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On a two-dimensional fractional thermoelastic system with nonlocal constraints describing a fractional Kirchhoff plate

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Abstract

We show herein the existence and uniqueness of solutions for coupled fractional order partial differential equations modeling a thermoelastic fractional Kirchhoff plate model associated with initial, Dirichlet, and nonlocal boundary conditions involving fractional Caputo derivative. Some efficient results of existence and uniqueness are obtained by employing the energy inequality method.

Keywords: Fractional system; Nonlocal condition; Existence and uniqueness; Energy inequality; Kirchhoff plate

1 Introduction

The systems of differential equations of time fractional order have been studied by many authors, and several results have been obtained. These types of systems have been successfully used in modeling many problems in different processes and systems such as physical and biological ones. Fractional calculus (fractional derivatives) can be used to describe viscoelastic materials much better than using ordinary derivatives, since for the ordinary derivatives the solution of the system predicts an instantaneous response, but when using the fractional derivatives, the solution of the system predicts a retarded response that depends on the history of the applied causes (see [1]), see also [2–5]. Many generalizations of thermoelasticity coupled theory were investigated (see [6]), they model heat conduction in solids as a wave propagation phenomenon. In this regard, the reader can see also [7–13], where the authors studied other models of fractional order thermoelasticity. Some new and recent results on fractional Caputo and Riemann–Liouville operators and their applications can be found in [14–19]. The reader also could refer to some recent thermoelasticity problems investigated by [20–22]. It is important to mention that fractional nonlocal problems are much harder to deal with, and this is because of the nonlocal nature of the fractional derivative and the nonlocal nature of the boundary condition (boundary integral condition). It seems that the functional analysis method we apply in this paper is very efficient to solve some nonlocal fractional initial boundary value problems for single and systems of some different classes of partial differential equations. We can find only a few papers that use the previous method in the literature, and we can cite, for example, [23–28].

Motivated by the above papers, in this work we deal with the existence and uniqueness of solutions for a fractional order initial boundary value problem for a two-dimensional coupled linear thermoelastic system of fourth order with nonlocal conditions defined by problem (2.1)–(2.4), which models a thermoelastic fractional Kirchhoff plate. If in our fractional thermoelasticity model (2.1) we let α to approach 1, we obtain some classes of classical thermoelastic models which all describe vibrations of some thin thermoelastic plate, and these vibrations are described by the Kirchhoff plate. See for example the model studied in [29]: In the domain $Q = \Omega \times (0,T)$, with Ω a bounded, open, connected set in $\mathbb{R}^2$, with boundary $\partial Q = \partial\Omega \times (0,T)$, the authors consider the following nonhomogeneous controlled system associated with some initial and boundary conditions:

$$\begin{cases} V_{tt} - \eta\Delta V_{tt} + \Delta^2 V = F, & \text{in } Q, \\ \beta U_t - \Delta U - C\Delta V_t = H, & \text{in } Q, \end{cases} \tag{1.1}$$

where η, β, and C, were positive constants. The authors considered the null controllability problem for system (1.1), which describes thermoelastic plates. The reader could also see [30], where the authors studied the uniform stability of an integer order thermoelastic plate, with some prescribed initial and boundary conditions taken from [31], which reads as follows:

$$\begin{cases} V_{tt} - \gamma\Delta V_{tt} + \Delta^2 V + \alpha\Delta U = 0, & \text{in } (0,\infty) \times \Omega, \\ \beta U_t - \eta\Delta U + \sigma U - \alpha\Delta V_t = 0, & \text{in } (0,\infty) \times \Omega, \end{cases} \tag{1.2}$$

where α, β, η are positive constants and γ is a nonnegative constant. For other nonfractional models, the reader could refer to [32–34] and the references therein.

This paper is structured as follows. After a short introduction in section one, in section two, the problem to investigate is reformulated and some function spaces are introduced. In section three, the main result of uniqueness of the solution of the posed problem is given. In section four, we establish the proof of the main result concerning the solvability of the posed problem.

2 Problem setting

Let $\Omega = (0,a) \times (0,b)$ be a bounded open subset of $\mathbb{R}^2$ with sufficiently smooth boundary, and let $T > 0$ be the terminal time. We consider on $(0,T) \times \Omega$ the following inhomogeneous fractional thermoelastic system with the control functions $f \in L^2(0,T;L^2(\Omega))$ (external force), and $g \in L^2(0,T;L^2(\Omega))$ (external thermal influence), which reads as follows:

$$\begin{cases} \mathcal{L}_1(\mathcal{M},\theta) = \partial_t^{\alpha+1}\mathcal{M} + \Delta^2\mathcal{M} - \gamma\partial_t^{\alpha+1}(\Delta\mathcal{M}) + d\Delta\theta \\ \qquad\qquad\quad = f(x,y,t), \\ \mathcal{L}_2(\mathcal{M},\theta) = \beta\partial_t^{\alpha}\theta - \eta\Delta\theta + \delta\theta - d\Delta\mathcal{M}_t \\ \qquad\qquad\quad = g(x,y,t), \end{cases} \tag{2.1}$$

along with the initial conditions

$$\begin{cases} \ell_1\mathcal{M} = \mathcal{M}(x,y,0) = \mathcal{M}_0(x,y), \\ \ell_2\mathcal{M} = \mathcal{M}_t(x,y,0) = \mathcal{M}_1(x,y), \\ \ell_3\theta = \theta(x,y,0) = \theta_0(x,y), \end{cases} \tag{2.2}$$

boundary Dirichlet conditions on the displacement $\mathcal{M}$

$$\left\{ \begin{array}{l} \mathcal{M}(0,y,t) = \mathcal{M}(a,y,t) = 0, \\ \mathcal{M}(x,0,t) = \mathcal{M}(x,b,t) = 0, \end{array} \right| , \tag{2.3}$$

and boundary integral conditions on the displacement $\mathcal{M}$ and on the thermal damping θ

$$\begin{cases} \int_0^a x^k \mathcal{M}\,dx = \int_0^b y^k \mathcal{M}\,dy = 0, \\ \int_0^1 x^k \theta\,dx = \int_0^1 y^k \theta\,dy = 0, \quad k = 0,1, \end{cases} \tag{2.4}$$

where ℓ_1, ℓ_2, ℓ_3 designate the trace operators and $\partial_t^{\alpha+1}\mathcal{M}$ is the time fractional Caputo derivative of order $1+\alpha$ with $\alpha \in (0,1)$ for the function $\mathcal{M}$ [14], and it is given by the formula

$$\partial_t^{\alpha+1}\mathcal{M}(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathcal{M}_{\tau\tau}(x,\tau)}{(t-\tau)^\alpha}\,d\tau,$$

and β, δ, γ, d, and η are strictly positive constants. In the elastic differential equation in (2.1), the term $\partial_t^{\alpha+1}(\Delta\mathcal{M})$ accounts for rotational inertia for $\gamma > 0$, where γ is proportional to the thickness of the plate, the constant d stands for the thermoelastic coupling parameter, the parameters δ and η are thermal coefficients, and β is considered as the heat capacity. The given model (2.1)–(2.4) mathematically describes a fractional Kirchhoff plate, the displacement of which is represented by the function $\mathcal{M}$ subjected to a thermal damping as quantified by θ. The boundary integral conditions may be interpreted as the average and weighted average of the displacement and the thermal damping. We mention here that some of the hinged conditions are replaced by nonlocal conditions (2.4), this may be due to the fact that some of the data cannot be measured on the boundary.

For establishing the existence and uniqueness of solution of problems (2.1)–(2.4), we reformulate them in an operator form, which allows us to obtain some energy estimates needed for our proofs. The solution of problems (2.1)–(2.4) can be regarded as the solution of the operator equation

$$\mathcal{A}U = W = \bigl(\{f,u_0,u_1\},\{g,\theta_0\}\bigr), \tag{2.5}$$

where $\mathcal{A}:\mathcal{E} \longrightarrow \mathcal{F}$ is an unbounded operator with domain $\mathcal{D}(\mathcal{A})$ consisting of all functions $(\mathcal{M},\theta)$ belonging to $L^2(Q^T)\times L^2(Q^T)$ for which $\mathcal{M}_{xxxx}$, $\mathcal{M}_{yyyy}$, $\partial_t^\alpha\mathcal{M}_t$, $\mathcal{M}_{xxtt}$, $\mathcal{M}_{yytt}$, $\mathcal{M}_{xxt}$, $\mathcal{M}_{yyt}$, $\mathcal{M}_t$, θ_{xx}, θ_{yy}, $\partial_t^\alpha\theta$ are in $L^2(Q^T)$, and satisfying conditions (2.3)–(2.4). Let $\mathcal{E}$ be a Banach space of functions $U = (\mathcal{M},\theta) \in (L^2(Q^T))^2$ endowed with the finite norm

$$\begin{aligned} \|U\|_{\mathcal{E}}^2 = {}& \bigl\|\mathcal{M}_x(\cdot,\cdot,\tau)\bigr\|^2_{C(0,T;B_2^{1,y}(\Omega))} + \bigl\|\mathcal{M}_y(\cdot,\cdot,\tau)\bigr\|^2_{C(0,T;B_2^{1,x}(\Omega))} \\ & + \bigl\|\mathcal{M}(\cdot,\cdot,\tau)\bigr\|^2_{C(0,T;L^2(\Omega))}, \end{aligned} \tag{2.6}$$

and let $\mathcal{F}$ be a Hilbert space constituting of the elements $W = (\{f, \mathcal{M}_0, \mathcal{M}_1\}, \{g, \theta_0\})$ equipped with the norm

$$\begin{aligned}\|\mathcal{A}U\|_{\mathcal{F}}^2 &= \|W\|_{\mathcal{F}}^2 \\ &= \|\mathcal{M}_0\|_{H_0^1(\Omega)}^2 + \|\mathcal{M}_1\|_{L^2(\Omega)}^2 + \|\theta_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q^T)}^2 + \|g\|_{L^2(Q^T)}^2. \end{aligned} \quad (2.7)$$

We introduce here the following function spaces: Let $H^1(\Omega)$ be the usual Sobolev space with the inner product

$$(Z, Y)_{H^1(\Omega)} = (Z, Y)_{L^2(\Omega)} + (Z_x, Y_x)_{L^2(\Omega)},$$

where $L^2(\Omega)$ is the space of square integrable functions. And let $C(0, T; B_2^N(\Omega))$, $C(0, T; B_2^{1,x}(\Omega))$, $C(0, T; B_2^{1,y}(\Omega))$, $C(0, T; B_2^{1,x,y}(\Omega))$ be the set of continuous mappings from the interval $[0, T]$ into the Hilbert spaces $B_2^N(\Omega))$, $B_2^{1,x}(\Omega)$, $B_2^{1,y}(\Omega)$, $B_2^{1,x,y}(\Omega)$, respectively, having the inner products

$$(Z, Y)_{B_2^N(\Omega)} = \int_\Omega \Im_x^N Z.\Im_x^N Y\,dx\,dy, \qquad (Z, Y)_{B_2^{1,x}(\Omega)} = \int_\Omega \Im_x Z.\Im_x Y\,dx\,dy,$$
$$(Z, Y)_{B_2^{1,y}(\Omega)} = \int_\Omega \Im_y Z.\Im_y Y\,dx\,dy, \qquad (Z, Y)_{B_2^{1,x,y}(\Omega)} = \int_\Omega \Im_{xy} Z.\Im_{xy} Y\,dx\,dy,$$

with $B_2^N(\Omega))$ (see [35]) being the set of function Z such that $\Im_\theta^N Z = \frac{1}{(N-1)!}\int_0^\theta (\theta - \nu)^{N-1} \times Z(\nu, t)\,d\nu \in L^2(\Omega)$ for $N \in \mathbb{N}^*$ and $Z \in L^2(\Omega)$ for $N = 0$.

The following crucial lemmas are needed to be used in different proofs of our results.

Lemma 2.1 ([23]) *For any absolutely continuous function* $L(t)$ *on the interval* $[0, T]$, *the following inequality holds*:

$$L(t)\,{}^C\partial_t^\beta L(t) \geq \frac{1}{2}\,{}^C\partial_t^\beta L^2(t), \quad 0 < \beta < 1. \quad (2.8)$$

Lemma 2.2 ([36]) *Let* $\mathcal{N}(s)$ *be nonnegative and absolutely continuous on* $[0, T]$ *and for almost all* $s \in [0, T]$ *satisfy the inequality*

$$\frac{d\mathcal{N}}{ds} \leq A_1(s)\mathcal{N}(t) + B_1(s), \quad (2.9)$$

where the functions $A_1(s)$ *and* $B_1(s)$ *are summable and nonnegative on* $[0, T]$. *Then*

$$\mathcal{N}(s) \leq \exp\left(\int_0^s A_1(t)\,dt\right)\left(\mathcal{N}(0) + \int_0^s B_1(t)\,dt\right). \quad (2.10)$$

Lemma 2.3 ([23]) *Let a nonnegative absolutely continuous function* $Q(t)$ *satisfy the inequality*

$$^C\partial_t^\beta Q(t) \leq b_1 Q(t) + b_2(t), \quad 0 < \beta < 1, \quad (2.11)$$

for almost all $t \in [0,T]$, where b_1 is a positive constant and $b_2(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$Q(t) \le Q(0)E_\beta\big(b_1 t^\beta\big) + \Gamma(\beta)E_{\beta,\beta}\big(b_1 t^\beta\big)D_t^{-\beta} b_2(t), \tag{2.12}$$

where

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \quad \text{and} \quad E_{\beta,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \mu)},$$

are the Mittag-Leffler functions and

$$D_t^{-\alpha} h(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(x,\tau)}{(t-\tau)^{1-\alpha}}\, d\tau$$

is the Riemann–Liouville integral of order $0 < \alpha < 1$ of the function h [37].

3 The energy inequality (uniqueness of solution)

Theorem 3.1 *For any function $U = (\mathcal{M},\theta)$ belonging to $\mathcal{D}(\mathcal{A})$, there exists a positive constant $\mathcal{D}^*$, independent of $\mathcal{M}$ and θ, such that the following a priori estimate holds*:

$$\begin{aligned} &\big\|\mathcal{M}_x(\cdot,\cdot,\tau)\big\|^2_{C(0,T;B_2^{1,y}(\Omega))} + \big\|\mathcal{M}_y(\cdot,\cdot,\tau)\big\|^2_{C(0,T;B_2^{1,x}(\Omega))} \\ &\quad + \big\|\mathcal{M}(\cdot,\cdot,\tau)\big\|^2_{C(0,T;L^2(\Omega))} \\ &\le \mathcal{D}^*\big(\|\mathcal{M}_0\|^2_{H_0^1(\Omega)} + \|\mathcal{M}_1\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} \\ &\quad + \|f\|^2_{L^2(Q^T)} + \|g\|^2_{L^2(Q^T)}\big), \end{aligned} \tag{3.1}$$

where

$$\mathcal{D}^* = \max\left\{2\mathcal{H}, \frac{\mathcal{H}\omega T^\alpha}{\Gamma(\alpha+1)} + 1\right\}$$

with

$$\begin{aligned} &\mathcal{H} = \max\left\{1, C^*, \frac{T^{1-\alpha}ab}{2(1-\alpha)\Gamma(1-\alpha)} + \frac{a}{2} + \frac{b}{2}\right\}, \\ &\omega = \big(\Gamma(\alpha).E_{\alpha,\alpha}\big(\mathcal{H}.T^\alpha\big)\big)\mathcal{H}\max\left(1, \frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\right), \end{aligned}$$

and

$$C^* = \frac{\max\{1, \frac{ab}{4\delta}, \frac{ab}{8}\}}{\min\{1,2,\beta,\gamma,2\eta,\delta\}}.$$

Proof Taking the scalar product in $L^2(\Omega)$ of partial differential (2.1) equations and the operators $M_1(\mathcal{M}) = \Im^2_{xy}\mathcal{M}_t$ and $M_2(\theta) = \Im^2_{xy}\theta$, respectively, where

$$\Im^2_{xy} v(x,y,t) = \int_0^x \int_0^{\xi_1} \int_0^y \int_0^{\eta_1} v(\gamma,\rho,t)\, d\rho\, d\eta_1\, d\gamma\, d\xi_1,$$

then we have

$$
\begin{aligned}
&\left(\partial_t^\alpha \mathcal{M}_t, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} + \left(\mathcal{M}_{xxxx}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} \\
&\quad + \left(\mathcal{M}_{yyyy}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} + 2\left(\mathcal{M}_{xxyy}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} \\
&\quad - \gamma\left(\partial_t^{\alpha+1}\mathcal{M}_{xx}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} - \gamma\left(\partial_t^{\alpha+1}\mathcal{M}_{yy}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} \\
&\quad + d\left(\theta_{xx}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} + d\left(\theta_{yy}, \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} \\
&\quad + \beta\left(\partial_t^\alpha\theta, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} - \eta\left(\theta_{xx}, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} \\
&\quad - \eta\left(\theta_{yy}, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} + \delta\left(\theta, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} \\
&\quad - d\left(\mathcal{M}_{xxt}, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} - d\left(\mathcal{M}_{yyt}, \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)} \\
&= \left(f(x,y,t), \Im_x^2\Im_y^2\mathcal{M}_t\right)_{L^2(\Omega)} + \left(g(x,y,t), \Im_x^2\Im_y^2\theta\right)_{L^2(\Omega)}.
\end{aligned}
\tag{3.2}
$$

We separately consider the inner products in (3.2). Integrating by parts and taking into account boundary and initial conditions (2.2)–(2.4), we obtain

$$\left(\partial_t^\alpha \mathcal{M}_t, \Im_{xxyy}\mathcal{M}_t\right)_{L^2(\Omega)} = \left(\partial_t^\alpha(\Im_{xy}\mathcal{M}_t), \Im_{xy}\mathcal{M}_t\right)_{L^2(\Omega)}, \tag{3.3}$$

$$(\mathcal{M}_{xxxx}, \Im_{xxyy}\mathcal{M}_t)_{L^2(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|\Im_y\mathcal{M}_x\|^2_{L^2(\Omega)}, \tag{3.4}$$

$$(\mathcal{M}_{yyyy}, \Im_{xxyy}\mathcal{M}_t)_{L^2(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|\Im_x\mathcal{M}_y\|^2_{L^2(\Omega)}, \tag{3.5}$$

$$2(\mathcal{M}_{xxyy}, \Im_{xxyy}\mathcal{M}_t)_{L^2(\Omega)} = \frac{\partial}{\partial t}\|\mathcal{M}\|^2_{L^2(\Omega)}, \tag{3.6}$$

$$-\gamma\left(\partial_t^{\alpha+1}\mathcal{M}_{xx}, \Im_{xxyy}\mathcal{M}_t\right)_{L^2(\Omega)} = \gamma\left(\partial_t^\alpha\Im_y\mathcal{M}_t, \Im_y\mathcal{M}_t\right)_{L^2(\Omega)}, \tag{3.7}$$

$$-\gamma\left(\partial_t^{\alpha+1}\mathcal{M}_{yy}, \Im_{xxyy}\mathcal{M}_t\right)_{L^2(\Omega)} = \gamma\left(\partial_t^\alpha\Im_x\mathcal{M}_t, \Im_x\mathcal{M}_t\right)_{L^2(\Omega)}, \tag{3.8}$$

$$d(\theta_{xx}, \Im_{xxyy}\mathcal{M}_t)_{L^2(\Omega)} = -d(\Im_y\theta, \Im_y\mathcal{M}_t)_{L^2(\Omega)}, \tag{3.9}$$

$$d(\theta_{yy}, \Im_{xxyy}\mathcal{M}_t)_{L^2(\Omega)} = -d(\Im_x\theta, \Im_x\mathcal{M}_t)_{L^2(\Omega)}, \tag{3.10}$$

$$\beta\left(\partial_t^\alpha\theta, \Im_{xxyy}\theta\right)_{L^2(\Omega)} = \beta\left(\partial_t^\alpha(\Im_{xy}\theta), \Im_{xy}\theta\right)_{L^2(\Omega)}, \tag{3.11}$$

$$-\eta(\theta_{xx}, \Im_{xxyy}\theta)_{L^2(\Omega)} = \eta\|\Im_y\theta\|^2_{L^2(\Omega)}, \tag{3.12}$$

$$-\eta(\theta_{yy}, \Im_{xxyy}\theta)_{L^2(\Omega)} = \eta\|\Im_x\theta\|^2_{L^2(\Omega)}, \tag{3.13}$$

$$\delta(\theta, \Im_{xxyy}\theta)_{L^2(0,1)} = \delta\|\Im_{xy}\theta\|^2_{L^2(\Omega)}, \tag{3.14}$$

$$-d(\mathcal{M}_{xxt}, \Im_{xxyy}\theta)_{L^2(\Omega)} = d(\Im_y\theta, \Im_y\mathcal{M}_t)_{L^2(\Omega)}, \tag{3.15}$$

$$-d(\mathcal{M}_{yyt}, \Im_{xxyy}\theta)_{L^2(\Omega)} = d(\Im_x\theta, \Im_x\mathcal{M}_t)_{L^2(\Omega)}. \tag{3.16}$$

Substitution of equations (3.3)–(3.16) into (3.2) yields

$$
\begin{aligned}
&\left(\partial_t^\alpha(\Im_{xy}\mathcal{M}_t), \Im_{xy}\mathcal{M}_t\right)_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|\Im_y\mathcal{M}_x\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\frac{\partial}{\partial t}\|\Im_x\mathcal{M}_y\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\|\mathcal{M}\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\left(\partial_t^\alpha \Im_y \mathcal{M}_t, \Im_y \mathcal{M}_t\right)_{L^2(\Omega)}+\gamma\left(\quad \partial_t^\alpha \Im_x \mathcal{M}_t, \Im_x \mathcal{M}_t\right)_{L^2(\Omega)} \\
&+\beta\left(\partial_t^\alpha\left(\Im_{xy} \theta\right), \Im_{xy} \theta\right)_{L^2(\Omega)}+\eta\left\|\Im_y \theta\right\|_{L^2(\Omega)}^2 \\
&+\eta\left\|\Im_x \theta\right\|_{L^2(\Omega)}^2+\delta\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 \\
=&\left(f(x, y, t), \Im_{xxyy} \mathcal{M}_t\right)_{L^2(\Omega)}+\left(g(x, y, t), \Im_{xxyy} \theta\right)_{L^2(\Omega)} .
\end{aligned}
\tag{3.17}
$$

By using Lemma 2.1, Cauchy ϵ inequality $\alpha\beta \le \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2$, and a Poincare type inequality [38], we obtain

$$
\begin{aligned}
&\frac{1}{2} \partial_t^\alpha\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\frac{\beta}{2} \partial_t^\alpha\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{1}{2} \frac{\partial}{\partial t}\left\|\Im_y \mathcal{M}_x\right\|_{L^2(\Omega)}^2+\frac{1}{2} \frac{\partial}{\partial t}\left\|\Im_x \mathcal{M}_y\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\partial}{\partial t}\|\mathcal{M}\|_{L^2(\Omega)}^2+\frac{\gamma}{2} \partial_t^\alpha\left\|\Im_y \mathcal{M}_t\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\gamma}{2} \partial_t^\alpha\left\|\Im_x \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\eta\left\|\Im_y \theta\right\|_{L^2(\Omega)}^2 \\
&\quad+\eta\left\|\Im_x \theta\right\|_{L^2(\Omega)}^2+\delta\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 \\
&\le \frac{\epsilon_1}{2}\|f\|_{L^2(\Omega)}^2+\frac{ab}{8 \epsilon_1}\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\epsilon_2}{2}\|g\|_{L^2(\Omega)}^2+\frac{ab}{8 \epsilon_2}\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 .
\end{aligned}
\tag{3.18}
$$

If in (3.18) we let $\epsilon_1 = 1$, $\epsilon_2 = \frac{ab}{4\delta}$, it follows that

$$
\begin{aligned}
&\partial_t^\alpha\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\beta \partial_t^\alpha\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2+\frac{\partial}{\partial t}\left\|\Im_y \mathcal{M}_x\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\partial}{\partial t}\left\|\Im_x \mathcal{M}_y\right\|_{L^2(\Omega)}^2+2 \frac{\partial}{\partial t}\|\mathcal{M}\|_{L^2(\Omega)}^2+\gamma \partial_t^\alpha\left\|\Im_y \mathcal{M}_t\right\|_{L^2(\Omega)}^2 \\
&\quad+\gamma \partial_t^\alpha\left\|\Im_x \mathcal{M}_t\right\|_{L^2(\Omega)}^2+2 \eta\left\|\Im_y \theta\right\|_{L^2(\Omega)}^2+2 \eta\left\|\Im_x \theta\right\|_{L^2(\Omega)}^2+\delta\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 \\
&\le\|f\|_{L^2(\Omega)}^2+\frac{ab}{8}\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\frac{ab}{4 \delta}\|g\|_{L^2(\Omega)}^2 .
\end{aligned}
\tag{3.19}
$$

We now discard the last three terms of the left-hand side (3.19) and get the inequality

$$
\begin{aligned}
&\partial_t^\alpha\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\partial_t^\alpha\left\|\Im_{xy} \theta\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\partial}{\partial t}\left\|\Im_y \mathcal{M}_x\right\|_{L^2(\Omega)}^2+\frac{\partial}{\partial t}\left\|\Im_x \mathcal{M}_y\right\|_{L^2(\Omega)}^2 \\
&\quad+\frac{\partial}{\partial t}\|\mathcal{M}\|_{L^2(\Omega)}^2+\partial_t^\alpha\left\|\Im_y \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\partial_t^\alpha\left\|\Im_x \mathcal{M}_t\right\|_{L^2(\Omega)}^2 \\
&\le C\left(\|f\|_{L^2(\Omega)}^2+\left\|\Im_{xy} \mathcal{M}_t\right\|_{L^2(\Omega)}^2+\|g\|_{L^2(\Omega)}^2\right),
\end{aligned}
\tag{3.20}
$$

where

$$
C^* = \frac{\max\{1, \frac{ab}{4\delta}, \frac{ab}{8}\}}{\min\{1, 2, \beta, \gamma, 2\eta, \delta\}}.
\tag{3.21}
$$

Now replacing t by τ and integrating with respect to τ from zero to t, we obtain

$$\begin{aligned} &D_t^{\alpha-1}\|\Im_{xy}\mathcal{M}_t\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_{xy}\theta\|^2_{L^2(\Omega)} + \|\Im_y\mathcal{M}_x\|^2_{L^2(\Omega)} \\ &\quad + \|\Im_x\mathcal{M}_y\|^2_{L^2(\Omega)} + \|\mathcal{M}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_y\mathcal{M}_t\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_x\mathcal{M}_t\|^2_{L^2(\Omega)} \\ &\quad \le \mathcal{H}\biggl(\int_0^t \|f\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|g\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_{xy}\mathcal{M}_\tau\|^2_{L^2(\Omega)}\,d\tau \\ &\quad + \|\mathcal{M}_1\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\mathcal{M}_0\|^2_{H_0^1(\Omega)}\biggr), \end{aligned} \tag{3.22}$$

where

$$\mathcal{H} = \max\biggl\{1, C^*, \frac{T^{1-\alpha}ab}{2(1-\alpha)\Gamma(1-\alpha)} + \frac{a}{2} + \frac{b}{2}\biggr\}. \tag{3.23}$$

Now, by dropping the last six terms from the left-hand side of (3.22) and applying Lemma 2.3 with

$$\begin{aligned} &Q(t) = \int_0^t \|\Im_{xy}\mathcal{M}_\tau\|^2_{L^2(\Omega)}\,d\tau, \quad Q(0) = 0, \\ &\partial_t^\alpha Q(t) = D_t^{\alpha-1}\|\Im_{xy}\mathcal{M}_t\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.24}$$

we have

$$\begin{aligned} \int_0^t \|\Im_{xy}\mathcal{M}_\tau\|^2_{L^2(\Omega)}\,d\tau &\le \omega\bigl(D_t^{-\alpha-1}\|f\|^2_{L^2(\Omega)} + D_t^{-\alpha-1}\|g\|^2_{L^2(\Omega)} \\ &\quad + \|\mathcal{M}_1\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\mathcal{M}_0\|^2_{H_0^1(\Omega)}\bigr), \end{aligned} \tag{3.25}$$

where

$$\omega = \bigl(\Gamma(\alpha).E_{\alpha,\alpha}\bigl(\mathcal{H}.T^\alpha\bigr)\bigr)\mathcal{H}\max\biggl(1, \frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\biggr). \tag{3.26}$$

By virtue of the inequality

$$\begin{aligned} &D_t^{-\alpha-1}\|f\|^2_{L^2(\Omega)} + D_t^{-\alpha-1}\|g\|^2_{L^2(\Omega)} \\ &\quad \le \frac{t^\alpha}{\Gamma(\alpha+1)}\biggl(\int_0^t \|f\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|g\|^2_{L^2(\Omega)}\,d\tau\biggr), \end{aligned} \tag{3.27}$$

it follows from inequalities (3.22), (3.25), and (3.27) that

$$\begin{aligned} &D_t^{\alpha-1}\|\mathcal{M}_t\|^2_{B_2^{1,x,y}(\Omega)} + D_t^{\alpha-1}\|\theta\|^2_{B_2^{1,x,y}(\Omega)} + D_t^{\alpha-1}\|\mathcal{M}_t\|^2_{B_2^{1,x}(\Omega)} \\ &\quad + D_t^{\alpha-1}\|\mathcal{M}_t\|^2_{B_2^{1,y}(\Omega)} + \|\mathcal{M}_y\|^2_{B_2^{1,x}(\Omega)} + \|\mathcal{M}\|^2_{L^2(\Omega)} + \|\mathcal{M}_x\|^2_{B_2^{1,y}(\Omega)} \\ &\quad \le \mathcal{D}^*\bigl(\|\mathcal{M}_0\|^2_{H_0^1(\Omega)} + \|\mathcal{M}_1\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} \\ &\quad + \|f\|^2_{L^2(Q^T)} + \|g\|^2_{L^2(Q^T)}\bigr), \end{aligned} \tag{3.28}$$

where

$$\mathcal{D}^* = \max\left\{2\mathcal{H}, \frac{\mathcal{H}\omega T^\alpha}{\Gamma(\alpha+1)} + 1\right\}. \tag{3.29}$$

If we discard the first four terms on the left-hand side of (3.28) and then pass to the supremum with respect to t from 0 to T, the a priori bound (3.1) follows. □

4 Solvability of the posed problem

As we only know that the range of the operator $\mathcal{A} : \mathcal{E} \to \mathcal{F}$, $R(\mathcal{A})$ is a subset of $\mathcal{F}$, we extend $\mathcal{A}$ in such a way that $\|U\|_{\mathcal{E}} \le C\|\overline{\mathcal{A}}\,U\|_{\mathcal{F}}$ for all $U \in D(\overline{\mathcal{A}})$ and $R(\overline{\mathcal{A}}) = \mathcal{F}$. For this purpose, we prove the following.

Theorem 4.1 *The unbounded operator $\mathcal{A} : \mathcal{E} \to \mathcal{F}$ admits a closure $\overline{\mathcal{A}}$ with the domain of definition $D(\overline{\mathcal{A}})$.*

Proof The proof is analogous to [39]. □

Theorem 4.2 *For any $(f, g) \in (L^2(Q^T))^2$ and any $(\mathcal{M}_0, \mathcal{M}_1, \theta_0) \in H_0^1(\Omega) \times (L^2(Q^T))^2$, there exists a unique strong solution $U = (\mathcal{M}, \theta) = (\overline{\mathcal{A}})^{-1}(W) = (\overline{\mathcal{A}^{-1}})(W)$ of problem (2.1)–(2.4).*

Proof To prove that problem (2.1)–(2.4) has a unique strong solution for all $W = (\{f, \mathcal{M}_0, \mathcal{M}_1\}, \{g, \theta_0\}) \in \mathcal{F}$, it suffices to prove that the range of the operator $\mathcal{A}$ is dense in $\mathcal{F}$. We first prove it in the case when $\mathcal{D}(\mathcal{A}) = \mathcal{D}_0(\mathcal{A}) = \{U \in \mathcal{D}(\mathcal{A}) : \ell_1\mathcal{M} = \ell_2\mathcal{M} = \ell_3\theta = 0\}$. For this purpose we need to prove the following result. □

Theorem 4.3 *If for some function $G = (G_1, G_2)$ belongs to $(L^2(\Omega))^2$ and for any $U = (\mathcal{M}, \theta) \in \mathcal{D}_0(\mathcal{A})$ we have*

$$\big(\mathcal{L}_1(\mathcal{M},\theta), G_1\big)_{L^2(Q^T)} + \big(\mathcal{L}_2(\mathcal{M},\theta), G_2\big)_{L^2(Q^T)} = 0, \tag{4.1}$$

then $G = (G_1, G_2) = (0, 0)$ almost everywhere in Q^T.

Assume that the proof of Theorem 4.3 is achieved. We suppose that, for some element $G = (G_1, G_2) = (\{f, \sigma_1, \sigma_2\}, \{g, \sigma_3\}) \in R(\mathcal{A})^\perp$ and for all $U \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}
(AU, \chi)_{\mathcal{F}} &= \Big(\big\{\big\{\mathcal{L}_1(\mathcal{M},\theta), \ell_1\mathcal{M}, \ell_2\mathcal{M}\big\}, \big\{\mathcal{L}_2(\mathcal{M},\theta), l_3\theta\big\}\big\}, \big\{\{f, \sigma_1, \sigma_2\}, \{g, \sigma_3\}\big\}\Big)_{\mathcal{E}} \\
&= \big(\mathcal{L}_1(\mathcal{M},\theta), f\big)_{L^2(Q^T)} + (\ell_1\mathcal{M}, \sigma_1)_{H_0^1(\Omega)} \\
&\quad + (\ell_2\mathcal{M}, \sigma_2)_{L^2(\Omega)} + \big(\mathcal{L}_2(\mathcal{M},\theta), g\big)_{L^2(Q^T)} + (\ell_3\theta, \sigma_3)_{L^2(\Omega)} \\
&= 0,
\end{aligned} \tag{4.2}$$

we must prove that $G = 0$. Taking $U \in \mathcal{D}_0(\mathcal{A})$ in (4.2), we get

$$\big(\mathcal{L}_1(\mathcal{M},\theta), f\big)_{L^2(Q^T)} + \big(\mathcal{L}_2(\mathcal{M},\theta), g\big)_{L^2(Q^T)} = 0. \tag{4.3}$$

Hence, by virtue of Theorem 4.3, it follows from (4.3) that $f = g = 0$. Thus (4.2) takes the form

$$(\ell_1\mathcal{M},\sigma_1)_{H_0^1(\Omega)} + (\ell_2\mathcal{M},\sigma_2)_{L^2(\Omega)} + (\ell_3\theta,\sigma_3)_{L^2(\Omega)} = 0. \tag{4.4}$$

By the fact that the ranges of the trace operators ℓ_1, ℓ_2, and ℓ_3 are, respectively, dense in the spaces $H_0^1(\Omega)$, $L^2(\Omega)$, $L^2(\Omega)$, we conclude from (4.4) that $\sigma_1 = \sigma_2 = \sigma_3 = 0$. Consequently, $G = (G_1, G_2) = (0, 0)$, that is, $R(\mathcal{A})^{\perp} = \{0\}$, thus $\overline{R(\mathcal{A})} = \mathcal{F}$.

To complete the proof of Theorem 4.2, we prove Theorem 4.3.

Proof of Theorem 4.3 Equation (4.1) implies

$$\begin{aligned}
&\bigl(\partial_t^{\alpha+1}\mathcal{M} + \mathcal{M}_{xxxx} + \mathcal{M}_{yyyy} + 2\mathcal{M}_{xxyy}\\
&\quad - \gamma\,\partial_t^{\alpha+1}\mathcal{M}_{xx} - \gamma\,\partial_t^{\alpha+1}\mathcal{M}_{yy} + d\theta_{xx} + d\theta_{yy}, G_1\bigr)_{L^2(Q^T)}\\
&\quad + \bigl(\beta\partial_t^{\alpha}\theta - \eta\theta_{xx} - \eta\theta_{yy} + \delta\theta\\
&\quad - d\mathcal{M}_{xxt} - d\mathcal{M}_{yyt}, G_2\bigr)_{L^2(Q^T)} = 0.
\end{aligned} \tag{4.5}$$

Let $\mathcal{Z}(x,t)$ be a function satisfying the initial, boundary, and integral conditions such that $\mathcal{Z}$, $\mathcal{Z}_x$, $\mathcal{Z}_y$, $\Im_t\mathcal{Z}$, $\Im_t\mathcal{Z}_x$, $\Im_t\mathcal{Z}_y$, $\partial_t^{\alpha+1}\mathcal{Z} \in L^2(Q^T)$. We set $\mathcal{M} = \Im_x^2\Im_y^2\Im_t^2\mathcal{Z}$, $\theta = \Im_x^4\Im_y^4\Im_t^2\mathcal{Z}$, and define $G_1 = \Im_t\mathcal{Z}$, $G_2 = \Im_t^2\mathcal{Z}$. Clearly, G_1 and G_2 belong to $L^2(Q^T)$. Relations (4.1) and (4.2) imply that

$$\begin{aligned}
&\bigl(\bigl(\partial_t^{\alpha+1}\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr), \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)} + \bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xxxx}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad + \bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{yyyy}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)} + 2\bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xxyy}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad - \gamma\bigl(\partial_t^{\alpha+1}\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xx}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)} - \gamma\bigl(\partial_t^{\alpha+1}\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{yy}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad + d\bigl(\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}\bigr)_{xx}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)} + d\bigl(\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}\bigr)_{yy}, \Im_t\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad + \beta\bigl(\partial_t^{\alpha}\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}\bigr), \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)} - \eta\bigl(\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}\bigr)_{xx}, \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad - \eta\bigl(\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}\bigr)_{yy}, \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)} + \delta\bigl(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}, \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\qquad - d\bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xxt}, \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)} - d\bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{yyt}, \Im_t^2\mathcal{Z}\bigr)_{L^2(Q^T)}\\
&\quad = 0.
\end{aligned} \tag{4.6}$$

We separately consider the terms in (4.6). Integrating by parts and taking into account that $\mathcal{Z}$ satisfies boundary and initial conditions (2.2)–(2.4), we obtain

$$\begin{aligned}
\bigl(\bigl(\partial_t^{\alpha+1}\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr), \Im_t\mathcal{Z}\bigr)_{L^2(\Omega)} &= \bigl(\bigl(\partial_t^{\alpha}\bigl(\Im_x^2\Im_y^2\Im_t\mathcal{Z}\bigr), \Im_t\mathcal{Z}\bigr)_{L^2(\Omega)}\\
&= \bigl(\bigl(\partial_t^{\alpha}(\Im_x\Im_y\Im_t\mathcal{Z}), \Im_x\Im_y\Im_t\mathcal{Z}\bigr)_{L^2(\Omega)},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\bigl(\bigl(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xxxx}, \Im_t\mathcal{Z}\bigr)_{L^2(\Omega)} &= \bigl(\bigl(\Im_y^2\Im_t^2\mathcal{Z}\bigr)_{xx}, \Im_t\mathcal{Z}\bigr)_{L^2(\Omega)}\\
&= \frac{1}{2}\frac{\partial}{\partial t}\bigl\|\Im_y\Im_t^2\mathcal{Z}_x\bigr\|_{L^2(\Omega)}^2,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\big((\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{yyyy},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} &= \big((\Im_x^2\Im_t^2\mathcal{Z})_{yy},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} \\
&= \frac{1}{2}\frac{\partial}{\partial t}\big\|\Im_x\Im_t^2\mathcal{Z}_y\big\|^2_{L^2(\Omega)},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
2\big((\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{xxyy},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} &= 2\big(\Im_t^2\mathcal{Z},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} \\
&= \frac{\partial}{\partial t}\big\|\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)},
\end{aligned} \tag{4.10}$$

$$-\gamma\big(\partial_t^{\alpha+1}(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{xx},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} = \gamma\big(\partial_t^{\alpha}(\Im_y\Im_t\mathcal{Z}),(\Im_y\Im_t\mathcal{Z})\big)_{L^2(\Omega)}, \tag{4.11}$$

$$-\gamma\big((\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{yy},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} = \gamma\big(\partial_t^{\alpha}(\Im_x\Im_t\mathcal{Z}),(\Im_x\Im_t\mathcal{Z})\big)_{L^2(\Omega)}, \tag{4.12}$$

$$\begin{aligned}
d\big((\Im_x^4\Im_y^4\Im_t^2\mathcal{Z})_{xx},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} &= d\big(\Im_x^2\Im_y^4\Im_t^2\mathcal{Z},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} \\
&= d\big(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z},\Im_y^2\Im_t\mathcal{Z}\big)_{L^2(\Omega)},
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&d\big((\Im_x^4\Im_y^4\Im_t^2\mathcal{Z})_{yy},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} \\
&\quad= d\big(\Im_x^4\Im_y^2\Im_t^2\mathcal{Z},\Im_t\mathcal{Z}\big)_{L^2(\Omega)} = d\big(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z},\Im_x^2\Im_t\mathcal{Z}\big)_{L^2(\Omega)},
\end{aligned} \tag{4.14}$$

$$\beta\big(\partial_t^{\alpha}(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z}),\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} = \beta\big(\partial_t^{\alpha}(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}),\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)}, \tag{4.15}$$

$$\begin{aligned}
-\eta\big((\Im_x^4\Im_y^4\Im_t^2\mathcal{Z})_{xx},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} &= -\eta\big(\Im_x^2\Im_y^4\Im_t^2\mathcal{Z},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} \\
&= \eta\big\|\Im_x\Im_y^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)},
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
-\eta\big((\Im_x^4\Im_y^4\Im_t^2\mathcal{Z})_{yy},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} &= -\eta\big(\Im_x^4\Im_y^2\Im_t^2\mathcal{Z},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} \\
&= \eta\big\|\Im_y\Im_x^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)},
\end{aligned} \tag{4.17}$$

$$\delta\big(\Im_x^4\Im_y^4\Im_t^2\mathcal{Z},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} = \delta\big\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)}, \tag{4.18}$$

$$\begin{aligned}
-d\big((\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{xxt},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} &= -d\big(\Im_y^2\Im_t\mathcal{Z},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} \\
&= d\big(\Im_y\Im_t\mathcal{Z},\Im_y\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)},
\end{aligned} \tag{4.19}$$

$$-d\big((\Im_x^2\Im_y^2\Im_t^2\mathcal{Z})_{yyt},\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} = d\big(\Im_x\Im_t\mathcal{Z},\Im_x\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)}. \tag{4.20}$$

Substitution of equations (4.7)–(4.20) into (4.6) yields

$$\begin{aligned}
&2\big(\big(\partial_t^{\alpha}(\Im_x\Im_y\Im_t\mathcal{Z}),\Im_x\Im_y\Im_t\mathcal{Z}\big)_{L^2(\Omega)} + 2\beta\big(\partial_t^{\alpha}(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}),\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} \\
&\quad+ \frac{\partial}{\partial t}\big\|\Im_x\Im_t^2\mathcal{Z}_y\big\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\big\|\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)} \\
&\quad+ \gamma\big(\partial_t^{\alpha}(\Im_y\Im_t\mathcal{Z}),(\Im_y\Im_t\mathcal{Z})\big)_{L^2(\Omega)} + \gamma\big(\partial_t^{\alpha}(\Im_x\Im_t\mathcal{Z}),(\Im_x\Im_t\mathcal{Z})\big)_{L^2(\Omega)} \\
&\quad+ \frac{\partial}{\partial t}\big\|\Im_y\Im_t^2\mathcal{Z}_x\big\|^2_{L^2(\Omega)} + 2\eta\big\|\Im_x\Im_y^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)} \\
&\quad+ 2\eta\big\|\Im_y\Im_x^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)} + 2\delta\big\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\big\|^2_{L^2(\Omega)} \\
&\quad\times 2d\big(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z},\Im_y^2\Im_t\mathcal{Z}\big)_{L^2(\Omega)} + 2d\big(\Im_x^2\Im_y^2\Im_t^2\mathcal{Z},\Im_x^2\Im_t\mathcal{Z}\big)_{L^2(\Omega)} \\
&\quad+ 2d\big(\Im_y\Im_t\mathcal{Z},\Im_y\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} + 2d\big(\Im_x\Im_t\mathcal{Z},\Im_x\Im_t^2\mathcal{Z}\big)_{L^2(\Omega)} \\
&\quad= 0.
\end{aligned} \tag{4.21}$$

By using Lemma 2.1, Cauchy ϵ inequality, and Poincare type inequality, we obtain

$$\begin{aligned}
&\partial_t^\alpha \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + \beta\partial_t^\alpha \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} + \gamma\partial_t^\alpha \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + \gamma\partial_t^\alpha \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} \\
&\quad + \frac{\partial}{\partial t}\left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\left\|\Im_y\Im_t^2\mathcal{Z}_x\right\|^2_{L^2(\Omega)} + 2\eta\left\|\Im_x\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\
&\quad + 2\eta\left\|\Im_y\Im_x^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} + 2\delta\left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\left\|\Im_x\Im_t^2\mathcal{Z}_y\right\|^2_{L^2(\Omega)} \\
&\le \frac{\epsilon_2 da^2b^2T^2}{8}\|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + \epsilon_1 d\left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\
&\quad + \left(\frac{db^2}{2\epsilon_1} + \epsilon_3 d\right)\|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + \left(\epsilon_4 d + \frac{da^2}{2\epsilon_2}\right)\|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} \\
&\quad + \left(\frac{db^2}{2\epsilon_3} + \frac{da^2}{2\epsilon_4}\right)\left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.22}$$

We now take $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$, drop the last five terms of the left-hand side of (4.22), replace t by τ, and integrate both sides with respect to τ from 0 to t, then we get

$$\begin{aligned}
&D_t^{\alpha-1}\|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\
&\quad + D_t^{\alpha-1}\|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + \left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\
&\le \Delta^*\left(\int_0^t \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau\right. \\
&\quad \left. + \int_0^t \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau\right),
\end{aligned} \tag{4.23}$$

where

$$\Delta^* = \frac{\max\{\frac{da^2b^2T^2}{8}, \frac{db^2}{2\epsilon_1} + d, d + \frac{da^2}{2}, \frac{db^2}{2} + \frac{da^2}{2}\}}{\min\{1, \beta, \gamma, 2\eta\}}. \tag{4.24}$$

Since (see (4.23))

$$\begin{aligned}
\left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} &\le \Delta^*\left(\int_0^t \left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau\right) \\
&\quad + \Delta^*\left(\int_0^t \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau\right. \\
&\quad \left. + \int_0^t \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau\right),
\end{aligned} \tag{4.25}$$

then if we set

$$\begin{aligned}
&\mathcal{N}(t) = \int_0^t \left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau, \\
&\frac{d\mathcal{N}(t)}{dt} = \left\|\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}, \\
&\mathcal{N}(0) = 0,
\end{aligned} \tag{4.26}$$

then, according to Lemma 2.2 (Gronwall–Bellman), we obtain

$$\mathcal{N}(t) \leq \Delta^* e^{\Delta^* T}\left(\int_0^t \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}d\tau + \int_0^t \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau\right). \tag{4.27}$$

Now, by omitting the last term on the left-hand side of (4.23) and by using inequality (4.27), we obtain

$$\begin{aligned} & D_t^{\alpha-1}\|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\ &\quad + D_t^{\alpha-1}\|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} \\ &\leq (\Delta^*)^2\left(1+e^{\Delta^* T}\right)\left(\int_0^t \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau \right. \\ &\quad + \int_0^t \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau \\ &\quad \left. + \int_0^t \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau\right). \end{aligned} \tag{4.28}$$

Lemma 2.3 can be applied by taking

$$\begin{aligned} & Q(t) = \int_0^t \|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)}\,d\tau \\ &\qquad + \int_0^t \|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau + \int_0^t \|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)}\,d\tau \\ & \partial_t^\alpha Q(t) = D_t^{\alpha-1}\|\Im_x\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\left\|\Im_x^2\Im_y^2\Im_t^2\mathcal{Z}\right\|^2_{L^2(\Omega)} \\ &\qquad + D_t^{\alpha-1}\|\Im_y\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} + D_t^{\alpha-1}\|\Im_x\Im_t\mathcal{Z}\|^2_{L^2(\Omega)} \\ & Q(0) = 0. \end{aligned} \tag{4.29}$$

It follows from (4.29) that

$$Q(t) \leq V.0 = 0, \tag{4.30}$$

where

$$V = \Gamma(\alpha)E_{\alpha,\alpha}\left(W^* T^\alpha\right)\max\left(1, \frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\right)$$

and

$$W^* = (\Delta^*)^2\left(1+e^{\Delta^* T}\right).$$

It follows from (4.30) that $G_1 = \Im_t\mathcal{Z} = 0$, $G_2 = \Im_t^2\mathcal{Z} = 0$ a.e. in Q^T. This achieves the proof of Theorem 4.2. □

5 Conclusion

The existence and uniqueness of solutions for a fractional order initial boundary value problem for a two-dimensional coupled linear thermoelastic system of fourth order with nonlocal conditions which models a thermoelastic fractional Kirchhoff plate are established. The method of energy inequalities is successfully applied for obtaining a priori estimates for the solution from which the uniqueness of the solution follows. Then, from Hilbert space theory, a density argument is employed to establish the solvability of the given problem. It is found that the application of the functional analysis method to systems of fractional order is very efficient in spite of the difficulties of choosing the appropriate multipliers and functions space solutions as well as the different hard computations.

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References

1. Jou, D., Casas-Vázquez, J., Lebon, G.: Extended irreversible thermodynamics. Rep. Prog. Phys. **51**, 1105–1179 (1988)
2. Caputo, M., Mainardi, F.: A new dissipation model based on memory mechanism. Pure Appl. Geophys. **91**, 134–147 (1971)
3. Caputo, M., Mainardi, F.: Linear model of dissipation in anelastic solids. Riv. Nuovo Cimento **1**, 161–198 (1971)
4. Caputo, M.: Vibrations on an infinite viscoelastic layer with a dissipative memory. J. Acoust. Soc. Am. **56**, 897–904 (1974)
5. Sherief, H., El-Sayed, A.M.A., Abd El-Latief, A.M.: Fractional order theory of thermoelasticity. Int. J. Solids Struct. **47**, 269–275 (2010)
6. Hetnarski, R.B., Ignaczak, J.: Generalized thermoelasticity. J. Therm. Stresses **22**, 4–5 (1999)
7. Povstenko, Y.Z.: Thermoelasticity that uses fractional heat conduction equation. J. Math. Sci. **162**, 296–305 (2009)
8. Povstenko, Y.Z.: Fractional heat conduction and associated thermal stress. J. Therm. Stresses **28**, 83–102 (2005)
9. Povstenko, Y.Z.: Fractional Cattaneo-type equations and generalized thermoelasticity. J. Therm. Stresses **34**, 97–114 (2011)
10. Sherief, H., El-Sayed, A.M.A.: Abd el-latief A.M.: fractional order theory of thermoelasticity. Int. J. Solids Struct. **47**, 269–275 (2010)
11. Ezzat, M.A.: Thermoelectric MHD non-Newtonian fluid with fractional derivative heat transfer. Physica B, Condens. Matter **405**, 4188–4194 (2010)
12. Jumarie, G.: Derivation and solutions of some fractional Black–Scholes equations in coarse-grained space and time. Application to Merton's optimal portfolio. Comput. Math. Appl. **59**, 1142–1164 (2010)
13. Hamza, F., Abdou, M., Abd El-Latief, A.M.: Generalized fractional thermoelasticity associated with two relaxation times. J. Therm. Stresses **37**(9), 1080–1098 (2014)
14. Agarwal, P., Berdyshev, A., Erkinjon, K.: Further extended Caputo fractional derivative operator and its applications. Russ. J. Math. Phys. **24**(4), 415–425 (2017)
15. Adel El-Sayed, A., Agarwal, P.: Numerical solution of multiterm variable-order fractional differential equations via shifted Legendre polynomials. Math. Methods Appl. Sci. **42**(11), 3978–3991 (2019)
16. Agarwal, P., Nieto, J.J., Luo, M.J.: Extended Riemann-Liouville type fractional derivative operator with applications. Open Math. **15**, 1667–1681 (2017)
17. Agarwal, P., Berdyshev, A., Erkinjon, K.: Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative. Results Math. **71**, 1235–1257 (2017)
18. Agarwal, P., Deniz, S., Jain, S., Alderremy, A.A., Shaban, A.: A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques. Phys. A, Stat. Mech. Appl. **542**, 122769 (2020)

19. Baltaeva, U., Agarwal, P.: Boundary value problems for the third order loaded equation with noncharacteristic type-change boundaries. Math. Methods Appl. Sci. **41**(9), 3307–3315 (2018)
20. Marin, M., Baleanu, D.: On vibrations in thermoelasticity without energy dissipation for micropolar bodies. Bound. Value Probl. **2016**, 111 (2016)
21. Marin, M., Baleanu, D., Vlase, S.: Effect of microtemperatures for micropolar thermoelastic bodies. Struct. Eng. Mech. **61**(3), 381–387 (2017)
22. Rahimi, Z., Sumelka, W., Rash, S.A.: Study and control of thermoelastic damping of in-plane vibration of the functionally graded nano-plate. J. Vib. Control **25**(23–24), 2850–2862 (2019)
23. Alikhanov, A.A.A.: Priori estimates for solutions of boundary value problems for fractional order equations. Differ. Equ. **46**, 660–666 (2010)
24. Mesloub, S.: Existence and uniqueness results for a fractional two-times evolution problem with constraints of purely integral type. Math. Methods Appl. Sci. **39**, 1558–1567 (2016)
25. Mesloub, S., Aldosari, F.: Even higher order fractional initial boundary value problem with nonlocal constraints of purely integral type. Symmetry **11**, 305 (2019)
26. Akilandeeswari, A., Balachandran, K., Annapoorani, N.: Solvability of hyperbolic fractional partial differential equations. J. Appl. Anal. Comput. **7**(4), 1570–1585 (2017)
27. Mesloub, S., Bachar, I.: On a nonlocal 1-d initial value problem for a singular fractional-order parabolic equation with Bessel operator. Adv. Differ. Equ. **2019**(1), 254 (2019)
28. Kasmi, L., Guerfi, A., Mesloub, S.: Existence of solution for 2-D time-fractional differential equations with a boundary integral condition. Adv. Differ. Equ. **2019**, 511 (2019)
29. Avalos, G., Lasiecka, I.: The null controllability of thermoelastic plates and singularity of the associated minimal energy function. J. Math. Anal. Appl. **294**, 34–61 (2004)
30. Avalos, G., Lasiecka, I.: Exponential stability of a thermoelastic system without mechanical dissipation. Rend. Istit. Mat. Univ. Trieste Suppl. **XXVIII**, 1–28 (1997)
31. Lagnese, J.: Boundary stabilization of thin plates. Siam Stud. Appl. Math. Vol. 10 (1989)
32. Lasiecka, I., Thomas, I.S.: Blowup estimates for observability of a thermoelastic system. Asymptot. Anal. **50**, 93–120 (2006)
33. Kumar, N.T., Sukavanam, N.: Exact controllability of a semilinear thermoelastic system with control solely in thermal equation. Numer. Funct. Anal. Optim. **29**(9–10), 1171–1179 (2008)
34. Hansen, S., Zhang, B.: Boundary control of thermoelastic beam. J. Math. Anal. Appl. **210**, 182–205 (1997)
35. Bouziani, A.: Mixed problem with integral conditions for a certain parabolic equation. J. Appl. Math. Stoch. Anal. **9**(3), 323–330 (1996)
36. Ladyzhenskaya, O.A.: The Boundary Value Problems of Mathematical Physics. Springer, New York (1985)
37. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
38. Mesloub, S.: A nonlinear nonlocal mixed problem for a second order parabolic equation. J. Math. Anal. Appl. **316**, 189–209 (2006)
39. Mesloub, S., Bouziani, A.: On a class of singular hyperbolic equations with a weighted integral condition. Int. J. Math. Math. Sci. **22**(3), 511–519 (1999)