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Non-classical symmetry and analytic self-similar solutions for a non-homogenous time-fractional vector NLS system

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Abstract

The complex PDEs are a very important and interesting task in nonlinear quantum science. Although there have been extensive studies on the classical complex models, solving the fractional complex models still has a lot of shortcomings, especially for the non-homogenous ones. Therefore, the present study focuses on solving the two-component non-homogenous time-fractional NLS system, our method is to solve a prolonged fractional system derived from the governed model. We first establish non-classical symmetries of this new enlarged system by using the fractional Lie group method. Then, with the help of fractional Erdélyi–Kober operator, we reduce this new system into fractional ODEs, the self-similar solutions are obtained via the power series expansion. The convergence of these solutions are proven as all the variable coefficients are analytic. Finally, we generalize our methods to handle the multi-component case. We conclude that this way may also bring some convenience for solving other complex systems.

Keywords: Non-classical symmetry; Vector NLS system; Erdélyi–Kober operator; Self-similar solutions

1 Introduction

The vector complex systems have attracted more and more attention in many different fields of nonlinear science during the past few years. To well describe the spins and kinetics of micro-particles, the partial differential equations for these complex systems were set up and widely used in the related ranges of particle physics, quantum mechanics, the condensed matter physics [1–3], and many other subjects. One of the most famous models is the nonlinear Schrödinger equations whose general version is governed as

$$iu_t^j + r_j(t,x)u_{xx}^j + f^j\left(t,x,\left|u^1\right|,\ldots,\left|u^m\right|\right)u^j = 0 \quad (j=1,\ldots,m). \tag{1.1}$$

Here, t, x are temporal and spatial independent variables, u^j represents the wave function which describes velocity envelope for multi-particles, the subscripts show the derivatives of corresponding variables, all coefficients $r_j(t,x)$ and $f^j(t,x,|u^1|,\ldots,|u^m|)$ are real analytic mean the ratios of non-homogenous diffusion and the intensity of nonlinear interac-

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tions. There has been abundant research on model (1.1) which explained the kinetics and diffusions of particles in the multi-body quantum regimes. To the best of our knowledge, a lot of soliton waves, breather waves, rogue waves, and periodic waves of Eq. (1.1) were studied by taking advantage of Darboux transformation [4–9], inverse scattering method [10, 11], Hirota's bilinear transformation [12–14], nonlocal symmetry method [15], and many other ways [1–3, 16, 17] in both mathematical and physical points of view. Some mixed type solutions, especially breather-soliton-rogue wave solutions [4, 7, 9, 17], were obtained and used to understand how the quantum waves interact in local excitation patterns.

Recently, models governed by the time-fractional PDEs have been considered in many fields of mechanics and physics [18, 19, 29–31]. Indeed, the fractional models are more precise than the integer-order ones. For many physical phenomena, different time memories are often represented by different integral kernels of several definitions [20, 21], two of the most influence and popularity are Riemann–Liouville type and Caputo type [18–21, 29–31] which include the singular kernel, and other definitions may contain the nonsingular kernel. The singular kernel (general kernel), for instant power kernel which was derived by Cauchy integral, describes how the quantity process obeys a singular law by empirical observation in many real problems. The power memory has many good mathematical properties such as self-similarity, semi-group property, Laplace transformation, but the disadvantage is the lack of elaborate statistical tests and empirical support. Thus it should be natural to consider the nonsingular kernel which can show the fading memories with relaxation. The typical type for nonsingular kernel is Caputo–Fabrizio definition [20, 21] of exponential memory that may be applied to well understand the stochastic process of empirical distribution, but this expression is more difficult to compute. In short, the singular kernel can more generally characterize the real nonlocal nonlinear phenomenon and is more convenient for calculating, thus it should take precedence to use for solving fractional differential equations. In physical point of view, some micro-structures may often lead to the short time memories effect, the smaller α decides the faster time memory. In addition, the Riemann–Liouville derivative has stronger singularity than Caputo derivative, thus the Riemann–Liouville definition can be often used without initial-boundary conditions. Therefore, in the present work we mainly investigate the following non-homogenous fractional NLS system with Riemann–Liouville time derivatives:

$$\begin{aligned} &i\frac{\partial^\alpha u}{\partial t^\alpha} + r(t,x)u_{xx} + f\big(t,x,|u|,|v|\big)u = 0,\\ &i\frac{\partial^\alpha v}{\partial t^\alpha} + s(t,x)v_{xx} + g\big(t,x,|u|,|v|\big)v = 0 \quad (0<\alpha\le 1), \end{aligned} \tag{1.2}$$

where Riemann–Liouville derivative is defined as

$${}_0^{RL}\partial_t^\alpha u(x,y,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-\tau)^{n-\alpha-1}u(x,y,\tau)\,d\tau & (n=[\alpha]+1),\\ \frac{\partial^n u(x,y,t)}{\partial t^n} & (\alpha = n). \end{cases}$$

This fractional system more precisely characterizes the Bose–Einstein concentration and phase transition behaviors of critical states than the integer one in the two-body quantum regimes, where the fractional derivatives $\frac{\partial^\alpha u}{\partial t^\alpha}$, $\frac{\partial^\alpha v}{\partial t^\alpha}$ describe two wave functions with nonlocal time memories, and $r(t,x)$, $s(t,x)$, $f(t,x,|u|,|v|)$, $g(t,x,|u|,|v|)$ are variable coefficients as (1.1).

However, solving fractional system (1.2) is really a new and difficult work. On one hand, since integrability of the fractional models is much poorer than that of the classical ones, the compound function solutions of (1.2), for typical traveling wave solutions, were hardly obtained by adopting some direct methods. On the other hand, there have been abundant studies on Lie symmetries, conservation laws, and exact explicit solutions for many integer and fractional real PDEs [22–28, 32–43]. However, few symmetries of the time-fractional complex system have been discussed until now, even non-homogenous ones. For the classical n-component complex PDE systems, the common method is to split the real and imaginary parts of two complex variables u, v and compute the symmetries of $2n$ equations with $2n$ variable coefficients r, s, f, g, this may cause some difficulties. To solve this problem in a concise way, we introduce the complex conjugations u^*, v^* and regard functions f, g as two new functions. Here, in order to close the system, we also need to relate f, g to u, v, u^*, v^*. Noting that the expression $f = f(t, x, |u|, |v|)$, $g = g(t, x, |u|, |v|)$ is equivalent to the differential system $uf_u - u^* f_{u^*} = 0$, $vf_v - v^* f_{v^*} = 0$, $ug_u - u^* g_{u^*} = 0$, $vg_v - v^* g_{v^*} = 0$, we can enlarge the vector fNLS model to a new closed fPDE system and only consider solving the new prolonged system. It is novel to construct the symmetries of the prolonged fractional equations since the non-classical symmetries of prolonged system always contain the classical symmetries of the governed model. We also verify that our results can be extended to the more general N-component case by introducing $f^i = f^i(t, x, |u^1|, |u^m|)$, $(i = 1, \dots, N)$) and differential system $u^j f^j_{u^j} - u^{j*} f_{u^{j*}} = 0$, $(j = 1, \dots, N)$.

The rest of the paper is organized as follows. The non-classical symmetries of prolonged complex system are discussed in Sect. 2. Then, in Sect. 3, this system is reduced by virtue of the Eydélyi–Kober fractional differential operator, and self-similar solutions are acquired by the power expanding method in the de-focused case. We also verify the convergence of solutions in Sect. 4 by using induction as all the coefficients are analytic. Finally, our results are extended to the multi-component case. The concluding remark of our work is put in the last section.

2 Non-classical symmetry for two-component fractional NLS system

This section considers the non-classical symmetry of system (1.2). By introducing two new conjugate variables u^*, v^*, we consider the following enlarged complex system:

$$\begin{aligned}
&iD_t^\alpha u + ru_{xx} + fu = 0,\\
&iD_t^\alpha v + sv_{xx} + gv = 0,\\
&uf_u - u^* f_{u^*} = 0,\\
&vf_v - v^* f_{v^*} = 0,\\
&ug_u - u^* g_{u^*} = 0,\\
&vg_v - v^* g_{v^*} = 0.
\end{aligned} \tag{2.1}$$

Here, we regard f, g as two new functions. Under the continuous transformation group

$$\begin{aligned}
&\bar{t} = t + \epsilon\tau\left(t, x, u, v, u^*, v^*\right) + o\left(\epsilon^2\right),\\
&\bar{x} = x + \epsilon\xi\left(t, x, u, v, u^*, v^*\right) + o\left(\epsilon^2\right),
\end{aligned}$$

$$\begin{aligned}
&\bar{u} = u + \epsilon\Phi\left(t,x,u,v,u^*,v^*\right) + o\left(\epsilon^2\right),\\
&\bar{v} = v + \epsilon\Psi\left(t,x,u,v,u^*,v^*\right) + o\left(\epsilon^2\right),\\
&\bar{u^*} = u^* + \epsilon\Phi^*\left(t,x,u,v,u^*,v^*\right) + o\left(\epsilon^2\right),\\
&\bar{v^*} = v^* + \epsilon\Psi^*\left(t,x,u,v,u^*,v^*\right) + o\left(\epsilon^2\right),\\
&\bar{f} = f + \epsilon F\left(t,x,u,v,u^*,v^*,f,g\right) + o\left(\epsilon^2\right),\\
&\bar{g} = g + \epsilon G\left(t,x,u,v,u^*,v^*,f,g\right) + o\left(\epsilon^2\right),
\end{aligned} \tag{2.2}$$

with infinitesimal generators ξ, τ, Φ, Φ^*, Ψ, Ψ^*, F, G, the vector field of the generators of Lie group is given by

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial u} + \Psi\frac{\partial}{\partial v} + \Phi^*\frac{\partial}{\partial u^*} + \Psi^*\frac{\partial}{\partial v^*} + F\frac{\partial}{\partial f} + G\frac{\partial}{\partial g}, \tag{2.3}$$

and the α, 2-order prolonged vector field is shown as

$$\begin{aligned}
pr^{\alpha,2}V = {} & V + \Phi^{\alpha}\frac{\partial}{\partial D_t^{\alpha}u} + \Psi^{\alpha}\frac{\partial}{\partial D_t^{\alpha}v} + \Phi^{xx}\frac{\partial}{\partial u_{xx}} + \Psi^{xx}\frac{\partial}{\partial v_{xx}} + F^{u}\frac{\partial}{\partial f_u} + F^{u^*}\frac{\partial}{\partial f_{u^*}}\\
& + F^{v}\frac{\partial}{\partial f_v} + F^{v^*}\frac{\partial}{\partial f_{v^*}} + G^{u}\frac{\partial}{\partial g_u} + G^{u^*}\frac{\partial}{\partial g_{u^*}} + G^{v}\frac{\partial}{\partial g_v} + G^{v^*}\frac{\partial}{\partial g_{v^*}},
\end{aligned} \tag{2.4}$$

where τ, ξ, F, G are real functions and Φ, Ψ are complex ones.

Applying the Lie symmetry method to system (1.2) yields the following results.

Theorem 1 *Under the continuous group transformation* (2.2), *invariance of system* (2.1) *admits the following infinitesimal generators*:

$$\begin{aligned}
&\xi = \xi(x),\\
&\tau = \tau(t), \tau'''(t) = 0,\\
&\Phi = \left(\frac{\alpha-1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_3\right)u,\\
&\Psi = \left(\frac{\alpha-1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_4\right)v,\\
&\Phi^* = \left(\frac{\alpha-1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_3\right)u^*,\\
&\Psi^* = \left(\frac{\alpha-1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_4\right)v^*,\\
&F = -\alpha\tau'(t)f - \frac{r}{2}\xi'''(x),\\
&G = -\alpha\tau'(t)g - \frac{s}{2}\xi'''(x),
\end{aligned}$$

where the diffusion coefficients solve the linear equations

$$\begin{aligned}
&\tau r_t + \xi r_x + (\alpha\tau_t - 2\xi_x)r = 0,\\
&\tau s_t + \xi s_x + (\alpha\tau_t - 2\xi_x)s = 0.
\end{aligned} \tag{2.5}$$

Notation In the following proof, we denote by C_α^n a combination number where $C_\alpha^n = \frac{\alpha!}{n!(\alpha-n)!}$.

Proof By adopting the fractional Lie group method, the invariance of system (2.1) is determined by the following linear equations:

$$\begin{aligned}
&i\Phi^{\alpha,t} + r\Phi^{xx} + (\tau r_t + \xi r_x)u_{xx} + Fu + f\Phi = 0,\\
&i\Psi^{\alpha,t} + s\Psi^{xx} + (\tau s_t + \xi s_x)v_{xx} + Gv + g\Psi = 0,\\
&\Phi f_u + uF^u - \Phi^* f_{u^*} - u^* F^{u^*} = 0,\\
&\Psi f_v + vF^v - \Psi^* f_{v^*} - v^* F^{v^*} = 0,\\
&\Phi g_u + uG^u - \Phi^* g_{u^*} - u^* G^{u^*} = 0,\\
&\Psi g_v + vG^v - \Psi^* g_{v^*} - v^* G^{v^*} = 0,
\end{aligned} \tag{2.6}$$

with the prolonged generators

$$\begin{aligned}
\Phi^{xx} &= D_x^2(\Phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt},\\
\Psi^{xx} &= D_x^2(\Psi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt},\\
F^u &= D_u\big(F - \xi f_x - \tau f_t - \Phi f_u - \Psi f_v - \Phi^* f_{u^*} - \Psi^* f_{v^*}\big) + \xi f_{xu} + \tau f_{tu}\\
&\quad + \Phi f_{uu} + \Psi f_{vu} + \Phi^* f_{u^*u} + \Psi f_{v^*u},\\
G^u &= D_u\big(G - \xi g_x - \tau g_t - \Phi g_u - \Psi g_v - \Phi^* g_{u^*} - \Psi^* g_{v^*}\big) + \xi g_{xu} + \tau g_{tu}\\
&\quad + \Phi g_{uu} + \Psi g_{vu} + \Phi^* g_{u^*u} + \Psi g_{v^*u},\\
F^v &= D_v\big(F - \xi f_x - \tau f_t - \Phi f_u - \Psi f_v - \Phi^* f_{u^*} - \Psi^* f_{v^*}\big) + \xi f_{xv} + \tau f_{tv}\\
&\quad + \Phi f_{uv} + \Psi f_{vv} + \Phi^* f_{u^*v} + \Psi f_{v^*v},\\
G^v &= D_v\big(G - \xi g_x - \tau g_t - \Phi g_u - \Psi g_v - \Phi^* g_{u^*} - \Psi^* g_{v^*}\big) + \xi g_{xv} + \tau g_{tv}\\
&\quad + \Phi g_{uv} + \Psi g_{vv} + \Phi^* g_{u^*v} + \Psi g_{v^*v},\\
F^{u^*} &= D_{u^*}\big(F - \xi f_x - \tau f_t - \Phi f_u - \Psi f_v - \Phi^* f_{u^*} - \Psi^* f_{v^*}\big) + \xi f_{xu^*} + \tau f_{tu^*}\\
&\quad + \Phi f_{uu^*} + \Psi f_{vu^*} + \Phi^* f_{u^*u^*} + \Psi f_{v^*u^*},\\
G^{u^*} &= D_{u^*}\big(G - \xi g_x - \tau g_t - \Phi g_u - \Psi g_v - \Phi^* g_{u^*} - \Psi^* g_{v^*}\big) + \xi g_{xu^*} + \tau g_{tu^*}\\
&\quad + \Phi g_{uu^*} + \Psi g_{vu^*} + \Phi^* g_{u^*u^*} + \Psi g_{v^*u^*},\\
F^{v^*} &= D_{v^*}\big(F - \xi f_x - \tau f_t - \Phi f_u - \Psi f_v - \Phi^* f_{u^*} - \Psi^* f_{v^*}\big) + \xi f_{xv^*} + \tau f_{tv^*}\\
&\quad + \Phi f_{uv^*} + \Psi f_{vv^*} + \Phi^* f_{u^*v^*} + \Psi f_{v^*v^*},\\
G^{v^*} &= D_{v^*}\big(G - \xi g_x - \tau g_t - \Phi g_u - \Psi g_v - \Phi^* g_{u^*} - \Psi^* g_{v^*}\big) + \xi g_{xv^*} + \tau g_{tv^*}\\
&\quad + \Phi g_{uv^*} + \Psi g_{vv^*} + \Phi^* g_{u^*v^*} + \Psi g_{v^*v^*},
\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
\Phi^{\alpha,t} &= \frac{\partial^\alpha \Phi}{\partial t^\alpha} + (\Phi_u - \alpha D_t \tau)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \Phi_u}{\partial t^\alpha} - \sum_{n=1}^{\infty} C_\alpha^n D_t^n \xi D_t^{\alpha-n} u_x \\
&\quad + \sum_{n=1}^{\infty}\left[C_\alpha^n \frac{\partial^n \Phi_u}{\partial t^n} - C_\alpha^{n+1} D_t^{n+1}\tau\right] D_t^{\alpha-n} u + \left(\Phi_v \frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \Phi_v}{\partial t^\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Phi_v}{\partial t^n} D_t^{\alpha-n} v + \left(\Phi_u^* \frac{\partial^\alpha u^*}{\partial t^\alpha} - u^* \frac{\partial^\alpha \Phi_u^*}{\partial t^\alpha}\right) + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Phi_u^*}{\partial t^n} D_t^{\alpha-n} u^* \\
&\quad + \left(\Phi_v^* \frac{\partial^\alpha v^*}{\partial t^\alpha} - v^* \frac{\partial^\alpha \Phi_v^*}{\partial t^\alpha}\right) + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Phi_v^*}{\partial t^n} D_t^{\alpha-n} v^* + \mu_{\Phi_1} + \mu_{\Phi_2} + \mu_{\Phi_3} + \mu_{\Phi_4}, \\
\Psi^{\alpha,t} &= \frac{\partial^\alpha \Psi}{\partial t^\alpha} + (\Psi_v - \alpha D_t \tau)\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \Psi_v}{\partial t^\alpha} - \sum_{n=1}^{\infty} C_\alpha^n D_t^n \xi D_t^{\alpha-n} v_x \\
&\quad + \sum_{n=1}^{\infty}\left[C_\alpha^n \frac{\partial^n \Psi_v}{\partial t^n} - C_\alpha^{n+1} D_t^{n+1}\tau\right] D_t^{\alpha-n} v + \left(\Psi_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \Psi_u}{\partial t^\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Psi_u}{\partial t^n} D_t^{\alpha-n} u + \left(\Psi_u^* \frac{\partial^\alpha u^*}{\partial t^\alpha} - u^* \frac{\partial^\alpha \Psi_u^*}{\partial t^\alpha}\right) + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Psi_u^*}{\partial t^n} D_t^{\alpha-n} u^* \\
&\quad + \left(\Psi_v^* \frac{\partial^\alpha v^*}{\partial t^\alpha} - v^* \frac{\partial^\alpha \Psi_v^*}{\partial t^\alpha}\right) + \sum_{n=1}^{\infty} C_\alpha^n \frac{\partial^n \Psi_v^*}{\partial t^n} D_t^{\alpha-n} v^* + \mu_{\Psi_1} + \mu_{\Psi_2} + \mu_{\Psi_3} + \mu_{\Psi_4},
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{\Phi_1} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^l \frac{\partial^m u^{k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial u^k}, \\
\mu_{\Phi_2} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-v)^l \frac{\partial^m v^{k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial v^k}, \\
\mu_{\Phi_3} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(-u^*\right)^l \frac{\partial^m u^{*k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial u^{*k}}, \\
\mu_{\Phi_4} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(-v^*\right)^l \frac{\partial^m v^{*k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial v^{*k}}, \\
\mu_{\Psi_1} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^l \frac{\partial^m u^{k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Psi}{\partial t^{n-m}\partial u^k}, \\
\mu_{\Psi_2} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-v)^l \frac{\partial^m v^{k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Psi}{\partial t^{n-m}\partial v^k}, \\
\mu_{\Psi_3} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(-u^*\right)^l \frac{\partial^m u^{*k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Psi}{\partial t^{n-m}\partial u^{*k}}, \\
\mu_{\Psi_4} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} C_\alpha^n C_n^m C_k^l \frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(-v^*\right)^l \frac{\partial^m v^{*k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Psi}{\partial t^{n-m}\partial v^{*k}}.
\end{aligned}
$$

Substituting (2.7) into (2.6) with the help of prolonged system (2.1), after equaling the coefficients of all derivatives of u, v, we have admitted conditions as follows:

$$
\begin{aligned}
&\Phi_{uu} = \Phi_{vv} = \Phi_{u^*} = \Phi_{v^*v^*} = \Phi_{uv} = \Phi_{uv^*} = \Phi_{vv^*} = \Phi_{xv} = \Phi_{xv^*} = \Phi_{vt} = \Phi_{v^*t} = 0,\\
&\Psi_{uu} = \Psi_{vv} = \Psi_{u^*u^*} = \Psi_{v^*} = \Psi_{uv} = \Psi_{uu^*} = \Psi_{vu^*} = \Psi_{xu} = \Psi_{xu^*} = \Psi_{ut} = \Psi_{u^*t} = 0,\\
&\xi_t = \xi_u = \xi_v = \xi_{u^*} = \xi_{v^*} = 0, \qquad \tau_x = \tau_u = \tau_v = \tau_{u^*} = \tau_{v^*} = 0, \quad \tau|_{t=0} = 0,\\
&\frac{\partial^n \Phi_u}{\partial t^n} = \frac{\partial^n \Psi_v}{\partial t^n} = \frac{\alpha - n}{n+1} D_t^{n+1}\tau \quad (n = 1, 2, \ldots),\\
&\Phi_{xu} = \Psi_{xv} = \frac{1}{2}\xi_{xx}, \quad \tau'''(t) = 0,\\
&(r-s)\Phi_v = 0, \qquad (r+s)\Phi_{v^*} = 0, \qquad (s-r)\Psi_u = 0, \qquad (r+s)\Psi_{u^*} = 0,\\
&\tau r_t + \xi r_x + (\alpha\tau_t - 2\xi_x)r = 0,\\
&\tau s_t + \xi s_x + (\alpha\tau_t - 2\xi_x)s = 0,\\
&F = F\big(t, x, |u|, |v|, f, g\big), \qquad G = G\big(t, x, |u|, |v|, f, g\big),\\
&i\left[\frac{\partial^\alpha \Phi}{\partial t^\alpha} - u\frac{\partial^\alpha \Phi_u}{\partial t^\alpha} - v\frac{\partial^\alpha \Phi_v}{\partial t^\alpha} - v^*\frac{\partial^\alpha \Phi_{v^*}}{\partial t^\alpha}\right]\\
&\quad - \big(\Phi_u - \alpha\tau'(t)\big)fu - \Phi_v gv + \Phi_{v^*} gv^* + f\Phi + Fu + r\Phi_{xx} = 0,\\
&i\left[\frac{\partial^\alpha \Psi}{\partial t^\alpha} - u\frac{\partial^\alpha \Psi_u}{\partial t^\alpha} - v\frac{\partial^\alpha \Psi_v}{\partial t^\alpha} - u^*\frac{\partial^\alpha \Psi_{u^*}}{\partial t^\alpha}\right]\\
&\quad - \big(\Psi_v - \alpha\tau'(t)\big)gv - \Psi_u fu + \Psi_{u^*} fu^* + g\Psi + Gv + s\Psi_{xx} = 0.
\end{aligned}
\tag{2.8}
$$

Solving the linear PDEs (2.8) one by one leads to the desired results. □

The next result shows the self-similar reduction.

Lemma 1 *If* $\alpha\tau'(t) - 2\xi'(x) = c_2$, *then we get the infinitesimal generators as follows*:

$$
\begin{aligned}
&\xi = \frac{c_1 - c_2}{2}x,\\
&\tau = \frac{c_1}{\alpha}t,\\
&\Phi = \left(\frac{c_1(\alpha-1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_3\right)u,\\
&\Psi = \left(\frac{c_1(\alpha-1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_4\right)v,\\
&\Phi^* = \left(\frac{c_1(\alpha-1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_3\right)u^*,\\
&\Psi^* = \left(\frac{c_1(\alpha-1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_4\right)v^*,\\
&F = -c_1 f,\\
&G = -c_1 g,
\end{aligned}
$$

with the coefficients

$$r = t^{-\frac{\alpha c_2}{c_1}} R\big(xt^{-\frac{\alpha(c_1-c_2)}{2c_1}}\big),$$
$$s = t^{-\frac{\alpha c_2}{c_1}} S\big(xt^{-\frac{\alpha(c_1-c_2)}{2c_1}}\big),$$

and four-dimensional Lie algebra $V = \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3 + \sigma_4 V_4$ *generated from the vector fields*

$$\begin{aligned}
V_1 &= \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{3\alpha-2}{4\alpha}u\frac{\partial}{\partial u} + \frac{3\alpha-2}{4\alpha}v\frac{\partial}{\partial v} + \frac{3\alpha-2}{4\alpha}u^*\frac{\partial}{\partial u^*} \\
&\quad + \frac{3\alpha-2}{4\alpha}v^*\frac{\partial}{\partial v^*} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \\
V_2 &= \frac{x}{2}\frac{\partial}{\partial x} + \frac{u}{4}\frac{\partial}{\partial u} + \frac{v}{4}\frac{\partial}{\partial v} + \frac{u^*}{4}\frac{\partial}{\partial u^*} + \frac{v^*}{4}\frac{\partial}{\partial v^*}, \\
V_3 &= u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}, \\
V_4 &= v\frac{\partial}{\partial v} + v^*\frac{\partial}{\partial v^*}.
\end{aligned} \tag{2.9}$$

Proof We obtain (2.9) by directly calculating. □

3 Self-similar solution for two-component fractional NLS system

Let us consider the scaling action $\overline{V} = V_1 + \sigma V_2$, where the parameter is chosen as $\sigma = -\frac{c_2}{c_1}$. In this section, we search for the self-similar solutions of system (1.2).

Theorem 2 *When we take* $\xi = x^{\frac{2}{1+\sigma}}t^{-\alpha}$, *under the scaling action* $\overline{V}$, *system* (2.1) *can be reduced to the following fractional ODEs*:

$$\begin{aligned}
&i\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha}U(\xi) + \left(\frac{2}{1+\sigma}\right)^2 \xi R\big(\xi^{\frac{1+\sigma}{2}}\big)\left(\frac{1-\sigma}{2}U'(\xi) + \xi U''(\xi)\right) \\
&\quad + \Theta_1\big(\xi, \big|U(\xi)\big|, \big|V(\xi)\big|\big)U(\xi) = 0, \\
&i\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha}V(\xi) + \left(\frac{2}{1+\sigma}\right)^2 \xi S\big(\xi^{\frac{1+\sigma}{2}}\big)\left(\frac{1-\sigma}{2}V'(\xi) + \xi V''(\xi)\right) \\
&\quad + \Theta_2\big(\xi, \big|U(\xi)\big|, \big|V(\xi)\big|\big)V(\xi) = 0.
\end{aligned} \tag{3.1}$$

Here, the Erdélyi–Kober fractional differential operator is defined as

$$\mathcal{P}_{\varrho}^{\vartheta,\alpha}f(y) = \prod_{k}^{a-1}\left(\vartheta + k - \frac{1}{\varrho}y\frac{d}{dy}\right)\big(\mathcal{K}_{\varrho}^{\vartheta+\alpha,a-\alpha}f\big)(y) \quad (y>0, \alpha>0, \varrho>0),$$

and

$$\mathcal{K}_{\varrho}^{\vartheta,\alpha}f(y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (\rho-1)^{\alpha-1}\rho^{-(\vartheta+\alpha)}f(y\rho^{\frac{1}{\varrho}})\,d\rho & (\alpha>0), \\ f(y) & (\alpha=0). \end{cases}$$

Proof Herein we use the invariance to construct self-similar solutions. First, solving the following characteristic

$$\begin{aligned}\frac{2dx}{(1+\sigma)x} &= \frac{\alpha\,dt}{t} = \frac{4\alpha\,du}{((\sigma+3)\alpha-2)u} = \frac{4\alpha\,dv}{((\sigma+3)\alpha-2)v}\\ &= \frac{4\alpha\,du^*}{((\sigma+3)\alpha-2)u^*} = \frac{4\alpha\,dv^*}{((\sigma+3)\alpha-2)v^*} = \frac{df}{-f} = \frac{dg}{-g}\end{aligned}$$

gives rise to

$$\begin{aligned}&u = t^{\frac{(\sigma+3)\alpha-2}{4}}U\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}\big),\\ &v = t^{\frac{(\sigma+3)\alpha-2}{4}}V\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}\big),\\ &u^* = t^{\frac{(\sigma+3)\alpha-2}{4}}U^*\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}\big),\\ &v^* = t^{\frac{(\sigma+3)\alpha-2}{4}}V^*\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}\big),\\ &f = t^{-\alpha}\Theta_1\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}, t^{\frac{(\sigma+3)\alpha-2}{4}}|U|, t^{\frac{(\sigma+3)\alpha-2}{4}}|V|\big),\\ &g = t^{-\alpha}\Theta_2\big(x^{\frac{2}{1+\sigma}}t^{-\alpha}, t^{\frac{(\sigma+3)\alpha-2}{4}}|U|, t^{\frac{(\sigma+3)\alpha-2}{4}}|V|\big),\end{aligned} \tag{3.2}$$

with

$$\begin{aligned}&r = t^{\sigma\alpha}R\big(xt^{-\frac{(1+\sigma)\alpha}{2}}\big),\\ &s = t^{\sigma\alpha}S\big(xt^{-\frac{(1+\sigma)\alpha}{2}}\big).\end{aligned} \tag{3.3}$$

Then, by using the chain rule, the prolonged parts of system (2.1) become

$$\begin{aligned}&U\Theta_{1U} = U^*\Theta_{1U^*}, \qquad V\Theta_{1V} = V^*\Theta_{1V^*},\\ &U\Theta_{2U} = U^*\Theta_{2U^*}, \qquad V\Theta_{2V} = V^*\Theta_{2V^*},\end{aligned}$$

solving these four linear PDEs yields

$$\Theta_1 = \Theta_1\big(\xi, |U|, |V|\big), \qquad \Theta_2 = \Theta_2\big(\xi, |U|, |V|\big). \tag{3.4}$$

On the other hand, from the definition of fractional Erdélyi–Kober differential operator, we obtain the fractional derivatives as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}\tau^{\frac{(\sigma+3)\alpha-2}{4}}U\big(x^{\frac{2}{1+\sigma}}\tau^{-\alpha}\big)\,d\tau\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\bigg[t^{n+\frac{(\sigma-1)\alpha-2}{4}}\int_1^\infty (s-1)^{n-\alpha-1}s^{-\frac{(\sigma-1)\alpha+2}{4}-n}U\big(\xi s^\alpha\big)\,ds\bigg]\\ &= \frac{d^n}{dt^n}\big[t^{n+\frac{(\sigma-1)\alpha-2}{4}}\mathcal{K}_{\frac{1}{\alpha}}^{\frac{(\sigma+3)\alpha+2}{4},n-\alpha}U(\xi)\big]\\ &= \frac{d^{n-1}}{dt^{n-1}}\bigg[t^{n-1+\frac{(\sigma-1)\alpha-2}{4}}\bigg(n+\frac{(\sigma-1)\alpha-2}{4}-\alpha\xi\frac{d}{d\xi}\bigg)\mathcal{K}_{\frac{1}{\alpha}}^{\frac{(\sigma+3)\alpha+2}{4},n-\alpha}U(\xi)\bigg] = \cdots\end{aligned}$$

$$= t^{\frac{(\sigma-1)\alpha-2}{4}} \prod_{k=0}^{n-1} \left(\frac{(\sigma-1)\alpha-2}{4} + k - \alpha\xi \frac{d}{d\xi} \right) \mathcal{K}_{\frac{1}{\alpha}}^{\frac{(\sigma+3)\alpha+2}{4},n-\alpha} U(\xi)$$

$$= t^{\frac{(\sigma-1)\alpha-2}{4}} P_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} U(\xi). \tag{3.5}$$

In the same way we have

$$\frac{\partial^\alpha v}{\partial t^\alpha} = t^{\frac{(\sigma-1)\alpha-2}{4}} P_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} V(\xi), \tag{3.6}$$

where

$$\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} = \prod_{k=0}^{n-1} \left(\frac{(\sigma-1)\alpha-2}{4} + k - \alpha\xi \frac{d}{d\xi} \right) \mathcal{K}_{\frac{1}{\alpha}}^{\frac{(\sigma+3)\alpha+2}{4},n-\alpha}.$$

In addition, other terms of system (2.1) become

$$\begin{aligned} &r(t,x)u_{xx} = t^{\frac{(\sigma-1)\alpha-2}{4}} \left(\frac{2}{1+\sigma} \right)^2 \xi R\left(\xi^{\frac{1+\sigma}{2}}\right) \left(\frac{1-\sigma}{2} U'(\xi) + \xi U''(\xi) \right), \\ &fu = t^{\frac{(\sigma-1)\alpha-2}{4}} \Theta_1\left(\xi, \left|U(\xi)\right|, \left|V(\xi)\right|\right) U(\xi), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &s(t,x)v_{xx} = t^{\frac{(\sigma-1)\alpha-2}{4}} \left(\frac{2}{1+\sigma} \right)^2 \xi S\left(\xi^{\frac{1+\sigma}{2}}\right) \left(\frac{1-\sigma}{2} V'(\xi) + \xi V''(\xi) \right), \\ &gv = t^{\frac{(\sigma-1)\alpha-2}{4}} \Theta_2\left(\xi, \left|U(\xi)\right|, \left|V(\xi)\right|\right) V(\xi). \end{aligned} \tag{3.8}$$

Injecting (3.2)–(3.8) into system (2.1) leads to the desired result. □

Theorem 3 *Under the assumption of Theorem 2, when the de-focusing coefficients are chosen as* $f(t,x,|u|,|v|) = t^{-\alpha} A(x^{\frac{2}{1+\sigma}} t^{-\alpha})(|u|^2 - |v|^2)$, $g(t,x,|u|,|v|) = t^{-\alpha} B(x^{\frac{2}{1+\sigma}} t^{-\alpha})(|u|^2 - |v|^2)$ *and the real function* $R(\xi)$, $S(\xi)$, $A(\xi)$, $B(\xi)$ *are all analytic in* $\xi \neq 0$, *then the nontrivial analytic self-similar solutions of Eqs.* (1.2) *are given by*

$$\begin{aligned} &u(t,x) \\ &= u_0 t^{\frac{(\sigma+3)\alpha-2}{4}} + \frac{u_0(a_0(|v_0|^2 - |u_0|^2) - i\Omega\frac{(\sigma-1)\alpha+6}{4})}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2})r_0} x^{\frac{2}{1+\sigma}} t^{\frac{(\sigma-1)\alpha-2}{4}} \\ &\quad + \frac{[i\Omega\alpha + a_0(|v_0|^2 - |u_0|^2) + \frac{2(\sigma-1)}{(1+\sigma)^2} r_1]u_1 + [a_0(v_0 v_1^* + v_1 v_0^* - u_0 u_1^* - u_1 u_0^*) + a_1(|v_0|^2 - |u_0|^2)]u_0}{r_0(3-\sigma)(\frac{2}{1+\sigma})^2} \\ &\quad \times x^{\frac{4}{1+\sigma}} t^{\frac{(\sigma-5)\alpha-2}{4}} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \frac{i\Omega(\alpha n u_{n-1} - \frac{(\sigma-1)\alpha+6}{4} u_n) - (\frac{2}{1+\sigma})^2[\frac{1-\sigma}{2} r_n u_1 + \sum_{k=1}^{n-1}(\frac{1-\sigma}{2} + k)(k+1) r_{n-k} u_{k+1}]}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2} + n)(n+1)r_0} \right. \\ &\quad \left. + \frac{\sum_{m=1}^{n-1} a_m \sum_{l=0}^{m-1} u_l \sum_{k=0}^{m-l}(v_k v_{m-l-k}^* - u_k u_{m-l-k}^*) + (a_0 u_n + a_n u_0)(|v_0|^2 - |u_0|^2)}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2} + n)(n+1)r_0} \right\} \\ &\quad \times x^{\frac{2(n+1)}{1+\sigma}} t^{\frac{(\sigma-4n-1)\alpha-2}{4}}, \end{aligned}$$

and

$$
\begin{aligned}
&v(t,x)\\
&= v_0 t^{\frac{(\sigma+3)\alpha-2}{4}} + \frac{v_0(b_0(|v_0|^2-|u_0|^2)-i\Omega\frac{(\sigma-1)\alpha+6}{4})}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2})s_0} x^{\frac{2}{1+\sigma}} t^{\frac{(\sigma-1)\alpha-2}{4}}\\
&\quad + \frac{[i\Omega\alpha + b_0(|v_0|^2-|u_0|^2) + \frac{2(\sigma-1)}{(1+\sigma)^2}s_1]v_1 + [b_0(v_0v_1^* + v_1v_0^* - u_0u_1^* - u_1u_0^*) + b_1(|v_0|^2-|u_0|^2)]v_0}{s_0(3-\sigma)(\frac{2}{1+\sigma})^2}\\
&\quad \times x^{\frac{4}{1+\sigma}} t^{\frac{(\sigma-5)\alpha-2}{4}}\\
&\quad + \sum_{n=2}^{\infty}\Bigg\{ \frac{i\Omega(\alpha n v_{n-1} - \frac{(\sigma-1)\alpha+6}{4}v_n) - (\frac{2}{1+\sigma})^2[\frac{1-\sigma}{2}s_n v_1 + \sum_{k=1}^{n-1}(\frac{1-\sigma}{2}+k)(k+1)s_{n-k}v_{k+1}]}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2}+n)(n+1)s_0}\\
&\quad + \frac{\sum_{m=1}^{n-1} b_m \sum_{l=0}^{m-1} v_l \sum_{k=0}^{m-l}(v_k v_{m-l-k}^* - u_k u_{m-l-k}^*) + (b_0u_n + b_n u_0)(|v_0|^2-|u_0|^2)}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2}+n)(n+1)s_0}\Bigg\}\\
&\quad \times x^{\frac{2(n+1)}{1+\sigma}} t^{\frac{(\sigma-4n-1)\alpha-2}{4}},
\end{aligned}
\tag{3.9}
$$

where a_n, b_n, r_n, s_n *are expanding coefficients of* $A, B, \overline{R} = \xi R, \overline{S} = \xi S$, *and* $\Omega_n = \frac{\Gamma(\frac{1}{2}+(\frac{\sigma+3}{4}-n)\alpha)}{\Gamma(\frac{3}{2}+(\frac{\sigma-1}{4}-n)\alpha)}$ *are the parameters*.

Proof Under the analytic assumptions, according to (3.2) and (3.3), (3.1) can be rewritten as

$$
\begin{aligned}
&i\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} U(\xi) + \left(\frac{2}{1+\sigma}\right)^2 \overline{R(\xi)}\left(\frac{1-\sigma}{2}U'(\xi) + \xi U''(\xi)\right)\\
&\quad + A(\xi)\left(\left|U(\xi)\right|^2 - \left|V(\xi)\right|^2\right)U(\xi) = 0,\\
&i\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} V(\xi) + \left(\frac{2}{1+\sigma}\right)^2 \overline{S(\xi)}\left(\frac{1-\sigma}{2}V'(\xi) + \xi V''(\xi)\right)\\
&\quad + B(\xi)\left(\left|U(\xi)\right|^2 - \left|V(\xi)\right|^2\right)V(\xi) = 0.
\end{aligned}
\tag{3.10}
$$

We suppose that the solutions of (3.10) are formed as follows:

$$
U(\xi) = \sum_{n=0}^{\infty} u_n \xi^n, \qquad V(\xi) = \sum_{n=0}^{\infty} v_n \xi^n,
$$

and

$$
U^*(\xi) = \sum_{n=0}^{\infty} u_n^* \xi^n, \qquad V^*(\xi) = \sum_{n=0}^{\infty} v_n^* \xi^n,
\tag{3.11}
$$

where u_n, v_n are unknown expanding coefficients.

Substituting (3.11) into the first term of (3.10)to simplify the fractional terms of (3.1), we have

$$\begin{aligned}
&\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} U(\xi) \\
&= \left[1+\frac{(\sigma-1)\alpha-2}{4}-\alpha\xi\frac{d}{d\xi}\right] K_{\frac{1}{\alpha}}^{\frac{(\sigma+3)\alpha+2}{4},1-\alpha} \\
&= \left[1+\frac{(\sigma-1)\alpha-2}{4}-\alpha\xi\frac{d}{d\xi}\right] \sum_{n=0}^{\infty}\left(\int_1^{\infty}(s-1)^{-\alpha}s^{-\frac{(\sigma-1)\alpha-2}{4}-1}s^{n\alpha}\,ds\right)u_n\xi^n \\
&= \left[\frac{(\sigma-1)\alpha+6}{4}-\alpha\xi\frac{d}{d\xi}\right] \sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+(\frac{\sigma+3}{4}-n)\alpha)}{\Gamma(\frac{3}{2}+(\frac{\sigma-1}{4}-n)\alpha)}u_n\xi^n,
\end{aligned} \tag{3.12}$$

and

$$\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha} V(\xi) = \left[\frac{(\sigma-1)\alpha+6}{4}-\alpha\xi\frac{d}{d\xi}\right] \sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+(\frac{\sigma+3}{4}-n)\alpha)}{\Gamma(\frac{3}{2}+(\frac{\sigma-1}{4}-n)\alpha)}v_n\xi^n, \tag{3.13}$$

where we use the integral

$$\int_1^{\infty}(s-1)^{-\alpha}s^{-\frac{(\sigma-1)\alpha-2}{4}-1}s^{n\alpha}\,ds = \frac{\Gamma(\frac{1}{2}+(\frac{\sigma+3}{4}-n)\alpha)}{\Gamma(\frac{3}{2}+(\frac{\sigma-1}{4}-n)\alpha)}.$$

The following expressions show the derivatives of U, V:

$$\begin{aligned}
&U'(\xi) = \sum_{n=1}^{\infty} nu_n\xi^{n-1}, \qquad V'(\xi) = \sum_{n=1}^{\infty} nv_n\xi^{n-1}, \\
&U''(\xi) = \sum_{n=2}^{\infty} n(n-1)u_n\xi^{n-2}, \qquad V''(\xi) = \sum_{n=2}^{\infty} n(n-1)v_n\xi^{n-2},
\end{aligned} \tag{3.14}$$

Equaling the coefficients of ξ-power by plugging (3.11)–(3.14) into (3.10) and $\overline{R} = \sum_{n=0}^{\infty} r_n\xi^n$, $\overline{S} = \sum_{n=0}^{\infty} s_n\xi^n$, $A(\xi) = \sum_{n=0}^{\infty} a_n\xi^n$, $B(\xi) = \sum_{n=0}^{\infty} b_n\xi^n$, we obtain the following inductions:

for 0-power:

$$\begin{aligned}
&i\Omega\frac{(\sigma-1)\alpha+6}{4}u_0 + \left(\frac{2}{1+\sigma}\right)^2\frac{1-\sigma}{2}r_0u_1 + a_0\left(|u_0|^2-|v_0|^2\right)u_0 = 0, \\
&i\Omega\frac{(\sigma-1)\alpha+6}{4}v_0 + \left(\frac{2}{1+\sigma}\right)^2\frac{1-\sigma}{2}s_0v_1 + b_0\left(|u_0|^2-|v_0|^2\right)v_0 = 0,
\end{aligned} \tag{3.15}$$

for ξ-power:

$$\begin{aligned}
&-i\Omega\alpha u_1 + \left(\frac{2}{1+\sigma}\right)^2\left(r_0u_2(3-\sigma)+\frac{1-\sigma}{2}r_1u_1\right) \\
&\quad + a_0\left[\left(u_0u_1^*+u_1u_0^*-v_0v_1^*-v_1v_0^*\right)u_0 + \left(|u_0|^2-|v_0|^2\right)u_1\right] \\
&\quad + a_1\left(|u_0|^2-|v_0|^2\right)u_0 = 0,
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&-i\Omega\alpha v_1+\left(\frac{2}{1+\sigma}\right)^2\left(s_0v_2(3-\sigma)+\frac{1-\sigma}{2}s_1v_1\right)\\
&\quad+b_0\left[\left(u_0u_1^*+u_1u_0^*-v_0v_1^*-v_1v_0^*\right)v_0+\left(|u_0|^2-|v_0|^2\right)v_1\right]\\
&\quad+b_1\left(|u_0|^2-|v_0|^2\right)v_0=0,
\end{aligned}
$$

$\cdots$,

for ξ^n-power:

$$
\begin{aligned}
&i\Omega\left(\frac{(\sigma-1)\alpha+6}{4}u_n-\alpha nu_{n-1}\right)\\
&\quad+\left(\frac{2}{1+\sigma}\right)^2\left[\frac{1-\sigma}{2}r_nu_1+\sum_{k=1}^{n}\left(\frac{1-\sigma}{2}+k\right)(k+1)r_{n-k}u_{k+1}\right]\\
&\quad+\sum_{m=1}^{n-1}a_m\sum_{l=0}^{m-1}u_l\sum_{k=0}^{m-l}\left(u_ku_{m-l-k}^*-v_kv_{m-l-k}^*\right)\\
&\quad+(a_0u_n+a_nu_0)\left(|u_0|^2-|v_0|^2\right)=0,\\
&i\Omega\left(\frac{(\sigma-1)\alpha+6}{4}b_n-\alpha nv_{n-1}\right)\\
&\quad+\left(\frac{2}{1+\sigma}\right)^2\left[\frac{1-\sigma}{2}s_nv_1+\sum_{k=1}^{n}\left(\frac{1-\sigma}{2}+k\right)(k+1)s_{n-k}v_{k+1}\right]\\
&\quad+\sum_{m=1}^{n-1}b_m\sum_{l=0}^{m-1}v_l\sum_{k=0}^{m-l}\left(u_ku_{m-l-k}^*-v_kv_{m-l-k}^*\right)\\
&\quad+(b_0u_n+b_nu_0)\left(|u_0|^2-|v_0|^2\right)=0,
\end{aligned}
\tag{3.17}
$$

From (3.15)–(3.17) we know that

$$
\begin{aligned}
u_1&=\frac{u_0(a_0(|v_0|^2-|u_0|^2)-i\Omega\frac{(\sigma-1)\alpha+6}{4})}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2})r_0},\\
v_1&=\frac{v_0(b_0(|v_0|^2-|u_0|^2)-i\Omega\frac{(\sigma-1)\alpha+6}{4})}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2})s_0},\\
u_2&=\frac{[i\Omega\alpha+a_0(|v_0|^2-|u_0|^2)+\frac{2(\sigma-1)}{(1+\sigma)^2}r_1]u_1+[a_0(v_0v_1^*+v_1v_0^*-u_0u_1^*-u_1u_0^*)+a_1(|v_0|^2-|u_0|^2)]u_0}{r_0(3-\sigma)(\frac{2}{1+\sigma})^2},\\
v_2&=\frac{[i\Omega\alpha+b_0(|v_0|^2-|u_0|^2)+\frac{2(\sigma-1)}{(1+\sigma)^2}s_1]v_1+[b_0(v_0v_1^*+v_1v_0^*-u_0u_1^*-u_1u_0^*)+b_1(|v_0|^2-|u_0|^2)]v_0}{s_0(3-\sigma)(\frac{2}{1+\sigma})^2},\\
&\cdots,\\
u_{n+1}&=\frac{i\Omega(\alpha nu_{n-1}-\frac{(\sigma-1)\alpha+6}{4}u_n)-(\frac{2}{1+\sigma})^2[\frac{1-\sigma}{2}r_nu_1+\sum_{k=1}^{n-1}(\frac{1-\sigma}{2}+k)(k+1)r_{n-k}u_{k+1}]}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2}+n)(n+1)r_0}\\
&\quad+\frac{\sum_{m=1}^{n-1}a_m\sum_{l=0}^{m-1}u_l\sum_{k=0}^{m-l}(v_kv_{m-l-k}^*-u_ku_{m-l-k}^*)+(a_0u_n+a_nu_0)(|v_0|^2-|u_0|^2)}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2}+n)(n+1)r_0},
\end{aligned}
$$

$$v_{n+1} = \frac{i\Omega(\alpha n v_{n-1} - \frac{(\sigma-1)\alpha+6}{4} v_n) - (\frac{2}{1+\sigma})^2[\frac{1-\sigma}{2} s_n v_1 + \sum_{k=1}^{n-1}(\frac{1-\sigma}{2} + k)(k+1)s_{n-k}v_{k+1}]}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2} + n)(n+1)s_0}$$
$$+ \frac{\sum_{m=1}^{n-1} b_m \sum_{l=0}^{m-1} v_l \sum_{k=0}^{m-l}(v_k v^*_{m-l-k} - u_k u^*_{m-l-k}) + (b_0 u_n + b_n u_0)(|v_0|^2 - |u_0|^2)}{(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2} + n)(n+1)s_0}.$$

These give the nontrivial self-similar solutions (3.9). □

4 Convergence analysis for self-similar solution

We prove the convergence of solutions (3.9) in this section.

Theorem 4 *Solutions* (3.9) *converge on the region* $0 < |\xi| < 1$ *as the functions* $\overline{R}(\xi)$, $\overline{S}(\xi)$, $A(\xi)$, $B(\xi)$ *are all analytic*.

Proof The key scheme of the proof is to construct the majorant series by using the induction.

We divide the function (u_n, v_n) into real part (u_{nR}, v_{nR}) and imaginary part (u_{nI}, v_{nI}).

Assume two new analytic functions as follows:

$$P(\xi) = \sum_{n=0}^{\infty} p_n \xi^n = \sum_{n=0}^{\infty}(p_{nR} + ip_{nI})\xi^n, \qquad Q(\xi) = \sum_{n=0}^{\infty} q_n \xi^n = \sum_{n=0}^{\infty}(q_{nR} + iq_{nI})\xi^n \tag{4.1}$$

with positive real part p_{nR}, q_{nR} and imaginary part p_{nI}, q_{nI} satisfying $|u_{nR}| \le p_{nR}$, $|u_{nI}| \le p_{nI}$, $|v_{nR}| \le q_{nR}$, $|v_{nI}| \le q_{nI}$.

We choose

$$|u_{0R}| = p_{0R}, \qquad |u_{0I}| = p_{0I}, \qquad |v_{0R}| = q_{0R}, \qquad |v_{0I}| = q_{0I}. \tag{4.2}$$

For $n = 1, 2$, it is shown that

$$\begin{aligned}
|u_{1R}| &\le \frac{1}{2|\frac{r_0(1-\sigma)}{(1+\sigma^2)}|}\left(|u_{0R}||a_0|||v_0|^2 - |u_0|^2| + |u_{0I}||\Omega_0|\left|\frac{(\sigma-1)\alpha+6}{4}\right|\right) \\
&\le M_1\big(|u_{0R}| + |u_{0I}|\big) \le \sqrt{2}M_1|u_0| = \widetilde{M}_1\big(\Omega_0, |u_0|, |v_0|, |a_0|, |r_0|\big) = p_{1R}, \\
|u_{1I}| &\le \frac{1}{2|\frac{r_0(1-\sigma)}{(1+\sigma^2)}|}\left(|u_{0I}||a_0|||v_0|^2 - |u_0|^2| + |u_{0R}||\Omega_0|\left|\frac{(\sigma-1)\alpha+6}{4}\right|\right) \\
&\le M_1\big(|u_{0R}| + |u_{0I}|\big) \le \widetilde{M}_1\big(\Omega_0, |u_0|, |v_0|, |a_0|, |r_0|\big) = p_{1I},
\end{aligned} \tag{4.3}$$

$(M_1 = \max \frac{||v_0|^2-|u_0|^2|(|\frac{(\sigma-1)\alpha+2}{4}|\Omega_0)}{2|\frac{(1-\sigma)r_0}{(1+\sigma)^2}|}, \frac{||v_0|^2-|u_0|^2||a_0|}{2|\frac{(1-\sigma)r_0}{(1+\sigma)^2}|})$,

$$\begin{aligned}
|v_{1R}| &\le \frac{1}{2|\frac{s_0(1-\sigma)}{(1+\sigma^2)}|}\left(|v_{0R}||b_0|||v_0|^2 - |u_0|^2| + |v_{0I}||\Omega_0|\left|\frac{(\sigma-1)\alpha+6}{4}\right|\right) \\
&\le N_1\big(|v_{0R}| + |v_{0I}|\big) \le \sqrt{2}N_1|v_0| = \widetilde{M}_1\big(\Omega_0, |u_0|, |v_0|, |b_0|, |s_0|\big) = q_{1R}, \\
|v_{1I}| &\le \frac{1}{2|\frac{s_0(1-\sigma)}{(1+\sigma^2)}|}\left(|v_{0I}||b_0|||v_0|^2 - |u_0|^2| + |v_{0R}||\Omega_0|\left|\frac{(\sigma-1)\alpha+6}{4}\right|\right) \\
&\le N_1\big(|v_{0R}| + |v_{0I}|\big) \le \widetilde{N}_1\big(\Omega_0, |u_0|, |v_0|, |b_0|, |s_0|\big) = q_{1I},
\end{aligned} \tag{4.4}$$

$(N_1 = \max \frac{||v_0|^2-|u_0|^2|(|\frac{(\sigma-1)\alpha+2}{4}|\Omega_0)}{2|\frac{(1-\sigma)s_0}{(1+\sigma)^2}|}, \frac{||v_0|^2-|u_0|^2||b_0|}{2|\frac{(1-\sigma)s_0}{(1+\sigma)^2}|})$, and

$$
\begin{aligned}
|u_{2R}| &\le \frac{|u_{1R}|(|a_0|||v_0|^2-|u_0|^2||+2|\frac{\sigma-1}{(1+\sigma)^2}||r_1|)+\alpha\Omega_1|u_{1I}|}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\quad+\frac{|u_{0R}|[|a_1|||v_0|^2-|u_0|^2|+2|a_0|(|v_{0R}||v_{1R}|+|v_{0I}||v_{1I}|+|u_{0R}||u_{1R}|+|u_{0I}||u_{1I}|)]}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\le \frac{(|a_0|||v_0|^2-|u_0|^2||+2|\frac{\sigma-1}{(1+\sigma)^2}||r_1|+\alpha\Omega_1)M_1(|u_{1R}|+|u_{0I}|)}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\quad+\frac{|u_{0R}|[|a_1|||v_0|^2-|u_0|^2|+2|a_0|(M_1(|u_{0R}|+|u_{0I}|)^2+N_1(|v_{0R}|+|v_{0I}|)^2)]}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\le M_2\big[\big(\big(|u_{0R}|+|u_{0I}|\big)^2+\big(|v_{0R}|+|v_{0I}|\big)^2+1\big)|u_{0R}|+|u_{0I}|\big] \\
&\le M_2\big[\big(2\big(|u_0|^2+|v_0|^2\big)+\sqrt{2}\big)|u_0|\big] \\
&= \widetilde{M}_2\big(\Omega_0,\Omega_1,|u_0|,|v_0|,|a_0|,|r_0|,|a_1|,|r_1|\big)=p_{2R}, \qquad (4.5)\\
|u_{2I}| &\le \frac{|u_{1I}|(|a_0|||v_0|^2-|u_0|^2||+2|\frac{\sigma-1}{(1+\sigma)^2}||r_1|)+\alpha\Omega_1|u_{1R}|}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\quad+\frac{|u_{0I}|[|a_1|||v_0|^2-|u_0|^2|+2|a_0|(|v_{0R}||v_{1R}|+|v_{0I}||v_{1I}|+|u_{0R}||u_{1R}|+|u_{0I}||u_{1I}|)]}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\le \frac{(|a_0|||v_0|^2-|u_0|^2||+2|\frac{\sigma-1}{(1+\sigma)^2}||r_1|+\alpha\Omega_1)M_1(|u_{1R}|+|u_{0I}|)}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\quad+\frac{|u_{0I}|[|a_1|||v_0|^2-|u_0|^2|+2|a_0|(M_1(|u_{0R}|+|u_{0I}|)^2+N_1(|v_{0R}|+|v_{0I}|)^2)]}{|r_0||3-\sigma|(\frac{2}{1+\sigma})^2} \\
&\le M_2\big[\big(\big(|u_{0R}|+|u_{0I}|\big)^2+\big(|v_{0R}|+|v_{0I}|\big)^2+1\big)|u_{0R}|+|u_{0I}|\big] \\
&\le M_2\big[\big(2\big(|u_0|^2+|v_0|^2\big)+\sqrt{2}\big)|u_0|\big] \\
&= \widetilde{M}_2\big(\Omega_0,\Omega_1,|u_0|,|v_0|,|a_0|,|r_0|,|a_1|,|r_1|\big)=p_{2I}, \qquad (4.6)
\end{aligned}
$$

$(M_2 = \max\{\frac{M_1(|a_0|||v_0|^2-|u_0|^2|+2\frac{(\sigma-1)r_1}{(1+\sigma)^2}+\alpha\Omega_1)}{(\frac{2}{1+\sigma})^2|(3-\sigma)r_0|}, \frac{M_1\alpha\Omega_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)r_0|}, \frac{2|a_0|M_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)r_0|}, \frac{2|a_0|N_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)r_0|}, \frac{|a_1|||v_0|^2-|u_0|^2|}{(\frac{2}{1+\sigma})^2|(3-\sigma)r_0|}\})$.

Similarly, for v, we arrive at

$$
\begin{aligned}
|v_{2R}| &\le N_2\big[\big(\big(|u_{0R}|+|u_{0I}|\big)^2+\big(|v_{0R}|+|v_{0I}|\big)^2+1\big)|v_{0R}|+|v_{0I}|\big] \\
&\le N_2\big[\big(2\big(|u_0|^2+|v_0|^2\big)+\sqrt{2}\big)|v_0|\big] \\
&= \widetilde{N}_2\big(\Omega_0,\Omega_1,|u_0|,|v_0|,|b_0|,|s_0|,|b_1|,|s_1|\big)=q_{2R}, \qquad (4.7)\\
|v_{2I}| &\le N_2\big[\big(\big(|u_{0R}|+|u_{0I}|\big)^2+\big(|v_{0R}|+|v_{0I}|\big)^2+1\big)|v_{0R}|+|v_{0I}|\big] \\
&\le N_2\big[\big(2\big(|u_0|^2+|v_0|^2\big)+\sqrt{2}\big)|v_0|\big] \\
&= \widetilde{N}_2\big(\Omega_0,\Omega_1,|u_0|,|v_0|,|b_0|,|s_0|,|b_1|,|s_1|\big)=q_{2I}, \qquad (4.8)
\end{aligned}
$$

$$(N_2 = \max\{\frac{N_1(|b_0|||v_0|^2-|u_0|^2|+2\frac{(\sigma-1)s_1}{(1+\sigma)^2}+\alpha\Omega_1)}{(\frac{2}{1+\sigma})^2|(3-\sigma)s_0|}, \quad \frac{N_1\alpha\Omega_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)s_0|}, \quad \frac{2|b_0|N_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)s_0|}, \quad \frac{2|b_0|M_1}{(\frac{2}{1+\sigma})^2|(3-\sigma)s_0|},$$
$$\frac{|b_1|||v_0|^2-|u_0|^2|}{(\frac{2}{1+\sigma})^2|(3-\sigma)s_0|}\}).$$

If $n+1=3$, it is not hard to verify that

$$\begin{aligned}
&|u_{3R}| \le \widetilde{M}_3\big(\Omega_0,\Omega_1,\Omega_2,|u_0|,|v_0|,|a_0|,|r_0|,|a_1|,|r_1|,|a_2|,|r_2|\big) = p_{3R},\\
&|u_{3I}| \le \widetilde{M}_3\big(\Omega_0,\Omega_1,\Omega_2,|u_0|,|v_0|,|a_0|,|r_0|,|a_1|,|r_1|,|a_2|,|r_2|\big) = p_{3I},\\
&|v_{3R}| \le \widetilde{N}_3\big(\Omega_0,\Omega_1,\Omega_2,|u_0|,|v_0|,|b_0|,|s_0|,|b_1|,|s_1|,|b_2|,|s_2|\big) = q_{3R},\\
&|v_{3I}| \le \widetilde{N}_3\big(\Omega_0,\Omega_1,\Omega_2,|u_0|,|v_0|,|b_0|,|s_0|,|b_1|,|s_1|,|b_2|,|s_2|\big) = q_{3I}.
\end{aligned} \tag{4.9}$$

Then, for $n+1>3$, we assume

$$\begin{aligned}
&|u_{n+1,R}| \le \widetilde{M_{n+1}}\big(\Omega_0,\Omega_1,\dots,\Omega_n,|u_0|,|v_0|,|a_0|,|r_0|,\dots,|a_n|,|r_n|\big) = p_{n+1,R},\\
&|u_{n+1,I}| \le \widetilde{M_{n+1}}\big(\Omega_0,\Omega_1,\dots,\Omega_n,|u_0|,|v_0|,|a_0|,|r_0|,\dots,|a_n|,|r_n|\big) = p_{n+1,I},\\
&|v_{n+1,R}| \le \widetilde{N_{n+1}}\big(\Omega_0,\Omega_1,\dots,\Omega_n,|u_0|,|v_0|,|b_0|,|s_0|,\dots,|b_n|,|s_n|\big) = q_{n+1,R},\\
&|v_{n+1,I}| \le \widetilde{N_{n+1}}\big(\Omega_0,\Omega_1,\dots,\Omega_n,|u_0|,|v_0|,|b_0|,|s_0|,\dots,|b_n|,|s_n|\big) = q_{n+1,I}.
\end{aligned} \tag{4.10}$$

It is suffice to prove that the case $n+2$ also satisfies (4.10).

Since

$$\begin{aligned}
&|u_{n+2,R}|\\
&\le \frac{|\Omega_{n+1}|(|\frac{(\sigma-1)\alpha+6}{4}||u_{n,I}| + \alpha(n+1)|u_{n,I}|) + (\frac{2}{1+\sigma})^2(|\frac{1-\sigma}{2}||r_{n+1}||u_{1R}| + \sum_{k=1}^{n}|\frac{1-\sigma}{2}+k|(k+1)|r_{n+1-k}||u_{k+1,R}|)}{(\frac{2}{1+\sigma})^2||r_0|\frac{1-\sigma}{2}+n+1|(n+2)}\\
&\quad\times\frac{\sum_{m=1}^{n}|a_m|\sum_{l=0}^{m-1}|u_{lR}|\sum_{k=0}^{m-l}(|v_{kR}||v_{m-l-k,R}| + |v_{kI}||v_{m-l-k,I}| + |u_{kR}||u_{m-l-k,R}| + |u_{kI}||u_{m-l-k,I}|)}{(\frac{2}{1+\sigma})^2|r_0||\frac{1-\sigma}{2}+n|(n+1)}\\
&\quad\times\frac{\sum_{m=1}^{n}|a_m|\sum_{l=0}^{m-1}|u_{lI}|\sum_{k=0}^{m-l}(|v_{kR}||v_{m-l-k,R}| + |v_{kI}||v_{m-l-k,I}| + |u_{kR}||u_{m-l-k,R}| + |u_{kI}||u_{m-l-k,I}|)}{(\frac{2}{1+\sigma})^2|r_0||\frac{1-\sigma}{2}+n|(n+1)}\\
&\quad+\frac{||v_0|^2-|u_0|^2|(|a_0||u_{n+1,R}| + |a_{n+1}||u_{0R}|)}{|(\frac{2}{1+\sigma})^2(\frac{1-\sigma}{2}+n+1)(n+2)r_0|},
\end{aligned} \tag{4.11}$$

substituting (4.2)–(4.10) into (4.11), it is not hard to obtain that $|u_{n+2,R}|$ satisfies (4.10) by using induction.

In the same manner, we can also get the uniform bound of $|u_{n+2,I}|$, $|v_{n+2,R}|$, $|v_{n+2,I}|$ as well as $|u_{n+2,R}|$.

By virtue of the analytic assumption of R, S, A, B, now we select $p_{n+1,R}, p_{n+1,I}, q_{n+1,R}, q_{n+1,I}$ as the right-hand side of (4.10) and notice that all $\widetilde{M}_1, \widetilde{M}_2, \dots, \widetilde{M}_{n+1}, \dots, \widetilde{N}_1, \widetilde{N}_2, \dots, \widetilde{N}_{n+1}, \dots$ are bounded. Thus we can assume the uniform bound as $\widetilde{M} = \max\{\widetilde{M}_2, \dots, \widetilde{M}_{n+1}\} = \widetilde{M}(|u_0|, |v_0|, |a_0|, |r_0|, \dots |a_n|, |r_n|)$, $\widetilde{N} = \max\{\widetilde{N}_2, \dots, \widetilde{N}_{n+1}\} = \widetilde{N}(|u_0|, |v_0|, |b_0|, |s_0|, \dots |b_n|, |s_n|)$.

Finally, we can set up four majorant functions as follows:

$$\begin{aligned}
&P_R = p_{0R} + p_{1R}\xi + p_{2R}\xi^2 + \sum_{n=2}^{\infty} p_{n+1,R}\xi^{n+1} \le p_{0R} + \widetilde{M}_1\xi + \widetilde{M}\sum_{n=1}^{\infty}\xi^{n+1},\\
&P_I = p_{0I} + p_{1I}\xi + p_{2I}\xi^2 + \sum_{n=2}^{\infty} p_{n+1,I}\xi^{n+1} \le p_{0I} + \widetilde{M}_1\xi + \widetilde{M}\sum_{n=1}^{\infty}\xi^{n+1},\\
&Q_R = q_{0R} + q_{1R}\xi + q_{2R}\xi^2 + \sum_{n=2}^{\infty} q_{n+1,R}\xi^{n+1} \le q_{0R} + \widetilde{N}_1\xi + \widetilde{N}\sum_{n=1}^{\infty}\xi^{n+1},\\
&Q_I = q_{0I} + q_{1I}\xi + q_{2I}\xi^2 + \sum_{n=2}^{\infty} q_{n+1,I}\xi^{n+1} \le q_{0I} + \widetilde{N}_1\xi + \widetilde{N}\sum_{n=1}^{\infty}\xi^{n+1}.
\end{aligned} \tag{4.12}$$

From (4.2)–(4.8) we have bounded all the first three terms of (4.12). On the interval $0 < |\xi| < 1$, the series $\sum_{n=1}^{\infty}\xi^{n+1}$ converges to $\frac{\xi^2}{1-\xi}$, this ends the proof. □

5 Extension to m-component case

In this section, we verify that the above results of system (1.2) can also be extended to the m-component fractional NLS model

$$i\frac{\partial^\alpha u^j}{\partial t^\alpha} + r^j(t,x)u^j_{xx} + f^j\left(t,x,\left|u^1\right|,\dots,\left|u^m\right|\right)u^j = 0 \quad (j = 1,\dots,m), \tag{5.1}$$

which describes the kinetics of multi-body quantums with time-memories and nonlinear interactions.

By introducing $|u^j| = \sqrt{u^j u^{j*}}$, we discuss the following prolonged system:

$$\begin{aligned}
&iD_t^\alpha u^j + r^j(t,x)u^j_{xx} + f^j\left(t,x,\left|u^1\right|,\dots,\left|u^m\right|\right)u^j = 0,\\
&u^j f^j_{u^j} - u^{j*} f^j_{u^{j*}} = 0 \quad (j = 1,\dots,m),
\end{aligned} \tag{5.2}$$

where f, g are also regarded as two new functions.

Under the continuous group transformation

$$\begin{aligned}
&\bar{t} = t + \epsilon\tau\left(t,x,u^1,\dots,u^m,u^{1*},\dots,u^{m*}\right) + o\left(\epsilon^2\right),\\
&\bar{x} = x + \epsilon\xi\left(t,x,u^1,\dots,u^m,u^{1*},\dots,u^{m*}\right) + o\left(\epsilon^2\right),\\
&\bar{u^j} = u^j + \epsilon\Phi^j\left(t,x,u^1,\dots,u^m,u^{1*},\dots,u^{m*}\right) + o\left(\epsilon^2\right),\\
&\bar{u^{j*}} = u^{j*} + \epsilon\Phi^{j*}\left(t,x,u^1,\dots,u^m,u^{1*},\dots,u^{m*}\right) + o\left(\epsilon^2\right),\\
&\bar{f^j} = f^j + \epsilon F^j\left(t,x,u^1,\dots,u^m,u^{1*},\dots,u^{m*},f^1,\dots,f^m\right) + o\left(\epsilon^2\right) \quad (j = 1,\dots m)
\end{aligned} \tag{5.3}$$

the vector field of infinitesimal generators of Lie group is given by

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \sum_{j=1}^{m}\Phi^j\frac{\partial}{\partial u^j} + \sum_{j=1}^{m}\Phi^{j*}\frac{\partial}{\partial u^{j*}} + \sum_{j=1}^{m}F^j\frac{\partial}{\partial f^j}, \tag{5.4}$$

and the prolonged vector field is shown as

$$pr^{\alpha,2}V = V + \sum_{j=1}^{m} \Phi^{j,\alpha} \frac{\partial}{\partial D_t^\alpha w^j} + \sum_{j=1}^{m} \Phi^{jxx} \frac{\partial}{\partial u_{xx}^j} + \sum_{j=1}^{m} F^{j,w^j} \frac{\partial}{\partial f_{w^j}^j} + \sum_{j=1}^{m} F^{j,w^{j*}} \frac{\partial}{\partial f_{w^{j*}}^j} \quad (j = 1, \dots, m), \tag{5.5}$$

where τ, ξ, F^j, G^j are real functions and Φ^j are complex ones.

Similar to the two-component case, we have the following results for m-component fractional NLS system.

Theorem 5 *Under the continuous group transformation* (5.3), *the invariance of system* (5.2) *admits the following infinitesimal generators*:

$$\begin{aligned}
&\xi = \xi(x),\\
&\tau = \tau(t), \tau'''(t) = 0,\\
&\Phi^j = \left(\frac{\alpha - 1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_{j+2}\right)u^j,\\
&\Phi^{j*} = \left(\frac{\alpha - 1}{2}\tau'(t) + \frac{1}{2}\xi'(x) + c_{j+2}\right)u^{j*},\\
&F^j = -\alpha\tau'(t)f^j - \frac{r^j}{2}\xi'''(x),
\end{aligned}$$

where the diffusion coefficients solve

$$\tau r_t^j + \xi r_x^j + (\alpha\tau_t - 2\xi_x)r^j = 0 \quad (j = 1, \dots, m). \tag{5.6}$$

Theorem 6 *If* $\alpha\tau'(t) - 2\xi'(x) = c_2$, *then we obtain the following infinitesimal generators*:

$$\begin{aligned}
&\xi = \frac{c_1 - c_2}{2}x,\\
&\tau = \frac{c_1}{\alpha}t,\\
&\Phi^j = \left(\frac{c_1(\alpha - 1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_{j+2}\right)u^j,\\
&\Phi^{j*} = \left(\frac{c_1(\alpha - 1)}{2\alpha} + \frac{c_1 - c_2}{4} + c_{j+2}\right)u^{j*},\\
&F^j = -c_1 f^j,
\end{aligned} \tag{5.7}$$

with the coefficients

$$r^j = t^{-\frac{\alpha c_2}{c_1}} R^j\left(xt^{-\frac{\alpha(c_1 - c_2)}{2c_1}}\right), \tag{5.8}$$

and $m+2$*-dimensional Lie algebra* $V=\sigma_1 V_1+\sigma_2 V_2+\sum_{j=1}^{m}\sigma_{j+2}V_{j+2}$ *generated from the following vector fields*:

$$\begin{aligned}
&V_1=\frac{x}{2}\frac{\partial}{\partial x}+\frac{t}{\alpha}\frac{\partial}{\partial t}+\sum_{j=1}^{m}\frac{3\alpha-2}{4\alpha}u^j\frac{\partial}{\partial u^j}+\sum_{j=1}^{m}\frac{3\alpha-2}{4\alpha}u^{j*}\frac{\partial}{\partial u^{j*}}-\sum_{j=1}^{m}f^j\frac{\partial}{\partial f^j},\\
&V_2=\frac{x}{2}\frac{\partial}{\partial x}+\sum_{j-1}^{m}\frac{u^j}{4}\frac{\partial}{\partial u^j}+\sum_{j-1}^{m}\frac{u^{j*}}{4}\frac{\partial}{\partial u^{j*}},\\
&V_{j+2}=u^j\frac{\partial}{\partial u^j}+u^{j*}\frac{\partial}{\partial u^{j*}}\quad (j=1,\dots,m).
\end{aligned}\tag{5.9}$$

Theorem 7 *When taking* $\xi=x^{\frac{2}{1+\sigma}}t^{-\alpha}$*, under the scaling group* $\overline{V}=V_1+\sigma V_2(\sigma=-\frac{c_2}{c_1})$*, we have that nontrivial self-similar solutions* $u^j=t^{\frac{(\sigma+3)\alpha-2}{4}}U^j(x^{\frac{2}{1+\sigma}}t^{-\alpha})$ *solve the following fractional ODEs*:

$$\begin{aligned}
&i\mathcal{P}_{\frac{1}{\alpha}}^{\frac{(\sigma-1)\alpha-2}{4},\alpha}U^j(\xi)+\left(\frac{2}{1+\sigma}\right)^2\xi R^j\left(\xi^{\frac{1+\sigma}{2}}\right)\left(\frac{1-\sigma}{2}\frac{dU^j(\xi)}{d\xi}+\xi\frac{d^2U^j(\xi)}{d\xi^2}\right)\\
&\quad+\Theta^j\left(\xi,\left|U^1(\xi)\right|,\dots,\left|U^m(\xi)\right|\right)U^j(\xi)=0,\quad (j=1,\dots,m).
\end{aligned}\tag{5.10}$$

Remark The proof of Theorems 5–7 can be achieved in a similar manner, we omit it here.

6 Concluding remarks

A new method of solving two-component non-homogenous fractional NLS system is proposed in the present work. We first consider non-classical Lie symmetry for an enlarged PDE system by introducing new complex conjugate functions u^*, v^* and regard f, g as new functions related to u, v, u^*, v^*. Next, by reducing this new system in terms of scaling transformation and fractional Erdélyi–Kober operator, we acquire self-similar solutions. Meanwhile, we have proved the convergence of solutions as all the coefficients are analytic. Finally, we can extend these results to the multi-component fractional NLS model. It is more novel and convenient to apply this new method to solve fractional complex PDE problems rather than the classical symmetry method. The corresponding results are remarkably different from the previous work. Due to analyticity of the variant coefficients, the solutions of this fractional model are more general.

In addition, it is interesting to develop our improved method to study some other nonlinear complex fPDEs in mathematical physics. To obtain more new type solutions, we will explore more effective way in the future.

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RR was the major contributor in writing the manuscript. SZ checked the manuscript. All authors read and approved the final manuscript.

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