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Robust stability analysis of impulsive quaternion-valued neural networks with distributed delays and parameter uncertainties

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Abstract

In this paper, an impulsive quaternion-valued neural networks (QVNNs) model with leakage, discrete, and distributed delays is considered. Based on the homeomorphic mapping method, Lyapunov stability theorem, and linear matrix inequality (LMI) approach, sufficient conditions for the existence, uniqueness, and global robust stability of the equilibrium point of the impulsive QVNNs are provided. A numerical example is provided to confirm the obtained results. A conclusion is presented in the end.

Keywords: Quaternion-valued neural networks; Distributed delays; Global robust stability; Lyapunov stability theorem

Neural networks have been proposed in the 1940s by psychologist MuCulloch and mathematician Pitts and since then have been paid extensive attention of researchers due to their wide applications in scientific and technological fields, such as image processing, signal processing, fault diagnosis, and associative memory [1–4]. Real-valued neural network (RVNN) is the most typical representative of a neural network system, and different kinds of stability are studied by researchers, for example, the power-rate global stability of the equilibrium is proposed in [5]. In [3], the authors propose a new concept of global μ-stability, which unifies the exponential stability, power-rate stability, and log-stability of neural networks. In 1992, Hirose proposed complex-valued neural networks (CVNNs) in which the state, output, weight, and domain of attraction are all complex values [6]. Compared with RVNNs, the advantage of CVNNs is that they can directly deal with 2D data. Since then, numerical researches pass the theoretical analysis of the basic properties of real- and complex-valued neurons, summarize the differences between the two, and clarify some inherent properties of complex-valued neurons at the boundary [7–14]. However, there are still some problems that the RVNNs and the CVNNs cannot deal with easily, such as 4D signals, body images, which are four- or more dimensional, new methods or theories have to be put forward, the theory of QVNNs thus emerges as required [4].

In practical life, QVNNs have many applications in various areas. One practical application by QVNNs is the 3D geometrical affine transformation, especially spatial rotation, which can be represented efficiently and compactly by QVNNs. Other practical applications of QVNNs are image impression, color night vision, etc. [15–22]. One specific example in reality is reconstructing gray and color images using designed QVNNs, which can possess high storage capacity in applications of associative memory and pattern recognition. For this class of QVNNs, via eigenstructure method, the results developed by researchers enable us to synthesize neural networks with specified equilibrium points. Other applications of parameter uncertainty for the integer-order neural network can be seen in the design of PI controller, the stability region of systems with parameter uncertainty is essential to the design of PI controller. In fact, in the actual operation process, the measured values of the characteristics or parameters will deviate from their actual values due to the inaccurate measurement or calculation. And system parameters will be changed. The system may not be stable after the parameter changes. Once the stability is lost, the images will be difficult to reconstruct.

To a general QVNN in real life, the delay is inevitable owing to the delay of transmission line, equivalent circuit of some components, integration and communication, etc. Usually, there exit three types of common delay: leakage delay, discrete delay, and distributed delay. For a simple circuit with a small number of neurons, it can be described by a time-delay feedback system with a fixed time delay [23–26]. However, since the neural networks are composed of a lot of neurons, with a large number of parallel channels, and the neural networks have the characteristics of time and space, we can introduce the distribution delay to describe some characteristics of the neural networks. In [24], the leakage delay is introduced into the model to explore its stability. In [25, 27], the influence of leakage and discrete delays on CVNNs are studied. In [28], the effects of leakage and discrete delays in QVNNs are considered. To the best of our knowledge, the results of QVNNs with leakage, discrete, and distributed delays are few. Besides time delays, impulse also exists in the application of neural networks, which makes the model more realistic [29–32]. For example, during the implementation of an electronic network, the state of the network will be interfered by transients, and sudden changes will occur at a certain time, which may be caused by switching phenomena, frequency changes, or other accidents. Therefore, it is necessary to consider the effect of impulsive effects and delays on the dynamic behavior of neural networks.

For the neural network model with different number fields, researchers all over the world have extended the model in different directions: considering different types of time delays, such as leakage delay, discrete delay, distributed delay, etc.; considering different types of equations, such as impulsive differential equation, stochastic differential equation, etc.; to study the existence, boundedness of equilibrium of a system, different types of stability, synchronization, and bifurcation of a neural network. The research scope has also been extended from integer-order neural networks to fractional-order neural networks [33, 34]. Recently, in [35], the authors presented a delayed QVNN with parameter uncertainties and investigated the robust stability of the system. In [36], the authors established sufficient conditions on the existence, uniqueness, and global stability of the equilibrium of a delayed QVNN with interval parameter uncertainties through constructing a couple of LMIs. Sequently, a sufficient criterion was obtained to ensure global robust stability of QVNNs by applying Lyapunov function method and inequality techniques. But in its judging criteria, two negative definite matrices are needed, and the terms in these matrices are the maximum values, which are determined by the absolute values of the upper and lower bounds of the elements of the connection weight matrix, ignoring the sign of the connection weight value. To overcome this shortcoming, Wang in [28] revisited the system and obtained a criterion whose elements rely on both the lower and upper bounds of the interval parameters.

Based on the above discussion, we propose a network model with mixed delays and impulsive effects. By means of quaternion-valued inequality, Lyapunov function, and homeomorphic mapping, a new sufficient condition to ensure the existence, uniqueness, and global robust stability of an equilibrium point for the QVNN is derived, which can be checked numerically using the effective YALMIP toolbox in MATLAB. Comparing to other references, our contributions lie in two aspects: firstly, the traditional model is extended in the article, the distributed delay and impulsive effect are considered in our model, which make the model more practical and is the substantial extension of the study of [8, 28], because the existence of the distributed delay and impulsive effect increases the dimension of the system, which produces great difficulties in numerical simulation and for the proof of the negative definiteness for V function. Secondly, in the judging criteria of reference [13, 36], the negative definiteness of two matrices is needed, however, in this paper, only one negative definite judging matrix is enough to guarantee the existence, uniqueness, and global robust stability of the equilibrium point. Moreover, the elements of the given criterion matrix in our proposed results depend not only on the lower bounds but also on the upper bounds of the interval parameters, which is less conservative and extend some previous contributions; for details, see [13, 14, 36].

The structure of this paper is as follows. In Sect. 2, the model of impulsive neural networks with three kinds of time delays involving leakage delay, discrete delay, and distributed delay is proposed, and some basic preparations are introduced. Sufficient conditions for the existence, uniqueness, and global robust stability of equilibrium point are established in Sect. 3. To illustrate the validity of the main results, numerical simulation is carried out in Sect. 4. Finally, we make a brief conclusion at the end.

1 Problem formulation and preliminaries

At the beginning, we give some notations used throughout the paper before proposing our model.

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ be the real field, complex field, and the skew field of quaternions, respectively. Let $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ be the n-dimensional vectors with entries from $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively; $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$ and $\mathbb{H}^{n\times m}$ represent $n \times m$ real-valued matrices, complex-valued matrices, and quaternion-valued matrices, respectively; $\bar{A}$, A^T, and A^* denote the conjugate, transpose, and conjugate transpose of a matrix A, respectively. For any $z \in \mathbb{C}^n$, let $\|z\| = \sqrt{z^* z}$ be the norm of z. The notation $X \geq Y$ (or $X > Y$) means that $X - Y$ is positive semidefinite (or positive definite). For a positive definite Hermitian matrix H, $\lambda_{\max}(H)$ and $\lambda_{\min}(H)$ represent the maximum and minimum eigenvalues of H. Symbol I denotes an identity matrix with an appropriate dimension, and symbol $*$ shows the conjugate transpose of a suitable block in a Hermitian matrix.

Next, we introduce some preliminaries on quaternion-valued notations and operations. If $p \in \mathbb{H}$, then it can be expressed as $p = p_0 + p_1\imath + p_2\jmath + p_3\kappa$, where p_0 is the real part of the quaternion, and p_1, p_2, and p_3 are the imaginary parts of the quaternion. When $a, b \in \mathbb{H}$ with $a = a_0 + a_1\imath + a_2\jmath + a_3\kappa$, $b = b_0 + b_1\imath + b_2\jmath + b_3\kappa$, then $a \preceq b$ means $a_i \leq b_i$ $(i = 1,2,3)$. Also, if $A, B \in \mathbb{H}^{n\times n}$, then $A \preceq B$ means $a_{ij} \leq b_{ij}$, $i,j = 1,2,\ldots,n$, where $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$.

1.1 Model description

In this paper, we consider the following impulsive QVNNs model with distributed delay:

$$\begin{cases} \dot{q}(t) = -Dq(t-\delta) + Ag(q(t)) + Bg(q(t-\tau)) \\ \qquad + C\int_t^{+\infty} K(t-s)g(q(s))\,ds + J, \quad t > 0, t \neq t_k, \\ \Delta q(t_k) = q(t_k) - q(t_k^-) = M_k(q(t_k^-), q_{t_k^-}), \quad k = 1,2,\ldots, \end{cases} \tag{1}$$

where $q(t) = (q_1(t), q_2(t), \ldots, q_n(t))^T \in \mathbb{H}^n$ is the state vector of the neural neuron of the neural network at time t; $g(q(t)) = (g(q_1(t)), g(q_2(t)), \ldots, g(q_n(t)))^T \in \mathbb{H}^n$ represents the activation function of neurons; $D = \operatorname{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n\times n}$ is the self-feedback connection weight matrix with $d_j > 0$ $(j = 1,2,\ldots,n)$; $A \in \mathbb{H}^{n\times n}$ is the connection weight matrix; $B \in \mathbb{H}^{n\times n}$ is the delayed connection weight matrix; $C \in \mathbb{H}^{n\times n}$ is the distributively delayed connection weight matrix; $J = (J_1, J_2, \ldots, J_n)^T \in \mathbb{H}^n$ denotes the input vector; $\delta > 0$ refers to the leakage delay; $0 < \tau < \rho$ refers to the transmission delay, and $0 < t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$. Also $K(\cdot) : [0, +\infty) \to [0, +\infty)$ is the delay kernel and M_k is the impulsive function.

Assume that system (1) satisfies the initial conditions given by

$$q(s) = \vartheta(s), \quad s \in [-\rho, 0], \tag{2}$$

here $\vartheta(\cdot)$ is a quaternion field continuous function defined on $[-\rho, 0]$, $\vartheta(s) = (\vartheta_1(s), \vartheta_2(s), \ldots, \vartheta_n(s))^T \in C([-\rho, 0], \mathbb{H}^n)$ and the norm

$$\|\vartheta(s)\| = \sup_{s\in[-\rho,0]} \sqrt{\sum_{i=1}^n |\vartheta_i(t)|^2}.$$

This paper will require the following assumptions:

$(\mathbf{A}_1)$ For $i = 1,2,\ldots,n$, the neuron activation function g_i is continuous and satisfies

$$|g_i(q_1) - g_i(q_2)| \leq l_i|q_1 - q_2|, \quad \forall q_1, q_2 \in \mathbb{H},$$

where l_i is a real-valued positive constant. In addition, we define $L = \operatorname{diag}\{L_1, L_2, \ldots, L_n\}$ for the convenience of the next proof.

$(\mathbf{A}_2)$ The matrices D, C, B, A, and J in (1) belong to the following sets, respectively:

$$\begin{aligned} D_I &= \{D \in \mathbb{R}^{n\times n} : 0 \prec \check{D} \preceq D \preceq \hat{D}, \check{D}, \hat{D} \in \mathbb{R}^{n\times n}\}, \\ C_I &= \{C \in \mathbb{H}^{n\times n} : \check{C} \preceq C \preceq \hat{C}, \check{C}, \hat{C} \in \mathbb{H}^{n\times n}\}, \\ B_I &= \{B \in \mathbb{H}^{n\times n} : \check{B} \preceq B \preceq \hat{B}, \check{B}, \hat{B} \in \mathbb{H}^{n\times n}\}, \\ A_I &= \{A \in \mathbb{H}^{n\times n} : \check{A} \preceq A \preceq \hat{A}, \check{A}, \hat{A} \in \mathbb{H}^{n\times n}\}, \\ J &= \{J \in \mathbb{H}^n : \check{J} \preceq J \preceq \hat{J}, \check{J}, \hat{J} \in \mathbb{H}^n\}, \end{aligned}$$

where $\check{D} = \operatorname{diag}\{\check{d}_1, \check{d}_2, \ldots, \check{d}_n\}$, $\hat{D} = \operatorname{diag}\{\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n\}$, $\check{C} = (\check{c}_{ij})_{n\times n}$, $\hat{C} = (\hat{c}_{ij})_{n\times n}$, $\check{B} = (\check{b}_{ij})_{n\times n}$, $\hat{B} = (\hat{b}_{ij})_{n\times n}$, $\check{A} = (\check{a}_{ij})_{n\times n}$, $\hat{A} = (\hat{a}_{ij})_{n\times n}$.

($\mathbf{A}_3$) K is a real-valued nonnegative continuous function defined on $[0, +\infty]$ and satisfies

$$\int_0^{+\infty} K(s)\,ds = 1, \qquad \int_0^{+\infty} sK(s)\,ds < +\infty.$$

For every $A_i = A_i^R + \imath A_i^I + \jmath A_i^J + \kappa A_i^K$, $B_i = B_i^R + \imath B_i^I + \jmath B_i^J + \kappa B_i^K$, $C_i = C_i^R + \imath C_i^I + \jmath C_i^J + \kappa C_i^K \in \mathbb{H}^{n\times n}$, $i = 0, 1$ let

$$\begin{aligned}
&A_0^R = \frac{1}{2}(\hat{A}^R + \check{A}^R), \qquad A_1^R = \frac{1}{2}(\hat{A}^R - \check{A}^R) = (\phi_{ij}^R)_{n\times n},\\
&A_0^I = \frac{1}{2}(\hat{A}^I + \check{A}^I), \qquad A_1^I = \frac{1}{2}(\hat{A}^I - \check{A}^I) = (\phi_{ij}^I)_{n\times n},\\
&A_0^J = \frac{1}{2}(\hat{A}^J + \check{A}^J), \qquad A_1^J = \frac{1}{2}(\hat{A}^J - \check{A}^J) = (\phi_{ij}^J)_{n\times n},\\
&A_0^K = \frac{1}{2}(\hat{A}^K + \check{A}^K), \qquad A_1^K = \frac{1}{2}(\hat{A}^K - \check{A}^K) = (\phi_{ij}^K)_{n\times n},\\
&B_0^R = \frac{1}{2}(\hat{B}^R + \check{B}^R), \qquad B_1^R = \frac{1}{2}(\hat{B}^R - \check{B}^R) = (\varphi_{ij}^R)_{n\times n},\\
&B_0^I = \frac{1}{2}(\hat{B}^I + \check{B}^I), \qquad B_1^I = \frac{1}{2}(\hat{B}^I - \check{B}^I) = (\varphi_{ij}^I)_{n\times n},\\
&B_0^J = \frac{1}{2}(\hat{B}^J + \check{B}^J), \qquad B_1^J = \frac{1}{2}(\hat{B}^J - \check{B}^J) = (\varphi_{ij}^J)_{n\times n},\\
&B_0^K = \frac{1}{2}(\hat{B}^K + \check{B}^K), \qquad B_1^K = \frac{1}{2}(\hat{B}^K - \check{B}^K) = (\varphi_{ij}^K)_{n\times n},\\
&C_0^R = \frac{1}{2}(\hat{C}^R + \check{C}^R), \qquad C_1^R = \frac{1}{2}(\hat{C}^R - \check{C}^R) = (\psi_{ij}^R)_{n\times n},\\
&C_0^I = \frac{1}{2}(\hat{C}^I + \check{C}^I), \qquad C_1^I = \frac{1}{2}(\hat{C}^I - \check{C}^I) = (\psi_{ij}^I)_{n\times n},\\
&C_0^J = \frac{1}{2}(\hat{C}^J + \check{C}^J), \qquad C_1^J = \frac{1}{2}(\hat{C}^J - \check{C}^J) = (\psi_{ij}^J)_{n\times n},\\
&C_0^K = \frac{1}{2}(\hat{C}^K + \check{C}^K), \qquad C_1^K = \frac{1}{2}(\hat{C}^K - \check{C}^K) = (\psi_{ij}^K)_{n\times n},\\
&D_0 = \frac{1}{2}(\hat{D} + \check{D}),\\
&D_1 = \frac{1}{2}(\hat{D} - \check{D}) = (\eta_{ij})_{n\times n} = \frac{1}{2}\operatorname{diag}\{\hat{d}_1 - \check{d}_1, \hat{d}_1 - \check{d}_2, \ldots, \hat{d}_n - \check{d}_n\}.
\end{aligned}$$

In addition, $A_1^X \succeq 0$, $B_1^X \succeq 0$, $C_1^X \succeq 0$, and $D_1 \succeq 0$ with X expressing R, I, J, K, respectively.

Let $e_i = (0, 0, \ldots, 1, \ldots, 0)_{n\times 1}^T$ be the vector with the ith entry 1. We define:

$$\begin{aligned}
&U_A^R = \left(\sqrt{\phi_{11}^R}e_1, \ldots, \sqrt{\phi_{1n}^R}e_1, \sqrt{\phi_{21}^R}e_2, \ldots, \sqrt{\phi_{2n}^R}e_2, \ldots, \sqrt{\phi_{n1}^R}e_n, \ldots, \sqrt{\phi_{nn}^R}e_n\right)_{n\times n^2},\\
&U_A^I = \left(\sqrt{\phi_{11}^I}e_1, \ldots, \sqrt{\phi_{1n}^I}e_1, \sqrt{\phi_{21}^I}e_2, \ldots, \sqrt{\phi_{2n}^I}e_2, \ldots, \sqrt{\phi_{n1}^I}e_n, \ldots, \sqrt{\phi_{nn}^I}e_n\right)_{n\times n^2},\\
&U_A^J = \left(\sqrt{\phi_{11}^J}e_1, \ldots, \sqrt{\phi_{1n}^J}e_1, \sqrt{\phi_{21}^J}e_2, \ldots, \sqrt{\phi_{2n}^J}e_2, \ldots, \sqrt{\phi_{n1}^J}e_n, \ldots, \sqrt{\phi_{nn}^J}e_n\right)_{n\times n^2},\\
&U_A^K = \left(\sqrt{\phi_{11}^K}e_1, \ldots, \sqrt{\phi_{1n}^K}e_1, \sqrt{\phi_{21}^K}e_2, \ldots, \sqrt{\phi_{2n}^K}e_2, \ldots, \sqrt{\phi_{n1}^K}e_n, \ldots, \sqrt{\phi_{nn}^K}e_n\right)_{n\times n^2},
\end{aligned}$$

$$
\begin{aligned}
U_B^R &= \left(\sqrt{\varphi_{11}^R}e_1,\dots,\sqrt{\varphi_{1n}^R}e_1,\sqrt{\varphi_{21}^R}e_2,\dots,\sqrt{\varphi_{2n}^R}e_2,\dots,\sqrt{\varphi_{n1}^R}e_n,\dots,\sqrt{\varphi_{nn}^R}e_n\right)_{n\times n^2},\\
U_B^I &= \left(\sqrt{\varphi_{11}^I}e_1,\dots,\sqrt{\varphi_{1n}^I}e_1,\sqrt{\varphi_{21}^I}e_2,\dots,\sqrt{\varphi_{2n}^I}e_2,\dots,\sqrt{\varphi_{n1}^I}e_n,\dots,\sqrt{\varphi_{nn}^I}e_n\right)_{n\times n^2},\\
U_B^J &= \left(\sqrt{\varphi_{11}^J}e_1,\dots,\sqrt{\varphi_{1n}^J}e_1,\sqrt{\varphi_{21}^J}e_2,\dots,\sqrt{\varphi_{2n}^J}e_2,\dots,\sqrt{\varphi_{n1}^J}e_n,\dots,\sqrt{\varphi_{nn}^J}e_n\right)_{n\times n^2},\\
U_B^K &= \left(\sqrt{\varphi_{11}^K}e_1,\dots,\sqrt{\varphi_{1n}^K}e_1,\sqrt{\varphi_{21}^K}e_2,\dots,\sqrt{\varphi_{2n}^K}e_2,\dots,\sqrt{\varphi_{n1}^K}e_n,\dots,\sqrt{\varphi_{nn}^K}e_n\right)_{n\times n^2},\\
U_C^R &= \left(\sqrt{\psi_{11}^R}e_1,\dots,\sqrt{\psi_{1n}^R}e_1,\sqrt{\psi_{21}^R}e_2,\dots,\sqrt{\psi_{2n}^R}e_2,\dots,\sqrt{\psi_{n1}^R}e_n,\dots,\sqrt{\psi_{nn}^R}e_n\right)_{n\times n^2},\\
U_C^I &= \left(\sqrt{\psi_{11}^I}e_1,\dots,\sqrt{\psi_{1n}^I}e_1,\sqrt{\psi_{21}^I}e_2,\dots,\sqrt{\psi_{2n}^I}e_2,\dots,\sqrt{\psi_{n1}^I}e_n,\dots,\sqrt{\psi_{nn}^I}e_n\right)_{n\times n^2},\\
U_C^J &= \left(\sqrt{\psi_{11}^J}e_1,\dots,\sqrt{\psi_{1n}^J}e_1,\sqrt{\psi_{21}^J}e_2,\dots,\sqrt{\psi_{2n}^J}e_2,\dots,\sqrt{\psi_{n1}^J}e_n,\dots,\sqrt{\psi_{nn}^J}e_n\right)_{n\times n^2},\\
U_C^K &= \left(\sqrt{\psi_{11}^K}e_1,\dots,\sqrt{\psi_{1n}^K}e_1,\sqrt{\psi_{21}^K}e_2,\dots,\sqrt{\psi_{2n}^K}e_2,\dots,\sqrt{\psi_{n1}^K}e_n,\dots,\sqrt{\psi_{nn}^K}e_n\right)_{n\times n^2},\\
U_D &= (\sqrt{\eta_{11}}e_1,\dots,\sqrt{\eta_{1n}}e_1,\sqrt{\eta_{21}}e_2,\dots,\sqrt{\eta_{2n}}e_2,\dots,\sqrt{\eta_{n1}}e_n,\dots,\sqrt{\eta_{nn}}e_n)_{n\times n^2},\\
V_A^R &= \left(\sqrt{\phi_{11}^R}e_1,\dots,\sqrt{\phi_{1n}^R}e_n,\sqrt{\phi_{21}^R}e_1,\dots,\sqrt{\phi_{2n}^R}e_n,\dots,\sqrt{\phi_{n1}^R}e_1,\dots,\sqrt{\phi_{nn}^R}e_n\right)_{n^2\times n}^T,\\
V_A^I &= \left(\sqrt{\phi_{11}^I}e_1,\dots,\sqrt{\phi_{1n}^I}e_n,\sqrt{\phi_{21}^I}e_1,\dots,\sqrt{\phi_{2n}^I}e_n,\dots,\sqrt{\phi_{n1}^I}e_1,\dots,\sqrt{\phi_{nn}^I}e_n\right)_{n^2\times n}^T,\\
V_A^J &= \left(\sqrt{\phi_{11}^J}e_1,\dots,\sqrt{\phi_{1n}^J}e_n,\sqrt{\phi_{21}^J}e_1,\dots,\sqrt{\phi_{2n}^J}e_n,\dots,\sqrt{\phi_{n1}^J}e_1,\dots,\sqrt{\phi_{nn}^J}e_n\right)_{n^2\times n}^T,\\
V_A^K &= \left(\sqrt{\phi_{11}^K}e_1,\dots,\sqrt{\phi_{1n}^K}e_n,\sqrt{\phi_{21}^K}e_1,\dots,\sqrt{\phi_{2n}^K}e_n,\dots,\sqrt{\phi_{n1}^K}e_1,\dots,\sqrt{\phi_{nn}^K}e_n\right)_{n^2\times n}^T,\\
V_B^R &= \left(\sqrt{\varphi_{11}^R}e_1,\dots,\sqrt{\varphi_{1n}^R}e_n,\sqrt{\varphi_{21}^R}e_1,\dots,\sqrt{\varphi_{2n}^R}e_n,\dots,\sqrt{\varphi_{n1}^R}e_1,\dots,\sqrt{\varphi_{nn}^R}e_n\right)_{n^2\times n}^T,\\
V_B^I &= \left(\sqrt{\varphi_{11}^I}e_1,\dots,\sqrt{\varphi_{1n}^I}e_n,\sqrt{\varphi_{21}^I}e_1,\dots,\sqrt{\varphi_{2n}^I}e_n,\dots,\sqrt{\varphi_{n1}^I}e_1,\dots,\sqrt{\varphi_{nn}^I}e_n\right)_{n^2\times n}^T,\\
V_B^J &= \left(\sqrt{\varphi_{11}^J}e_1,\dots,\sqrt{\varphi_{1n}^J}e_n,\sqrt{\varphi_{21}^J}e_1,\dots,\sqrt{\varphi_{2n}^J}e_n,\dots,\sqrt{\varphi_{n1}^J}e_1,\dots,\sqrt{\varphi_{nn}^J}e_n\right)_{n^2\times n}^T,\\
V_B^K &= \left(\sqrt{\varphi_{11}^K}e_1,\dots,\sqrt{\varphi_{1n}^K}e_n,\sqrt{\varphi_{21}^K}e_1,\dots,\sqrt{\varphi_{2n}^K}e_n,\dots,\sqrt{\varphi_{n1}^K}e_1,\dots,\sqrt{\varphi_{nn}^K}e_n\right)_{n^2\times n}^T,\\
V_C^R &= \left(\sqrt{\psi_{11}^R}e_1,\dots,\sqrt{\psi_{1n}^R}e_n,\sqrt{\psi_{21}^R}e_1,\dots,\sqrt{\psi_{2n}^R}e_n,\dots,\sqrt{\psi_{n1}^R}e_1,\dots,\sqrt{\psi_{nn}^R}e_n\right)_{n^2\times n}^T,\\
V_C^I &= \left(\sqrt{\psi_{11}^I}e_1,\dots,\sqrt{\psi_{1n}^I}e_n,\sqrt{\psi_{21}^I}e_1,\dots,\sqrt{\psi_{2n}^I}e_n,\dots,\sqrt{\psi_{n1}^I}e_1,\dots,\sqrt{\psi_{nn}^I}e_n\right)_{n^2\times n}^T,\\
V_C^J &= \left(\sqrt{\psi_{11}^J}e_1,\dots,\sqrt{\psi_{1n}^J}e_n,\sqrt{\psi_{21}^J}e_1,\dots,\sqrt{\psi_{2n}^J}e_n,\dots,\sqrt{\psi_{n1}^J}e_1,\dots,\sqrt{\psi_{nn}^J}e_n\right)_{n^2\times n}^T,\\
V_C^K &= \left(\sqrt{\psi_{11}^K}e_1,\dots,\sqrt{\psi_{1n}^K}e_n,\sqrt{\psi_{21}^K}e_1,\dots,\sqrt{\psi_{2n}^K}e_n,\dots,\sqrt{\psi_{n1}^K}e_1,\dots,\sqrt{\psi_{nn}^K}e_n\right)_{n^2\times n}^T,\\
V_D &= (\sqrt{\eta_{11}}e_1,\dots,\sqrt{\eta_{1n}}e_n,\sqrt{\eta_{21}}e_1,\dots,\sqrt{\eta_{2n}}e_n,\dots,\sqrt{\eta_{n1}}e_1,\dots,\sqrt{\eta_{nn}}e_n)_{n^2\times n}^T.
\end{aligned}
$$

We can get:

$$
\begin{aligned}
U_A^R\left(U_A^R\right)^T &= \operatorname{diag}\left(\sum_{j=1}^n \phi_{1j}^R,\dots,\sum_{j=1}^n \phi_{nj}^R\right), \qquad U_A^I\left(U_A^I\right)^T = \operatorname{diag}\left(\sum_{j=1}^n \phi_{1j}^I,\dots,\sum_{j=1}^n \phi_{nj}^I\right),\\
U_A^J\left(U_A^J\right)^T &= \operatorname{diag}\left(\sum_{j=1}^n \phi_{1j}^J,\dots,\sum_{j=1}^n \phi_{nj}^J\right), \qquad U_A^K\left(U_A^K\right)^T = \operatorname{diag}\left(\sum_{j=1}^n \phi_{1j}^K,\dots,\sum_{j=1}^n \phi_{nj}^K\right),
\end{aligned}
$$

$$U_B^R(U_B^R)^T = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{1j}^R, \dots, \sum_{j=1}^n \varphi_{nj}^R\right), \qquad U_B^I(U_B^I)^T = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{1j}^I, \dots, \sum_{j=1}^n \varphi_{nj}^I\right),$$

$$U_B^J(U_B^J)^T = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{1j}^J, \dots, \sum_{j=1}^n \varphi_{nj}^J\right), \qquad U_B^K(U_B^K)^T = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{1j}^K, \dots, \sum_{j=1}^n \varphi_{nj}^K\right),$$

$$U_C^R(U_C^R)^T = \operatorname{diag}\left(\sum_{j=1}^n \psi_{1j}^R, \dots, \sum_{j=1}^n \psi_{nj}^R\right), \qquad U_C^I(U_C^I)^T = \operatorname{diag}\left(\sum_{j=1}^n \psi_{1j}^I, \dots, \sum_{j=1}^n \psi_{nj}^I\right),$$

$$U_C^J(U_C^J)^T = \operatorname{diag}\left(\sum_{j=1}^n \psi_{1j}^J, \dots, \sum_{j=1}^n \psi_{nj}^J\right), \qquad U_C^K(U_C^K)^T = \operatorname{diag}\left(\sum_{j=1}^n \psi_{1j}^K, \dots, \sum_{j=1}^n \psi\psi_{nj}^K\right),$$

$$U_D(U_D)^T = \operatorname{diag}\left(\sum_{j=1}^n \eta_{1j}, \dots, \sum_{j=1}^n \eta_{nj}\right), \qquad (V_A^R)^T V_A^R = \operatorname{diag}\left(\sum_{j=1}^n \phi_{j1}^R, \dots, \sum_{j=1}^n \phi_{jn}^R\right),$$

$$(V_A^I)^T V_A^I = \operatorname{diag}\left(\sum_{j=1}^n \phi_{j1}^I, \dots, \sum_{j=1}^n \phi_{jn}^I\right), \qquad (V_A^J)^T V_A^J = \operatorname{diag}\left(\sum_{j=1}^n \phi_{j1}^J, \dots, \sum_{j=1}^n \phi_{jn}^J\right),$$

$$(V_A^K)^T V_A^K = \operatorname{diag}\left(\sum_{j=1}^n \phi_{j1}^K, \dots, \sum_{j=1}^n \phi_{jn}^K\right), \qquad (V_B^R)^T V_B^R = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{j1}^R, \dots, \sum_{j=1}^n \varphi_{jn}^R\right),$$

$$(V_B^I)^T V_B^I = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{j1}^I, \dots, \sum_{j=1}^n \varphi_{jn}^I\right), \qquad (V_B^J)^T V_B^J = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{j1}^J, \dots, \sum_{j=1}^n \varphi_{jn}^J\right),$$

$$(V_B^K)^T V_B^K = \operatorname{diag}\left(\sum_{j=1}^n \varphi_{j1}^K, \dots, \sum_{j=1}^n \varphi_{jn}^K\right), \qquad (V_C^R)^T V_C^R = \operatorname{diag}\left(\sum_{j=1}^n \psi_{j1}^R, \dots, \sum_{j=1}^n \psi_{jn}^R\right),$$

$$(V_C^I)^T V_C^I = \operatorname{diag}\left(\sum_{j=1}^n \psi_{j1}^I, \dots, \sum_{j=1}^n \psi_{jn}^I\right), \qquad (V_C^J)^T V_C^J = \operatorname{diag}\left(\sum_{j=1}^n \psi_{j1}^J, \dots, \sum_{j=1}^n \psi_{jn}^J\right),$$

$$(V_C^K)^T V_C^K = \operatorname{diag}\left(\sum_{j=1}^n \psi_{j1}^K, \dots, \sum_{j=1}^n \psi_{jn}^K\right), \qquad (V_D)^T V_D = \operatorname{diag}\left(\sum_{j=1}^n \eta_{j1}, \dots, \sum_{j=1}^n \eta_{jn}\right).$$

Then, we can define:

$$N_A^R = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \phi_{1j}^R}, \dots, \sqrt{\sum_{j=1}^n \phi_{nj}^R}\right), \qquad N_A^I = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \phi_{1j}^I}, \dots, \sqrt{\sum_{j=1}^n \phi_{nj}^I}\right),$$

$$N_A^J = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \phi_{1j}^J}, \dots, \sqrt{\sum_{j=1}^n \phi_{nj}^J}\right), \qquad N_A^K = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \phi_{1j}^K}, \dots, \sqrt{\sum_{j=1}^n \phi_{nj}^K}\right),$$

$$N_B^R = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \varphi_{1j}^R}, \dots, \sqrt{\sum_{j=1}^n \varphi_{nj}^R}\right), \qquad N_B^I = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \varphi_{1j}^I}, \dots, \sqrt{\sum_{j=1}^n \varphi_{nj}^I}\right),$$

$$N_B^J = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \varphi_{1j}^J}, \dots, \sqrt{\sum_{j=1}^n \varphi_{nj}^J}\right), \qquad N_B^K = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \varphi_{1j}^K}, \dots, \sqrt{\sum_{j=1}^n \varphi_{nj}^K}\right),$$

$$N_C^R = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \psi_{1j}^R}, \dots, \sqrt{\sum_{j=1}^n \psi_{nj}^R}\right), \qquad N_C^I = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \psi_{1j}^I}, \dots, \sqrt{\sum_{j=1}^n \psi_{nj}^I}\right),$$

$$N_C^J = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \psi_{1j}^J}, \dots, \sqrt{\sum_{j=1}^n \psi_{nj}^J}\right), \qquad N_C^K = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \psi_{1j}^K}, \dots, \sqrt{\sum_{j=1}^n \psi_{nj}^K}\right),$$

$$N_D = \operatorname{diag}\left(\sqrt{\sum_{j=1}^n \eta_{1j}}, \dots, \sqrt{\sum_{j=1}^n \eta_{nj}}\right).$$

1.2 Basic definitions and lemmas

For the next work, we introduce the following definitions and lemmas.

Definition 1 A function $g(t) \in C((-\infty, +\infty), \mathbb{H}^n)$ is a solution of system (1) satisfying the initial value condition (2), if the following conditions are satisfied:

(i) $g(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}) \subset (-\infty, +\infty)$, $k = 1, 2, \dots$,

(ii) for any $t_k \in [0, +\infty)$, $k = 1, 2, \dots$, $g(t_k^+)$ and $(g(t_k^-))$ exist and $g(t_k^+) = g(t_k)$.

Lemma 1 ([28]) *Let* $\Upsilon^* = \{\Upsilon \in \mathbb{R}^{n^2 \times n^2} : \Upsilon = \operatorname{diag}(\gamma_{11}, \dots, \gamma_{1n}, \dots, \gamma_{n1}, \dots, \gamma_{nn})$, *where* $|\gamma_{ij}| \leq 1$, $i, j = 1, 2, \dots, n\}$, *then* $\Upsilon^T \Upsilon \leq I$. *Furthermore, let*

$$\begin{aligned}
\widetilde{D} &= \{D = D_0 + U_D \Upsilon_D V_D\}, \\
\widetilde{C} &= \{C = C_0 + U_C^R \Upsilon_C^R V_C^R + \imath U_C^I \Upsilon_C^I V_C^I + \jmath U_C^J \Upsilon_C^J V_C^J + \kappa U_C^K \Upsilon_C^K V_C^K\}, \\
\widetilde{B} &= \{B = B_0 + U_B^R \Upsilon_B^R V_B^R + \imath U_B^I \Upsilon_B^I V_B^I + \jmath U_B^J \Upsilon_B^J V_B^J + \kappa U_B^K \Upsilon_B^K V_B^K\}, \\
\widetilde{A} &= \{A = A_0 + U_A^R \Upsilon_A^R V_A^R + \imath U_A^I \Upsilon_A^I V_A^I + \jmath U_A^J \Upsilon_A^J V_A^J + \kappa U_A^K \Upsilon_A^K V_A^K\}.
\end{aligned}$$

Then for $\Upsilon_C, \Upsilon_A^R, \Upsilon_A^I, \Upsilon_A^J, \Upsilon_A^K, \Upsilon_B^R, \Upsilon_B^I, \Upsilon_B^J, \Upsilon_B^K, \Upsilon_C^R, \Upsilon_C^I, \Upsilon_C^J, \Upsilon_C^K \in \Upsilon^*$, $D_I = \widetilde{D}$, $C_I = \widetilde{C}$, $A_I = \widetilde{A}$, $B_I = \widetilde{B}$.

Lemma 2 ([8]) *If* U_i, V_i *and* W_i $(i = 1, 2, \dots, m)$ *are complex-valued matrices of appropriate dimension with* M *satisfying* $M^* = M$, *then*

$$M + \sum_{i=1}^m \left(U_i V_i W_i + W_i^* V_i^* U_i^*\right) < 0,$$

for all $V_i^* V_i \leq I$ $(i = 1, 2, \dots, m)$, *if and only if there exist positive constants* ε_i $(i = 1, 2, \dots, m)$ *such that*

$$M + \sum_{i=1}^m \left(\varepsilon_i^{-1} U_i U_i^* + \varepsilon_i W_i^* W_i\right) < 0.$$

Lemma 3 ([13]) *For a given Hermitian matrix,*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} < 0,$$

where $S_{11}^* = S_{11}$, $S_{12}^* = S_{21}$, *and* $S_{22}^* = S_{22}$, *is equivalent to the following conditions:*

(i) $S_{22} < 0$ *and* $S_{11} - S_{12} S_{22}^{-1} S_{21} < 0$,

(ii) $S_{11} < 0$ *and* $S_{22} - S_{21} S_{11}^{-1} S_{12} < 0$.

Lemma 4 ([35]) *For any $a, b \in \mathbb{H}^n$, if $P \in \mathbb{H}^{n\times n}$ is a positive definite Hermitian matrix, then*

$$a^*b + b^*a \le a^*Pa + b^*P^{-1}b.$$

Lemma 5 ([35]) *For any positive definite constant Hermitian matrix $W \in \mathbb{H}^{n\times n}$ and any scalar function $\omega(s)$: $[a,b] \to \mathbb{H}^n$ with scalars $a < b$ such that the integrals concerned are well defined,*

$$\left(\int_a^b \omega(s)\,ds\right)^* W\left(\int_a^b \omega(s)\,ds\right) \le (b-a)\int_a^b \omega^*(s)W\omega(s)\,ds.$$

Lemma 6 ([35]) *If $G(q)$: $\mathbb{H}^n \to \mathbb{H}^n$ is a continuous map and satisfies the following conditions:*

(i) *$G(q)$ is injective on $\mathbb{H}^n$,*

(ii) $\lim_{\|q\|\to\infty} \|G(q)\| = \infty$,

then $G(q)$ is a homeomorphism of $\mathbb{H}^n$ onto itself.

Lemma 7 ([37]) *Let $A = A_1 + A_2\jmath$ and $B = B_1 + B_2\jmath$, where $A_1, A_2, B_1, B_2 \in \mathbb{C}^{n\times n}$ and $A, B \in \mathbb{H}^{n\times n}$. Then*

(1) $A^* = A_1^* - A_2\jmath^T$;

(2) $AB = (A_1B_1 - A_2\bar{B}_2) + (A_1B_2 + A_2\bar{B}_1)\jmath$,

where $\bar{B}_1$ and $\bar{B}_2$ denote the conjugate matrices of B_1 and B_2, respectively.

Lemma 8 ([37]) *Let $Q \in \mathbb{H}^{n\times n}$ be a Hermite matrix, $Q = Q_1 + Q_2\jmath$, where $Q_1, Q_2 \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{H}^{n\times n}$. Then $Q < 0$ is equivalent to*

$$\begin{pmatrix} Q_1 & -Q_2 \\ \bar{Q}_2 & \bar{Q}_1 \end{pmatrix} < 0,$$

where $\bar{Q}_1$ and $\bar{Q}_2$ denote the conjugate matrices of Q_1 and Q_2, respectively.

2 Main results

In this section, we study the existence and uniqueness of an equilibrium point of system (1) and analyze the global robust stability of the unique equilibrium point for system (1) under assumptions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, and $(\mathbf{A}_3)$.

Theorem 1 *Under assumptions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, and $(\mathbf{A}_3)$, QVNN (1) has a unique equilibrium point, which is globally robust stable, if there are a positive definite Hermitian matrix P_1, four positive diagonal matrices P_i $(i = 2,3,4)$ and R, and positive constants ε_i $(i = 1,2,\ldots,34)$ such that the following linear matrix inequality holds:*

$$\Xi = (\Xi_{ij})_{41\times 41} < 0, \tag{3}$$

where

$$\Xi_{11} = -P_1D_0 - D_0^TP_1 + P_2 + \delta^2P_3 + L(2P_2 + R)L + \varepsilon_1(V_D)^TV_D$$
$$+ \varepsilon_{15}(V_D)^TV_D, \qquad \Xi_{12} = \frac{3}{2}P_1A_0, \qquad \Xi_{13} = \frac{3}{2}P_1B_0,$$

$$\Xi_{14} = \frac{3}{2}P_1C_0, \qquad \Xi_{15} = (D_0)^TP_1D_0, \qquad \Xi_{16} = -\frac{1}{2}P_1D_0, \qquad \Xi_{18} = P_1N_D,$$

$$\Xi_{19} = \frac{3}{2}P_1N_A^R, \qquad \Xi_{1,10} = \frac{3}{2}P_1N_A^I, \qquad \Xi_{1.11} = \frac{3}{2}P_1N_A^J, \qquad \Xi_{1.12} = \frac{3}{2}P_1N_A^K,$$

$$\Xi_{1.13} = \frac{3}{2}P_1N_B^R, \qquad \Xi_{1.14} = \frac{3}{2}P_1N_B^I, \qquad \Xi_{1.15} = \frac{3}{2}P_1N_B^J, \qquad \Xi_{1.16} = \frac{3}{2}P_1N_B^K,$$

$$\Xi_{1.17} = \frac{3}{2}P_1N_C^R, \qquad \Xi_{1.18} = \frac{3}{2}P_1N_C^I, \qquad \Xi_{1.19} = \frac{3}{2}P_1N_C^J, \qquad \Xi_{1.20} = \frac{3}{2}P_1N_C^K,$$

$$\Xi_{1.21} = D_0^TP_1N_D, \qquad \Xi_{1.23} = (U_D\Upsilon_DV_D)^TP_1N_D,$$

$$\Xi_{1.24} = \frac{1}{2}P_1N_D, \qquad \Xi_{21} = \frac{3}{2}(A_0)^*P_1,$$

$$\begin{aligned}\Xi_{22} &= P_4 - 2P_2 + \varepsilon_2\big(V_A^R\big)^TV_A^R + \varepsilon_3\big(V_A^I\big)^TV_A^I + \varepsilon_4\big(V_A^J\big)^TV_A^J + \varepsilon_5\big(V_A^K\big)^TV_A^K \\ &\quad + \varepsilon_{18}\big(V_A^R\big)^TV_A^R + \varepsilon_{19}\big(V_A^I\big)^TV_A^I + \varepsilon_{20}\big(V_A^J\big)^TV_A^J + \varepsilon_{21}\big(V_A^K\big)^TV_A^K,\end{aligned}$$

$$\Xi_{27} = (A_0)^*P_1, \qquad \Xi_{31} = \frac{3}{2}(B_0)^*P_1,$$

$$\begin{aligned}\Xi_{33} &= -P_4 + \varepsilon_6\big(V_B^R\big)^TV_B^R + \varepsilon_7\big(V_B^I\big)^TV_B^I + \varepsilon_8\big(V_B^J\big)^TV_B^J + \varepsilon_9\big(V_B^K\big)^TV_B^K \\ &\quad + \varepsilon_{22}\big(V_B^R\big)^TV_B^R + \varepsilon_{23}\big(V_B^I\big)^TV_B^I + \varepsilon_{24}\big(V_B^J\big)^TV_B^J + \varepsilon_{25}\big(V_B^K\big)^TV_B^K,\end{aligned}$$

$$\Xi_{37} = (B_0)^*P_1, \qquad \Xi_{41} = \frac{3}{2}(C_0)^*P_1,$$

$$\begin{aligned}\Xi_{44} &= -R + \varepsilon_{10}\big(V_C^R\big)^TV_C^R + \varepsilon_{11}\big(V_C^I\big)^TV_C^I + \varepsilon_{12}\big(V_C^J\big)^TV_C^J + \varepsilon_{13}\big(V_C^K\big)^TV_C^K \\ &\quad + \varepsilon_{26}\big(V_C^R\big)^TV_C^R + \varepsilon_{27}\big(V_C^I\big)^TV_C^I + \varepsilon_{28}\big(V_C^J\big)^TV_C^J + \varepsilon_{29}\big(V_C^K\big)^TV_C^K,\end{aligned}$$

$$\Xi_{47} = (C_0)^*P_1, \qquad \Xi_{51} = (D_0)^TP_1D_0,$$

$$\Xi_{55} = -P_3 + \varepsilon_{14}(V_D)^TV_D + \varepsilon_{16}(V_D)^TV_D + \varepsilon_{33}(V_D)^TV_D,$$

$$\Xi_{56} = -(D_0)^TP_1D_0, \qquad \Xi_{57} = -(D_0)^TP_1, \qquad \Xi_{5.22} = D_0^TP_1N_D, \qquad \Xi_{5.37} = P_1N_D,$$

$$\Xi_{5.39} = D_0^TP_1N_D, \qquad \Xi_{5.41} = (U_D\Upsilon_DV_D)^TP_1N_D, \qquad \Xi_{61} = -\frac{1}{2}D_0^TP_1,$$

$$\Xi_{65} = -(D_0)^TP_1D_0, \qquad \Xi_{66} = -P_2 + \varepsilon_{17}(V_D)^TV_D + \varepsilon_{32}(V_D)^TV_D + \varepsilon_{34}(V_D)^TV_D,$$

$$\Xi_{67} = -(D_0)^TP_1, \qquad \Xi_{6.38} = P_1N_D, \qquad \Xi_{6.40} = D_0^TP_1N_D,$$

$$\Xi_{72} = P_1A_0, \qquad \Xi_{73} = P_1B_0, \qquad \Xi_{74} = P_1C_0, \qquad \Xi_{75} = -P_1D_0, \qquad \Xi_{76} = -P_1D_0,$$

$$\Xi_{77} = -P_1 - P_1 + \varepsilon_{30}(V_D)^TV_D + \varepsilon_{31}(V_D)^TV_D,$$

$$\Xi_{7.25} = P_1N_A^R, \qquad \Xi_{7.26} = P_1N_A^I, \qquad \Xi_{7.27} = P_1N_A^J, \qquad \Xi_{7.28} = P_1N_A^K,$$

$$\Xi_{7.29} = P_1N_B^R, \qquad \Xi_{7.30} = P_1N_B^I, \qquad \Xi_{7.31} = P_1N_B^J, \qquad \Xi_{7.32} = P_1N_B^K,$$

$$\Xi_{7.33} = P_1N_C^R, \qquad \Xi_{7.34} = P_1N_C^I, \qquad \Xi_{7.35} = P_1N_C^J, \qquad \Xi_{7.36} = P_1N_C^K,$$

$$\Xi_{81} = (N_D)^TP_1, \qquad \Xi_{88} = -\varepsilon_1I, \qquad \Xi_{91} = \frac{3}{2}\big(N_A^R\big)^TP_1,$$

$$\Xi_{99} = -\varepsilon_2I, \qquad \Xi_{10.1} = \frac{3}{2}\big(N_A^I\big)^TP_1, \qquad \Xi_{10.10} = -\varepsilon_3I, \qquad \Xi_{11.1} = \frac{3}{2}\big(N_A^J\big)^TP_1,$$

$$\Xi_{11.11} = -\varepsilon_4I, \qquad \Xi_{12.1} = \frac{3}{2}\big(N_A^K\big)^TP_1, \qquad \Xi_{12.12} = -\varepsilon_5I, \qquad \Xi_{13.1} = \frac{3}{2}\big(N_B^R\big)^TP_1,$$

$$\Xi_{13.13} = -\varepsilon_6 I, \qquad \Xi_{14.1} = \frac{3}{2}\left(N_B^I\right)^T P_1, \qquad \Xi_{14.14} = -\varepsilon_7 I, \qquad \Xi_{15.1} = \frac{3}{2}\left(N_B^J\right)^T P_1,$$
$$\Xi_{15.15} = -\varepsilon_8 I, \qquad \Xi_{16.1} = \frac{3}{2}\left(N_B^K\right)^T P_1, \qquad \Xi_{16.16} = -\varepsilon_9 I, \qquad \Xi_{17.1} = \frac{3}{2}\left(N_C^R\right)^T P_1,$$
$$\Xi_{17.17} = -\varepsilon_{10} I, \qquad \Xi_{18.1} = \frac{3}{2}\left(N_C^I\right)^T P_1, \qquad \Xi_{18.18} = -\varepsilon_{11} I, \qquad \Xi_{19.1} = \frac{3}{2}\left(N_C^J\right)^T P_1,$$
$$\Xi_{19.19} = -\varepsilon_{12} I, \qquad \Xi_{20.1} = \frac{3}{2}\left(N_C^K\right)^T P_1, \qquad \Xi_{20.20} = -\varepsilon_{13} I, \qquad \Xi_{21.1} = (N_D)^T P_1 D_0,$$
$$\Xi_{21.21} = -\varepsilon_{14} I, \qquad \Xi_{22.5} = (N_D)^T P_1 D_0, \qquad \Xi_{22.22} = -\varepsilon_{15} I,$$
$$\Xi_{23.1} = (N_D)^T P_1 U_D \Upsilon_D V_D, \qquad \Xi_{23.23} = -\varepsilon_{16} I, \qquad \Xi_{24.1} = \frac{1}{2}(N_D)^T P_1,$$
$$\Xi_{24.24} = -\varepsilon_{17} I, \qquad \Xi_{25.7} = \left(N_A^R\right)^T P_1, \qquad \Xi_{25.25} = -\varepsilon_{18} I, \qquad \Xi_{26.7} = \left(N_A^I\right)^T P_1,$$
$$\Xi_{26.26} = -\varepsilon_{19} I, \qquad \Xi_{27.7} = \left(N_A^J\right)^T P_1, \qquad \Xi_{27.27} = -\varepsilon_{20} I, \qquad \Xi_{28.7} = \left(N_A^K\right)^T P_1,$$
$$\Xi_{28.28} = -\varepsilon_{21} I, \qquad \Xi_{29.7} = \left(N_B^R\right)^T P_1, \qquad \Xi_{29.29} = -\varepsilon_{22} I, \qquad \Xi_{30.7} = \left(N_B^I\right)^T P_1,$$
$$\Xi_{30.30} = -\varepsilon_{23} I, \qquad \Xi_{31.7} = \left(N_B^J\right)^T P_1, \qquad \Xi_{31.31} = -\varepsilon_{24} I, \qquad \Xi_{32.7} = \left(N_B^K\right)^T P_1,$$
$$\Xi_{32.32} = -\varepsilon_{25} I, \qquad \Xi_{33.7} = \left(N_C^R\right)^T P_1, \qquad \Xi_{33.33} = -\varepsilon_{26} I, \qquad \Xi_{347} = \left(N_C^I\right)^T P_1,$$
$$\Xi_{34.34} = -\varepsilon_{27} I, \qquad \Xi_{35.7} = \left(N_C^J\right)^T P_1, \qquad \Xi_{35.35} = -\varepsilon_{28} I, \qquad \Xi_{36.7} = \left(N_C^K\right)^T P_1,$$
$$\Xi_{36.36} = -\varepsilon_{29} I, \qquad \Xi_{37.5} = (N_D)^T P_1 \qquad \Xi_{37.37} = -\varepsilon_{30} I, \qquad \Xi_{38.6} = (N_D)^T P_1,$$
$$\Xi_{38.38} = -\varepsilon_{31} I, \qquad \Xi_{39.5} = (N_D)^T P_1 D_0, \qquad \Xi_{39.39} = -\varepsilon_{32} I, \qquad \Xi_{40.6} = (N_D)^T P_1 D_0,$$
$$\Xi_{40.40} = -\varepsilon_{33} I, \qquad \Xi_{41.5} = (N_D)^T P_1 U_D \Upsilon_D V_D, \qquad \Xi_{41.41} = -\varepsilon_{34} I,$$

and the other entries in Ξ *are zero.*

Proof We prove this theorem in three steps:

Step 1: Matrices Ξ and M are equivalent in negative definiteness.

$$\mathrm{M} = \begin{pmatrix} M_{11} & \frac{3}{2}P_1 A & \frac{3}{2}P_1 B & \frac{3}{2}P_1 C & D^T P_1 D & -\frac{1}{2}P_1 D & 0 \\ \star & P_4 - 2P_2 & 0 & 0 & 0 & 0 & A^* P_1 \\ \star & \star & -P_4 & 0 & 0 & 0 & B^* P_1 \\ \star & \star & \star & -R & 0 & 0 & C^* P_1 \\ \star & \star & \star & \star & -P_3 & -D^T P_1 D & -D^T P_1 \\ \star & \star & \star & \star & \star & -P_2 & -D^T P_1 \\ \star & \star & \star & \star & \star & \star & -P_1 - P_1 \end{pmatrix}, \tag{4}$$

where $M_{11} = -P_1 D - D^T P_1 + \delta^2 P_3 + P_2 + L(2P_2 + R)L$.

From Lemma 1, for any $D \in D_I$, $C \in C_I$, $A \in A_I$, $B \in B_I$, we can obtain that

$$D = D_0 + U_d \Upsilon_d V_d, \tag{5}$$
$$A = A_0 + U_A^R \Upsilon_A^R V_A^R + \imath U_A^I \Upsilon_A^I V_A^I + \jmath U_A^J \Upsilon_A^J V_A^J + \kappa U_A^K \Upsilon_A^K V_A^K, \tag{6}$$
$$B = B_0 + U_B^R \Upsilon_B^R V_B^R + \imath U_B^I \Upsilon_B^I V_B^I + \jmath U_B^J \Upsilon_B^J V_B^J + \kappa U_B^K \Upsilon_B^K V_B^K, \tag{7}$$
$$C = C_0 + U_C^R \Upsilon_C^R V_C^R + \imath U_C^I \Upsilon_C^I V_C^I + \jmath U_C^J \Upsilon_C^J V_C^J + \kappa U_C^K \Upsilon_C^K V_C^K, \tag{8}$$

and for (Υ_D), (Υ_A^X), (Υ_B^X), $(\Upsilon_C^X) \in \Upsilon^*$, where X denotes R, I, J, K, respectively. We have $(\Upsilon_A^X)^T\Upsilon_A^X \le I$, $(\Upsilon_B^X)^T\Upsilon_B^X \le I$, $(\Upsilon_C^X)^T\Upsilon^X \le I$, $(\Upsilon_D)^T\Upsilon_D \le I$.

By substituting (5), (6), (7), and (8) into (4), we can get

$$
\begin{aligned}
\mathrm{M} = {} & \begin{pmatrix}
M_{11} & \frac{3}{2}P_1A_0 & \frac{3}{2}P_1B_0 & \frac{3}{2}P_1C_0 & D_0^TP_1D_0 & -\frac{1}{2}P_1D_0 & 0 \\
\star & P_4-2P_2 & 0 & 0 & 0 & 0 & (A_0)^*P_1 \\
\star & \star & -P_4 & 0 & 0 & 0 & (B_0)^*P_1 \\
\star & \star & \star & -R & 0 & 0 & (C_0)^*P_1 \\
\star & \star & \star & \star & -P_3 & -D_0^TP_1D_0 & -D_0^TP_1 \\
\star & \star & \star & \star & \star & -P_2 & -D_0^TP_1 \\
\star & \star & \star & \star & \star & \star & -P_1-P_1
\end{pmatrix} \\
& + \big((-U_D^TP_1)\lambda_1\big)^T\Upsilon_D\big((V_D)\lambda_1\big) + \big((V_D)\lambda_1\big)^T(\Upsilon_D)^T\big((-U_D^TP_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((U_A^R)^TP_1)\lambda_1\big)^T(\Upsilon_A^R)\big((V_A^R)\lambda_2\big) + \frac{3}{2}\big((V_A^R)\lambda_2\big)^T(\Upsilon_A^R)^T\big(((U_A^R)^TP_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((iU_A^I)^*P_1)\lambda_1\big)^*(\Upsilon_A^I)\big((V_A^I)\lambda_2\big) + \frac{3}{2}\big((V_A^I)\lambda_2\big)^T(\Upsilon_A^R)^T\big(((iU_A^I)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((jU_A^J)^*P_1)\lambda_1\big)^*(\Upsilon_A^J)\big((V_A^J)\lambda_2\big) + \frac{3}{2}\big((V_A^J)\lambda_2\big)^T(\Upsilon_A^J)^T\big(((jU_A^J)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((kU_A^K)^*P_1)\lambda_1\big)^*(\Upsilon_A^K)\big((V_A^K)\lambda_2\big) + \frac{3}{2}\big((V_A^K)\lambda_2\big)^T(\Upsilon_A^K)^T\big(((kU_A^K)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((U_B^R)^TP_1)\lambda_1\big)^T(\Upsilon_A^R)\big((V_B^R)\lambda_3\big) + \frac{3}{2}\big((V_B^R)\lambda_3\big)^T(\Upsilon_B^R)^T\big(((U_B^R)^TP_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((iU_B^I)^*P_1)\lambda_1\big)^*(\Upsilon_B^I)\big((V_B^I)\lambda_3\big) + \frac{3}{2}\big((V_B^I)\lambda_3\big)^T(\Upsilon_B^I)^T\big(((iU_B^I)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((jU_B^J)^*P_1)\lambda_1\big)^*(\Upsilon_B^J)\big((V_B^J)\lambda_3\big) + \frac{3}{2}\big((V_B^J)\lambda_3\big)^T(\Upsilon_B^J)^T\big(((jU_B^J)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((kU_B^K)^*P_1)\lambda_1\big)^*(\Upsilon_B^K)\big((V_B^K)\lambda_3\big) + \frac{3}{2}\big((V_B^K)\lambda_3\big)^T(\Upsilon_B^K)^T\big(((kU_B^K)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((U_C^R)^TP_1)\lambda_1\big)^T(\Upsilon_C^R)\big((V_C^R)\lambda_4\big) + \frac{3}{2}\big((V_C^R)\lambda_4\big)^T(\Upsilon_C^R)^T\big(((U_C^R)^TP_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((iU_C^I)^*P_1)\lambda_1\big)^*(\Upsilon_C^I)\big((V_C^I)\lambda_4\big) + \frac{3}{2}\big((V_C^I)\lambda_4\big)^T(\Upsilon_C^I)^T\big(((iU_C^I)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((jU_C^J)^*P_1)\lambda_1\big)^*(\Upsilon_C^J)\big((V_C^J)\lambda_4\big) + \frac{3}{2}\big((V_C^J)\lambda_4\big)^T(\Upsilon_C^J)^T\big(((jU_C^J)^*P_1)\lambda_1\big) \\
& + \frac{3}{2}\big(((kU_C^K)^*P_1)\lambda_1\big)^*(\Upsilon_C^K)\big(V_C^K\lambda_4\big) + \frac{3}{2}\big((V_C^K)\lambda_4\big)^T(\Upsilon_C^K)^T\big(((kU_C^K)^*P_1)\lambda_1\big) \\
& + \big((U_D^TP_1D_0)\lambda_1\big)^T\Upsilon_D\big((V_D)\lambda_5\big) + \big((V_D)\lambda_5\big)^T\Upsilon_D^T\big((U_D^TP_1D_0)\lambda_1\big) \\
& + \big((U_D^TP_1D_0)\lambda_5\big)^T\Upsilon_D\big((V_D)\lambda_1\big) + \big((V_D)\lambda_1\big)^T\Upsilon_D^T\big((U_D^TP_1D_0)\lambda_5\big) \\
& + \big((U_D^TP_1U_D\Upsilon_DV_D)\lambda_1\big)^T\Upsilon_D\big((V_D)\lambda_5\big) + \big((V_D)\lambda_5\big)^T\Upsilon_D^T\big((U_D^TP_1U_D\Upsilon_DV_D)\lambda_1\big) \\
& + \frac{1}{2}\big((-U_D^TP_1)\lambda_1\big)^T\Upsilon_D\big((V_D)\lambda_6\big) + \frac{1}{2}\big((V_D)\lambda_6\big)^T\Upsilon_D^T\big((-U_D^TP_1)\lambda_1\big) \\
& + \big(((U_A^R)^TP_1)\lambda_7\big)^T(\Upsilon_A^R)\big((V_A^R)\lambda_2\big) + \big((V_A^R)\lambda_2\big)^T(\Upsilon_A^R)^T\big(((U_A^R)^TP_1)\lambda_7\big) \\
& + \big(((iU_A^I)^*P_1)\lambda_7\big)^*(\Upsilon_A^I)\big((V_A^I)\lambda_2\big) + \big((V_A^I)\lambda_2\big)^T(\Upsilon_A^I)^T\big(((iU_A^I)^*P_1)\lambda_7\big) \\
& + \big(((jU_A^J)^*P_1)\lambda_7\big)^*(\Upsilon_A^J)\big((V_A^J)\lambda_2\big) + \big((V_A^J)\lambda_2\big)^T(\Upsilon_A^J)^T\big(((jU_A^J)^*P_1)\lambda_7\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\big(\big(\big(kU_A^K\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_A^K\big)\big(\big(V_A^K\big)\lambda_2\big)+\big(\big(V_A^K\big)\lambda_2\big)^T\big(\Upsilon_A^K\big)^T\big(\big(\big(kU_A^K\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(U_B^R\big)^TP_1\big)\lambda_7\big)^T\big(\Upsilon_B^R\big)\big(\big(V_B^R\big)\lambda_3\big)+\big(\big(V_B^R\big)\lambda_3\big)^T\big(\Upsilon_B^R\big)^T\big(\big(\big(U_B^R\big)^TP_1\big)\lambda_7\big)\\
&+\big(\big(\big(iU_B^I\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_B^I\big)\big(\big(V_B^I\big)\lambda_3\big)+\big(\big(V_B^I\big)\lambda_3\big)^T\big(\Upsilon_B^I\big)^T\big(\big(\big(iU_B^I\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(jU_B^J\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_B^J\big)\big(\big(V_B^J\big)\lambda_3\big)+\big(\big(V_B^J\big)\lambda_3\big)^T\big(\Upsilon_B^J\big)^T\big(\big(\big(jU_B^J\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(kU_B^K\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_B^K\big)\big(\big(V_B^K\big)\lambda_3\big)+\big(\big(V_B^K\big)\lambda_3\big)^T\big(\Upsilon_B^K\big)^T\big(\big(\big(kU_B^K\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(U_C^R\big)^TP_1\big)\lambda_7\big)^T\big(\Upsilon_C^R\big)\big(\big(V_C^R\big)\lambda_4\big)+\big(\big(V_C^R\big)\lambda_4\big)^T\big(\Upsilon_C^R\big)^T\big(\big(\big(U_C^R\big)^TP_1\big)\lambda_7\big)\\
&+\big(\big(\big(iU_C^I\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_C^I\big)\big(\big(V_C^I\big)\lambda_4\big)+\big(\big(V_C^I\big)\lambda_4\big)^T\big(\Upsilon_C^I\big)^T\big(\big(\big(iU_C^I\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(jU_C^J\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_C^J\big)\big(\big(V_C^J\big)\lambda_4\big)+\big(\big(V_C^J\big)\lambda_4\big)^T\big(\Upsilon_C^J\big)^T\big(\big(\big(jU_C^J\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(\big(kU_C^K\big)^*P_1\big)\lambda_7\big)^*\big(\Upsilon_C^K\big)\big(\big(V_C^K\big)\lambda_4\big)+\big(\big(V_C^K\big)\lambda_4\big)^T\big(\Upsilon_C^K\big)^T\big(\big(\big(kU_C^K\big)^*P_1\big)\lambda_7\big)\\
&+\big(\big(-U_D^TP_1\big)\lambda_5\big)^T+\Upsilon_D\big((V_D)\lambda_7\big)+\big((V_D)\lambda_7\big)^T\Upsilon_D^T\big(\big(-U_D^TP_1\big)\lambda_5\big)\\
&+\big(\big(-U_D^TP_1\big)\lambda_6\big)^T+\Upsilon_D\big((V_D)\lambda_7\big)+\big((V_D)\lambda_7\big)^T\Upsilon_D^T\big(\big(-U_D^TP_1\big)\lambda_6\big)\\
&+\big(\big(-U_D^TP_1D_0\big)\lambda_5\big)^T\Upsilon_D\big((V_D)\lambda_6\big)+\big((V_D)\lambda_6\big)^T\Upsilon_D^T\big(\big(-U_D^TP_1D_0\big)\lambda_5\big)\\
&+\big(\big(-U_D^TP_1D_0\big)\lambda_6\big)^T\Upsilon_D\big((V_D)\lambda_5\big)+\big((V_D)\lambda_5\big)^T\Upsilon_D^T\big(\big(-U_D^TP_1D_0\big)\lambda_6\big)\\
&+\big(\big(-U_D^TP_1U_D\Upsilon_DV_D\big)\lambda_5\big)^T\Upsilon_D\big((V_D)\lambda_6\big)+\big((V_D)\lambda_6\big)^T\Upsilon_D^T\big(\big(-U_D^TP_1U_D\Upsilon_DV_D\big)\lambda_5\big),
\end{aligned}
$$

where $(X\lambda_i)_{n^2\times 7n}$ represents the block matrix, and the matrix $X_{n^2\times n}$ is in the ith column, here $i=1,2,\ldots,7$, $M_{11}=-P_1D_0-D_0^TP_1+\delta^2P_3+P_2+L(2P_2+R)L$.

Using Lemma 2, we can only get $M<0$ if and only if there exist positive constants ε_i ($i=1,2,\ldots,34$) such that

$$
\begin{aligned}
\bar{M}={}&\begin{pmatrix}
M_{11} & \frac{3}{2}P_1A_0 & \frac{3}{2}P_1B_0 & \frac{3}{2}P_1C_0 & D_0^TP_1D_0 & -\frac{1}{2}P_1D_0 & 0\\
\star & P_4-2P_2 & 0 & 0 & 0 & 0 & (A_0)^*P_1\\
\star & \star & -P_4 & 0 & 0 & 0 & (B_0)^*P_1\\
\star & \star & \star & -R & 0 & 0 & (C_0)^*P_1\\
\star & \star & \star & \star & -P_3 & -D_0^TP_1D_0 & -D_0^TP_1\\
\star & \star & \star & \star & \star & -P_2 & -D_0^TP_1\\
\star & \star & \star & \star & \star & \star & -P_1-P_1
\end{pmatrix}\\
&+\varepsilon_1^{-1}\big(\big(-U_D^TP_1\big)\lambda_1\big)^T\big(\big(-U_D^TP_1\big)\lambda_1\big)+\varepsilon_1\big((V_D)\lambda_1\big)^T\big((V_D)\lambda_1\big)\\
&+\varepsilon_2^{-1}\frac{9}{4}\big(\big(\big(U_A^R\big)^TP_1\big)\lambda_1\big)^T\big(\big(\big(U_A^R\big)^TP_1\big)\lambda_1\big)+\varepsilon_2\big(\big(V_A^R\big)\lambda_2\big)^T\big(\big(V_A^R\big)\lambda_2\big)\\
&+\varepsilon_3^{-1}\frac{9}{4}\big(\big(\big(iU_A^I\big)^*P_1\big)\lambda_1\big)^*\big(\big(\big(iU_A^I\big)^*P_1\big)\lambda_1\big)+\varepsilon_3\big(\big(V_A^I\big)\lambda_2\big)^T\big(\big(V_A^I\big)\lambda_2\big)\\
&+\varepsilon_4^{-1}\frac{9}{4}\big(\big(\big(jU_A^J\big)^*P_1\big)\lambda_1\big)^*\big(\big(\big(jU_A^J\big)^*P_1\big)\lambda_1\big)+\varepsilon_4\big(\big(V_A^J\big)\lambda_2\big)^T\big(\big(V_A^J\big)\lambda_2\big)\\
&+\varepsilon_5^{-1}\frac{9}{4}\big(\big(\big(kU_A^K\big)^*P_1\big)\lambda_1\big)^*\big(\big(\big(kU_A^K\big)^*P_1\big)\lambda_1\big)+\varepsilon_5\big(\big(V_A^K\big)\lambda_2\big)^T\big(\big(V_A^K\big)\lambda_2\big)\\
&+\varepsilon_6^{-1}\frac{9}{4}\big(\big(\big(U_B^R\big)^TP_1\big)\lambda_1\big)^T\big(\big(\big(U_B^R\big)^TP_1\big)\lambda_1\big)+\varepsilon_6\big(\big(V_B^R\big)\lambda_3\big)^T\big(\big(V_B^R\big)\lambda_3\big)\\
&+\varepsilon_7^{-1}\frac{9}{4}\big(\big(\big(iU_B^I\big)^*P_1\big)\lambda_1\big)^*\big(\big(\big(iU_B^I\big)^*P_1\big)\lambda_1\big)+\varepsilon_7\big(\big(V_B^I\big)\lambda_3\big)^T\big(\big(V_B^I\big)\lambda_3\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon_8^{-1}\frac{9}{4}\big(\big((jU_B^J)^*P_1\big)\lambda_1\big)^*\big(\big((jU_B^J)^*P_1\big)\lambda_1\big)+\varepsilon_8\big((V_B^J)\lambda_3\big)^T\big((V_B^J)\lambda_3\big)\\
&+\varepsilon_9^{-1}\frac{9}{4}\big(\big((kU_B^K)^*P_1\big)\lambda_1\big)^*\big(\big((kU_B^K)^*P_1\big)\lambda_1\big)+\varepsilon_9\big((V_B^K)\lambda_3\big)^T\big((V_B^K)\lambda_3\big)\\
&+\varepsilon_{10}^{-1}\frac{9}{4}\big(\big((U_C^R)^TP_1\big)\lambda_1\big)^T\big(\big((U_C^R)^TP_1\big)\lambda_1\big)+\varepsilon_{10}\big((V_C^R)\lambda_4\big)^T\big((V_C^R)\lambda_4\big)\\
&+\varepsilon_{11}^{-1}\frac{9}{4}\big(\big((iU_C^I)^*P_1\big)\lambda_1\big)^*\big(\big((iU_C^I)^*P_1\big)\lambda_1\big)+\varepsilon_{11}\big((V_C^I)\lambda_4\big)^T\big((V_C^I)\lambda_4\big)\\
&+\varepsilon_{12}^{-1}\frac{9}{4}\big(\big((jU_C^J)^*P_1\big)\lambda_1\big)^*\big(\big((jU_C^J)^*P_1\big)\lambda_1\big)+\varepsilon_{12}\big((V_C^J)\lambda_4\big)^T\big((V_C^J)\lambda_4\big)\\
&+\varepsilon_{13}^{-1}\frac{9}{4}\big(\big((kU_C^K)^*P_1\big)\lambda_1\big)^*\big(\big((kU_C^K)^*P_1\big)\lambda_1\big)+\varepsilon_{13}\big((V_C^K)\lambda_4\big)^T\big((V_C^K)\lambda_4\big)\\
&+\varepsilon_{14}^{-1}\big((U_D^TP_1D_0)\lambda_1\big)^T\big((U_D^TP_1D_0)\lambda_1\big)+\varepsilon_{14}\big((V_D)\lambda_5\big)^T\big((V_D)\lambda_5\big)\\
&+\varepsilon_{15}^{-1}\big((U_D^TP_1D_0)\lambda_5\big)^T\big((U_D^TP_1D_0)\lambda_5\big)+\varepsilon_{15}\big((V_D)\lambda_1\big)^T\big((V_D)\lambda_1\big)\\
&+\varepsilon_{16}^{-1}\big((U_D^TP_1U_D\Upsilon_DV_D)\lambda_1\big)^T\big((U_D^TP_1U_D\Sigma_DV_D)\lambda_1\big)+\varepsilon_{16}\big((V_D)\lambda_5\big)^T\big((V_D)\lambda_5\big)\\
&+\varepsilon_{17}^{-1}\frac{1}{4}\big((-U_D^TP_1)\lambda_1\big)^T\big((-U_D^TP_1)\lambda_1\big)+\varepsilon_{17}\big((V_D)\lambda_6\big)^T\big((V_D)\lambda_6\big)\\
&+\varepsilon_{18}^{-1}\big(\big((U_A^R)^TP_1\big)\lambda_7\big)^T\big(\big((U_A^R)^TP_1\big)\lambda_7\big)+\varepsilon_{18}\big((V_A^R)\lambda_2\big)^T\big((V_A^R)\lambda_2\big)\\
&+\varepsilon_{19}^{-1}\big(\big((iU_A^I)^*P_1\big)\lambda_7\big)^*\big(\big((iU_A^I)^*P_1\big)\lambda_7\big)+\varepsilon_{19}\big((V_A^I)\lambda_2\big)^T\big((V_A^I)\lambda_2\big)\\
&+\varepsilon_{20}^{-1}\big(\big((jU_A^J)^*P_1\big)\lambda_7\big)^*\big(\big((jU_A^J)^*P_1\big)\lambda_7\big)+\varepsilon_{20}\big((V_A^J)\lambda_2\big)^T\big((V_A^J)\lambda_2\big)\\
&+\varepsilon_{21}^{-1}\big(\big((kU_A^K)^*P_1\big)\lambda_7\big)^*\big(\big((kU_A^K)^*P_1\big)\lambda_7\big)+\varepsilon_{21}\big((V_A^K)\lambda_2\big)^T\big((V_A^K)\lambda_2\big)\\
&+\varepsilon_{22}^{-1}\big(\big((U_B^R)^TP_1\big)\lambda_7\big)^T\big(\big((U_B^R)^TP_1\big)\lambda_7\big)+\varepsilon_{22}\big((V_B^R)\lambda_3\big)^T\big((V_B^R)\lambda_3\big)\\
&+\varepsilon_{23}^{-1}\big(\big((iU_B^I)^*P_1\big)\lambda_7\big)^*\big(\big((iU_B^I)^*P_1\big)\lambda_7\big)+\varepsilon_{23}\big((V_B^I)\lambda_3\big)^T\big((V_B^I)\lambda_3\big)\\
&+\varepsilon_{24}^{-1}\big(\big((jU_B^J)^*P_1\big)\lambda_7\big)^*\big(\big((jU_B^J)^*P_1\big)\lambda_7\big)+\varepsilon_{24}\big((V_B^J)\lambda_3\big)^T\big((V_B^J)\lambda_3\big)\\
&+\varepsilon_{25}^{-1}\big(\big((kU_B^K)^*P_1\big)\lambda_7\big)^*\big(\big((kU_B^K)^*P_1\big)\lambda_7\big)+\varepsilon_{25}\big((V_B^K)\lambda_3\big)^T\big((V_B^K)\lambda_3\big)\\
&+\varepsilon_{26}^{-1}\big(\big((U_C^R)^TP_1\big)\lambda_7\big)^T\big(\big((U_C^R)^TP_1\big)\lambda_7\big)+\varepsilon_{26}\big((V_C^R)\lambda_4\big)^T\big((V_C^R)\lambda_4\big)\\
&+\varepsilon_{27}^{-1}\big(\big((iU_C^I)^*P_1\big)\lambda_7\big)^*\big(\big((iU_C^I)^*P_1\big)\lambda_7\big)+\varepsilon_{27}\big((V_C^I)\lambda_4\big)^T\big((V_C^I)\lambda_4\big)\\
&+\varepsilon_{28}^{-1}\big(\big((jU_C^J)^*P_1\big)\lambda_7\big)^*\big(\big((jU_C^J)^*P_1\big)\lambda_7\big)+\varepsilon_{28}\big((V_C^J)\lambda_4\big)^T\big((V_C^J)\lambda_4\big)\\
&+\varepsilon_{29}^{-1}\big(\big((kU_C^K)^*P_1\big)\lambda_7\big)^*\big(\big((kU_C^K)^*P_1\big)\lambda_7\big)+\varepsilon_{29}\big((V_C^K)\lambda_4\big)^T\big((V_C^K)\lambda_4\big)\\
&+\varepsilon_{30}^{-1}\big((-U_D^TP_1)\lambda_5\big)^T\big((-U_D^TP_1)\lambda_5\big)+\varepsilon_{30}\big((V_D)\lambda_7\big)^T\big((V_D)\lambda_7\big)\\
&+\varepsilon_{31}^{-1}\big((-U_D^TP_1)\lambda_6\big)^T\big((-U_D^TP_1)\lambda_6\big)+\varepsilon_{31}\big((V_D)\lambda_7\big)^T\big((V_D)\lambda_7\big)\\
&+\varepsilon_{32}^{-1}\big((-U_D^TP_1D_0)\lambda_5\big)^T\big((-U_D^TP_1D_0)\lambda_5\big)+\varepsilon_{32}\big((V_D)\lambda_6\big)^T\big((V_D)\lambda_6\big)\\
&+\varepsilon_{33}^{-1}\big((-U_D^TP_1D_0)\lambda_6\big)^T\big((-U_D^TP_1D_0)\lambda_6\big)+\varepsilon_{33}\big((V_D)\lambda_5\big)^T\big((V_D)\lambda_5\big)\\
&+\varepsilon_{34}^{-1}\big((-U_D^TP_1U_D\Sigma_DV_D)\lambda_5\big)^T\big((-U_D^TP_1U_D\Sigma_DV_D)\lambda_5\big)+\varepsilon_{34}\big((V_D)\lambda_6\big)^T\big((V_D)\lambda_6\big)\\
&<0. \qquad (9)
\end{aligned}
$$

Applying Lemma 3, we obtain that (9) is equivalent to (3), and we know $\bar{M} = S_{11} - S_{12}S_{22}^{-1}S_{21}$, also $\Xi = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. It is clear that $S_{22} < 0$, where $(S_{11})_{7n\times 7n}$, $(S_{12})_{7n\times 34n}$, $(S_{21})_{34n\times 7n}$,

$(S_{22})_{34n\times 34n}$ are the partitioned matrices of the matrix Ξ. Therefore, the proof of this theorem is turned into the proof that model (1) has the global robust stability of the unique equilibrium point if $\bar{M} < 0$.

Step 2: Under the condition $M < 0$, we prove that model (1) has a unique equilibrium point. Let $\check{q}$ be an equilibrium point of the system (1), and $\check{q}$ satisfies

$$-D\check{q} + Ag(\check{q}) + Bg(\check{q}) + Cg(\check{q}) + J = 0. \tag{10}$$

Let $G(q) = -Dq + Ag(q) + Bg(q) + Cg(q) + J$.

In the following, we prove that map G is a homeomorphism, where $G : \mathbb{H}^n \to \mathbb{H}^n$.

On the one hand, we prove that $G(q)$ is an injective map on $\mathbb{H}^n$.

Assuming there are $q_1, q_2 \in \mathbb{H}^n$ with $q_1 \neq q_2$ such that $G(q_1) = G(q_2)$, we can get

$$-D(q_1 - q_2) + A\bigl(g(q_1) - g(q_2)\bigr) + B\bigl(g(q_1) - g(q_2)\bigr) + C\bigl(g(q_1) - g(q_2)\bigr) = 0. \tag{11}$$

Left-multiplying both sides of (11) by $\frac{9}{4}(q_1 - q_2)^* P_1$, we get

$$\begin{aligned} 0 = {}& -(q_1 - q_2)^* \frac{9}{4} P_1 D(q_1 - q_2) + (q_1 - q_2)^* \frac{9}{4} P_1 A\bigl(g(q_1) - g(q_2)\bigr) \\ & + (q_1 - q_2)^* \frac{9}{4} P_1 B\bigl(g(q_1) - g(q_2)\bigr) + (q_1 - q_2)^* \frac{9}{4} P_1 C\bigl(g(q_1) - g(q_2)\bigr). \end{aligned} \tag{12}$$

By taking the conjugate transpose on both sides of (12) leads to

$$\begin{aligned} 0 = {}& -(q_1 - q_2)^* \frac{9}{4} D^* P_1 (q_1 - q_2) + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} A^* P_1 (q_1 - q_2) \\ & + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} B^* P_1 (q_1 - q_2) \\ & + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} C^* P_1 (q_1 - q_2). \end{aligned} \tag{13}$$

Adding (12) and (13) brings about

$$\begin{aligned} 0 = {}& -(q_1 - q_2)^* \biggl(-\frac{9}{4} P_1 D - \frac{9}{4} D^* P_1\biggr)(q_1 - q_2) \\ & + (q_1 - q_2)^* \frac{9}{4} P_1 A + \bigl(g(q_1) - g(q_2)\bigr) + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} A^* P_1 (q_1 - q_2) \\ & + (q_1 - q_2)^* \frac{9}{4} P_1 B + \bigl(g(q_1) - g(q_2)\bigr) + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} B^* P_1 (q_1 - q_2) \\ & + (q_1 - q_2)^* \frac{9}{4} P_1 C + \bigl(g(q_1) - g(q_2)\bigr) \\ & + \bigl(g(q_1) - g(q_2)\bigr)^* \frac{9}{4} C^* P_1 (q_1 - q_2). \end{aligned} \tag{14}$$

From $M < 0$, we know

$$P_4 - P_2 < 0, \qquad -P_4 < 0, \qquad -\bigl(P_2 + \delta^2 P_3\bigr) < 0, \tag{15}$$

$$\begin{pmatrix} -P_1D-D^TP_1+P_2+\delta^2P_3+L(R+2P_2)L & \frac{3}{2}P_1A & \frac{3}{2}P_1B & \frac{3}{2}P_1C \\ \star & P_4-2P_2 & 0 & 0 \\ \star & \star & -P_4 & 0 \\ \star & \star & \star & -R \end{pmatrix} < 0. \tag{16}$$

It follows from (15) and (16) that

$$\begin{pmatrix} -P_1D-D^TP_1+L(R+2P_2)L & \frac{3}{2}P_1A & \frac{3}{2}P_1B & \frac{3}{2}P_1C \\ \star & P_4-2P_2 & 0 & 0 \\ \star & \star & -P_4 & 0 \\ \star & \star & \star & -R \end{pmatrix} < 0. \tag{17}$$

By Lemma 3, we have

$$\begin{aligned} &-P_1D-D^TP_1+L(2P_2+R)L+P_1A\frac{9}{4}(2P_2-P_4)^{-1}A^*P_1 \\ &\quad+P_1B\frac{9}{4}P_4^{-1}B^*P_1+P_1C\frac{9}{4}R^{-1}C^*P_1<0. \end{aligned} \tag{18}$$

Using Lemma 4 and Assumption ($\mathbf{A}_1$), for the positive definiteness of $2P_2-P_4$, P_4 and R, we have from (14) that

$$\begin{aligned} 0 &\le (q_1-q_2)^*\bigg[-\frac{9}{4}P_1D-\frac{9}{4}D^*P_1+P_1A\frac{81}{16}(2P_2-P_4)^{-1}A^*P_1 \\ &\quad+P_1B\frac{81}{16}P_4^{-1}B^*P_1+P_1C\frac{81}{16}R^{-1}C^*P_1\bigg](q_1-q_2) \\ &\quad+\big(g(q_1)-g(q_2)\big)^*(2P_2+R)\big(g(q_1)-g(q_2)\big) \\ &\le (q_1-q_2)^*\bigg[-\frac{9}{4}P_1D-\frac{9}{4}D^*P_1+P_1A\frac{81}{16}(2P_2-P_4)^{-1}A^*P_1 \\ &\quad+P_1B\frac{81}{16}P_4^{-1}B^*P_1+P_1C\frac{81}{16}R^{-1}C^*P_1+L(2P_2+R)L\bigg](q_1-q_2) \\ &= (q_1-q_2)^*\bigg[-P_1D-D^*P_1+P_1A\frac{9}{4}(2P_2-P_4)^{-1}A^*P_1 \\ &\quad+P_1B\frac{9}{4}P_4^{-1}B^*P_1+P_1C\frac{9}{4}R^{-1}C^*P_1+\frac{4}{9}L(2P_2+R)L\bigg](q_1-q_2). \end{aligned} \tag{19}$$

Since $L(R+2P_2)L>0$ and from (18), we have

$$\begin{aligned} &-P_1D-D^*P_1+\frac{4}{9}L(2P_2+R)L+P_1A\frac{9}{4}(2P_2-P_4)^{-1}A^*P_1 \\ &\quad+P_1B\frac{9}{4}P_4^{-1}B^*P_1+P_1C\frac{9}{4}R^{-1}C^*P_1<0. \end{aligned} \tag{20}$$

From (19) and (20), we get $q_1=q_2$, which contradicts our assumption. Therefore, $G(q)$ is an injective map on $\mathbb{H}^n$.

On the other hand, we prove that $\|G(q)\|\to+\infty$ as $\|q\|\to+\infty$. Indeed,

$$G(q)-G(0)=-Dq+A\big(g(q)-g(0)\big)+B\big(g(q)-g(0)\big)+C\big(g(q)-g(0)\big). \tag{21}$$

Left-multiplying by $\frac{9}{4}q^*P_1$ both sides of (21) leads to

$$\frac{9}{4}q^*P_1\big(G(q)-G(0)\big) = -\frac{9}{4}q^*P_1Dq + \frac{9}{4}q^*P_1A\big(g(q)-g(0)\big) + \frac{9}{4}q^*P_2B\big(g(q)-g(0)\big) + \frac{9}{4}q^*P_2C\big(g(q)-g(0)\big). \tag{22}$$

Taking the conjugate transpose of equality (22), we obtain that

$$\frac{9}{4}\big(G(q)-G(0)\big)^*P_1q = -\frac{9}{4}q^*D^*P_1q + \big(g(q)-g(0)\big)^*A^*P_1 + \frac{9}{4}\big(g(q)-g(0)\big)^*B^*P_1q + \frac{9}{4}\big(g(q)-g(0)\big)^*C^*P_1q. \tag{23}$$

Summing (22) and (23) leads to

$$\begin{aligned} &q^*P_1\big(G(q)-G(0)\big) + \big(G(q)-G(0)\big)^*P_1q \\ &\quad= q^*\left(-\frac{9}{4}P_1D - \frac{9}{4}D^*P_1\right)q \\ &\qquad+ q^*\frac{9}{4}P_1A\big(g(q)-g(0)\big) + \big(g(q)-g(0)\big)^*\frac{9}{4}A^*P_1q \\ &\qquad+ q^*\frac{9}{4}P_1B\big(g(q)-g(0)\big) + \big(g(q)-g(0)\big)^*\frac{9}{4}B^*P_1q \\ &\qquad+ q^*\frac{9}{4}P_1C\big(g(q)-g(0)\big) + \big(g(q)-g(0)\big)^*\frac{9}{4}C^*P_1q. \end{aligned}$$

Similar to proving the injectivity of the map, we can get

$$\begin{aligned} &q^*P_1\big(G(q)-G(0)\big) + \big(G(q)-G(0)\big)^*P_1q \\ &\quad\le q^*\bigg[-P_1D - D^*P_1 + \frac{9}{4}P_1A(2P_2-P_4)^{-1}A^*P_1 \\ &\qquad+ \frac{9}{4}P_1BP_4^{-1}B^*P_1 + 4P_1CR^{-1}C^*P_1 + \frac{4}{9}L(R+2P_2)L\bigg]q \\ &\quad\le -\lambda_{\min}(-\Theta)\|q\|^2, \end{aligned}$$

where

$$\begin{aligned} \Theta = {}& -P_1D - D^*P_1 + \frac{9}{4}P_1A(2P_2-P_4)^{-1}A^*P_1 + \frac{9}{4}P_1BP_4^{-1}B^*P_1 \\ &+ \frac{9}{4}P_1CR^{-1}C^*P_1 + \frac{4}{9}L(R+2P_2)L < 0. \end{aligned}$$

According to Cauchy–Schwarz inequality,

$$\begin{aligned} \lambda_{\min}(-\Theta)\|q\|^2 &\le -q^*P_1\big(G(q)-G(0)\big) + \big(G(q)-G(0)\big)^*P_1q \\ &= -2\operatorname{Re}\big(q^*P_1\big(G(q)-G(0)\big)\big) \\ &\le 2\big|q^*P_1\big(G(q)-G(0)\big)\big| \end{aligned}$$

$$\leq 2\|q\| \cdot \|P_1\| \cdot \big\|G(q) - G(0)\big\|$$
$$\leq 2\|q\| \cdot \|P_2\| \cdot \big(\big\|G(q)\big\| + \big\|G(0)\big\|\big). \tag{24}$$

Therefore, $\|G(q)\| \to +\infty$ as $\|q\| \to +\infty$, so we know that $G(q)$ is a homeomorphic map on $\mathbb{H}^n$ by Lemma 5. Thus, system (1) has a unique equilibrium point.

Step 3: We prove that the equilibrium point enjoys global asymptotical robust stability.

From the previous proof, we know that system (1) has a unique equilibrium point $\check{q}$. For convenience letting $\tilde{q(t)} = q(t) - \check{q}$, system (1) can be rewritten as

$$\begin{cases} \dot{\tilde{q}}(t) = -D\tilde{q}(t-\delta) + Af(\tilde{q}(t)) + Bf(\tilde{q}(t-\tau)) \\ \qquad + C\int_{-\infty}^{t} K(t-s)f(\tilde{q}(s))\,ds, \quad t > 0, t \neq t_k. \\ \Delta\tilde{q} = M_k(\tilde{q}(t_k^-)), \quad k = 1, 2, \ldots, \end{cases} \tag{25}$$

where $f(\tilde{q}(t)) = g(q(t)) - g(\check{q})$, $f(\tilde{q}(t-\tau)) = g(q(t-\tau)) - g(\check{q})$. In the meantime, the initial condition (2) can be transformed into

$$\tilde{q}(s) = \tilde{\vartheta}(s), \quad s \in [-\rho, 0], \tag{26}$$

where $\tilde{\vartheta}(s) = \vartheta(s) - \check{q} \in C([-\rho, 0], \mathbb{H}^n)$.

Consider the following Lyapunov function:

$$V\big(\tilde{q}(t)\big) = V_1\big(\tilde{q}(t)\big) + V_2\big(\tilde{q}(t)\big) + V_3\big(\tilde{q}(t)\big) + V_4\big(\tilde{q}(t)\big) + V_5\big(\tilde{q}(t)\big) + V_6\big(\tilde{q}(t)\big),$$

where

$$V_1\big(\tilde{q}(t)\big) = \left(\tilde{q}(t) - D\int_{t-\delta}^{t} \tilde{q}(s)\,ds\right)^* P_1\left(\tilde{q}(t) - D\int_{t-\delta}^{t} \tilde{q}(s)\,ds\right), \tag{27}$$

$$V_2\big(\tilde{q}(t)\big) = \int_{t-\delta}^{t} \tilde{q}^*(s)P_2\tilde{q}(s)\,ds, \tag{28}$$

$$V_3\big(\tilde{q}(t)\big) = \delta\int_{0}^{\delta}\int_{t-u}^{t} \tilde{q}^*(s)P_3\tilde{q}(s)\,ds\,du, \tag{29}$$

$$V_4\big(\tilde{q}(t)\big) = \int_{t-\tau}^{t} f^*\big(\tilde{q}(s)\big)P_4 f\big(\tilde{q}(s)\big)\,ds, \tag{30}$$

$$V_5\big(\tilde{q}(t)\big) = \sum_{j=1}^{n} r_j \int_{0}^{\infty} K(s) \int_{t-s}^{t} f_j^*\big(\tilde{q}_j(t)\big) f_j\big(\tilde{q}_j(t)\big)\,dt\,ds, \tag{31}$$

$$V_6\big(\tilde{q}(t)\big) = \frac{1}{2}\tilde{q}^*(t)P_1\tilde{q}(t), \tag{32}$$

where r_j is the principal diagonal element of R and $R = \operatorname{diag}(r_1, r_2, \ldots, r_n)$.

When $t \neq t_k$, taking the time derivative of $V_1(\tilde{q}(t))$, $V_2(\tilde{q}(t))$, $V_3(\tilde{q}(t))$, $V_4(\tilde{q}(t))$, $V_5(\tilde{q}(t))$, and $V_6(\tilde{q}(t))$, we can get

$$\dot{V}_1\big(\tilde{q}(t)\big) = \big(\tilde{q}(t)\big) - D\int_{t-\delta}^{t} \tilde{q}(s)\,ds)^* P_1\big(\dot{\tilde{q}}(t) - D\tilde{q}(t) + D\tilde{q}(t-\delta)\big)$$
$$+ \left(\dot{\tilde{q}}(t) - D\tilde{q}(t) + D\tilde{q}(t-\delta)^* P_1\big(\tilde{q}(t)\big) - D\int_{t-\delta}^{t} \tilde{q}(s)\,ds\right)$$

$$
\begin{aligned}
&= \big(\tilde{q}(t)\big) - D\int_{t-\delta}^{t}\tilde{q}(s)\,ds)^{*}P_1\Big(Af\big(\tilde{q}(t)\big) \\
&\quad + Bf\big(\tilde{q}(t-\tau)\big) + C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds \\
&\quad - D\tilde{q}(t)\Big) + \Big(Af\big(\tilde{q}(t)\big) + Bf\big(\tilde{q}(t-\tau)\big) + C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds \\
&\quad - D\tilde{q}(t)\Big)^{*}P_1\Big(\tilde{q}(t) - D\int_{t-\delta}^{t}\tilde{q}(s)\,ds\Big) \\
&= \tilde{q}^{*}(t)P_1Af\big(\tilde{q}(t)\big) + \tilde{q}^{*}(t)P_1Bf\big(\tilde{q}(t-\tau)\big) \\
&\quad + \tilde{q}^{*}(t)P_1C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds - \tilde{q}^{*}(t)P_1D\tilde{q}(t) \\
&\quad + f^{*}\big(\tilde{q}(t)\big)A^{*}P_1\tilde{q}(t) + f^{*}\big(\tilde{q}(t-\tau)\big)B^{*}P_1\tilde{q}(t) \\
&\quad + \Big(\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\Big)^{*}C^{*}P_1\tilde{q}(t) - \tilde{q}^{*}(t)D^{T}P_1\tilde{q}(t) \\
&\quad - \Big(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\Big)^{*}D^{T}P_1Af\big(\tilde{q}(t)\big) \\
&\quad - \Big(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\Big)^{*}D^{T}P_1Bf\big(\tilde{q}(t-\tau)\big) \\
&\quad - \Big(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\Big)^{*}D^{T}P_1C\int_{t}^{+\infty}K(t-s)f\big(\tilde{q}(s)\big)\,ds \\
&\quad + \Big(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\Big)^{*}D^{T}P_1D\tilde{q}(t) - f^{*}\big(\tilde{q}(t)\big)A^{*}P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds \\
&\quad - f^{*}\big(\tilde{q}(t-\tau)\big)B^{*}P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds \\
&\quad - \Big(\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\Big)^{*}C^{*}P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds \\
&\quad + \tilde{q}^{*}(t)D^{T}P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds,
\end{aligned}
\tag{33}
$$

$$
\dot{V}_2\big(\tilde{q}(t)\big) = \tilde{q}(t)^{*}(t)P_2\tilde{q}(t) - \tilde{q}^{*}(t-\delta)P_2\tilde{q}(t-\delta), \tag{34}
$$

$$
\begin{aligned}
\dot{V}_3\big(\tilde{q}(t)\big) &= \delta^2\tilde{q}(t)^{*}(t)P_3\tilde{q}(t) - \delta\int_{0}^{\delta}\tilde{q}^{*}(t-u)P_3\tilde{q}(t-u)\,du \\
&= \delta^2\tilde{q}(t)^{*}(t)P_3\tilde{q}(t) - \delta\int_{t-\delta}^{t}\tilde{q}^{*}(s)P_3\tilde{q}(s)\,ds.
\end{aligned}
$$

Using Lemma 5, we have

$$
\dot{V}_3\big(\tilde{q}(t)\big) \le \delta^2\tilde{q}(t)^{*}(t)P_3\tilde{q}(t) - \Big(\int_{t-\delta}^{t}\tilde{q}^{*}(s)\,ds\Big)^{*}P_3\Big(\int_{t-\delta}^{t}\tilde{q}^{*}(s)\,ds\Big), \tag{35}
$$

$$
\dot{V}_4\big(\tilde{q}(t)\big) = f^{*}\big(\tilde{q}(t)\big)P_4f\big(\tilde{q}(t)\big) - f^{*}\big(\tilde{q}(t-\tau)\big)P_4f\big(\tilde{q}(t-\tau)\big), \tag{36}
$$

$$
\begin{aligned}
\dot{V}_5\big(\tilde{q}(t)\big) &= \sum_{j=1}^{n} r_j\int_{0}^{\infty}K(s)f_j^{*}\big(\tilde{q}_j(t)\big)f_j\big(\tilde{q}_j(t)\big)\,ds \\
&\quad - \sum_{j=1}^{n} r_j\int_{0}^{\infty}K(s)f_j^{*}\big(\tilde{q}_j(t-s)\big)f_j\big(\tilde{q}_j(t-s)\big)\,ds.
\end{aligned}
$$

Magnifying the equation for $\dot{V}_5(\tilde{q}(t))$ by using assumption ($\mathbf{A}_1$) yields

$$\begin{aligned}\dot{V}_5(\tilde{q}(t)) &\le \tilde{q}^*(t)LRL\tilde{q}(t)\\&\quad-\sum_{j=1}^{n} r_j\int_0^{\infty}K(s)f_j^*(\tilde{q}_j(t-s))\,ds\int_0^{\infty}K(s)f_j(\tilde{q}_j(t-s))\,ds\\&\le \tilde{q}^*(t)LRL\tilde{q}(t)\\&\quad-\left(\int_{-\infty}^{t}K(s)f_j(\tilde{q}_j(t-s))\,ds\right)^*R\left(\int_{-\infty}^{t}K(s)f_j(\tilde{q}_j(t-s))\,ds\right),\end{aligned}\tag{37}$$

$$\begin{aligned}\dot{V}_6(\tilde{q}(t)) &= \frac{1}{2}\big[\dot{\tilde{q}}^*(t)P_1\tilde{q}(t)+\tilde{q}^*(t)P_1\dot{\tilde{q}}(t)\big]\\&=\frac{1}{2}\bigg[\bigg(-D\tilde{q}(t-\delta)+Af(\check{q}(t))+Bf(\tilde{q}(t-\tau))\\&\quad+C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\bigg)^*P_1\tilde{q}(t)+\tilde{q}^*(t)P_1\bigg(-D\tilde{q}(t-\delta)\\&\quad+Af(\check{q}(t))+Bf(\tilde{q}(t-\tau))+C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\bigg)\bigg]\\&=\frac{1}{2}\bigg[-\tilde{q}^*(t-\delta)D^TP_1\tilde{q}(t)+f^*(\check{q}(t))A^*P_1\tilde{q}(t)\\&\quad+f^*(\check{q}(t-\tau))B^*P_1\tilde{q}(t)+\left(\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\right)^*C^*P_1\tilde{q}(t)\\&\quad-\tilde{q}^*(t)P_1D\tilde{q}(t-\delta)+\tilde{q}^*(t)P_1Af(\tilde{q}(t))\\&\quad+\tilde{q}^*(t)P_1Bf(\tilde{q}(t-\tau))+\tilde{q}^*(t)P_1C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\bigg].\end{aligned}\tag{38}$$

In addition, for the real-valued diagonal matrix $2P_2$, using assumption ($\mathbf{A}_1$), we can get

$$0\le\tilde{q}^*(t)L2P_2L\tilde{q}(t)-f^*(\tilde{q}(t))2P_2f(\tilde{q}(t)).$$

From equation (25), we have

$$\begin{aligned}0 &= \left(P_1\dot{\tilde{q}}(t)+P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^*\bigg(-\dot{\tilde{q}}(t)-D\tilde{q}(t-\delta)+Af(\check{q}(t))\\&\quad+Bf(\tilde{q}(t-\tau))+C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\bigg)\\&\quad+\bigg(-\dot{\tilde{q}}(t)-D\tilde{q}(t-\delta)+Af(\check{q}(t))+Bf(\tilde{q}(t-\tau))\\&\quad+C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds\bigg)^*\left(P_1\dot{\tilde{q}}(t)+P_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)\\&=\dot{\tilde{q}}^*(t)P_1\dot{\tilde{q}}(t)-\dot{\tilde{q}}^*(t)P_1D\tilde{q}(t-\delta)+\dot{\tilde{q}}^*(t)P_1Af(\check{q}(t))+\dot{\tilde{q}}^*(t)P_1Bf(\tilde{q}(t-\tau))\\&\quad+\dot{\tilde{q}}^*(t)P_1C\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds-\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^*D^TP_1\dot{\tilde{q}}(t)\\&\quad-\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^*D^TP_1D\tilde{q}(t-\delta)+\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^*D^TP_1Af(\check{q}(t))\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^{*}D^{T}P_{1}Bf\big(\tilde{q}(t-\tau)\big)\\
&+\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^{*}D^{T}P_{1}C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\\
&-\dot{\tilde{q}}^{*}(t)P_{1}\dot{\tilde{q}}(t)-\dot{\tilde{q}}(t-\delta)D^{T}P_{1}\dot{\tilde{q}}(t)\\
&+f^{*}\big(\check{q}(t)\big)A^{*}P_{1}\dot{\tilde{q}}(t)+f^{*}\big(\tilde{q}(t-\tau)\big)B^{*}P_{1}\dot{\tilde{q}}(t)\\
&+\left(\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\right)^{*}C^{*}P_{1}\dot{\tilde{q}}(t)\\
&-\dot{\tilde{q}}^{*}(t)P_{1}D\int_{t-\delta}^{t}\tilde{q}(s)\,ds-\tilde{q}^{*}(t-\delta)D^{T}P_{1}D\int_{t-\delta}^{t}\tilde{q}(s)\,ds\\
&+f^{*}\big(\check{q}(t)\big)A^{*}P_{1}\int_{t-\delta}^{t}\tilde{q}(s)\,ds+f^{*}\big(\tilde{q}(t-\tau)\big)B^{*}P_{1}\int_{t-\delta}^{t}\tilde{q}(s)\,ds\\
&+\left(\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\right)^{*}C^{*}P_{1}D\int_{t-\delta}^{t}\tilde{q}(s)\,ds. \qquad (39)
\end{aligned}
$$

So from (33)–(39) it follows that

$$
\begin{aligned}
\dot{V}\big(\tilde{q}(t)\big)\le{}&\tilde{q}^{*}(t)\frac{3}{2}P_{1}Af\big(\tilde{q}(t)\big)+\tilde{q}^{*}(t)\frac{3}{2}P_{1}Bf\big(\tilde{q}(t-\tau)\big)\\
&+\tilde{q}^{*}(t)\frac{3}{2}P_{1}C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds-\tilde{q}^{*}(t)P_{1}D\tilde{q}(t)\\
&+f^{*}\big(\tilde{q}(t)\big)\frac{3}{2}A^{*}P_{1}\tilde{q}(t)+f^{*}\big(\tilde{q}(t-\tau)\big)\frac{3}{2}B^{*}P_{1}\tilde{q}(t)\\
&+\left(\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\right)^{*}\frac{3}{2}C^{*}P_{1}\tilde{q}(t)-\tilde{q}^{*}(t)D^{T}P_{1}\tilde{q}(t)\\
&+\left(\int_{t-\delta}^{t}\tilde{q}(t)\,ds\right)^{*}D^{T}P_{1}D\tilde{q}(t)+\tilde{q}^{*}(t)D^{T}P_{1}D\int_{t-\delta}^{t}\tilde{q}(s)\,ds\\
&+\tilde{q}(t)^{*}(t)P_{2}\tilde{q}(t)-\tilde{q}^{*}(t-\delta)P_{2}\tilde{q}(t-\delta)\\
&+\delta^{2}\tilde{q}(t)^{*}(t)P_{3}\tilde{q}(t)-\left(\int_{t-\delta}^{t}\tilde{q}^{*}(s)\,ds\right)^{*}P_{3}\left(\int_{t-\delta}^{t}\tilde{q}^{*}(s)\,ds\right)\\
&+f^{*}\big(\tilde{q}(t)\big)P_{4}f\big(\tilde{q}(t)\big)-f^{*}\big(\tilde{q}(t-\tau)\big)P_{4}f\big(\tilde{q}(t-\tau)\big)\\
&+\tilde{q}^{*}(t)LRL\tilde{q}(t)-\frac{1}{2}\tilde{q}^{*}(t-\delta)D^{T}P_{1}\tilde{q}(t)\\
&-\left(\int_{-\infty}^{t}K(s)f_{j}\big(\tilde{q}_{j}(t-s)\big)\,ds\right)^{*}R\left(\int_{-\infty}^{t}K(s)f_{j}\big(\tilde{q}_{j}(t-s)\big)\,ds\right)\\
&-\frac{1}{2}\tilde{q}^{*}(t)P_{1}D\tilde{q}(t-\delta)-\dot{\tilde{q}}^{*}(t)P_{1}\dot{\tilde{q}}(t)-\dot{\tilde{q}}^{*}(t)P_{1}D\tilde{q}(t-\delta)\\
&+\dot{\tilde{q}}^{*}(t)P_{1}Af\big(\check{q}(t)\big)+\dot{\tilde{q}}^{*}(t)P_{1}Bf\big(\tilde{q}(t-\tau)\big)\\
&+\dot{\tilde{q}}^{*}(t)P_{1}C\int_{-\infty}^{t}K(t-s)f\big(\tilde{q}(s)\big)\,ds\\
&-\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^{*}D^{T}P_{1}\dot{\tilde{q}}(t)-\left(\int_{t-\delta}^{t}\tilde{q}(s)\,ds\right)^{*}D^{T}P_{1}D\tilde{q}(t-\delta)\\
&-\dot{\tilde{q}}^{*}(t)P_{1}\dot{\tilde{q}}(t)-\dot{\tilde{q}}(t-\delta)D^{T}P_{1}\dot{\tilde{q}}(t)-\dot{\tilde{q}}^{*}(t)P_{1}D\int_{t-\delta}^{t}\tilde{q}(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+f^*\big(\tilde{q}(t)\big)A^*P_1\int_{t-\delta}^{t}\tilde{q}(s)\,ds+f^*\big(\tilde{q}(t-\tau)\big)B^*P_1\int_{t-\delta}^{t}\tilde{q}(s)\,ds\\
&-\tilde{q}^*(t-\delta)D^TP_1D\int_{t-\delta}^{t}\tilde{q}(s)\,ds+\tilde{q}^*(t)L2P_2L\tilde{q}(t)\\
&-f^*\big(\tilde{q}(t)\big)2P_2f\big(\tilde{q}(t)\big)\\
=\ &\xi^*\big(\tilde{q}(t)\big)M\xi\big(\tilde{q}(t)\big),
\end{aligned}
\tag{40}
$$

where $\xi^*(\tilde{q}(t)) = [\tilde{q}^*(t), f^*(\tilde{q}(t)), f^*(\tilde{q}(t-\tau)), (\int_{-\infty}^{t}K(t-s)f(\tilde{q}(s))\,ds)^*, (\int_{t-\delta}^{t}\tilde{q}(s)\,ds)^*, (\tilde{q}^*(t-\delta)), \dot{\tilde{q}}^*(t)]$.

Since $M<0$, we know

$$
\dot{V}\big(\tilde{q}(t)\big)\le\xi^*\big(\tilde{q}(t)\big)M\xi\big(\tilde{q}(t)\big)\le 0.
\tag{41}
$$

When $t=t_k$, $k=1,2,\dots$, we define the impulsive function M_k as follows:

$$
M_k\big(q(t_k^-),q_{t_k^-}\big)=E_k\bigg[q\big(t_k^-\big)-\check{q}-D\int_{t_k-\delta}^{t_k}\big(q(s)-\check{q}\big)\,ds\bigg],
$$

where $k=1,2,\dots,E_k\in\mathbb{H}^{n\times n}$. Meanwhile, we let

$$
\begin{pmatrix} P_1 & (I+E_k)^*P_1\\ \star & P_1\end{pmatrix}>0,
$$

and then can compute

$$
\begin{aligned}
\begin{pmatrix} P_1 & (I+E_k)^*P_1\\ \star & P_1\end{pmatrix}>0
\quad&\Leftrightarrow\quad
\begin{pmatrix} I & 0_1\\ 0 & P_1^{-1}\end{pmatrix}
\begin{pmatrix} P_1 & (I+E_k)^*P_1\\ \star & P_1\end{pmatrix}
\begin{pmatrix} I & 0\\ 0 & P_1^{-1}\end{pmatrix}>0\\
&\Leftrightarrow\quad
\begin{pmatrix} P_1 & (I+E_k)^*\\ \star & P_1^{-1}\end{pmatrix}>0\\
&\Leftrightarrow\quad P_1-(I+E_k)^*P_1(I+E_k)>0,
\end{aligned}
\tag{42}
$$

in which the last equivalence comes from Lemma 3. Thus, it yields

$$
\begin{aligned}
V_1\big(\tilde{q}(t_k)\big)=\ &\bigg(\tilde{q}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)^*P_1\bigg(\tilde{q}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)\\
=\ &\bigg(\tilde{q}\big(t_k^-\big)+E_k\bigg(\tilde{q}\big(t_k^-\big)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)\\
&-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)^*\\
&+P_1\bigg(\tilde{q}\big(t_k^-\big)+E_k\bigg(\tilde{q}\big(t_k^-\big)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)\\
=\ &\bigg(\tilde{q}\big(t_k^-\big)-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)^*(I+E_k)^*P_1(I+E_k)\bigg(\tilde{q}\big(t_k^-\big)\\
&-D\int_{t_k-\delta}^{t_k}\tilde{q}(s)\,ds\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\tilde{q}(t_k^-) - D\int_{t_k-\delta}^{t_k} \tilde{q}(s)\,ds\right)^* P_1 \left(\tilde{q}(t_k^-) - D\int_{t_k-\delta}^{t_k} \tilde{q}(s)\,ds\right) \\
&= V_1\big(\tilde{q}(t_k^-)\big).
\end{aligned}
\tag{43}
$$

Hence, we can infer that

$$
V\big(\tilde{q}(t_k)\big) \leq V\big(\tilde{q}(t_k^-)\big), \quad k = 1, 2, \ldots. \tag{44}
$$

It follows from (41)–(44) that $V(\tilde{q}(t))$ is nonincreasing for $t \geq 0$. On the basis of assumption ($\mathbf{A}_3$), letting $\int_0^{+\infty} sK(s)\,ds = \beta$, where β is a nonnegative real constant, then, from the definition of $V(\tilde{q}(t))$, by assumption ($\mathbf{A}_1$), together with Lemmas 4 and 5, we can infer

$$
\begin{aligned}
V\big(\tilde{q}(t)\big) &\leq V\big(\tilde{q}(0)\big) = \sum_{i=1}^{6} V_i\big(\tilde{q}(0)\big) \\
&= \left(\tilde{q}(0) - D\int_{-\delta}^{0} \tilde{q}(s)\,ds\right)^* P_1 \left(\tilde{q}(0) - D\int_{-\delta}^{0} \tilde{q}(s)\,ds\right) \\
&\quad + \int_{-\delta}^{0} \tilde{q}^*(s) P_2 \tilde{q}(s)\,ds + \delta \int_0^{\delta}\int_{-u}^{0} \tilde{q}^*(s) P_3 \tilde{q}(s)\,ds\,du \\
&\quad + \int_{-\tau}^{0} f^*\big(\tilde{q}(s)\big) P_4 f\big(\tilde{q}(s)\big)\,ds \\
&\quad + \int_{-\tau}^{0} \sum_{j=1}^{n} r_j \int_0^{\infty} K(s) \int_{-s}^{0} f_j^*\big(\tilde{q}_j(t)\big) f_j\big(\tilde{q}_j(t)\big)\,dt\,ds + \frac{1}{2}\tilde{q}^*(0) P_1 \tilde{q}(0) \\
&= \tilde{q}^*(0) P_1 \tilde{q}(0) - \tilde{q}^*(0) P_1 D \int_{-\delta}^{0} \tilde{q}(s)\,ds - \left(\int_{-\delta}^{0} \tilde{q}(s)\,ds\right)^* D^T P_1 \tilde{q}(0) \\
&\quad + \left(\int_{-\delta}^{0} \tilde{q}(s)\,ds\right)^* D^T P_1 D \int_{-\delta}^{0} \tilde{q}(s)\,ds + \int_{-\delta}^{0} \tilde{q}^*(s) P_2 \tilde{q}(s)\,ds \\
&\quad + \delta \int_0^{\delta}\int_{-u}^{0} \tilde{q}^*(s) P_3 \tilde{q}(s)\,ds\,du + \int_{-\tau}^{0} f^*\big(\tilde{q}(s)\big) P_4 f\big(\tilde{q}(s)\big)\,ds \\
&\quad + \sum_{j=1}^{n} r_j \int_0^{\infty} K(s) \int_{-s}^{0} f_j^*\big(\tilde{q}_j(t)\big) f_j\big(\tilde{q}_j(t)\big)\,dt\,ds + \frac{1}{2}\tilde{q}^*(0) P_1 \tilde{q}(0) \\
&\leq \Big(2\lambda_{\max}(P_1)\big(1 + \delta^2\big) + \delta\lambda_{\max}(P_2) + \frac{1}{2}\delta^3\lambda_{\max}(P_3) \\
&\quad + \tau\lambda_{\max}\big(L^2\big)\lambda_{\max}(P_4) + \beta\lambda_{\max}\big(L^2\big)\lambda_{\max}(R) \\
&\quad + \frac{1}{2}\lambda_{\max}(P_1)\Big) \big\|\tilde{\vartheta}(t)\big\|^2.
\end{aligned}
\tag{45}
$$

Furthermore, by the definition of $V(\tilde{q}(t))$, we know

$$
V\big(\tilde{q}(t)\big) \geq V_6\big(\tilde{q}(t)\big) \geq \lambda_{\min}(P_1)\big\|\tilde{q}(t)\big\|^2. \tag{46}
$$

According to (45) and (46), we obtain

$$
\big\|\tilde{q}(t)\big\| \leq \sqrt{\frac{U}{\lambda_{\min}(P_1)}}\big\|\tilde{\vartheta}(t)\big\|,
$$

where $U = 2\lambda_{\max}(P_1)(1 + \delta^2) + \delta\lambda_{\max}(P_2) + \frac{1}{2}\delta^3\lambda_{\max}(P_3) + \tau\lambda_{\max}(L^2)\lambda_{\max}(P_4) + \beta\lambda_{\max}(L^2)\lambda_{\max}(R) + \frac{1}{2}\lambda_{\max}(P_1)$.

Taking advantage of Lyapunov theory, we show that the equilibrium point $\check{q}$ of system (1) is globally robust stable.

The proof is completed. □

Remark 1 Note that RVNNs and CVNNs are special cases of QVNNs. Therefore, RVNNs and CVNNs can also be considered in the results of this paper in the form of (1).

As LMI (3) is quaternion-valued, we should transform the quaternion LMI into complex one in order that it can be directly handled via the Matlab LMI toolbox. Hence, if we express the parameters as pairs of complex parts, i.e., $A_0^R + \imath A_0^R + \jmath A_0^J + \kappa A_0^K = A_0^R + \imath A_0^R + (A_0^J + \imath A_0^K)\jmath = A_1 + A_{2\jmath}$, $B_0^R + \imath B_0^R + \jmath B_0^J + \kappa B_0^K = B_0^R + \imath B_0^R + (B_0^J + \imath B_0^K)\jmath = B_1 + B_{2\jmath}$, and $C_0^R + \imath C_0^R + \jmath C_0^J + \kappa C_0^K = C_0^R + \imath C_0^R + (C_0^J + \imath C_0^K)\jmath = C_1 + C_{2\jmath}$, where $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{C}^{n\times n}$, we can get the following result from Lemmas 7 and 8.

Corollary 1 *Suppose assumptions* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, *and* $(\mathbf{A}_3)$ *are satisfied. If there exist a positive definite Hermitian matrix* $P_{11} \in \mathbb{C}^{n\times n}$, *skew-symmetric matrix* $P_{12} \in \mathbb{C}^{n\times n}$, *real positive diagonal matrices* $P_i \in \mathbb{R}^{n\times n}$ $(i = 2,3,4)$ *and R, positive constants* λ_i $(i = 1,2,\dots,27)$, *such that following CVLMIs hold*:

$$\begin{pmatrix} P_{11} & -P_{12} \\ \bar{P}_{12} & \bar{P}_1 \end{pmatrix} > 0, \qquad \begin{pmatrix} \Xi_1 & -\Xi_2 \\ \bar{\Xi}_2 & \bar{\Xi}_1 \end{pmatrix} < 0, \tag{47}$$

where Ξ_1 *and* Ξ_2 *are defined in* (48) *and* (49),

$$\Xi_1 = (\Xi_{ij})_{41\times 41} < 0, \tag{48}$$

where

$$\begin{aligned}
\Xi_{11} &= -P_{11}D_0 - D_0^T P_{11} + P_2 + \delta^2 P_3 + L(2P_2 + R)L + \varepsilon_1(V_D)^T V_D \\
&\quad + \varepsilon_{15}(V_D)^T V_D, \qquad \Xi_{12} = \frac{3}{2}(P_{11}A_1 - P_{12}\bar{A}_2), \\
\Xi_{13} &= \frac{3}{2}(P_{11}B_1 - P_{12}\bar{B}_2), \qquad \Xi_{14} = \frac{3}{2}(P_{11}C_1 - P_{12}\bar{C}_2), \\
\Xi_{15} &= (D_0)^T P_{11}D_0, \qquad \Xi_{16} = -\frac{1}{2}P_{11}D_0, \qquad \Xi_{18} = P_{11}N_D, \qquad \Xi_{19} = \frac{3}{2}P_{11}N_A^R, \\
\Xi_{1.10} &= \frac{3}{2}P_{11}N_A^I, \qquad \Xi_{1.11} = \frac{3}{2}P_{11}N_A^J, \qquad \Xi_{1.12} = \frac{3}{2}P_{11}N_A^K, \qquad \Xi_{1.13} = \frac{3}{2}P_{11}N_B^R, \\
\Xi_{1.14} &= \frac{3}{2}P_{11}N_B^I, \qquad \Xi_{1.15} = \frac{3}{2}P_{11}N_B^J, \qquad \Xi_{1.16} = \frac{3}{2}P_{11}N_B^K, \qquad \Xi_{1.17} = \frac{3}{2}P_{11}N_C^R, \\
\Xi_{1.18} &= \frac{3}{2}P_{11}N_C^I, \qquad \Xi_{1.19} = \frac{3}{2}P_{11}N_C^J, \qquad \Xi_{1.20} = \frac{3}{2}P_{11}N_C^K, \qquad \Xi_{1.21} = D_0^T P_{11}N_D, \\
\Xi_{1.23} &= (U_D\Upsilon_D V_D)^T P_{11}N_D, \qquad \Xi_{1.24} = \frac{1}{2}P_{11}N_D, \qquad \Xi_{21} = \frac{3}{2}A_1^* P_{11}^* - A_2^T P_{12}^*, \\
\Xi_{22} &= P_4 - 2P_2 + \varepsilon_2\big(V_A^R\big)^T V_A^R + \varepsilon_3\big(V_A^I\big)^T V_A^I + \varepsilon_4\big(V_A^J\big)^T V_A^J + \varepsilon_5\big(V_A^K\big)^T V_A^K
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon_{18}(V_A^R)^T V_A^R + \varepsilon_{19}(V_A^I)^T V_A^I + \varepsilon_{20}(V_A^J)^T V_A^J + \varepsilon_{21}(V_A^K)^T V_A^K, \\
&\Xi_{27} = A_1^* P_{11} + A_2^T \bar{P}_{12}, \qquad \Xi_{31} = \frac{3}{2} B_1^* P_{11}^* - B_2^T P_{12}^*, \\
&\Xi_{33} = -P_4 + \varepsilon_6(V_B^R)^T V_B^R + \varepsilon_7(V_B^I)^T V_B^I + \varepsilon_8(V_B^J)^T V_B^J + \varepsilon_9(V_B^K)^T V_B^K \\
&\quad + \varepsilon_{22}(V_B^R)^T V_B^R + \varepsilon_{23}(V_B^I)^T V_B^I + \varepsilon_{24}(V_B^J)^T V_B^J + \varepsilon_{25}(V_B^K)^T V_B^K, \\
&\Xi_{37} = B_1^* P_{11} + B_2^T \bar{P}_{12}, \qquad \Xi_{41} = \frac{3}{2} C_1^* P_{11}^* - C_2^T P_{12}^*, \\
&\Xi_{44} = -R + \varepsilon_{10}(V_C^R)^T V_C^R + \varepsilon_{11}(V_C^I)^T V_C^I + \varepsilon_{12}(V_C^J)^T V_C^J + \varepsilon_{13}(V_C^K)^T V_C^K \\
&\quad + \varepsilon_{26}(V_C^R)^T V_C^R + \varepsilon_{27}(V_C^I)^T V_C^I + \varepsilon_{28}(V_C^J)^T V_C^J + \varepsilon_{29}(V_C^K)^T V_C^K, \\
&\Xi_{47} = C_1^* P_{11} + C_2^T \bar{P}_{12}, \qquad \Xi_{51} = (D_0)^T P_{11} D_0, \\
&\Xi_{55} = -P_3 + \varepsilon_{14}(V_D)^T V_D + \varepsilon_{16}(V_D)^T V_D + \varepsilon_{33}(V_D)^T V_D, \qquad \Xi_{56} = -(D_0)^T P_1 D_0, \\
&\Xi_{57} = -(D_0)^T P_{11}, \qquad \Xi_{5.22} = D_0^T P_{11} N_D, \qquad \Xi_{5.37} = P_{11} N_D, \qquad \Xi_{5.39} = D_0^T P_{11} N_D, \\
&\Xi_{5.41} = (U_D \Upsilon_D V_D)^T P_{11} N_D, \qquad \Xi_{61} = -\frac{1}{2} D_0^T P_{11}, \qquad \Xi_{65} = -(D_0)^T P_{11} D_0, \\
&\Xi_{66} = -P_2 + \varepsilon_{17}(V_D)^T V_D + \varepsilon_{32}(V_D)^T V_D + \varepsilon_{34}(V_D)^T V_D, \qquad \Xi_{67} = -(D_0)^T P_{11}, \\
&\Xi_{6.38} = P_{11} N_D, \qquad \Xi_{6.40} = D_0^T P_{11} N_D, \qquad \Xi_{72} = P_{11}^* A_1 + P_{12}^T \bar{A}_2, \\
&\Xi_{73} = P_{11}^* B_1 + P_{12}^T \bar{B}_2, \qquad \Xi_{74} = P_{11}^* C_1 + P_{12}^T \bar{C}_2, \qquad \Xi_{75} = -P_{11} D_0, \\
&\Xi_{76} = -P_{11} D_0, \qquad \Xi_{77} = -P_{11} - P_{11} + \varepsilon_{30}(V_D)^T V_D + \varepsilon_{31}(V_D)^T V_D, \\
&\Xi_{7.25} = P_{11} N_A^R, \qquad \Xi_{7.26} = P_{11} N_A^I, \qquad \Xi_{7.27} = P_{11} N_A^J, \qquad \Xi_{7.28} = P_{11} N_A^K, \\
&\Xi_{7.29} = P_{11} N_B^R, \qquad \Xi_{7.30} = P_{11} N_B^I, \qquad \Xi_{7.31} = P_{11} N_B^J, \qquad \Xi_{7.32} = P_{11} N_B^K, \\
&\Xi_{7.33} = P_{11} N_C^R, \qquad \Xi_{7.34} = P_{11} N_C^I, \qquad \Xi_{7.35} = P_{11} N_C^J, \qquad \Xi_{7.36} = P_{11} N_C^K, \\
&\Xi_{81} = (N_D)^T P_{11}, \qquad \Xi_{88} = -\varepsilon_1 I, \qquad \Xi_{91} = \frac{3}{2}(N_A^R)^T P_{11}, \\
&\Xi_{99} = -\varepsilon_2 I, \qquad \Xi_{10.1} = \frac{3}{2}(N_A^I)^T P_{11}, \qquad \Xi_{10.10} = -\varepsilon_3 I, \\
&\Xi_{11.1} = \frac{3}{2}(N_A^J)^T P_{11}, \qquad \Xi_{11.11} = -\varepsilon_4 I, \qquad \Xi_{12.1} = \frac{3}{2}(N_A^K)^T P_{11}, \qquad \Xi_{12.12} = -\varepsilon_5 I, \\
&\Xi_{13.1} = \frac{3}{2}(N_B^R)^T P_{11}, \qquad \Xi_{13.13} = -\varepsilon_6 I, \qquad \Xi_{14.1} = \frac{3}{2}(N_B^I)^T P_{11}, \qquad \Xi_{14.14} = -\varepsilon_7 I, \\
&\Xi_{15.1} = \frac{3}{2}(N_B^J)^T P_{11}, \qquad \Xi_{15.15} = -\varepsilon_8 I, \qquad \Xi_{16.1} = \frac{3}{2}(N_B^K)^T P_{11}, \qquad \Xi_{16.16} = -\varepsilon_9 I, \\
&\Xi_{17.1} = \frac{3}{2}(N_C^R)^T P_{11}, \qquad \Xi_{17.17} = -\varepsilon_{10} I, \qquad \Xi_{18.1} = \frac{3}{2}(N_C^I)^T P_{11}, \qquad \Xi_{18.18} = -\varepsilon_{11} I, \\
&\Xi_{19.1} = \frac{3}{2}(N_C^J)^T P_{11}, \qquad \Xi_{19.19} = -\varepsilon_{12} I, \qquad \Xi_{20.1} = \frac{3}{2}(N_C^K)^T P_{11}, \qquad \Xi_{20.20} = -\varepsilon_{13} I, \\
&\Xi_{21.1} = (N_D)^T P_{11} D_0, \qquad \Xi_{21.21} = -\varepsilon_{14} I, \qquad \Xi_{22.5} = (N_D)^T P_{11} D_0, \qquad \Xi_{22.22} = -\varepsilon_{15} I, \\
&\Xi_{23.1} = (N_D)^T P_{11} U_D \Upsilon_D V_D, \qquad \Xi_{23.23} = -\varepsilon_{16} I, \qquad \Xi_{24.1} = \frac{1}{2}(N_D)^T P_{11}, \\
&\Xi_{24.24} = -\varepsilon_{17} I, \qquad \Xi_{25.7} = (N_A^R)^T P_{11}, \qquad \Xi_{25.25} = -\varepsilon_{18} I, \qquad \Xi_{26.7} = (N_A^I)^T P_{11},
\end{aligned}
$$

$$\Xi_{26.26} = -\varepsilon_{19}I, \quad \Xi_{27.7} = \left(N_A^J\right)^T P_{11}, \quad \Xi_{27.27} = -\varepsilon_{20}I, \quad \Xi_{28.7} = \left(N_A^K\right)^T P_{11},$$
$$\Xi_{28.28} = -\varepsilon_{21}I, \quad \Xi_{29.7} = \left(N_B^R\right)^T P_{11}, \quad \Xi_{29.29} = -\varepsilon_{22}I, \quad \Xi_{30.7} = \left(N_B^I\right)^T P_{11},$$
$$\Xi_{30.30} = -\varepsilon_{23}I, \quad \Xi_{31.7} = \left(N_B^J\right)^T P_{11}, \quad \Xi_{31.31} = -\varepsilon_{24}I, \quad \Xi_{32.7} = \left(N_B^K\right)^T P_{11},$$
$$\Xi_{32.32} = -\varepsilon_{25}I, \quad \Xi_{33.7} = \left(N_C^R\right)^T P_{11}, \quad \Xi_{33.33} = -\varepsilon_{26}I, \quad \Xi_{34.7} = \left(N_C^I\right)^T P_{11},$$
$$\Xi_{34.34} = -\varepsilon_{27}I, \quad \Xi_{35.7} = \left(N_C^J\right)^T P_{11}, \quad \Xi_{35.35} = -\varepsilon_{28}I, \quad \Xi_{36.7} = \left(N_C^K\right)^T P_{11},$$
$$\Xi_{36.36} = -\varepsilon_{29}I, \quad \Xi_{37.5} = (N_D)^T P_{11}, \quad \Xi_{37.37} = -\varepsilon_{30}I, \quad \Xi_{38.6} = (N_D)^T P_{11},$$
$$\Xi_{38.38} = -\varepsilon_{31}I, \quad \Xi_{39.5} = (N_D)^T P_{11} D_0, \quad \Xi_{39.39} = -\varepsilon_{32}I, \quad \Xi_{40.6} = (N_D)^T P_{11} D_0,$$
$$\Xi_{40.40} = -\varepsilon_{33}I, \quad \Xi_{41.5} = (N_D)^T P_{11} U_D \Upsilon_D V_D, \quad \Xi_{41.41} = -\varepsilon_{34}I, \quad \textit{and}$$
$$\Xi_2 = (\Xi_{ij})_{41\times 41} < 0, \tag{49}$$

where

$$\Xi_{11} = -P_{12}D_0 - D_0^T P_{12}, \quad \Xi_{12} = \frac{3}{2}(Q_{11}A_2 + P_{12}\bar{A}_1), \quad \Xi_{13} = \frac{3}{2}(Q_{11}B_2 + P_{12}\bar{B}_1),$$
$$\Xi_{14} = \frac{3}{2}(Q_{11}C_2 + P_{12}\bar{C}_1), \quad \Xi_{15} = (D_0)^T P_{12} D_0, \quad \Xi_{16} = -\frac{1}{2}P_{12}D_0,$$
$$\Xi_{18} = P_{12}N_D, \quad \Xi_{19} = \frac{3}{2}P_{12}N_A^R, \quad \Xi_{1.10} = \frac{3}{2}P_{12}N_A^I, \quad \Xi_{1,11} = \frac{3}{2}P_{12}N_A^J,$$
$$\Xi_{1.12} = \frac{3}{2}P_{12}N_A^K, \quad \Xi_{1.13} = \frac{3}{2}P_{12}N_B^R, \quad \Xi_{1.14} = \frac{3}{2}P_{12}N_B^I, \quad \Xi_{1.15} = \frac{3}{2}P_{12}N_B^J,$$
$$\Xi_{1.16} = \frac{3}{2}P_{12}N_B^K, \quad \Xi_{1.17} = \frac{3}{2}P_{12}N_C^R, \quad \Xi_{1.18} = \frac{3}{2}P_{12}N_C^I, \quad \Xi_{1.19} = \frac{3}{2}P_{12}N_C^J,$$
$$\Xi_{1.20} = \frac{3}{2}P_{12}N_C^K, \quad \Xi_{1.21} = D_0^T P_{12}N_D, \quad \Xi_{1.23} = (U_D\Upsilon_D V_D)^T P_{12}N_D,$$
$$\Xi_{1.24} = \frac{1}{2}P_{12}N_D, \quad \Xi_{21} = -\frac{3}{2}A_2^T P_{11}^T - A_1^* P_{12}^T, \quad \Xi_{27} = A_1^* P_{12} - A_2^T \bar{P}_{11},$$
$$\Xi_{31} = -\frac{3}{2}B_2^T P_{11}^T - B_1^T P_{12}^T, \quad \Xi_{37} = B_1^* P_{11} + B_2^T \bar{P}_{12}, \quad \Xi_{41} = -\frac{3}{2}C_2^T P_{11}^T - C_1^T P_{12}^T,$$
$$\Xi_{47} = C_1^* P_{11} + C_2^T \bar{P}_{12}, \quad \Xi_{51} = (D_0)^T P_{12} D_0, \quad \Xi_{56} = -(D_0)^T P_{12} D_0,$$
$$\Xi_{57} = -(D_0)^T P_{12}, \quad \Xi_{5.22} = D_0^T P_{12} N_D, \quad \Xi_{5.37} = P_{12} N_D,$$
$$\Xi_{5.39} = D_0^T P_{12} N_D, \quad \Xi_{5.41} = (U_D\Upsilon_D V_D)^T P_{12} N_D, \quad \Xi_{61} = -\frac{1}{2}D_0^T P_{12},$$
$$\Xi_{65} = -(D_0)^T P_{12} D_0, \quad \Xi_{67} = -(D_0)^T P_{12}, \quad \Xi_{6.38} = P_{12}N_D, \quad \Xi_{6.40} = D_0^T P_{12} N_D,$$
$$\Xi_{72} = P_{11}^* A_1 + P_{12}^T \bar{A}_2, \quad \Xi_{73} = P_{11}^* B_1 + P_{12}^T \bar{B}_2, \quad \Xi_{74} = P_{11}^* C_1 + P_{12}^T \bar{C}_2,$$
$$\Xi_{75} = -P_{11}D_0, \quad \Xi_{76} = -P_{11}D_0, \quad \Xi_{77} = -P_{12} - P_{12}, \quad \Xi_{7.25} = P_{12}N_A^R,$$
$$\Xi_{7.26} = P_{12}N_A^I, \quad \Xi_{7.27} = P_{12}N_A^J, \quad \Xi_{7.28} = P_{12}N_A^K, \quad \Xi_{7.29} = P_{12}N_B^R,$$
$$\Xi_{7.30} = P_{12}N_B^I, \quad \Xi_{7.31} = P_{12}N_B^J, \quad \Xi_{7.32} = P_{12}N_B^K, \quad \Xi_{7.33} = P_{12}N_C^R,$$
$$\Xi_{7.34} = P_{12}N_C^I, \quad \Xi_{7.35} = P_{12}N_C^J, \quad \Xi_{7.36} = P_{12}N_C^K, \quad \Xi_{81} = -(N_D)^T P_{12},$$
$$\Xi_{91} = -\frac{3}{2}\left(N_A^R\right)^T P_{12}, \quad \Xi_{10.1} = -\frac{3}{2}\left(N_A^I\right)^T P_{12}, \quad \Xi_{11.1} = -\frac{3}{2}\left(N_A^J\right)^T P_{12},$$

$$\Xi_{12.1} = -\frac{3}{2}\bigl(N_A^K\bigr)^T P_{12}, \qquad \Xi_{13.1} = -\frac{3}{2}\bigl(N_B^R\bigr)^T P_{12}, \qquad \Xi_{14.1} = -\frac{3}{2}\bigl(N_B^I\bigr)^T P_{12},$$
$$\Xi_{15.1} = -\frac{3}{2}\bigl(N_B^J\bigr)^T P_{12}, \qquad \Xi_{16.1} = -\frac{3}{2}\bigl(N_B^K\bigr)^T P_{12}, \qquad \Xi_{17.1} = -\frac{3}{2}\bigl(N_C^R\bigr)^T P_{12},$$
$$\Xi_{18.1} = -\frac{3}{2}\bigl(N_C^I\bigr)^T P_{12}, \qquad \Xi_{19.1} = -\frac{3}{2}\bigl(N_C^J\bigr)^T P_{12}, \qquad \Xi_{20.1} = -\frac{3}{2}\bigl(N_C^K\bigr)^T P_{12},$$
$$\Xi_{21.1} = -(N_D)^T P_{12} D_0, \qquad \Xi_{22.5} = -(N_D)^T P_{12} D_0, \qquad \Xi_{23.1} = -(N_D)^T P_{12} U_D \Upsilon_D V_D,$$
$$\Xi_{24.1} = -\frac{1}{2}(N_D)^T P_{12}, \qquad \Xi_{25.7} = -\bigl(N_A^R\bigr)^T P_{12}, \qquad \Xi_{26.7} = -\bigl(N_A^I\bigr)^T P_{12},$$
$$\Xi_{27.7} = -\bigl(N_A^J\bigr)^T P_{12}, \qquad \Xi_{28.7} = -\bigl(N_A^K\bigr)^T P_{12}, \qquad \Xi_{29.7} = -\bigl(N_B^R\bigr)^T P_{12},$$
$$\Xi_{30.7} = -\bigl(N_B^I\bigr)^T P_{12}, \qquad \Xi_{31.7} = -\bigl(N_B^J\bigr)^T P_{12}, \qquad \Xi_{32.7} = -\bigl(N_B^K\bigr)^T P_{12},$$
$$\Xi_{33.7} = -\bigl(N_C^R\bigr)^T P_{12}, \qquad \Xi_{34.7} = -\bigl(N_C^I\bigr)^T P_{12}, \qquad \Xi_{35.7} = -\bigl(N_C^J\bigr)^T P_{12},$$
$$\Xi_{36.7} = -\bigl(N_C^K\bigr)^T P_{12}, \qquad \Xi_{37.5} = -(N_D)^T P_{12}, \qquad \Xi_{38.6} = -(N_D)^T P_{12},$$
$$\Xi_{39.5} = -(N_D)^T P_{12} D_0, \qquad \Xi_{40.6} = -(N_D)^T P_{12} D_0, \qquad \Xi_{41.5} = -(N_D)^T P_{12} U_D \Upsilon_D V_D,$$

and the other entries in Ξ_1 *and* Ξ_2 *are zeros.*

Proof According to Lemmas 7 and 8, as well as Theorem 1, we can easily have the result. □

3 Numerical example

In this section, we illustrate the validity of our results with the following example.

Example 1 Take the parameters of QVNN as follows:

$$\check{D} = \begin{pmatrix} 0.029 & 0 \\ 0 & 0.0305 \end{pmatrix}, \qquad \hat{D} = \begin{pmatrix} 0.032 & 0 \\ 0 & 0.0307 \end{pmatrix},$$
$$L = \begin{pmatrix} 0.005 & 0 \\ 0 & 0.005 \end{pmatrix}, \qquad \delta = 0.05, \qquad \tau = 0.1,$$
$$E_k = \begin{pmatrix} -0.3 + 0.2i & 0 \\ 0.1 - 0.6i & -0.2 + 0.2i \end{pmatrix}, \quad k \in 1, 2, \ldots,$$
$$\check{A} = (\check{a}_{ij})_{2\times 2}, \qquad \check{A} = (\check{a}_{ij})_{2\times 2}, \qquad \check{B} = (\check{b}_{ij})_{2\times 2},$$
$$\check{B} = (\check{b}_{ij})_{2\times 2}, \qquad \check{C} = (\check{c}_{ij})_{2\times 2}, \qquad \check{C} = (\check{c}_{ij})_{2\times 2},$$

where

$$\check{a}_{11} = -0.002 - 0.002\imath + 0.002\jmath + 0\kappa,$$
$$\check{a}_{12} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$
$$\check{a}_{21} = 0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$
$$\check{a}_{22} = -0.0015 - 0.002\imath - 0.002\jmath - 0.002\kappa,$$
$$\hat{a}_{11} = 0.001 + 0.001\imath + 0.001\jmath + 0.001\kappa, \qquad \hat{a}_{12} = 0 + 0\imath + 0\jmath + 0\kappa,$$

$$\hat{a}_{21} = 0.0015 + 0\imath + 0\jmath + 0\kappa,$$

$$\hat{a}_{22} = 0.001 + 0.001\imath + 0.001\jmath + 0.001\kappa,$$

$$\check{b}_{11} = -0.002 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\check{b}_{12} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\check{b}_{21} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\check{b}_{22} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\hat{b}_{11} = 0.001 + 0.001\imath + 0.001\jmath + 0.001\kappa, \qquad \hat{b}_{12} = 0 + 0\imath + 0\jmath + 0\kappa,$$

$$\hat{b}_{21} = 0 + 0\imath + 0\jmath + 0\kappa, \qquad \hat{b}_{22} = 0.001 + 0.001\imath + 0.001\jmath + 0.001\kappa,$$

$$\check{c}_{11} = 0.001 - 0.001\imath - 0.0015\jmath - 0.001\kappa,$$

$$\check{c}_{12} = -0.001 - 0.001\imath - 0.0014\jmath - 0.001\kappa,$$

$$\check{c}_{21} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\check{c}_{22} = -0.001 - 0.001\imath - 0.001\jmath - 0.001\kappa,$$

$$\hat{c}_{11} = 0.0015 + 0.0034\imath + 0.001\jmath + 0.0015\kappa, \qquad \hat{c}_{12} = 0 + 0.0012\imath + 0\jmath + 0\kappa,$$

$$\hat{c}_{21} = 0 + 0.0041\imath + 0.001\jmath + 0.0015\kappa, \qquad \hat{c}_{22} = 0.0012 + 0.001\imath + 0\jmath + 0.002\kappa.$$

In addition, we take activation and delay kernel functions as follows:

$$f_1(q) = f_2(q) = \big(|q+1| - |q-1|\big) \times 0.1, \quad \forall q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}.$$

$$K_1(s) = K_2(s) = e^{-s}, \quad s \in [0, +\infty).$$

Using MATLAB tools, we can get the results for the LMI (3) in Theorem 1:

$$P_1 = \begin{pmatrix} 44.3253 + 0.0000\imath & -1.0086 + 1.1303\imath \\ -1.0086 - 1.1303\imath & 46.9798 + 0.0000\imath \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1.2253 & 0 \\ 0 & 1.3223 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 7.7262 & 0 \\ 0 & 12.9308 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1.1407 & 0 \\ 0 & 1.1596 \end{pmatrix}, \qquad R = \begin{pmatrix} 79.5482 & 0 \\ 0 & 80.5706 \end{pmatrix},$$

$$\varepsilon_1 = 49.8632, \quad \varepsilon_2 = 112.5925, \quad \varepsilon_3 = 112.5168, \quad \varepsilon_4 = 112.5168,$$

$$\varepsilon_5 = 108.2435, \quad \varepsilon_6 = 112.5942, \quad \varepsilon_7 = 110.3803, \quad \varepsilon_8 = 110.3803,$$

$$\varepsilon_9 = 110.3803, \quad \varepsilon_{10} = 144.5979, \quad \varepsilon_{11} = 218.7847, \quad \varepsilon_{12} = 170.7464,$$

$$\varepsilon_{13} = 179.4117, \quad \varepsilon_{14} = 156.2501, \quad \varepsilon_{15} = 18.9679, \quad \varepsilon_{16} = 75.6929,$$

$$\varepsilon_{17} = 46.0796, \quad \varepsilon_{18} = 33.2054, \quad \varepsilon_{19} = 30.5856, \quad \varepsilon_{20} = 30.5856,$$

$$\varepsilon_{21} = 37.6041, \quad \varepsilon_{22} = 32.8740, \quad \varepsilon_{23} = 36.5195, \quad \varepsilon_{24} = 36.5195,$$

$$\varepsilon_{25} = 36.5195, \quad \varepsilon_{26} = 76.0829, \quad \varepsilon_{27} = 76.1625, \quad \varepsilon_{28} = 76.1027,$$

$$\varepsilon_{29} = 76.1166, \qquad \varepsilon_{30} = 76.1752, \qquad \varepsilon_{31} = 98.0675,$$
$$\varepsilon_{32} = 31.7173, \qquad \varepsilon_{33} = 75.7220, \qquad \varepsilon_{34} = 31.7172.$$

Obviously, it satisfies the conditions of Theorem 1, and there exists a unique equilibrium point which is globally robust and stable.

Next, we select the fixed neural network parameters:

$$D = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.0306 \end{pmatrix}, \qquad A = (a_{ij})_{2\times2}, \qquad B = (b_{ij})_{2\times2}, \qquad C = (c_{ij})_{2\times2},$$
$$J = \begin{pmatrix} 0.001 - 0.001\imath - 0.002\jmath + 0.005\kappa & 0 \\ 0 & -0.002 + 0.001\imath + 0\jmath - 0.001\kappa \end{pmatrix},$$

where

$$a_{11} = -0.001 - 0.001\imath + 0\jmath + 0.0005\kappa,$$
$$a_{12} = -0.001 - 0.0005\imath - 0.0007\jmath - 0.0005\kappa,$$
$$a_{21} = 0.01 - 0.0005\imath - 0.0005\jmath + 0\kappa, \qquad a_{22} = -0.001 + 0\imath + 0\jmath + 0\kappa,$$
$$b_{11} = -0.001 + 0.001\imath + 0\jmath + 0\kappa, \qquad b_{12} = 0 + 0\imath - 0.001\jmath - 0.001\kappa,$$
$$b_{21} = 0 - 0.0005\imath - 0.0005\jmath - 0.0005\kappa, \qquad b_{22} = 0.0005 + 0\imath + 0\jmath + 0\kappa,$$
$$c_{11} = 0.0012 + 0.002\imath + 0\jmath + 0.001\kappa,$$
$$c_{12} = -0.0005 - 0.0006\imath - 0.001\jmath - 0.001\kappa,$$
$$c_{21} = 0.0005 + 0.002\imath - 0\jmath - 0\kappa, \qquad c_{22} = 0.001 + 0\imath - 0.0005\jmath + 0.001\kappa.$$

By employing Quaternion Toolbox for Matlab and the fourth-order Runge–Kutta method, we perform numerical simulation of the network. Figures 1, 2, 3, and 4 depict the four parts of the states of the considered system, where the initial conditions are chosen by 10 random constant quaternion-valued vectors. It can be seen from these figures that each neuron state converges to the stable equilibrium point, which is $(0.0338 - 0.0322\imath - 0.0657\jmath + 0.1680\kappa, -0.0634 - 0.0323\imath - 0.0004\jmath - 0.0331\kappa)^T$.

4 Conclusions

In this paper, the issue of robust stability of pulse and delay QVNNs with interval parameter uncertainties is considered. A sufficient condition to guarantee the existence, uniqueness, and global robust stability of an equilibrium point has been deduced by using homomorphic mapping theorem, Lyapunov method, and inequality techniques. Finally, the effectiveness of the proposed theoretical condition is verified via a numerical example. It should be noted that the activation function is continuous in this article, however, the discontinuous case in QVNNs is our coming consideration, which is of great importance in real applications of everyday life.

Recently, some researchers investigated the Mittag-Leffler stability of multiple equilibrium points for fractional-order QVNNs with an impulsive term [33]. Other interesting

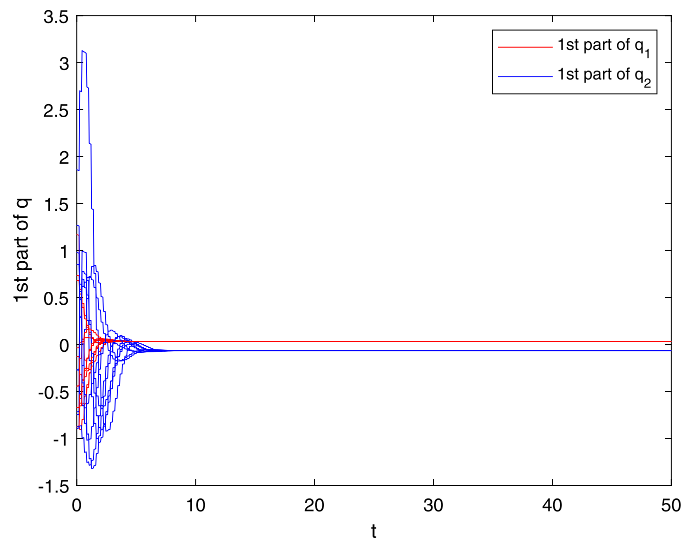

Figure 1 The first part of the state trajectories for system (1) with parameters

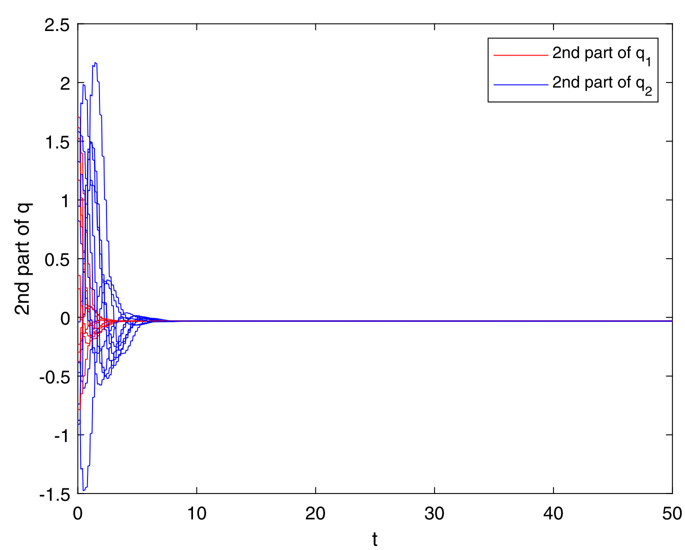

Figure 2 The second part of the state trajectories for system (1) with parameters

work involves the Hopf bifurcation of a fractional-order octonion-valued neural networks with time delay [34]. This important work is a substantial extension of traditional integer-order neural networks, which provide us a new way to extend our work. Note that, in references [33] and [34], to obtain the corresponding results, the authors transform their QVNNs into several RVNNs or CVNNs from research methods. In this article, we obtain the dynamical behaviors of QVNNs directly using the basic properties of QVNNs, instead of converting them into complex- or real-valued system, which avoids the increase of system dimension.

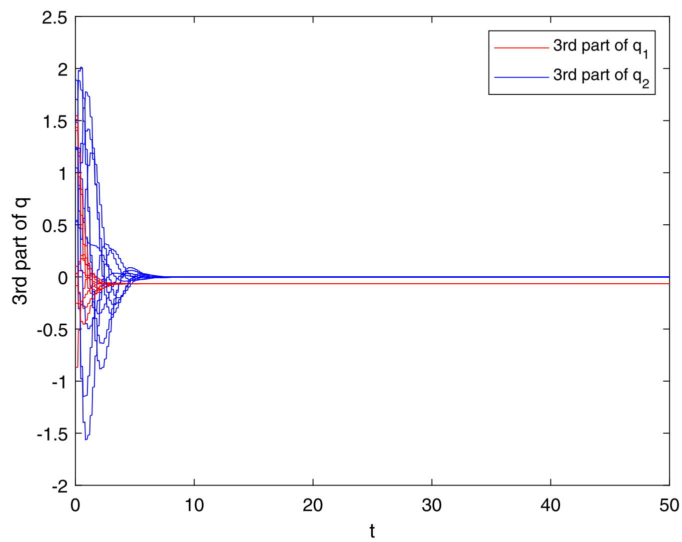

Figure 3 The third part of the state trajectories for system (1) with parameters

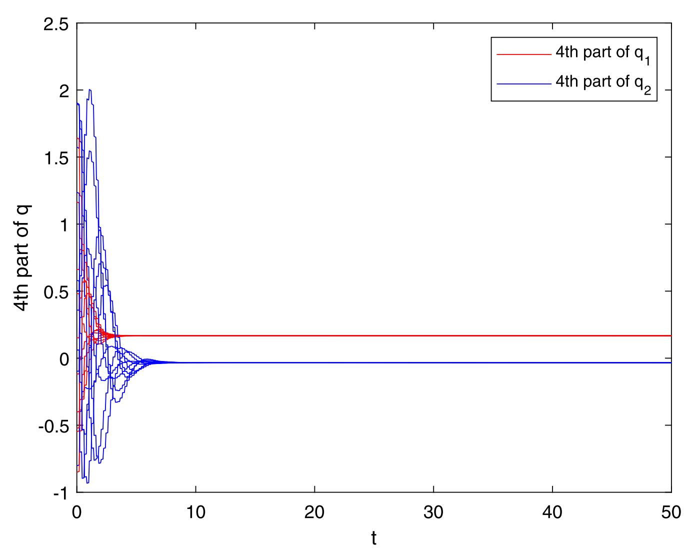

Figure 4 The fourth part of the state trajectories for system (1) with parameters

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Competing interests
The authors declare that they have no conflict of interest.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, read and approved the final manuscript.

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