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Some k -fractional extension of Grüss-type inequalities via generalized Hilfer–Katugampola derivative

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Abstract

In this paper, we prove several inequalities of the Grüss type involving generalized k -fractional Hilfer–Katugampola derivative. In 1935, Grüss demonstrated a fascinating integral inequality, which gives approximation for the product of two functions. For these functions, we develop some new fractional integral inequalities. Our results with this new derivative operator are capable of evaluating several mathematical problems relevant to practical applications.

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1 Introduction

In many problems, fractional derivatives accomplish a vital role. Fractional derivatives are used to solve many imperative real-world problems. In recent decades, this field has been highly considered by scientists and mathematicians. Fractional calculus is an important branch of applied mathematics that tackles derivatives and integrals of arbitrary orders. Fractional integral inequalities have demonstrated being one of the most significant and effective tools for the advancement of many areas of pure and applied mathematics. The latest formulations vary in various components from the existing ones. For example, classic partial derivatives are thus defined so that the classical derivatives in the sense of Newton and Leibniz are recovered within the limit, where the derivative order is an integer.

Different researchers have given numerous applications of integral inequalities in different fields of mathematics. Grüss-type inequalities have significant applications, which include the h -integral arithmetic mean, inner product spaces, and the Mellin transform of polynomials in Hilbert spaces. There are numerous significant integral inequalities, which include Jensen's, Hölder's, Minkowski's, and reverse Minkowski's inequalities; for these applications, see [1–4, 6, 7, 9, 11–13, 15, 17, 18, 20].

In recent years the inequalities involving fractional calculus play a very important role in all mathematical fields, which gave rise to important theories in mathematics, engineering, physics, and other fields of science. A remarkably large number of inequalities of the above

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type involving the special fractional integral (such as the Liouville, Riemann–Liouville, Erdelyi–Kober, Katugampola, Hadamard, and Weyl types) have been investigated by many researchers and received considerable attention: see Kiblas et al. [10].

Let $\Phi, \Psi : [a, b] \rightarrow \mathbb{R}$, be integrable functions such that

$$\varphi_1(z) \leq \Phi(z) \leq \varphi_2(z) \quad \text{and} \quad \psi_1(z) \leq \Psi(z) \leq \psi_2(z), \quad \forall z \in [a, b].$$

Grüss-type inequality is defined as [8]

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Phi(z)\Psi(z) dz - \frac{1}{(b-a)^2} \int_a^b \Phi(z) dz \int_a^b \Psi(z) dz \right| \\ & \leq \frac{1}{4}(\varphi_2 - \varphi_1)(\psi_2 - \psi_1), \end{aligned} \tag{1.1}$$

where the constant $\frac{1}{4}$ is the best value, not replaceable by any other value.

The paper is organized as follows. In Sect. 1, we give an introduction of the Grüss-type inequalities. In Sect. 2, we present the definition of the k -fractional integrals in the sense of Riemann–Liouville fractional integral and spaces needed for our research. In Sect. 3, we show the Grüss inequality by using the generalized k -fractional Hilfer–Katugampola derivative with the k -Riemann–Liouville integral operator. In Sect. 4, we show another inequality by using the generalized k -fractional Hilfer–Katugampola derivative with the k -Riemann–Liouville integral operator. By means of the given Grüss-type inequality we prove other inequalities. Concluding marks are given in Sect. 5.

2 Preliminaries

Firstly, we include some mandatory definitions and mathematical preliminaries of the fractional operators of calculus.

Definition 2.1 ([10]) Let $[a, b]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty, \infty)$. By $M_q(a, b)$ we denote the set of the complex-valued Lebesgue-measurable function ψ on $[a, b]$,

$$M_q(a, b) = \left\{ \psi : \|\psi\|_q = \sqrt[q]{\int_a^b |\psi(z)|^q dz} < +\infty \right\}, \quad 1 \leq q < \infty.$$

In case $q = 1$, we have $M(a, b) = M_q(a, b)$.

Definition 2.2 ([5]) Diaz et al. defined the k -gamma function as

$$\Gamma_\kappa(z) = \int_0^\infty t^{z-1} e^{-\frac{t^\kappa}{\kappa}} dt \tag{2.1}$$

with $z, \kappa > 0$. It has the following properties: $\Gamma_\kappa(z + \kappa) = z\Gamma_\kappa(z)$ and $\Gamma_\kappa(z) = \kappa^{\frac{z}{\kappa}-1}\Gamma(\frac{z}{\kappa})$.

Definition 2.3 ([19]) Sarikaya et al. presented the left and right generalized k -fractional integrals of order ω with $m - 1 < \omega \leq m, m \in \mathbb{N}, \rho > 0, \kappa > 0, \omega > 0$ as

$$({}_\kappa^{\rho} \mathfrak{I}_a^\omega \psi)(z) = \frac{\rho^{1-\frac{\omega}{\kappa}}}{\kappa \Gamma_\kappa(\frac{\omega}{\kappa})} \int_a^z (z^\rho - y^\rho)^{\frac{\omega}{\kappa}-1} y^{\rho-1} \psi(y) dy, \quad z > a, \tag{2.2}$$

$$({}_\kappa^\rho \mathfrak{S}_b^\omega \psi)(z) = \frac{\rho^{1-\frac{\omega}{\kappa}}}{\kappa \Gamma_\kappa(\omega)} \int_z^b (y^\rho - z^\rho)^{\frac{\omega}{\kappa}-1} y^{\rho-1} \psi(y) dy, \quad z < b. \tag{2.3}$$

Definition 2.4 ([14]) Nisar et al. presented the left and right generalized k -fractional derivatives of order ω in terms of the integral defined in Definition 2.3 as

$$({}_\kappa^\rho \mathcal{D}_a^\gamma \psi)(z) = \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_a^{\kappa m - \omega} \psi)(z), \quad z > a, \tag{2.4}$$

$$({}_\kappa^\rho \mathcal{D}_b^\gamma \psi)(z) = \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_b^{\kappa m - \omega} \psi)(z), \quad z < b. \tag{2.5}$$

Definition 2.5 ([16]) Let $m - 1 < \omega \leq m, 0 \leq \theta \leq 1, m \in \mathbb{N}, \rho > 0, \kappa > 0$, and $\psi \in M_q(a, b)$. The generalized k -fractional Hilfer–Katugampola derivatives (left-sided and right-sided) are defined as

$$({}_\kappa^\rho \mathcal{D}_a^{\omega, \theta} \psi)(z) = \left({}_\kappa^\rho \mathfrak{S}_a^{\theta(\kappa m - \omega)} \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_a^{(1-\theta)(\kappa m - \omega)} \psi) \right)(z), \tag{2.6}$$

$$({}_\kappa^\rho \mathcal{D}_b^{\omega, \theta} \psi)(z) = \left({}_\kappa^\rho \mathfrak{S}_b^{\theta(\kappa m - \omega)} \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_b^{(1-\theta)(\kappa m - \omega)} \psi) \right)(z), \tag{2.7}$$

where \mathfrak{S} is the integral from Definition 2.3.

Lemma 2.1 Let $m - 1 < \omega \leq m, 0 \leq \theta \leq 1, m \in \mathbb{N}, \rho > 0, \kappa > 0$, and $\psi \in M_q(a, b)$. Then

$$\begin{aligned} {}_\kappa^\rho \mathcal{D}_a^{\omega, \theta} \psi(z) &= \left({}_\kappa^\rho \mathfrak{S}_a^{\theta(\kappa m - \omega)} \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_a^{(1-\theta)(\kappa m - \omega)} \psi) \right)(z) \\ &= \left({}_\kappa^\rho \mathfrak{S}_a^{\theta(\kappa m - \omega)} \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_a^{\kappa m - \omega - \theta(\kappa m - \omega)}) \right) \psi(z) \\ &= \left({}_\kappa^\rho \mathfrak{S}_a^{\theta(\kappa m - \omega)} \left(z^{1-\rho} \frac{d}{dz} \right)^m ({}_\kappa^m \cdot {}_\kappa^\rho \mathfrak{S}_a^{\kappa m - \{\omega + \theta(\kappa m - \omega)\}} \psi) \right)(z) \\ &= ({}_\kappa^\rho \mathfrak{S}_a^{\theta(\kappa m - \omega)} {}_\kappa^\rho D_a^{\omega + \theta(\kappa m - \omega)} \psi)(z) \quad \text{by equation (2.4)} \\ &= ({}_\kappa^\rho \mathfrak{S}_a^{\gamma - \omega} {}_\kappa^\rho D_a^\gamma \psi)(z) \\ &= ({}_\kappa^\rho \mathfrak{S}_a^{\gamma - \omega} \psi^{(\gamma)})(z) \\ &= \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_\kappa(\gamma - \omega)} \int_a^z (z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \psi^{(\gamma)}(y) dy \quad \text{by equation (2.2)}, \end{aligned}$$

where $\gamma = \omega + \theta(\kappa m - \omega), \omega > 0$, and $\psi^{(\gamma)}$ is the derivative of ψ from Definition 2.4.

So the previously defined generalized k -fractional Hilfer–Katugampola derivative can be written as

$$({}_\kappa^\rho \mathcal{D}_a^{\omega, \gamma} \psi)(z) = \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_\kappa(\gamma - \omega)} \int_a^z (z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \psi^{(\gamma)}(y) dy, \quad z > a, \tag{2.8}$$

$$({}_\kappa^\rho \mathcal{D}_b^{\omega, \gamma} \psi)(z) = \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_\kappa(\gamma - \omega)} \int_z^b (y^\rho - z^\rho)^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \psi^{(\gamma)}(y) dy, \quad z < b. \tag{2.9}$$

3 Auxiliary results

In this section, we prove a Grüss-type inequality by using the generalized k -fractional Hilfer–Katugampola derivative.

Theorem 3.1 *Let $\rho, \delta, \omega, \gamma, \kappa, a > 0$, and let $\Phi, \Psi \in M_q[a, b]$ be positive integrable functions on $[a, b]$. Suppose that there exist $\varphi_1, \varphi_2 \in [a, b]$ such that*

$$\varphi_1(z) \leq \Phi(z) \leq \varphi_2(z), \quad z \in [a, b]. \tag{3.1}$$

Then we have the following inequality for the generalized k -fractional Hilfer–Katugampola derivative:

$$\begin{aligned} & {}^{\rho}_{\kappa} \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}^{\rho}_{\kappa} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq {}^{\rho}_{\kappa} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}^{\rho}_{\kappa} \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned} \tag{3.2}$$

Proof Applying condition (3.1), we obtain

$$(\varphi_2(y) - \Phi(y))(\Phi(\zeta) - \varphi_1(\zeta)) \geq 0.$$

By simplifying we get

$$\varphi_2(y)\Phi(\zeta) + \Phi(y)\varphi_1(\zeta) \geq \varphi_2(y)\varphi_1(\zeta) + \Phi(y)\Phi(\zeta).$$

Talking the γ th derivative of this inequality with respect to y , we obtain

$$\varphi_2^{(\gamma)}(y)\Phi(\zeta) + \Phi^{(\gamma)}(y)\varphi_1(\zeta) \geq \varphi_2^{(\gamma)}(y)\varphi_1(\zeta) + \Phi^{(\gamma)}(y)\Phi(\zeta). \tag{3.3}$$

Multiplying inequality (3.3) by $\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)}(z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}$ and integrating with respect to y from a to z , we get

$$\begin{aligned} & \Phi(\zeta) \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \varphi_2^{(\gamma)}(y) dy \\ & + \varphi_1(\zeta) \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \Phi^{(\gamma)}(\zeta) dy \\ & \geq \varphi_1(\zeta) \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \varphi_2^{(\gamma)}(y) dy \\ & + \Phi(\zeta) \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \Phi^{(\gamma)}(y) dy. \end{aligned}$$

By (2.9) we have

$$\begin{aligned} & \Phi(\zeta) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \varphi_1(\zeta) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq \varphi_1(\zeta) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \Phi(\zeta) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned} \tag{3.4}$$

Again taking the γ th derivative of (3.4) with respect to ζ , we obtain

$$\begin{aligned} & \Phi^{(\gamma)}(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \varphi_1^{(\gamma)}(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq \varphi_1^{(\gamma)}(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \Phi^{(\gamma)}(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned} \tag{3.5}$$

Multiplying (3.5) by $\frac{\rho^{1-\frac{\gamma-\delta}{\kappa}}}{\kappa \Gamma_{\kappa}(\gamma-\delta)} (z^{\rho} - \zeta^{\rho})^{\frac{\gamma-\delta}{\kappa}-1} \zeta^{\rho-1}$ and integrating with respect to ζ from a to z , we have

$$\begin{aligned} & {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \Phi(z), \end{aligned}$$

which is the desired inequality. □

Corollary 3.1 *If we take $\gamma = 0$, then (3.2) becomes*

$$\begin{aligned} & {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \Phi(z) {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \varphi_2(z) + {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \varphi_1(z) {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \Phi(z) \\ & \geq {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \varphi_1(z) {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \varphi_2(z) + {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \Phi(z) {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \Phi(z), \end{aligned}$$

which converts to inequality for the generalized k -Riemann–Liouville integral.

Corollary 3.2 *If we consider $\Phi(z) = z^{\gamma}$, then*

$${}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \Phi(z) = \frac{(z^{\rho} - a^{\rho})^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_{\kappa}(\gamma - \omega + \kappa)} \quad \text{and} \quad {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \Phi(z) = \frac{(z^{\rho} - a^{\rho})^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_{\kappa}(\gamma - \delta + \kappa)}.$$

Inequality (3.2) becomes

$$\begin{aligned} & \frac{(z^{\rho} - a^{\rho})^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_{\kappa}(\gamma - \delta + \kappa)} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \frac{(z^{\rho} - a^{\rho})^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_{\kappa}(\gamma - \omega + \kappa)} {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) \\ & \geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \frac{(z^{\rho} - a^{\rho})^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_{\kappa}(\gamma - \omega + \kappa) \Gamma_{\kappa}(\gamma - \delta + \kappa)}. \end{aligned}$$

Corollary 3.3 *For $\gamma = 0$ and $\Phi(z) = 1$, we have*

$${}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \Phi(z) = \frac{(z^{\rho} - a^{\rho})^{\frac{1-\omega}{\kappa}}}{\rho^{\frac{1-\omega}{\kappa}} \Gamma_{\kappa}(1 - \omega + \kappa)} \quad \text{and} \quad {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \Phi(z) = \frac{(z^{\rho} - a^{\rho})^{\frac{1-\delta}{\kappa}}}{\rho^{\frac{1-\delta}{\kappa}} \Gamma_{\kappa}(1 - \delta + \kappa)}.$$

By inequality (3.2) we obtain

$$\begin{aligned} & \frac{(z^{\rho} - a^{\rho})^{\frac{1-\delta}{\kappa}}}{\rho^{\frac{1-\delta}{\kappa}} \Gamma_{\kappa}(1 - \delta + \kappa)} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \varphi_2(z) + \frac{(z^{\rho} - a^{\rho})^{\frac{1-\omega}{\kappa}}}{\rho^{\frac{1-\omega}{\kappa}} \Gamma_{\kappa}(1 - \omega + \kappa)} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \varphi_1(z) \\ & \geq {}_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \varphi_1(z) {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \varphi_2(z) + \frac{(z^{\rho} - a^{\rho})^{\frac{2-\omega-\delta}{\kappa}}}{\rho^{\frac{2-\omega-\delta}{\kappa}} \Gamma_{\kappa}(1 - \omega + \kappa) \Gamma_{\kappa}(1 - \delta + \kappa)}. \end{aligned}$$

Corollary 3.4 For $nz^\gamma \leq \Phi(z) \leq Nz^\gamma$, $z \in [a, b]$, we have

$${}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) = \frac{N(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} \quad \text{and} \quad {}^\rho \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) = \frac{n(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)}.$$

Inequality (3.2) is

$$\begin{aligned} & \frac{N(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \Phi(z) + \frac{n(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq \frac{nN(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned} \tag{3.6}$$

Corollary 3.5 Further, if we take $\gamma = 0$ in (3.6), then we get

$$\begin{aligned} & \frac{N(z^\rho - a^\rho)^{\frac{1-\omega}{\kappa}}}{\rho^{\frac{1-\omega}{\kappa}} \Gamma_\kappa(1 - \omega + \kappa)} {}^\rho \mathfrak{I}_a^\delta \Phi(z) + \frac{n(z^\rho - a^\rho)^{\frac{1-\delta}{\kappa}}}{\rho^{\frac{1-\delta}{\kappa}} \Gamma_\kappa(1 - \delta + \kappa)} {}^\rho \mathfrak{I}_a^\omega \Phi(z) \\ & \geq \frac{nN(z^\rho - a^\rho)^{\frac{2-\omega-\delta}{\kappa}}}{\rho^{\frac{2-\omega-\delta}{\kappa}} \Gamma_\kappa(1 - \omega + \kappa) \Gamma_\kappa(1 - \delta + \kappa)} + {}^\rho \mathfrak{I}_a^\delta \Phi(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z), \end{aligned}$$

where \mathfrak{I} is the generalized k -Riemann–Liouville integral.

Theorem 3.2 Let $\rho, \delta, \omega, \gamma, \kappa, a > 0$, and let $\Phi, \Psi \in M_q[a, b]$ be positive integrable functions on $[a, b]$. Suppose that (3.1) holds and there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in [a, b]$ such that

$$\psi_1(z) \leq \Psi(z) \leq \psi_2(z), \quad z \in [a, b]. \tag{3.7}$$

Then we have the following inequalities for the generalized k -fractional Hilfer–Katugampola derivative:

$$\begin{aligned} & {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) \\ & \geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z), \end{aligned} \tag{3.8}$$

$$\begin{aligned} & {}^\rho \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Psi(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Phi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \psi_2(z) \\ & \geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \psi_2(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z), \end{aligned} \tag{3.9}$$

$$\begin{aligned} & {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_2(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_2(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z), \end{aligned} \tag{3.10}$$

$$\begin{aligned} & {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_1(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ & \geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_1(z) + {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned} \tag{3.11}$$

Proof Applying condition (3.1) and (3.7), we get

$$(\varphi_2(y) - \Phi(y))(\Psi(\zeta) - \psi_1(\zeta)) \geq 0.$$

It follows that

$$\varphi_2(y)\Psi(\zeta) + \Phi(y)\psi_1(\zeta) \geq \varphi_2(y)\psi_1(\zeta) + \Phi(y)\Psi(\zeta).$$

Taking the γ th derivative with respect to y , we get

$$\varphi_2^{(\gamma)}(y)\Psi(\zeta) + \Phi^{(\gamma)}(y)\psi_1(\zeta) \geq \varphi_2^{(\gamma)}(y)\psi_1(\zeta) + \Phi^{(\gamma)}(y)\Psi(\zeta). \tag{3.12}$$

Multiplying (3) by $\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\omega)}(z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}$ and then integrating from a to z with respect to y , we have

$$\begin{aligned} &\Psi(\zeta)\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\omega)}\int_a^z(z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}\varphi_2^{(\gamma)}(y)dy \\ &\quad + \psi_1(\zeta)\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\omega)}\int_a^z(z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}\Phi^{(\gamma)}(\zeta)dy \\ &\geq \psi_1(\zeta)\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\omega)}\int_a^z(z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}\varphi_2^{(\gamma)}(y)dy \\ &\quad + \Psi(\zeta)\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\omega)}\int_a^z(z^\rho - y^\rho)^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}\Phi^{(\gamma)}(y)dy. \end{aligned}$$

Using definition (2.9), we obtain

$$\Psi(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + \psi_1(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z) \geq \psi_1(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + \Psi(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z). \tag{3.13}$$

Taking the γ th derivative of (3.13), we have

$$\begin{aligned} &\Psi^{(\gamma)}(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + \psi_1^{(\gamma)}(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z) \\ &\geq \psi_1^{(\gamma)}(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + \Psi^{(\gamma)}(\zeta)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z).m \end{aligned} \tag{3.14}$$

Multiplying (3.14) by $\frac{\rho^{1-\frac{\gamma-\delta}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\delta)}(z^\rho - \zeta^\rho)^{\frac{\gamma-\delta}{\kappa}-1}\zeta^{\rho-1}$, then integrating with respect to ζ from a to z , we have

$$\begin{aligned} &{}_\kappa^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + {}_\kappa^\rho \mathcal{D}_a^{\delta,\gamma} \psi_1(z)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z) \\ &\geq {}_\kappa^\rho \mathcal{D}_a^{\delta,\gamma} \psi_1(z)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) + {}_\kappa^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z)_\kappa^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z), \end{aligned}$$

which is the desired inequality (3.8).

Now we prove the other inequalities.

To prove inequality (3.9), we follow the same steps as in the proof of inequality (3.8) by letting

$$(\psi_2(y) - \Phi(y))(\Psi(\zeta) - \varphi_1(\zeta)) \geq 0.$$

Similarly, the inequalities

$$(\varphi_2(y) - \Phi(y))(\Psi(\zeta) - \psi_2(\zeta)) \geq 0,$$

$$(\varphi_1(y) - \Phi(y))(\Psi(\zeta) - \psi_1(\zeta)) \geq 0$$

leads to inequalities (3.10) and (3.11), respectively. □

Corollary 3.6 *Let $\gamma = 0$. The inequalities in Theorem 3.2 lead to the inequalities for the generalized k -Riemann–Liouville integral:*

$$\begin{aligned} & {}^\rho \mathfrak{I}_a^\delta \psi_1(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z) + {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \varphi_2(z) \\ & \geq {}^\rho \mathfrak{I}_a^\delta \psi_1(z) {}^\rho \mathfrak{I}_a^\omega \varphi_2(z) + {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z), \\ & {}^\rho \mathfrak{I}_a^\delta \varphi_1(z) {}^\rho \mathfrak{I}_a^\omega \Psi(z) + {}^\rho \mathfrak{I}_a^\delta \Phi(z) {}^\rho \mathfrak{I}_a^\omega \psi_2(z) \\ & \geq {}^\rho \mathfrak{I}_a^\delta \varphi_1(z) {}^\rho \mathfrak{I}_a^\omega \psi_2(z) + {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z), \\ & {}^\rho \mathfrak{I}_a^\delta \psi_2(z) {}^\rho \mathfrak{I}_a^\omega \varphi_2(z) + {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z) \\ & \geq {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \varphi_2(z) + {}^\rho \mathfrak{I}_a^\delta \psi_2(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z), \\ & {}^\rho \mathfrak{I}_a^\delta \psi_1(z) {}^\rho \mathfrak{I}_a^\omega \varphi_1(z) + {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z) \\ & \geq {}^\rho \mathfrak{I}_a^\delta \Psi(z) {}^\rho \mathfrak{I}_a^\omega \varphi_1(z) + {}^\rho \mathfrak{I}_a^\delta \psi_1(z) {}^\rho \mathfrak{I}_a^\omega \Phi(z). \end{aligned}$$

Corollary 3.7 *Let $nz^\gamma \leq \Phi(z) \leq Nz^\gamma$ and $mz^\gamma \leq \Psi(z) \leq Mz^\gamma$. We have*

$$\begin{aligned} {}^\rho \mathcal{D}_a^{\omega,\gamma} \varphi_2(z) &= \frac{N(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)}, \\ {}^\rho \mathcal{D}_a^{\delta,\gamma} \varphi_1(z) &= \frac{n(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)}, \\ {}^\rho \mathcal{D}_a^{\omega,\gamma} \psi_2(z) &= \frac{M(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)}, \\ {}^\rho \mathcal{D}_a^{\delta,\gamma} \psi_1(z) &= \frac{m(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)}, \end{aligned}$$

which by Theorem 3.2 lead to the inequalities

$$\begin{aligned} & \frac{m(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z) + \frac{N(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z) \\ & \geq \frac{mN(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} + {}^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z), \\ & \frac{n(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega,\gamma} \Psi(z) + \frac{M(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta,\gamma} \Phi(z) \\ & \geq \frac{nM(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} + {}^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z), \\ & \frac{NM(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} + {}^\rho \mathcal{D}_a^{\delta,\gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z) \end{aligned}$$

$$\begin{aligned} &\geq \frac{N(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) + \frac{M(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z), \\ &\frac{nm(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} + {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) \\ &\geq \frac{n(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \Psi(z) + \frac{m(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z). \end{aligned}$$

Corollary 3.8 *Let $\Phi(z) = z^\gamma, \Psi(z) = z^\gamma$. Then we have*

$$\begin{aligned} {}^\rho \mathcal{D}_a^{\omega, \gamma} \Phi(z) &= \frac{(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)}, \\ {}^\rho \mathcal{D}_a^{\delta, \gamma} \Phi(z) &= \frac{(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)}. \end{aligned}$$

The inequalities in Theorem 3.2 lead to

$$\begin{aligned} &\frac{(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) + \frac{(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) \\ &\geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \frac{(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)}, \\ &\frac{(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) + \frac{(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \psi_2(z) \\ &\geq {}^\rho \mathcal{D}_a^{\delta, \gamma} \varphi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \psi_2(z) + \frac{(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)}, \\ &{}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_2(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \frac{(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} \\ &\geq \frac{(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_2(z) + \frac{(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_2(z), \\ &{}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z) {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_1(z) + \frac{(z^\rho - a^\rho)^{\frac{2\gamma-\omega-\delta}{\kappa}}}{\rho^{\frac{2\gamma-\omega-\delta}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa) \Gamma_\kappa(\gamma - \delta + \kappa)} \\ &\geq \frac{(z^\rho - a^\rho)^{\frac{\gamma-\delta}{\kappa}}}{\rho^{\frac{\gamma-\delta}{\kappa}} \Gamma_\kappa(\gamma - \delta + \kappa)} {}^\rho \mathcal{D}_a^{\omega, \gamma} \varphi_1(z) + \frac{(z^\rho - a^\rho)^{\frac{\gamma-\omega}{\kappa}}}{\rho^{\frac{\gamma-\omega}{\kappa}} \Gamma_\kappa(\gamma - \omega + \kappa)} {}^\rho \mathcal{D}_a^{\delta, \gamma} \psi_1(z). \end{aligned}$$

4 Other related integral inequalities via generalized k -fractional Hilfer–Katugampola derivative

In this section, we prove other related integral inequalities by using the generalized k -fractional Hilfer–Katugampola derivative.

Theorem 4.1 *Let $\rho, \delta, \omega, \gamma, \kappa, a > 0$, and let $\Phi, \Psi \in M_q[a, b]$ be positive integrable functions on $[a, b]$. Suppose that there exists $\varphi_1, \varphi_2 \in [a, b]$. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have the*

following inequalities for the generalized k -fractional Hilfer–Katugampola derivative:

$$\begin{aligned} & \frac{1}{p} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Psi(z))^p {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z))^p + \frac{1}{q} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Phi(z))^q {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Psi(z))^q \\ & \geq {}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi \Psi \} (z) {}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi \Psi \} (z), \end{aligned} \tag{4.1}$$

$$\begin{aligned} & \frac{1}{p} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Psi(z))^p {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z))^p + \frac{1}{q} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Phi(z))^q {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Psi(z))^q \\ & \geq {}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi^{p-1} \Psi^{q-1} \} (z) {}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi \Psi \} (z), \end{aligned} \tag{4.2}$$

$$\begin{aligned} & \frac{1}{p} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Psi(z))^p {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z))^2 + \frac{1}{q} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Phi(z))^q {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Psi(z))^2 \\ & \geq {}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi^{\frac{2}{p}} \Psi^{\frac{2}{q}} \} (z) {}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi \Psi \} (z), \end{aligned} \tag{4.3}$$

$$\begin{aligned} & \frac{1}{p} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Psi(z))^q {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z))^2 + \frac{1}{q} {}^{\rho} \mathcal{D}_a^{\delta, \gamma} (\Phi(z))^p {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Psi(z))^2 \\ & \geq {}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi^{p-1} \Psi^{q-1} \} (z) {}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^{\frac{2}{p}} \Psi^{\frac{2}{q}} \} (z). \end{aligned} \tag{4.4}$$

Proof By Young’s inequality we have

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab, \quad a, b > 0.$$

Now, letting $a = \Phi(y)\Psi(\zeta)$ and $b = \Phi(\zeta)\Psi(y)$, we get

$$\frac{1}{p} (\Phi(y)\Psi(\zeta))^p + \frac{1}{q} (\Phi(\zeta)\Psi(y))^q \geq (\Phi(y)\Psi(\zeta))(\Phi(\zeta)\Psi(y)). \tag{4.5}$$

Taking the γ th derivative with respect to y of inequality (4.5), we have

$$\frac{1}{p} \{ (\Phi(y)\Psi(\zeta))^p \}^{(\gamma)} + \frac{1}{q} \{ (\Phi(\zeta)\Psi(y))^q \}^{(\gamma)} \geq \{ (\Phi(y)\Psi(\zeta))(\Phi(\zeta)\Psi(y)) \}^{(\gamma)}.$$

Multiplying by $\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_{\kappa}(\gamma-\omega)} (z^{\rho} - y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1}$ and integrating with respect to y from a to z , we get

$$\begin{aligned} & \frac{1}{p} \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho} - y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ (\Phi(y)\Psi(\zeta))^p \}^{(\gamma)} dy \\ & \quad + \frac{1}{q} \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho} - y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ (\Phi(\zeta)\Psi(y))^q \}^{(\gamma)} dy \\ & \geq \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa \Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho} - y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ (\Phi(y)\Psi(\zeta))(\Phi(\zeta)\Psi(y)) \}^{(\gamma)} dy. \end{aligned} \tag{4.6}$$

Applying the definition in (2.9), we obtain

$$\frac{1}{p} {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z)\Psi(\zeta))^p + \frac{1}{q} {}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(\zeta)\Psi(z))^q \geq {}^{\rho} \mathcal{D}_a^{\omega, \gamma} ((\Phi(z)\Psi(\zeta))(\Phi(\zeta)\Psi(z))). \tag{4.7}$$

It follows that

$$\frac{(\Psi(\zeta))^p}{p} {}_k^\rho \mathcal{D}_a^{\omega,\gamma} (\Phi(z))^p + \frac{(\Phi(\zeta))^q}{q} {}_k^\rho \mathcal{D}_a^{\omega,\gamma} (\Psi(z))^q \geq \Phi(\zeta)\Psi(\zeta) {}_k^\rho \mathcal{D}_a^{\omega,\gamma} \Phi(z)\Psi(z).$$

Again taking the γ th derivative of this inequality and then multiplying by $\frac{\rho^{1-\frac{\gamma-\delta}{\kappa}}}{\kappa\Gamma_\kappa(\gamma-\delta)} \times (z^\rho - \zeta^\rho)^{\frac{\gamma-\delta}{\kappa}-1} \zeta^{\rho-1}$ and integrating with respect to ζ from a to z , we obtain

$$\begin{aligned} & \frac{1}{p} {}_k^\rho \mathcal{D}_a^{\delta,\gamma} (\Psi(z))^p {}_k^\rho \mathcal{D}_a^{\omega,\gamma} (\Phi(z))^p + \frac{1}{q} {}_k^\rho \mathcal{D}_a^{\delta,\gamma} (\Phi(z))^q {}_k^\rho \mathcal{D}_a^{\omega,\gamma} (\Psi(z))^q \\ & \geq {}_k^\rho \mathcal{D}_a^{\delta,\gamma} \{\Phi\Psi\}(z) {}_k^\rho \mathcal{D}_a^{\omega,\gamma} \{\Phi\Psi\}(z). \end{aligned}$$

Which is the desired inequality.

Now to prove the other inequalities.

To prove inequality (4.2), we follow the same steps as in the proof of inequality (4.1) by letting

$$a = \frac{\Phi(y)}{\Phi(\zeta)}, \quad b = \frac{\Psi(y)}{\Psi(\zeta)}, \quad \Phi(\zeta), \Psi(\zeta) \neq 0.$$

Similarly, the suppositions

$$\begin{aligned} a &= \Phi(y)\Psi^{\frac{2}{p}}(\zeta), & b &= \Phi^{\frac{2}{q}}(\zeta)\Psi(y), \\ a &= \Phi^{\frac{2}{p}}(y)\Psi(\zeta), & b &= \Phi^{\frac{2}{q}}(y)\Psi(\zeta) \end{aligned}$$

lead to inequalities (4.3) and (4.4), respectively. □

Corollary 4.1 *Letting $\gamma = 0$ in Theorem 4.1, we have*

$$\begin{aligned} & \frac{1}{p} {}_k^\rho \mathfrak{I}_a^\delta (\Psi(z))^{p\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Phi(z))^p + \frac{1}{q} {}_k^\rho \mathfrak{I}_a^\delta (\Phi(z))^{q\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Psi(z))^q \geq {}_k^\rho \mathfrak{I}_a^\delta \{\Phi\Psi\}(z) {}_k^\rho \mathfrak{I}_a^\omega \{\Phi\Psi\}(z), \\ & \frac{1}{p} {}_k^\rho \mathfrak{I}_a^\delta (\Psi(z))^{p\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Phi(z))^p + \frac{1}{q} {}_k^\rho \mathfrak{I}_a^\delta (\Phi(z))^{q\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Psi(z))^q \\ & \geq {}_k^\rho \mathfrak{I}_a^\delta \{\Phi^{p-1}\Psi^{q-1}\}(z) {}_k^\rho \mathfrak{I}_a^\omega \{\Phi\Psi\}(z), \\ & \frac{1}{p} {}_k^\rho \mathfrak{I}_a^\delta (\Psi(z))^{p\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Phi(z))^2 + \frac{1}{q} {}_k^\rho \mathfrak{I}_a^\delta (\Phi(z))^{q\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Psi(z))^2 \geq {}_k^\rho \mathfrak{I}_a^\delta \{\Phi^{\frac{2}{p}}\Psi^{\frac{2}{q}}\}(z) {}_k^\rho \mathfrak{I}_a^\omega \{\Phi\Psi\}(z), \\ & \frac{1}{p} {}_k^\rho \mathfrak{I}_a^\delta (\Psi(z))^{q\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Phi(z))^2 + \frac{1}{q} {}_k^\rho \mathfrak{I}_a^\delta (\Phi(z))^{p\rho} {}_k^\rho \mathfrak{I}_a^\omega (\Psi(z))^2 \\ & \geq {}_k^\rho \mathfrak{I}_a^\delta \{\Phi^{p-1}\Psi^{q-1}\}(z) {}_k^\rho \mathfrak{I}_a^\omega \{\Phi^{\frac{2}{p}}\Psi^{\frac{2}{q}}\}(z). \end{aligned}$$

The inequalities convert to the generalized k -fractional Riemann–Liouville integral.

Theorem 4.2 *Let $\rho, \delta, \omega, \gamma, \kappa, a > 0$, and let $\Phi, \Psi \in M_q[a, b]$ be positive integrable functions on $[a, b]$. Suppose that there exist $\varphi_1, \varphi_2 \in [a, b]$. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have the inequalities for the generalized k -fractional Hilfer–Katugampola derivative:*

$$p {}_k^\rho \mathcal{D}_a^{\delta,\gamma} \{\Psi(z)\} {}_k^\rho \mathcal{D}_a^{\omega,\gamma} \{\Phi(z)\} + q {}_k^\rho \mathcal{D}_a^{\delta,\gamma} \{\Phi(z)\} {}_k^\rho \mathcal{D}_a^{\omega,\gamma} \{\Psi(z)\}$$

$$\begin{aligned}
 &\geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi^q \Psi^p \} (z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^p \Psi^q \} (z), \\
 & p {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi(z) \Psi^q(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Phi(z))^{p-1} \} + q {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ (\Phi(z))^q \Psi(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Psi(z))^{q-1} \} \\
 &\geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Psi^q \} (z) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^p(z) \}, \\
 & p {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ (\Psi(z))^{\frac{2}{p}} \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Phi(z)) \} + q {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ (\Phi(z))^{\frac{2}{q}} \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Psi(z))^q \} \\
 &\geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Phi^p(z) \Psi(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \Psi^q(z) \}, \\
 & p {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ (\Psi(z))^{q-1} \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Phi(z))^{\frac{2}{p}} (\Psi(z))^q \} \\
 &\quad + q {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ (\Phi(z))^{\frac{2}{q}} (\Psi(z))^p \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Psi(z))^{q-1} \} \\
 &\geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\delta, \gamma} \{ \Psi^2(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \}.
 \end{aligned} \tag{4.8}$$

Proof By arithmetic mean–geometric mean inequality we obtain

$$pa + qb \geq a^p b^q, \quad p + q = 1. \tag{4.9}$$

Now substituting

$$a = \Phi(y)\Psi(\zeta) \quad \text{and} \quad b = \Phi(\zeta)\Psi(y)$$

into (4.9), we obtain

$$p\Phi(y)\Psi(\zeta) + \frac{1}{q}\Phi(\zeta)\Psi(y) \geq (\Phi(y)\Psi(\zeta))^p (\Phi(\zeta)\Psi(y))^q. \tag{4.10}$$

Taking the γ th derivative of inequality (4.10), we have

$$p\{\Phi(y)\Psi(\zeta)\}^{(\gamma)} + q\{\Phi(\zeta)\Psi(y)\}^{(\gamma)} \geq \{(\Phi(y)\Psi(\zeta))^p (\Phi(\zeta)\Psi(y))^q\}^{(\gamma)}. \tag{4.11}$$

Multiplying (4.11) by $\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)}(z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}$ and integrating with respect to y from a to z , we obtain

$$\begin{aligned}
 & p \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{\Phi(y)\Psi(\zeta)\}^{(\gamma)} dy \\
 &\quad + q \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{\Phi(\zeta)\Psi(y)\}^{(\gamma)} dy \\
 &\geq \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{(\Phi(y)\Psi(\zeta))^p (\Phi(\zeta)\Psi(y))^q\}^{(\gamma)} dy.
 \end{aligned} \tag{4.12}$$

Now by definition (2.9) we have

$$p {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z) \Psi(\zeta) \} + \frac{1}{q} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(\zeta) \Psi(z) \} \geq {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ (\Phi(y) \Psi(\zeta))^p (\Phi(\zeta) \Psi(y))^q \},$$

which can be written in simplified form as

$$p\Psi(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Phi(z)) + q\Phi(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} (\Psi(z)) \geq \Phi(\zeta)\Psi(\zeta) {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^p(z) \Psi^q(z) \}. \tag{4.13}$$

Again taking the γ th derivative of (4.13) and then multiplying by $\frac{\rho^{1-\frac{\gamma-\delta}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\delta)}(z^{\rho}-\zeta^{\rho})^{\frac{\gamma-\delta}{\kappa}-1}\zeta^{\rho-1}$ and integrating with respect to ζ from a to z , we have

$$p_{\kappa}^{\rho} \mathcal{D}_a^{\delta,\gamma}(\Psi(z))_{\kappa}^{\rho} \mathcal{D}_a^{\omega,\gamma}(\Phi(z)) + q_{\kappa}^{\rho} \mathcal{D}_a^{\delta,\gamma}(\Phi(z))_{\kappa}^{\rho} \mathcal{D}_a^{\omega,\gamma}(\Psi(z)) \geq_{\kappa}^{\rho} \mathcal{D}_a^{\delta,\gamma}\{\Phi^p(z)\Psi^q(z)\}_{\kappa}^{\rho} \mathcal{D}_a^{\omega,\gamma}\{\Phi^p(z)\Psi^q(z)\},$$

which is the desired inequality.

For the second inequality of (4.8), let

$$a = \frac{\Phi(y)}{\Phi(\zeta)}, \quad b = \frac{\Psi(y)}{\Psi(\zeta)}, \quad \Phi(\zeta), \Psi(\zeta) \neq 0.$$

Proceeding in the same way, as in the proof of the first part of inequality (4.8), we obtain the desired one.

Now for the third and fourth parts of inequality (4.8), let

$$a = \Phi(y)\Psi^{\frac{2}{p}}(\zeta), \quad b = \Phi^{\frac{2}{q}}(\zeta)\Psi(y),$$

$$a = \frac{\Phi^{\frac{2}{p}}(y)}{\Psi(\zeta)}, \quad b = \frac{\Phi^{\frac{2}{q}}(\zeta)}{\Psi(y)} \cdot \Psi(\zeta), \quad \Psi(y) \neq 0.$$

These substitutions lead to the desired results. □

Corollary 4.2 *Letting $\gamma = 0$ in (4.8), we obtain the inequalities for the generalized k -fractional Riemann–Liouville integral:*

$$p_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Psi(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Phi(z) \} + q_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Phi(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Psi(z) \} \geq_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Phi^q \Psi^p \}(z)_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Phi^p \Psi^q \}(z),$$

$$p_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Phi(z) \Psi^q(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Phi(z))^{p-1} \} + q_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ (\Phi(z))^q \Psi(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Psi(z))^{q-1} \} \geq_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Psi^q \}(z)_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Phi^p(z) \},$$

$$p_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ (\Psi(z))^{\frac{2}{p}} \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Phi(z)) \} + q_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ (\Phi(z))^{\frac{2}{q}} \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Psi(z))^q \} \geq_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Phi^p(z) \Psi(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Phi^2(z) \Psi^q(z) \},$$

$$p_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ (\Psi(z))^{q-1} \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Phi(z))^{\frac{2}{p}} (\Psi(z))^q \} + q_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ (\Phi(z))^{\frac{2}{q}} (\Psi(z))^p \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ (\Psi(z))^{q-1} \} \geq_{\kappa}^{\rho} \mathfrak{I}_a^{\delta} \{ \Psi^2(z) \}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega} \{ \Phi^2(z) \}.$$

Theorem 4.3 *Let $\rho, \delta, \omega, \gamma, \kappa, a > 0$, and let $\Phi, \Psi \in M_q[a, b]$ be positive integrable functions on $[a, b]$. Suppose that there exist $\varphi_1, \varphi_2 \in [a, b]$. Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let*

$$P = \min_{0 \leq y \leq z} \frac{\Phi(y)}{\Psi(y)} \quad \text{and} \quad Q = \max_{0 \leq y \leq z} \frac{\Phi(y)}{\Psi(y)}. \tag{4.14}$$

Then we have the following inequalities for the generalized k -fractional Hilfer–Katugampola derivative:

$$\begin{aligned}
 & \frac{(P+Q)^2}{4PQ} {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}^2 \geq {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi(z) \}^2 {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z) \}^2 \geq 0, \\
 & \frac{\sqrt{P}-\sqrt{Q}}{2\sqrt{PQ}} {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \\
 & \geq \sqrt{{}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi(z) \}^2 {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z) \}^2} - {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \geq 0, \\
 & \frac{P-Q}{4PQ} {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}^2 \\
 & \geq {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi(z) \}^2 {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z) \}^2 - [{}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}]^2 \geq 0.
 \end{aligned} \tag{4.15}$$

Proof Using the condition in (4.14), we have

$$\left(\frac{\Phi(y)}{\Psi(y)} - P \right) \left(Q - \frac{\Phi(y)}{\Psi(y)} \right) \Psi^2(y) \geq 0,$$

which can be written as

$$(P+Q)\Phi(y)\Psi(y) \geq \Phi^2(y) + PQ\Phi^2(y).$$

Taking the γ th derivative of this inequality, we get

$$(P+Q)\{ \Phi(y)\Psi(y) \}^{(\gamma)} \geq \{ \Phi^2(y) \}^{(\gamma)} + PQ\{ \Phi^2(y) \}^{(\gamma)}. \tag{4.16}$$

Multiplying (4.16) by $\frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)}(z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1}y^{\rho-1}$ and integrating from a to z with respect to y , we obtain

$$\begin{aligned}
 & (P+Q) \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ \Phi(y)\Psi(y) \}^{(\gamma)} dy \\
 & \geq \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ \Phi^2(y) \}^{(\gamma)} dy \\
 & \quad + PQ \frac{\rho^{1-\frac{\gamma-\omega}{\kappa}}}{\kappa\Gamma_{\kappa}(\gamma-\omega)} \int_a^z (z^{\rho}-y^{\rho})^{\frac{\gamma-\omega}{\kappa}-1} y^{\rho-1} \{ \Phi^2(y) \}^{(\gamma)} dy.
 \end{aligned} \tag{4.17}$$

By (2.9) we have

$$(P+Q) {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \geq {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \} + {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \}. \tag{4.18}$$

Now, since $PQ > 0$ and

$$\left(\sqrt{{}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \}} \sqrt{PQ {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \}} \right)^2 \geq 0,$$

we get

$${}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \} + PQ {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \} \geq 2 \sqrt{{}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \}} \sqrt{PQ {}^{\rho}_{\kappa} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \}}. \tag{4.19}$$

So, from inequalities (4.18) and (4.19) we get

$$(P + Q)^2 \left[{}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \right]^2 \geq 4PQ {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \}, \tag{4.20}$$

which is the required result.

From inequality (4.20) we obtain

$$\frac{P + Q}{2\sqrt{PQ}} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \geq \sqrt{{}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi^2(z) \} {}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Psi^2(z) \}}. \tag{4.21}$$

Subtracting ${}_{\kappa}^{\rho} \mathcal{D}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}$ from inequality (4.21) leads to the second part of inequality (4.15). Analogously, we can prove the third part of inequality (4.15). \square

Corollary 4.3 *Letting $\gamma = 0$ in inequality (4.15), the inequalities turn to inequalities for the generalized k -fractional Riemann–Liouville integral:*

$$\begin{aligned} \frac{(P + Q)^2}{4PQ} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}^2 &\geq {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Psi(z) \}^{2\rho} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z) \}^2 \geq 0, \\ \frac{\sqrt{P} - \sqrt{Q}}{2\sqrt{PQ}} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} &\geq \sqrt{{}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Psi(z) \}^{2\rho} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z) \}^2} - {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \} \geq 0, \\ \frac{P - Q}{4PQ} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}^2 &\geq {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Psi(z) \}^{2\rho} {}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z) \}^2 - [{}_{\kappa}^{\rho} \mathfrak{I}_a^{\omega, \gamma} \{ \Phi(z)\Psi(z) \}]^2 \geq 0. \end{aligned}$$

5 Conclusion

In this paper, we have presented the Grüss-type inequality via the generalized k -fractional Hilfer–Katugampola derivative. We also proved other related inequalities by using the given operator. The given derivative operator converts to the k -Riemann–Liouville fractional integral by taking $\gamma = 0$. The results are very significant and fascinating. Moreover, other related integral inequalities can be easily derived by using the given derivative operator.

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Authors' contributions

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