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Trigonometric approximation of functions $f(x,y)$ of generalized Lipschitz class by double Hausdorff matrix summability method

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Abstract

In this paper, we establish a new estimate for the degree of approximation of functions $f(x,y)$ belonging to the generalized Lipschitz class $Lip((\xi_1,\xi_2);r)$, $r \geq 1$, by double Hausdorff matrix summability means of double Fourier series. We also deduce the degree of approximation of functions from $Lip((\alpha,\beta);r)$ and $Lip(\alpha,\beta)$ in the form of corollary. We establish some auxiliary results on trigonometric approximation for almost Euler means and (C,γ,δ) means.

Keywords: Double Hausdorff matrix summability; Double Fourier series; Generalized Lipschitz class; Modulus of continuity; Cesàro summability; Almost Euler summability means; Degree of approximation

1 Introduction

The study of various summability means of double Fourier series have been done by several authors, for example, Chow [2], Sharma [11], Łenski [6], and Ustina [15]. Dealing with the first arithmetic means of double Fourier series, Hasegawa [4] obtained the following:

Theorem A *If a continuous function $f(x,y)$ of period 2π with respect to both x and y belongs to $Lip(\alpha,\beta)$, where $0<\alpha<l$ and $0<\beta<1$, then*

$$\left|\sigma_{m,n}(x,y)-f(x,y)\right| = O\left(m^{-\alpha}+n^{-\beta}\right)$$

uniformly in (x,y) as m and n independently tend to infinity.

If $\alpha=\beta=1$, then

$$\left|\sigma_{m,n}(x,y)-f(x,y)\right| = O\left(m^{-1}\log m+n^{-1}\log n\right)$$

uniformly in (x,y) as m and n independently tend to infinity.

Siddiqui and Mohammadzadeh [12] investigated the approximation by Cesàro and B means of double Fourier series. Stepanets [13, 14] has established estimates of approximation for certain classes of periodic functions and differentiable periodic functions of two variables by linear methods of summation of their Fourier sums. Móricz and Shi [8] proved the following result for the approximation to continuous functions by Cesàro means of double Fourier series.

Theorem B *If* $f \in E(\alpha,\beta)$, $0 < \alpha, \beta \le 1$, $\gamma, \delta \ge 0$, *then*

$$\begin{aligned}\left\|\sigma_{mn}^{\gamma\delta}(f,x,y) - f(x,y)\right\| &= O\left(\frac{1}{(m+1)^{\alpha}} + \frac{1}{(n+1)^{\beta}}\right) \quad \textit{if } 0 < \alpha, \beta \le 1, \\ &= O\left(\frac{1}{(m+1)^{\alpha}} + \frac{\log(n+2)}{(n+1)}\right) \quad \textit{if } 0 < \alpha < \beta = 1, \\ &= O\left(\frac{\log(m+2)}{(m+1)} + \frac{\log(n+2)}{(n+1)}\right) \quad \textit{if } \alpha = \beta = 1.\end{aligned}$$

The degree of approximation using Gauss–Weierstrass integrals was also investigated by Khan and Ram [5]. Recently, error and bounds of certain bivariate functions by almost Euler means of double Fourier series for the functions of Lipschitz and Zygmund classes was estimated by Rathor and Singh [9]. To find the approximation of functions of two-dimensional torus, in this paper, we obtain a new estimate for trigonometric approximation of functions $f(x,y)$ of generalized Lipschitz class by double Hausdorff matrix summability method of double Fourier series. For other summability methods of approximation, see [1] and [7].

2 Definitions and preliminaries

Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}$ be double series with the sequence of (m,n)th partial sums

$$s_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} g_{j,k}.$$

A double Hausdorff matrix has the entries

$$h_{m,n}^{j,k} = \binom{m}{j}\binom{n}{k}\Delta_1^{m-j}\Delta_2^{n-k}\mu_{j,k},$$

where $\{\mu_{j,k}\}$ is any real or complex sequence, and

$$\Delta_1^{m-j}\Delta_2^{n-k}\mu_{j,k} = \sum_{w=0}^{m-j}\sum_{z=0}^{n-k}(-1)^{j+k}\binom{m-j}{w}\binom{n-k}{z}\mu_{j+w,k+z}.$$

If $t_{m,n}^{H} = \sum_{j=0}^{m}\sum_{k=0}^{n} h_{m,n}^{j,k} s_{j,k} \to g$ as $m \to \infty$ and $n \to \infty$, then $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}$ is said to be summable to the sum g by the double Hausdorff matrix summability method [15].

A necessary and sufficient condition for double Hausdorff matrix summability method to be regular is there exists a function $\chi(s,t) \in BV[0,1] \times [0,1]$ such that

$$\int_0^1\int_0^1 \left|d\chi(s,t)\right| < \infty$$

and

$$\mu_{m,n} = \int_0^1\int_0^1 s^m t^n \, d\chi(s,t),$$

where $\chi(s,0) = \chi(s,0^+) = \chi(0^+,t) = \chi(0,t) = 0$, $0 \le s, t \le 1$, and $\chi(1,1) - \chi(1,0) - \chi(0,1) + \chi(0,0) = 1$ [10].

It is easy to see that the absolute value of the measure $d\chi(s,t)$ can me majorized by $K_1K_2\,ds\,dt$ for some constants K_1 and K_2 (see [16]).

The important particular cases of double Hausdorff matrix summability means are as follows:

1 Almost Euler summability means ((E,q_1,q_2) means) if $\mu_{m,n} = \frac{1}{(1+q_1)^m}\frac{1}{(1+q_2)^n}$.
2 $(E,1,1)$ means if $q_1 = 1$ and $q_2 = 1$ in (E,q_1,q_2) means.
3 (C,γ,δ) means if $\mu_{m,n} = \frac{1}{A_m^\gamma}\frac{1}{A_n^\delta}$, where $\gamma,\delta \ge -1$ and $A_m^\gamma = \binom{\gamma+m}{m}$, $A_n^\delta = \binom{\delta+n}{n}$.
4 $(C,1,1)$ means if $\gamma = \delta = 1$ in (C,γ,δ) means.

Let $f(x,y)$ be a Lebesgue-integrable function of period 2π with respect to both variables x and y and summable in the fundamental square $Q:(-\pi,\pi)\times(-\pi,\pi)$. The double Fourier series of $f(x,y)$ is given by

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}[a_{m,n}\cos mx\cos ny + b_{m,n}\sin mx\cos ny + c_{m,n}\cos mx\sin ny + d_{m,n}\sin mx\cos ny] \tag{1}$$

with (m,n)th partial sums $s_{m,n}(f;(x,y))$, where

$$\lambda_{m,n} = \begin{cases} 1/4 & \text{for } m = n = 0, \\ 1/2 & \text{for } m > 0, n = 0 \text{ and } m = 0, n > 0, \\ 1 & \text{for } m > 0, n > 0, \end{cases}$$

$$a_{m,n} = \pi^{-2}\iint_Q f(x,y)\cos mx\cos ny\,dx\,dy,$$

and similar expressions for $b_{m,n}$, $c_{m,n}$, and $d_{m,n}$ [3].

We define the L^r norm by

$$\|f\|_r = \begin{cases} \{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}|f(x,y)|^r\,dx\,dy\}^{1/r}, & r \ge 1, \\ \operatorname{ess\,sup}_{0\le x,y\le 2\pi}|f(x,y)|, & r = \infty. \end{cases}$$

The degree of approximation of a function $f:\mathbb{R}^2\to\mathbb{R}$ by a trigonometric polynomial [17]

$$t_{m,n}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_{m,n}[a_{j,k}\cos mx\cos ny + b_{j,k}\sin mx\cos ny + c_{j,k}\cos mx\sin ny + d_{j,k}\sin mx\cos ny]$$

of order $(m+n)$ is defined by

$$E_{m,n}(f,L^r) = \min_{0\le x,y\le 2\pi}\|t_{m,n} - f\|_r.$$

A function $f:\mathbb{R}^2 \to \mathbb{R}$ of two variables x and y is said to belong to the class $Lip(\alpha,\beta)$ [4] if

$$\bigl|f(x+u,y+v)-f(x,y)\bigr| = O\bigl(|u|^\alpha + |v|^\beta\bigr), \quad 0<\alpha\le 1, 0<\beta\le 1,$$

to the class $Lip((\alpha,\beta);r)$ if

$$\left\{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\bigl|f(x+u,y+v)-f(x,y)\bigr|^r\,dx\,dy\right\}^{1/r} = O\bigl(|u|^\alpha + |v|^\beta\bigr), \quad r\ge 1,$$

and to the class $Lip((\xi_1,\xi_2);r)$ if

$$\left\{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\bigl|f(x+u,y+v)-f(x,y)\bigr|^r\,dx\,dy\right\}^{1/r} = O\bigl(\xi_1(u)+\xi_2(v)\bigr), \quad r\ge 1,$$

where ξ_1 and ξ_2 are moduli of continuity, that is, nonnegative nondecreasing continuous functions such that $\xi_1(0)=\xi_2(0)=0$, $\xi_1(u_1+u_2)\le \xi_1(u_1)+\xi_1(u_2)$, and $\xi_2(v_1+v_2)\le \xi_2(v_1)+\xi_2(v_2)$.

If $\xi_1(u)=u^\alpha$ and $\xi_2(v)=v^\beta$, $0<\alpha\le 1$, $0<\beta\le 1$, then the class $Lip((\xi_1,\xi_2);r)$ coincides with $Lip((\alpha,\beta);r)$. As $r\to\infty$, $Lip((\alpha,\beta);r)$ reduces to $Lip(\alpha,\beta)$. Clearly, $Lip(\alpha,\beta)\subseteq Lip((\alpha,\beta);r)\subseteq Lip((\xi_1,\xi_2);r)$.

We define the forward difference operator Δ as $\Delta\mu_k = \mu_k - \mu_{k+1}$; also, $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$, $k\ge 0$. We denote

$$\begin{aligned}
\phi(u,v) &= (1/4)\bigl[f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)\\
&\quad - 4f(x,y)\bigr],\\
M_m^H(u) &= \frac{K_1}{2\pi}\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\,ds\frac{\sin(j+\frac{1}{2})u}{\sin\frac{u}{2}},\\
K_n^H(v) &= \frac{K_2}{2\pi}\sum_{k=0}^{n}\int_0^1\binom{N}{K}t^k(1-t)^{n-k}\,dt\frac{\sin(k+\frac{1}{2})v}{\sin\frac{v}{2}}.
\end{aligned}$$

3 Result

The object of this paper is obtaining the degree of approximation of functions $f(x,y)$ of generalized Lipschitz class by double Hausdorff matrix summability means of its double Fourier series:

Theorem 1 *If $f(x,y)$ is a 2π periodic function with respect to both variables x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belonging to the class $Lip((\xi_1,\xi_2);r)$ $(r\ge 1)$, then the degree of approximation of $f(x,y)$ by double Hausdorff matrix summability means*

$$t_{m,n}^H = \sum_{j=0}^{m}\sum_{k=0}^{n}\int_0^1\int_0^1\binom{m}{j}\binom{n}{k}s^j(1-s)^{m-j}t^k(1-t)^{n-k}\,d\chi(s,t)s_{j,k}$$

of double Fourier series (1) *satisfies*

$$\left\|t^H_{m,n} - f\right\|_r = O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right) \tag{2}$$

for $m, n = 0, 1, 2, \ldots$.

4 Lemmas

For the proof of our theorems, we need the following lemmas.

Lemma 1 $|M^H_m(u)| = O(m+1)$ *for* $0 < u \leq \frac{1}{m+1}$, *and* $|K^H_n(v)| = O(n+1)$ *for* $0 < v \leq \frac{1}{n+1}$.

Proof Since $|\sin mu| \leq mu$ for $0 < u \leq \frac{1}{m+1}$ and $\sin(u/2) \geq (u/\pi)$, we have

$$\begin{aligned}
\left|M^H_m(u)\right| &= \left|\frac{K_1}{2\pi}\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\,ds\frac{\sin(j+\frac{1}{2})u}{\sin\frac{u}{2}}\right| \\
&= \frac{K_1}{2\pi}\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\,ds\frac{|\sin(j+\frac{1}{2})u|}{|\sin\frac{u}{2}|} \\
&\leq \frac{K_1}{2\pi}\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\,ds\frac{(j+\frac{1}{2})u}{|\frac{u}{\pi}|} \\
&= K_1\pi\left(m+\frac{1}{2}\right)\int_0^1\sum_{j=0}^{m}\binom{m}{j}s^j(1-s)^{m-j}\,ds \\
&= K_1\pi\left(m+\frac{1}{2}\right)\int_0^1(s+1-s)^m\,ds \\
&= O(m+1).
\end{aligned}$$

Similarly, for $0 < v \leq \frac{1}{n+1}$,

$$\left|K^H_n(v)\right| = O(n+1).$$

□

Lemma 2 $|M^H_m(u)| = O(\frac{1}{(j+1)u^2})$ *for* $\frac{1}{m+1} < u \leq \pi$, *and* $|K^H_n(v)| = O(\frac{1}{(k+1)v^2})$ *for* $\frac{1}{n+1} < v \leq \pi$.

Proof Since $\sin(m+1)u \leq 1$ for $\frac{1}{m+1} < u \leq \pi$ and $\sin(u/2) \geq (u/\pi)$, we get

$$\begin{aligned}
\left|\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}e^{i(j+\frac{1}{2})u}\,ds\right| &= \int_0^1 e^{iu/2}\sum_{j=0}^{m}\binom{m}{j}s^j(1-s)^{m-j}e^{iju}\,ds \\
&= \int_0^1 e^{iu/2}\left(1-s+se^{iu}\right)^m\,ds \\
&= O\left(\frac{1}{(m+1)}\right)\left(\frac{e^{iu/2}(e^{i(m+1)u}-1)}{e^{iu}-1}\right).
\end{aligned}$$

Equating the imaginary parts of both sides, we get

$$\left|\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^k(1-s)^{m-j}\sin\left(k+\frac{1}{2}\right)ds\right| = O\left(\frac{1}{(m+1)u}\right).$$

Therefore

$$\begin{aligned}
\left|M_m^H(u)\right| &= \left|\frac{K_1}{2\pi}\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\frac{\sin(j+\frac{1}{2})u}{\sin\frac{u}{2}}\,ds\right| \\
&\le \frac{K_1}{2u}\left|\sum_{j=0}^{m}\int_0^1\binom{m}{j}s^j(1-s)^{m-j}\sin\left(j+\frac{1}{2}\right)u\,ds\right| \\
&= O\left(\frac{1}{(m+1)u^2}\right).
\end{aligned}$$

Similarly, for $\frac{1}{n+1} < v \le \pi$,

$$\left|K_n^H(v)\right| = O\left(\frac{1}{(n+1)v^2}\right). \qquad \square$$

Lemma 3 *If $f(x,y) \in Lip((\xi_1,\xi_2);r)$ $(r \ge 1)$, then $\|\phi(u,v))\|_r = O(\xi_1(u)+\xi_2(v))$.*

Proof Clearly,

$$\begin{aligned}
\left|\phi(u,v)\right| &= \frac{1}{4}\left|f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4f(x,y)\right| \\
&\le \frac{1}{4}\big[\left|f(x+u,y+v)-f(x,y)\right|+\left|f(x+u,y-v)-f(x,y)\right| \\
&\quad+\left|f(x-u,y+v)-f(x,y)\right|+\left|f(x-u,y-v)-f(x,y)\right|\big], \\
\left\|\phi(u,v)\right\|_r &\le \frac{1}{4}\big[\left\|f(x+u,y+v)-f(x,y)\right\|_r+\left\|f(x+u,y-v)-f(x,y)\right\|_r \\
&\quad+\left\|f(x-u,y+v)-f(x,y)\right\|_r+\left\|f(x-u,y-v)-f(x,y)\right\|_r\big] \\
&= O\big(\xi_1(u)+\xi_2(v)\big). \qquad \square
\end{aligned}$$

5 Proof of Theorem 1

The (m,n)th partial sum of the double Fourier series (1) is given by

$$s_{m,n}\big(f;(x,y)\big)-f(x,y) = \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi\phi(u,v)\frac{\sin(m+\frac{1}{2})u\sin(n+\frac{1}{2})v}{\sin\frac{u}{2}\sin\frac{v}{2}}\,du\,dv.$$

Denoting the double Hausdorff matrix sums of $s_{m,n}$ by $t_{m,n}^H$, we have

$$\begin{aligned}
t_{m,n}^H(x,y)-f(x,y) &= \sum_{j=0}^{m}\sum_{k=0}^{n}h_{m,n}^{j,k}\big\{s_{j,k}\big(f;(x,y)\big)-f(x,y)\big\} \\
&= \int_0^\pi\int_0^\pi\phi(u,v)\sum_{j=0}^{m}\sum_{k=0}^{n}h_{m,n}^{j,k}\frac{\sin(j+\frac{1}{2})u\sin(k+\frac{1}{2})v}{\sin\frac{u}{2}\sin\frac{v}{2}}\,du\,dv \\
&= \int_0^\pi\int_0^\pi\phi(u,v)M_m^H(u)K_n^H(v)\,du\,dv, \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\|t_{m,n}^{H}-f\|_{r} &= \int_{0}^{\pi}\int_{0}^{\pi}\|\phi(u,v)\|_{r} M_{m}^{H}(u)K_{n}^{H}(v)\,du\,dv \\
&= \left(\int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}} + \int_{0}^{\frac{1}{m+1}}\int_{\frac{1}{n+1}}^{\pi} + \int_{\frac{1}{m+1}}^{\pi}\int_{0}^{\frac{1}{n+1}} + \int_{\frac{1}{m+1}}^{\pi}\int_{\frac{1}{n+1}}^{\pi}\right) \qquad (4)\\
&\quad \|\phi(u,v)\|_{r} M_{m}^{H}(u)K_{n}^{H}(v)\,du\,dv \\
&= I_1 + I_2 + I_3 + I_4, \quad \text{say.} \qquad (5)
\end{aligned}$$

Using Lemmas 1 and 3, we obtain

$$\begin{aligned}
|I_1| &= \int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}}\|\phi(u,v)\|_{r} M_{m}^{H}(u)K_{n}^{H}(v)\,du\,dv \\
&= O\left(\int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}}\big(\xi_1(u)+\xi_2(v)\big)(m+1)(n+1)\,du\,dv\right) \\
&= O\left((m+1)(n+1)\int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}}\big(\xi_1(u)+\xi_2(v)\big)\,du\,dv\right) \\
&= O\left[(m+1)(n+1)\left(\int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}}\xi_1(u)\,du\,dv + \int_{0}^{\frac{1}{m+1}}\int_{0}^{\frac{1}{n+1}}\xi_2(v)\,du\,dv\right)\right] \\
&= O\left[(m+1)(n+1)\left(\int_{0}^{\frac{1}{m+1}}\frac{\xi_1(u)}{n+1}\,du + \int_{0}^{\frac{1}{m+1}}\frac{\xi_2(\frac{1}{(n+1)})}{n+1}\,dv\right)\right] \\
&= O\left[(m+1)(n+1)\left(\frac{\xi_1(\frac{1}{(m+1)})}{(m+1)(n+1)} + \frac{\xi_2(\frac{1}{(n+1)})}{(m+1)(n+1)}\right)\right] \\
&= O\left(\xi_1\left(\frac{1}{m+1}\right) + \xi_2\left(\frac{1}{n+1}\right)\right).
\end{aligned}$$

Again by Lemmas 1–3, we have

$$\begin{aligned}
|I_2| &= O\left[\int_{0}^{\frac{1}{m+1}}\int_{\frac{1}{n+1}}^{\pi}\big(\xi_1(u)+\xi_2(v)\big)\frac{(m+1)}{(n+1)v^2}\,du\,dv\right] \\
&= O\left[\frac{(m+1)}{(n+1)}\left(\int_{0}^{\frac{1}{m+1}}\xi_1(u)\,du\int_{\frac{1}{n+1}}^{\pi}\frac{dv}{v^2} + \int_{0}^{\frac{1}{m+1}}du\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right)\right] \\
&= O\left[\frac{(m+1)}{(n+1)}\left(\xi_1\left(\frac{1}{m+1}\right)\frac{1}{(m+1)}\left((n+1)-\frac{1}{\pi}\right) + \frac{1}{(m+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right)\right] \\
&= O\left(\xi_1\left(\frac{1}{m+1}\right) + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right). \qquad (6)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|I_3| &= O\left[\int_{\frac{1}{m+1}}^{\pi}\int_{0}^{\frac{1}{n+1}}\big(\xi_1(u)+\xi_2(v)\big)\frac{(n+1)}{(m+1)u^2}\,du\,dv\right] \\
&= O\left[\frac{(n+1)}{(m+1)}\left(\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du\int_{0}^{\frac{1}{n+1}}dv + \int_{\frac{1}{m+1}}^{\pi}\frac{du}{u^2}\int_{0}^{\frac{1}{n+1}}\xi_2(v)\,dv\right)\right] \\
&= O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du + \xi_2\left(\frac{1}{n+1}\right)\right). \qquad (7)
\end{aligned}$$

Also, using Lemmas 2 and 3, we get

$$\begin{aligned}|I_4| &= O\left[\int_{\frac{1}{m+1}}^{\pi}\int_{\frac{1}{n+1}}^{\pi}\bigl(\xi_1(u)+\xi_2(v)\bigr)\frac{1}{(m+1)u^2}\frac{1}{(n+1)v^2}\,du\,dv\right]\\
&= O\left[\frac{1}{(m+1)(n+1)}\left(\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1}{u^2}\,du\int_{\frac{1}{n+1}}^{\pi}\frac{1}{v^2}\,dv+\int_{\frac{1}{m+1}}^{\pi}\frac{1}{u^2}\,du\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2}{v^2}\,dv\right)\right]\\
&= O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du+\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right). \qquad (8)\end{aligned}$$

Next,

$$\begin{aligned}\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du &\geq \frac{1}{(m+1)}\xi_1\left(\frac{1}{m+1}\right)\int_{\frac{1}{m+1}}^{\pi}\frac{1}{u^2}\,dt\\
&= \frac{1}{(m+1)}\xi_1\left(\frac{1}{m+1}\right)\left\{-\frac{1}{u}\right\}_{\frac{1}{m+1}}^{\pi}\\
&= \xi_1\left(\frac{1}{m+1}\right)\left\{1-\frac{1}{(m+1)\pi}\right\}\\
&\geq \frac{1}{2}\xi_1\left(\frac{1}{m+1}\right),\end{aligned}$$

$$\text{or}\quad \xi_1\left(\frac{1}{m+1}\right) = O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,dt\right). \qquad (9)$$

Similarly,

$$\xi_2\left(\frac{1}{(n+1)}\right) = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dt\right). \qquad (10)$$

Combining equations (5)–(10), we have

$$\left\|t_{m,n}^{H}-f\right\|_r = O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du+\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right).$$

This completes the proof of Theorem 1.

6 Corollaries

From the main theorem we derive the following corollaries.

Corollary 1 *If $f(x,y)$ is a 2π periodic function with respect to both variables x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belonging to the class $Lip((\alpha,\beta);r)$ $(r\geq 1)$, then the degree of approximation of $f(x,y)$ by means $t_{m,n}^{H}$ of double Fourier series (1) satisfies*

$$\left\|t_{m,n}^{H}-f\right\|_r = \begin{cases} O((m+1)^{-\alpha}+(n+1)^{-\beta}), & 0<\alpha<1, 0<\beta<1,\\ O((m+1)^{-\alpha}+\frac{\log(n+1)\pi}{(n+1)}), & 0<\alpha<1, \beta=1,\\ O(\frac{\log(m+1)\pi}{(m+1)}+(n+1)^{-\beta}), & \alpha=1, 0<\beta<1,\\ O(\frac{\log(m+1)\pi}{(m+1)}+\frac{\log(n+1)\pi}{(n+1)}), & \alpha=\beta=1,\end{cases}$$

for $m,n=0,1,2,\ldots$.

Corollary 2 *If $f(x,y)$ is a 2π periodic function with respect to both variables x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belonging to the class $Lip(\alpha,\beta)$, then the degree of approximation of $f(x,y)$ by double Hausdorff matrix summability means $t^H_{m,n}$ of double Fourier series* (1) *satisfies*

$$\|t^H_{m,n}-f\|_\infty = \begin{cases} O((m+1)^{-\alpha}+(n+1)^{-\beta}), & 0<\alpha<1, 0<\beta<1,\\ O((m+1)^{-\alpha}+\frac{\log(n+1)\pi}{(n+1)}), & 0<\alpha<1, \beta=1,\\ O(\frac{\log(m+1)\pi}{(m+1)}+(n+1)^{-\beta}), & \alpha=1, 0<\beta<1,\\ O(\frac{\log(m+1)\pi}{(m+1)}+\frac{\log(n+1)\pi}{(n+1)}), & \alpha=\beta=1, \end{cases}$$

for $m,n=0,1,2,\ldots$.

Corollary 3 *If $f(x,y)$ is a 2π periodic function with respect to both variables x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belonging to the class $Lip((\xi_1,\xi_2);r)$, then the degree of approximation of $f(x,y)$ by almost Euler summability means*

$$t^E_{m,n}=\frac{1}{(1+q_1)^m}\frac{1}{(1+q_2)^n}\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}q_1^{m-j}q_2^{n-k}s_{j,k}$$

of double Fourier series (1) *satisfies*

$$\|t^E_{m,n}-f\|_r = O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du+\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right)$$

for $m,n=0,1,2,\ldots$.

Corollary 4 *For $\gamma,\delta\geq -1$, the Cesàro means $\sigma^{\gamma,\delta}_{m,n}$ of order γ and δ, that is, (C,γ,δ) means of double Fourier series, are given by*

$$\sigma^{\gamma,\delta}_{m,n}=\frac{1}{A^\gamma_m}\frac{1}{A^\delta_n}\sum_{j=0}^{m}\sum_{k=0}^{n}A^{\gamma-1}_{m-j}A^{\delta-1}_{n-k}s_{j,k},$$

where $A^\gamma_m=\binom{\gamma+m}{m}$ and $A^\delta_n=\binom{\delta+n}{n}$.

If $f(x,y)$ is a 2π periodic function with respect to both variables x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belonging to the class $Lip((\xi_1,\xi_2);r)$, then the degree of approximation of $f(x,y)$ by (C,γ,δ) means of double Fourier series (1), *satisfies*

$$\|\sigma^{\gamma,\delta}_{m,n}-f\|_r = O\left(\frac{1}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(u)}{u^2}\,du+\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_2(v)}{v^2}\,dv\right)$$

for $m,n=0,1,2,\ldots$.

7 Conclusion

We established the degree of approximation of a function $f(x,y)$ belonging to the generalized Lipschitz class by double Hausdorff matrix summability means of its double Fourier series in the form of equation (2). If $\xi_1 = u^\alpha$ and $\xi_2 = v^\beta$, then Theorem 1 reduces to Corollary 1, and as $r \to \infty$, Corollary 1 reduces to Corollary 2. Independent proofs of Corollaries 1–4 can be developed along the same lines as that of Theorem 1. Results similar to Corollaries 3 and 4 can be derived for $(E,1,1)$ means and $(C,1,1)$ means of its double Fourier series. In this way, we can obtain some more different results by changing ξ_1, ξ_2, and $\mu_{m,n}$ under given conditions. For functions $f(x,y)$ belonging to the Zygmund classes $Zyg(\alpha,\beta)$ and $Zyg(\alpha,\beta;p)$ discussed in [9], the degree of approximation using double Hausdorff matrix summability means and hence almost Euler means of its double Fourier series can be obtained similarly to Theorem 1.

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References

1. Acar, T., Mohiuddine, S.A.: Statistical $(C,1)(E,1)$ summability and Korovkin's theorem. Filomat **30**(2), 387–393 (2016)
2. Chow, Y.S.: On the Cesàro summability of double Fourier series. Tohoku Math. J. **5**, 277–283 (1935)
3. Gergen, J.J.: Convergence criteria for double Fourier series. Transl. Am. Math. Soc. **35**(1), 29–63 (1933)
4. Hasegawa, Y.: On summabilities of double Fourier series. Kodai Math. Semin. Rep. **15**, 226–238 (1963)
5. Khan, H.H., Ram, G.: On the degree of approximation by Gauss Weierstrass integrals. Int. J. Math. Math. Sci. **23**(9), 645–649 (2000)
6. Łenski, W., Topolewska, M.: On the rate of strong summability of double Fourier series. Math. Bohem. **123**(4), 337–363 (1998)
7. Mohiuddine, S.A., Acar, T.: Advances in Summability and Approximation Theory. Springer, Berlin (2018)
8. Móricz, F., Shi, X.: Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series. J. Approx. Theory **49**, 346–377 (1987)
9. Rathore, A., Singh, U.: Approximation of certain bivariate functions by almost Euler means of double Fourier series. J. Inequal. Appl. **2018**, 89 (2018)
10. Savaş, E., Rhoades, B.E.: Every conservative double Hausdorff matrix is a *k*th absolutely summable operator. Anal. Math. **35**, 249–256 (2009)
11. Sharma, P.L.: On the harmonic summability of double Fourier series. Proc. Am. Math. Soc. **91**, 979–986 (1958)
12. Siddiqui, A.H., Mohammadzadeh, M.: Approximation by Cesàro and B means of double Fourier series. Math. Jpn. **21**(4), 343–349 (1976)
13. Stepanets, A.I.: The approximation of certain classes of differentiable periodic functions of two variables by Fourier sums. Ukr. Math. J. **25**(5), 599–609 (1973)
14. Stepanets, A.I.: Approximation of certain classes of periodic functions of two variables by linear methods of summation of their Fourier series. Ukr. Math. J. **26**(2), 205–215 (1974)
15. Ustina, F.: The Hausdorff means for double sequences. Can. Math. Bull. **10**(3), 347–352 (1967)
16. Ustina, F.: The Hausdorff means of double Fourier series and the principle of localization. Pac. J. Math. **37**(1), 238–251 (1971)
17. Zygmund, A.: Trigonometric Series, 2nd rev. ed., I. Cambridge University Press, Cambridge (1968)