mu + \gamma + \alpha) v^* = 0$. We denote $C = C(\mathbb{R} \times [-\tau, 0])$ and $C_{v^*} = \{v \in C : 0 \leq v \leq v^*\}$. Applying the semigroup theory [42, 43], we know that system (50) generates a monotone semi-flow $Q_t : C_{v^*} \to C_{v^*}$ defined as follows:

$$Q_t(\psi)(x) = v(x, t+s), \quad x \in \mathbb{R}, t \geq 0, s \in [-\tau, 0], \psi \in C_{v^*},$$

where $v(x,t)$ is the unique solution of system (50) with the initial value $v(x,s) = \psi$.

Denote $\tilde{C} = C([-\tau, 0])$ and $\tilde{C}_{v^*} = \{v \in \tilde{C} : 0 \leq v \leq v^*\}$. Let $\tilde{Q}_t : \tilde{C}_{v^*} \to \tilde{C}_{v^*}$ be the solution semi-flow generated by the following delayed differential equation:

$$\frac{dv(t)}{dt} = \int_0^\tau h(s) b\big(v(t-s)\big)\, ds - (\mu + \gamma + \alpha) v(t), \quad t \geq 0,$$

with the initial value $v_0 = \psi_0 \in \tilde{C}_{v^*}$, where $v_t = v(t+s)$ for $s \in [-\tau, 0]$. From Corollary 5.3.5 in [44], we have that $\tilde{Q}_t$ is eventually strongly monotone on $\tilde{C}_{v^*}$. Furthermore, combining the Dancer–Hess connecting orbit lemma [45], we obtain that $\tilde{Q}_t$ also is a strongly monotone full orbit connecting 0 to v^*. Hence, hypothesis (A5) in [41] holds. In fact, we can easily see that for each $t > 0$, $\tilde{Q}_t$ satisfies all hypotheses (A1)–(A5) in [41]. It is clear that $\tilde{Q}_t$ also satisfies equation (50). Hence, $\tilde{Q}_t$ also is the restriction of Q_t to $\tilde{C}_{v^*}$. This implies that Theorem 2.17 in [41] can be applied. Therefore, we finally obtain that (51) holds.

Choosing $c_0 \in (c_1, c^*)$ and letting $x = -c_0 t$, it follows from (49) and (51) that

$$\liminf_{t\to\infty} u(x,t) \geq \lim_{t\to\infty} \inf_{|x|\leq c_0 t} v(x,t) > 0. \tag{52}$$

Since $\xi = x + c_1 t = (c_1 - c_0)t \to -\infty$ as $t \to \infty$, we finally obtain $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} I^*(x + c_1 t) = \lim_{t\to\infty} I^*((c_1 - c_0)t) = \lim_{\xi\to-\infty} I^*(\xi) = 0$. This is a contradiction to (52). This completes the proof. □

7 Numerical examples

In this section, we give some numerical simulations to verify the validity of our theoretical results obtained in Sect. 5. We directly simulate the traveling wave system, which is the system satisfied by the traveling wave solution of the model. We adopt the following kernel functions:

$$J(x) = \begin{cases} Ce^{\frac{1}{4x^2-1}}, & -0.5 < x < 0.5, \\ 0, & \text{otherwise}, \end{cases}$$

where C is a constant taken as 4.5046 such that $\int_{\mathbb{R}} J(x)\,dx = \int_{-0.5}^{0.5} J(x)\,dx \approx 1$. Similarly, the kernel function $h(s)$ is defined by

$$h(s) = 4.5046e^{\frac{1}{4(s-0.5)^2-1}}, \quad 0 < s < 1,$$

we can verify $\int_0^1 h(s)\,ds \approx 1$.

Example 1 In system (5), we set $d_1 = 0.01$, $d_2 = 0.1$, $d_3 = 0.01$, $\Lambda = 0.1$, $\mu = 0.15$, $\beta = 0.15$, $\gamma = 0.1$, $\alpha = 0.04$, $c = 1$, $\tau = 1$, and $f(S,I) = SI$. By simple calculation, we know that system (5) with the above parameters has a disease-free equilibrium $E_0 = (10, 0, 0)$ and an endemic equilibrium $E^* = (1, 0.6, 6)$. It follows from equation (9) that the minimal wave speed $c^* = 0.112167$ and the basic reproduction number $\mathcal{R}_0 = 10$. Since $\mathcal{R}_0 > 1$ and $c > c^*$, it follows from Theorem 5 that system (4) admits a nonnegative traveling wave solution $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting the disease-free equilibrium E_0 and the endemic equilibrium E^*. The dynamical behavior of $(S(\xi), I(\xi), R(\xi))$ is given in Fig. 1.

We can also observe that the dynamical behavior of traveling wave solution $(S(x+t), I(x+t), R(x+t))$ of system (4) by the numerical simulations is given in Fig. 2. From the numerical simulations, we see that the solution $(S(x+t), I(x+t), R(x+t))$ is a nonnegative and nontrivial traveling wave solution connecting the disease-free equilibrium E_0 and the endemic equilibrium E^*.

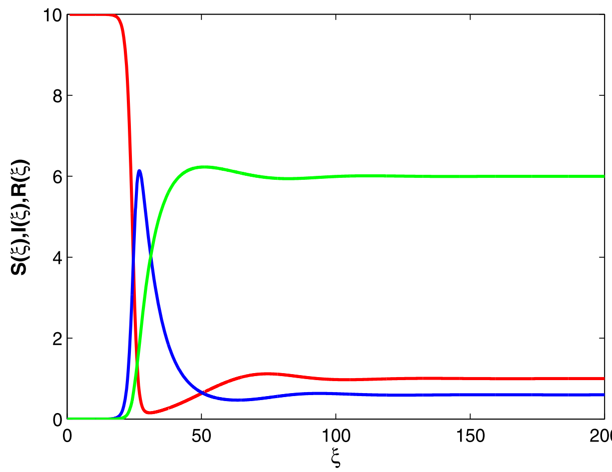

Figure 1 When the incidence function is bilinear, that is, $f(S,I) = SI$, then system (4) admits a solution $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting $E_0(S_0, 0, 0)$ with $E^*(S^*, I^*, R^*)$ for $c > c^*$

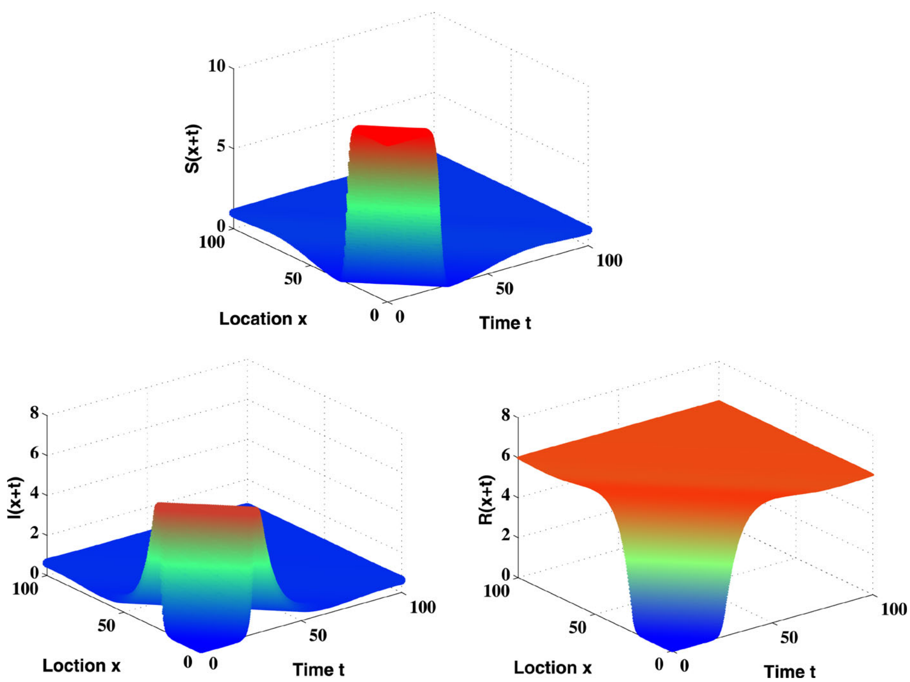

Figure 2 The traveling solution $(S(x+t), I(x+t), R(x+t))$ is observed for system (4), where the initial $(S(0), I(0), R(0)) = (10, 0.00000001, 0)$

Example 2 In system (5), we set $d_1 = d_2 = d_3 = 0.01$, $\Lambda = 0.1$, $\mu = 0.01$, $\beta = 0.15$, $\gamma = 0.1$, $\alpha = 0.04$, $c = 1$, $\tau = 1$ and an incidence function $f(S,I) = \frac{SI}{S+I}$. By a simple calculation, we know that system (5) with the above parameters has a disease-free equilibrium $E_0 = (10, 0, 0)$, an endemic equilibrium $E^* = (1.6667, 0.8333, 8.3333)$, the minimal wave speed $c^* = 0.065823$, and the basic reproduction number $\mathcal{R}_0 = 10$. Since $\mathcal{R}_0 > 1$ and the wave speed $c > c^*$, it follows from Theorem 5 that system (4) admits a nonnegative travel-

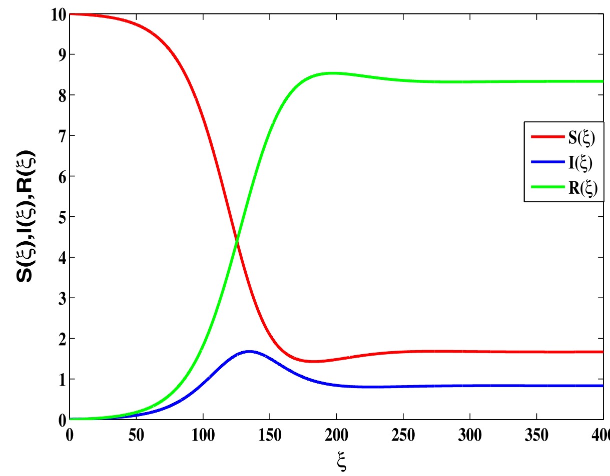

Figure 3 When the incidence function is standard, that is, $f(S,I) = \frac{SI}{S+I}$, then system (4) admits a traveling wave $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting $E_0(S_0, 0, 0)$ with $E^*(S^*, I^*, R^*)$ for $c > c^*$

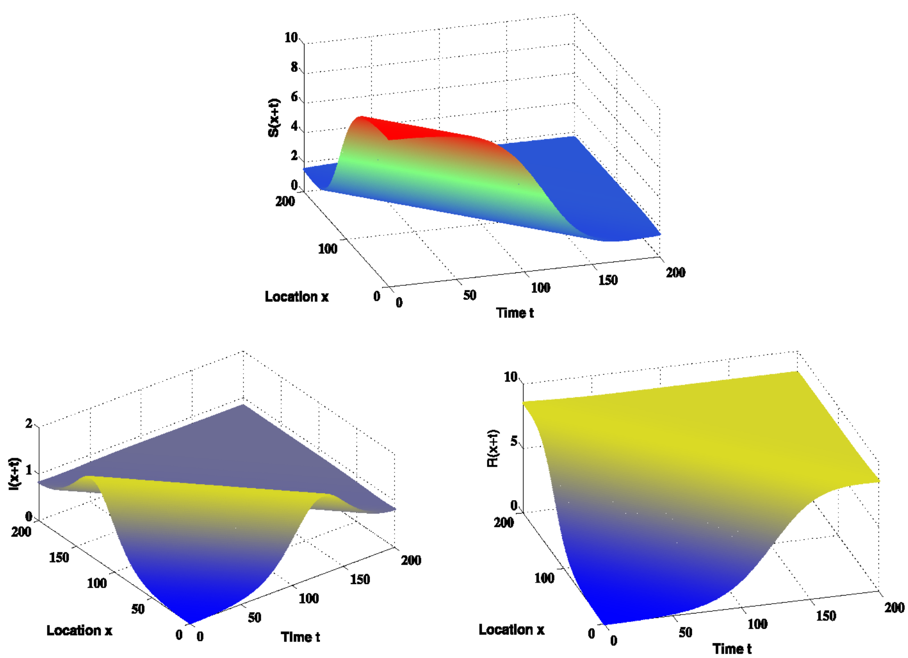

Figure 4 The traveling solution is observed in system (4), where the initial $(S(0), I(0), R(0)) = (10, 0.01.0)$

ing wave solution $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting E_0 and E^*. The dynamical behavior of $(S(\xi), I(\xi), R(\xi))$ by the numerical simulations is given in Fig. 3.

The dynamical behavior of traveling wave solution $(S(x+t), I(x+t), R(x+t))$ of system (4) by the numerical simulations is given in Fig. 4. From the numerical simulations, we see that the solution $(S(x+t), I(x+t), R(x+t))$ is a nonnegative and nontrivial traveling wave solution connecting E_0 and E^*.

In order to observe the effects of distributed time delay on disease propagation, we study the relationship between time delay τ and the critical wave speed c^* from equation (9). For

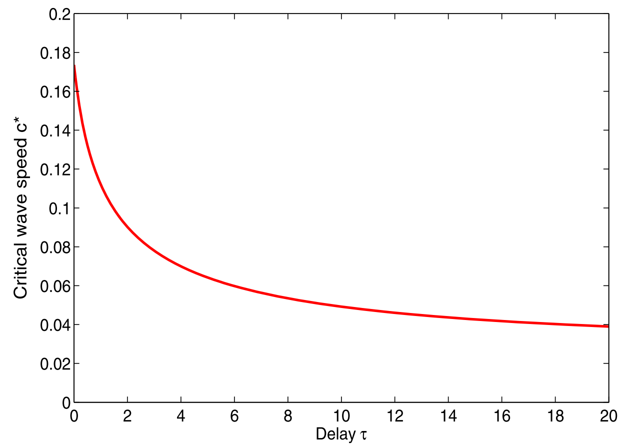

Figure 5 The relationship between time delay τ and the critical wave speed c^*

any given $\tau > 0$, we can choose the distributed delay kernel function $h(s)$ as follows:

$$h(s) = \frac{1.57079}{\tau} \sin\left(\frac{\pi s}{\tau}\right), \quad 0 < s < \tau.$$

We can verify $\int_0^\tau h(s)\,ds \approx 1$ for all $\tau > 0$. Suppose that all parameters and incidence function are as in Example 1 except the distributed delay kernel function and the wave speed. The relationship between the distributed time delay τ and the critical wave speed c^* is given in Fig. 5. The numerical simulation results show that the critical wave speed decreases with the time delay.

8 Conclusions

In this paper, we have dealt with the existence and nonexistence of traveling waves for a nonlocal dispersal SIR epidemic model with nonlinear incidence and distributed latent delay. In the model, the incidence function $f(S,I)$ is assumed to satisfy biologically reasonable criteria given in (A1). These criteria are sufficient for our main results to hold, but are probably not necessary. Our results are more general and more reasonable than previous work.

We define the basic reproduction number $\mathcal{R}_0$ and the minimal wave speed c^*, which is a very important quantity on the spread of disease. The existence of traveling waves $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting $E_0(S_0, 0, 0)$ with $E^*(S^*, I^*, R^*)$ is proved by introducing an auxiliary system, combining upper-lower solutions, applying Schauder's fixed point theorem, limiting arguments, and Lyapunov function methods. Our results show that, when $\mathcal{R}_0 > 1$ and $c > c^*$, then there exists a traveling wave satisfying asymptotic boundary conditions (6), and the existence of traveling waves reveals that the disease can spread. Furthermore, when $\mathcal{R}_0 > 1$ and $0 < c < c^*$, then our results show that there does not exist traveling waves connecting the equilibrium E_0 with E^* for system (5). Clearly, if $\mathcal{R}_0 \leq 1$, then system (5) has no endemic equilibrium. Therefore, there are no nontrivial traveling wave solutions $(S^*(\xi), I^*(\xi), R^*(\xi))$ connecting E_0 and E^* of system (5) for any wave speed $c > 0$.

Our model contains some classes of epidemic models found in the literature such as in Li et al. [22] and Li and Wang [46]. Moreover, according to Yang and Li [47], it is interesting to investigate traveling waves with $c = c^*$. However, as we all know, it is difficult to consider the critical wave speed. We have to leave this as future work.

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