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Construction of a new family of Fubini-type polynomials and its applications

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Abstract

This paper gives an overview of systematic and analytic approach of operational technique involves to study multi-variable special functions significant in both mathematical and applied framework and to introduce new families of special polynomials. Motivation of this paper is to construct a new class of generalized Fubini-type polynomials of the parametric kind via operational view point. The generating functions, differential equations, and other properties for these polynomials are established within the context of the monomiality principle. Using the generating functions, various interesting identities and relations related to the generalized Fubini-type polynomials are derived. Further, we obtain certain partial derivative formulas including the generalized Fubini-type polynomials. In addition, certain members belonging to the aforementioned general class of polynomials are considered. The numerical results to calculate the zeros and approximate solutions of these polynomials are given and their graphical representation are shown.

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1 Introduction

Special polynomials and numbers have significant roles in various branches of mathematics, theoretical physics, and engineering. The problems arising in mathematical physics and engineering are framed in terms of differential equations. Most of these equations can only be treated by using various families of special polynomials which provide new means of mathematical analysis. They are widely used in computational models of scientific and engineering problems. In addition, these special polynomials allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of special polynomials. Throughout this article, we use the following notations and definitions:

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Fubini-type polynomials appear in combinatorial mathematics and play an important role in the theory and applications of mathematics, thus many number theory and combinatorics experts have extensively studied their properties and obtained series of interesting results [7, 12, 14].

Kilar and Simsek [12] introduced the Fubini-type polynomials $F_n^{(\upsilon)}(x)$ of order υ, which are defined by the generating function

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}=\sum_{n=0}^{\infty}F_n^{(\upsilon)}(x)\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr).\tag{1.1}$$

For $x = 0$, Eq. (1.1) becomes

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}=\sum_{n=0}^{\infty}F_n^{(\upsilon)}\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr),\tag{1.2}$$

where $F_n^{(\upsilon)}$ denotes the Fubini-type numbers of order υ.

The Fubini-type numbers are related to Apostol–Bernoulli numbers and proven to be an effective tool in different topics in combinatorics and analysis.

For $x, y \in \mathbb{R}$, the Taylor–Maclaurin expansions of the two functions $e^{xt}\cos(yt)$ and $e^{xt}\sin(yt)$ are given, respectively, by (see [20])

$$e^{xt}\cos(yt)=\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!}\tag{1.3}$$

and

$$e^{xt}\sin(yt)=\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!},\tag{1.4}$$

where the functions $C_n(x,y)$ and $S_n(x,y)$ are defined as

$$C_n(x,y)=\sum_{\kappa=0}^{[\frac{n}{2}]}(-1)^{\kappa}\binom{n}{2\kappa}x^{n-2\kappa}y^{2\kappa}\tag{1.5}$$

and

$$S_n(x,y)=\sum_{\kappa=0}^{[\frac{n-1}{2}]}(-1)^{\kappa}\binom{n}{2\kappa+1}x^{n-2\kappa-1}y^{2\kappa+1}.\tag{1.6}$$

Recently, Srivastava and Kızılateş [22] introduced two parametric kinds of the Fubini-type polynomials which are defined by the generating functions

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}F_n^{(c,\upsilon)}(x,y)\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr)\tag{1.7}$$

and

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}F_n^{(s,\upsilon)}(x,y)\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr).\tag{1.8}$$

The Apostol-type polynomials and their numerous properties have been investigated in the literature extensively and widely [3, 5, 6, 13, 15–19].

The Apostol–Bernoulli polynomials $\mathfrak{B}_n^{(\upsilon)}(x;\lambda)$ [18] of order υ are defined by

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\upsilon} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\upsilon)}(x;\lambda)\frac{t^n}{n!} \tag{1.9}$$

$$\bigl(|t| < 2\pi \text{ when } \lambda = 1 \text{ and } |t| < |\log\lambda| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C}\bigr),$$

where $\mathfrak{B}_n^{(\upsilon)}(0;\lambda) := \mathfrak{B}_n^{(\upsilon)}(\lambda)$ denotes the Apostol–Bernoulli numbers of order υ.

The Apostol–Euler polynomials $\mathcal{E}_n^{(\upsilon)}(x;\lambda)$ [16] of order υ are defined by

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\upsilon} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\upsilon)}(x;\lambda)\frac{t^n}{n!} \tag{1.10}$$

$$\bigl(|t| < \pi \text{ when } \lambda = 1 \text{ and } |t| < \bigl|\log(-\lambda)\bigr| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C}\bigr),$$

where $\mathcal{E}_n^{(\upsilon)}(0;\lambda) := \mathcal{E}_n^{(\upsilon)}(\lambda)$ denotes the Apostol–Euler numbers of order υ.

The Apostol–Genocchi polynomials $\mathcal{G}_n^{(\upsilon)}(x;\lambda)$ [17] of order υ are defined by

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\upsilon} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\upsilon)}(x;\lambda)\frac{t^n}{n!} \tag{1.11}$$

$$\bigl(|t| < \pi \text{ when } \lambda = 1 \text{ and } |t| < \bigl|\log(-\lambda)\bigr| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C}\bigr),$$

where $\mathcal{G}_n^{(\upsilon)}(0;\lambda) := \mathcal{G}_n^{(\upsilon)}(\lambda)$ denotes the Apostol–Genocchi numbers of order υ.

In [23], Srivastava *et al.* introduced two parameter kinds of each of the Apostol–Bernoulli polynomials $\mathfrak{B}_n(x;\lambda)$, Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ and Apostol–Genocchi polynomials $\mathcal{G}_n(x;\lambda)$. Recently, the parametric kinds of each of the Apostol–Bernoulli polynomials $\mathfrak{B}_n^{(\upsilon)}(x;\lambda)$ of order υ, the Apostol–Euler polynomials $\mathcal{E}_n^{(\upsilon)}(x;\lambda)$ of order υ and the Apostol–Genocchi polynomials $\mathcal{G}_n^{(\upsilon)}(x;\lambda)$ of order υ were introduced by Srivastava and Kızılateş [22] as

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\upsilon} e^{xt}\cos(yt) = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(c,\upsilon)}(x,y;\lambda)\frac{t^n}{n!} \tag{1.12}$$

and

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\upsilon} e^{xt}\sin(yt) = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(s,\upsilon)}(x,y;\lambda)\frac{t^n}{n!}; \tag{1.13}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\upsilon} e^{xt}\cos(yt) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(c,\upsilon)}(x,y;\lambda)\frac{t^n}{n!} \tag{1.14}$$

and

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\upsilon} e^{xt}\sin(yt) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(s,\upsilon)}(x,y;\lambda)\frac{t^n}{n!}; \tag{1.15}$$

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\upsilon} e^{xt} \cos(yt) = \sum_{n=0}^{\infty} \mathcal{G}_n^{(c,\upsilon)}(x, y; \lambda) \frac{t^n}{n!} \tag{1.16}$$

and

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\upsilon} e^{xt} \sin(yt) = \sum_{n=0}^{\infty} \mathcal{G}_n^{(s,\upsilon)}(x, y; \lambda) \frac{t^n}{n!}. \tag{1.17}$$

In recent years, active research on multivariable and generalized forms of special functions of mathematical physics has been witnessed. Most of the multivariable and generalized special functions of mathematical physics arises in various branches of mathematics ranging from the theory of partial differential equations to the abstract group theory. We recall the following two variable form of special polynomials.

The 2-variable general polynomials (2VGP) $\mathbb{G}_n(x, y)$ [8] are defined by

$$e^{xt} \varphi(y, t) = \sum_{n=0}^{\infty} \mathbb{G}_n(x, y) \frac{t^n}{n!}, \quad \mathbb{G}_0(x, y) = 1, \tag{1.18}$$

where

$$\varphi(y, t) = \sum_{n=0}^{\infty} \varphi_n(y) \frac{t^n}{n!}, \quad \varphi_0(y) \neq 0. \tag{1.19}$$

Over the last few years, many authors have used the operational methods combined with the monomiality principle [1] to introduce and study new families of special polynomials [9, 21, 24–27]. Operational techniques are applicable to solve problems both in classical and quantum mechanics.

The 2VGP $\mathbb{G}_n(x, y)$ are quasi-monomial [8] with respect to the following multiplicative and derivative operators:

$$\hat{M}_{\mathbb{G}} = x + \frac{\varphi'(y, D_x)}{\varphi(y, D_x)} \quad \left(D_x := \frac{\partial}{\partial x}; \varphi'(y, t) := \frac{\partial}{\partial t} \varphi(y, t)\right) \tag{1.20}$$

and

$$\hat{P}_{\mathbb{G}} = D_x, \tag{1.21}$$

respectively.

In view of the monomiality principle, the 2VGP $\mathbb{G}_n(x, y)$ satisfy the following relations:

$$\hat{M}_{\mathbb{G}} \left\{ \mathbb{G}_n(x, y) \right\} = \mathbb{G}_{n+1}(x, y), \tag{1.22}$$

$$\hat{P}_{\mathbb{G}} \left\{ \mathbb{G}_n(x, y) \right\} = n \mathbb{G}_{n-1}(x, y), \tag{1.23}$$

$$\hat{M}_{\mathbb{G}} \hat{P}_{\mathbb{G}} \left\{ \mathbb{G}_n(x, y) \right\} = n \mathbb{G}_n(x, y), \tag{1.24}$$

$$\exp(\hat{M}_{\mathbb{G}} t)\{1\} = \sum_{n=0}^{\infty} \mathbb{G}_n(x, y) \frac{t^n}{n!} \quad \left(|t| < \infty\right). \tag{1.25}$$

Table 1 Certain members of 2VGP family $\mathbb{G}_n(x,y)$

S. no.	$\varphi(y,t)$	Generating functions	Polynomials
I.	e^{yt^r}	$e^{xt+yt^r}=\sum_{n=0}^{\infty}\mathcal{H}_n^{(r)}(x,y)\frac{t^n}{n!}$	The Gould–Hopper polynomials $\mathcal{H}_n^{(r)}(x,y)$ [4]
II.	$C_0(yt)$	$e^{xt}C_0(yt)=\sum_{n=0}^{\infty}\mathrm{L}_n(y,x)\frac{t^n}{n!}$	The 2-variable Laguerre polynomials $\mathrm{L}_n(y,x)$ [1]
III.	$\frac{1}{1-yt^s}$	$\frac{1}{1-yt^s}e^{xt}=\sum_{n=0}^{\infty}e_n^{(s)}(x,y)\frac{t^n}{n!}$,	The 2-variable truncated exponential of order s $e_n^{(s)}(x,y)$ [2]
VI.	$A(t)e^{yt^2}$	$A(t)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_{\mathcal{H}}A_n(x,y)\frac{t^n}{n!}$,	The Hermite–Appell polynomials ${}_{\mathcal{H}}A_n(x,y)$ [10]

The 2VGP family $\mathbb{G}_n(x,y)$ contains a number of significant 2-variable special polynomials. Based on suitable selection for the function $\varphi(y,t)$, different members belonging to the family of 2-variable general polynomials $\mathbb{G}_n(x,y)$ can be obtained. These members along with their notations, names and generating functions are mentioned in Table 1.

The article is organized as follows: In Sect. 2, two parametric types of the generalized Fubini-type polynomials are introduced via monomiality principle. The generating functions, multiplicative and derivative operators and differential equations for these families of polynomials are established. In Sect. 3, series definitions and certain other important relations for the generalized Fubini-type polynomials are derived. In Sect. 4, some partial derivative equations including these polynomials are obtained. In Sect. 5, certain members related to the generalized Fubini-type polynomials are considered as special cases.

2 Generalized Fubini-type polynomials via monomiality principle

In this section, with the help of the monomiality principle, two parametric types of the generalized Fubini-type polynomials are introduced by means of the generating functions. Further, quasi-monomial properties and differential equations satisfied by these polynomials are established.

Theorem 1 *The generating functions for the generalized Fubini-type polynomials* ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ *and* ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ *are given as follows*:

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\cos(zt)=\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr)\tag{2.1}$$

and

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\sin(zt)=\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\frac{t^n}{n!}\quad\bigl(\upsilon\in\mathbb{N}_0;|t|<\log 2\bigr),\tag{2.2}$$

respectively.

Proof In Eq. (1.7), replacing x and y by the multiplicative operator $\hat{M}_{\mathbb{G}}$ of the 2VGP $\mathbb{G}_n(x,y)$ and z, respectively, gives

$$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}\exp(\hat{M}_{\mathbb{G}}t)\cos(zt)=\sum_{n=0}^{\infty}F_n^{(c,\upsilon)}(\hat{M}_{\mathbb{G}},z)\frac{t^n}{n!},\tag{2.3}$$

which, upon using Eq. (1.25) in the left hand side and Eq. (1.20) in the right hand side, gives

$$\frac{2^{\upsilon}}{(2-e^{t})^{2\upsilon}}\left(\sum_{n=0}^{\infty}\mathbb{G}_{n}(x,y)\frac{t^{n}}{n!}\right)\cos(zt)=\sum_{n=0}^{\infty}F_{n}^{(c,\upsilon)}\left(x+\frac{\varphi'(y,D_{x})}{\varphi(y,D_{x})},z\right)\frac{t^{n}}{n!}. \tag{2.4}$$

Finally, using Eq. (1.18) in the left hand side and denoting the resultant generalized Fubini-type polynomials in the right hand side by ${}_{\mathbb{G}}F_{n}^{(c,\upsilon)}(x,y,z)$, that is,

$${}_{\mathbb{G}}F_{n}^{(c,\upsilon)}(x,y,z)=F_{n}^{(c,\upsilon)}\left(x+\frac{\varphi'(y,D_{x})}{\varphi(y,D_{x})},z\right), \tag{2.5}$$

we get the assertion in Eq. (2.1). Similarly, we can prove the assertion in Eq. (2.2). □

In order to derive the quasi-monomial properties of Fubini-type polynomials ${}_{\mathbb{G}}F_{n}^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_{n}^{(s,\upsilon)}(x,y,z)$, we prove the following results.

Theorem 2 *The generalized Fubini-type polynomials ${}_{\mathbb{G}}F_{n}^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_{n}^{(s,\upsilon)}(x,y,z)$ are quasi-monomial with respect to the following multiplicative and derivative operators*:

$$\hat{M}_{\mathbb{G}Fc}=x+\frac{\varphi'(y,D_{x})}{\varphi(y,D_{x})}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}-z\tan(zD_{x}) \tag{2.6}$$

and

$$\hat{P}_{\mathbb{G}Fc}=D_{x}; \tag{2.7}$$

$$\hat{M}_{\mathbb{G}Fs}=x+\frac{\varphi'(y,D_{x})}{\varphi(y,D_{x})}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}+z\cot(zD_{x}) \tag{2.8}$$

and

$$\hat{P}_{\mathbb{G}Fs}=D_{x}, \tag{2.9}$$

respectively.

Proof Differentiating Eq. (2.1) partially with respect to t gives

$$\begin{aligned}&\left(x+\frac{\varphi'(y,t)}{\varphi(y,t)}+\frac{2\upsilon e^{t}}{2-e^{t}}-z\tan(zt)\right)\left(\frac{2^{\upsilon}}{(2-e^{t})^{2\upsilon}}\right)e^{xt}\varphi(y,t)\cos(zt)\\&\quad=\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_{n+1}^{(c,\upsilon)}(x,y,z)\frac{t^{n}}{n!}\end{aligned} \tag{2.10}$$

which, in view of Eq. (2.1), becomes

$$\sum_{n=0}^{\infty}\left(x+\frac{\varphi'(y,t)}{\varphi(y,t)}+\frac{2\upsilon e^{t}}{2-e^{t}}-z\tan(zt)\right)\left\{{}_{\mathbb{G}}F_{n}^{(s,\upsilon)}(x,y,z)\frac{t^{n}}{n!}\right\}=\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_{n+1}^{(c,\upsilon)}(x,y,z)\frac{t^{n}}{n!}. \tag{2.11}$$

Now, applying the identity

$$D_x\big(e^{xt}\varphi(y,t)\big) = t\big(e^{xt}\varphi(y,t)\big) \tag{2.12}$$

in Eq. (2.11) and equating the coefficients of like powers of t in both sides of the resultant equation, we get

$$\left(x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}} - z\tan(zD_x)\right)\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\} = {}_{\mathbb{G}}F_{n+1}^{(c,\upsilon)}(x,y,z), \tag{2.13}$$

which in view of the monomiality principle given in Eq. (1.22) (for ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$) yields the assertion in Eq. (2.6).

Next, differentiating Eq. (2.1) partially with respect to x gives

$$D_x\left\{\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}\right\} = \sum_{n=0}^{\infty} {}_{\mathbb{G}}F_{n-1}^{(c,\upsilon)}(x,y,z)\frac{t^n}{(n-1)!}, \tag{2.14}$$

which, upon equating the coefficients of like powers of t together with the use of monomiality principle given in Eq. (1.23) (for ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$), yields the assertion in Eq. (2.7). Furthermore, using similar arguments as in the proof of Eqs. (2.6) and (2.7), we can prove the assertions in Eqs. (2.8) and (2.9). □

Theorem 3 *The generalized Fubini-type polynomials* ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ *and* ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ *satisfy the following differential equations*:

$$\left(xD_x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)}D_x + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x - z\tan(zD_x)D_x - n\right){}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z) = 0 \tag{2.15}$$

and

$$\left(xD_x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)}D_x + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x + z\cot(zD_x)D_x - n\right){}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z) = 0, \tag{2.16}$$

respectively.

Proof Using operators (2.6) and (2.7) and in view of the monomiality principle in Eq. (1.24), we get the assertion in Eq. (2.15). Similarly, we can prove the assertion in Eq. (2.16). □

The properties established in this section show that the operational technique provides a mechanism to obtain results for these polynomials as well as their generalizations and demonstrate the usefulness of method in problems of both physics and mathematics.

3 Identities and relations

In this section, by using generating functions (2.1) and (2.2), we establish various novel identities and relations including the generalized Fubini-type polynomials.

Theorem 4 *The generalized Fubini-type polynomials* ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ *and* ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ *are defined by the following series representations*:

$$ {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} F_{n-\kappa}^{(c,\upsilon)}(z)\mathbb{G}_\kappa(x,y) \tag{3.1} $$

and

$$ {}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} F_{n-\kappa}^{(s,\upsilon)}(z)\mathbb{G}_\kappa(x,y), \tag{3.2} $$

respectively.

Proof Utilizing generating relations (1.7) and (1.18) in generating relation (2.1) and making use of the Cauchy product rule in the resultant equation, we get

$$ \sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{n} \binom{n}{\kappa} F_{n-\kappa}^{(c,\upsilon)}(z)\mathbb{G}_\kappa(x,y)\frac{t^n}{n!}. \tag{3.3} $$

Comparing the coefficients of the analogous powers of t in both sides of the above equation, we get the assertion in Eq. (3.1). Similarly, we can get the assertion in Eq. (3.2). □

Theorem 5 *The following summation formulae for the generalized Fubini-type polynomials* ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ *and* ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ *hold true*:

$$ {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} \varphi_\kappa(y) F_{n-\kappa}^{(c,\upsilon)}(x,z), \tag{3.4} $$

$$ {}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} \varphi_\kappa(y) F_{n-\kappa}^{(s,\upsilon)}(x,z). \tag{3.5} $$

Proof Utilizing generating relations (1.7) and (1.19) in generating relation (2.1) and making use of the Cauchy product rule in the resultant equation, we get

$$ \sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{n} \binom{n}{\kappa} \varphi_k(y) F_{n-\kappa}^{(c,\upsilon)}(x,z)\frac{t^n}{n!}. \tag{3.6} $$

Comparing the coefficients of the analogous powers of t in both sides of the above equation, we get the assertion in Eq. (3.4). Similarly, we can get the assertion in Eq. (3.5). □

Theorem 6 *The following implicit summation formula for the generalized Fubini-type polynomials* ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ *holds true*:

$$ {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x+u,y,z) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} {}_{\mathbb{G}}F_{n-\kappa}^{(c,\upsilon)}(x,y,z)u^\kappa. \tag{3.7} $$

Proof In Eq. (2.1), replacing x by $x+u$ then making use of Eq. (2.1) together with the series expansion of e^{ut} in the resultant equation, we get

$$\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x+u,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)u^{\kappa}\frac{t^{n+\kappa}}{n!\kappa!}, \tag{3.8}$$

which, upon replacing n by $n-\kappa$ in the right hand side and then comparing the coefficients of the like powers of t in both sides of the resultant equation yields the assertion in Eq. (3.7). □

Corollary 1 *For $u = x$ in Eq.* (3.7), *we have*

$${}_{\mathbb{G}}F_n^{(c,\upsilon)}(2x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa} {}_{\mathbb{G}}F_{n-\kappa}^{(c,\upsilon)}(x,y,z)x^{\kappa}. \tag{3.9}$$

Corollary 2 *For $u = 1$ in Eq.* (3.7), *we have*

$${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x+1,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa} {}_{\mathbb{G}}F_{n-\kappa}^{(c,\upsilon)}(x,y,z). \tag{3.10}$$

Theorem 7 *The following implicit summation formula for the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ holds true*:

$${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x+u,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa} F_{n-\kappa}^{(s,\upsilon)}(x,z)\mathbb{G}_{\kappa}(u,y). \tag{3.11}$$

Proof In Eq. (2.1), replacing x by $x+u$ then making use of Eqs. (1.8) and (1.18) in the resultant equation, we get

$$\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(s,\upsilon)}(x+u,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{\infty} F_n^{(s,\upsilon)}(x,z)\mathbb{G}_{\kappa}(u,y)\frac{t^{n+\kappa}}{n!\kappa!}, \tag{3.12}$$

which, upon simplification, gives the desired result. □

Theorem 8 *The following implicit summation formula for the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ holds true*:

$${}_{\mathbb{G}}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z) = \sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}(\omega-x)^{l+m} {}_{\mathbb{G}}F_{n+\kappa-l-m}^{(c,\upsilon)}(x,y,z). \tag{3.13}$$

Proof Consider the following identity [11]:

$$\sum_{m=0}^{\infty} g(m)\frac{(x+y)^m}{m!} = \sum_{s,r=0}^{\infty} g(s+r)\frac{x^s y^r}{s!r!}. \tag{3.14}$$

Replacing t by $t+s$ in the generating function (2.1) and making use of the identity (3.14), we have

$$\frac{2^{\upsilon}}{(2-e^{(t+s)})^{2\upsilon}}\varphi(y,t+s)\cos\bigl(z(t+s)\bigr)=e^{-x(t+s)}\sum_{n,\kappa=0}^{\infty}{}_{\mathbb{G}}F_{n+\kappa}^{(c,\upsilon)}(x,y,z)\frac{t^n s^\kappa}{n!\kappa!}. \tag{3.15}$$

Replacing x by ω in Eq. (3.15), equating the resultant equation to Eq. (3.15) and then expanding the exponential function, we get

$$\sum_{n,\kappa=0}^{\infty}{}_{\mathbb{G}}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z)\frac{t^n s^\kappa}{n!\kappa!}=\sum_{\chi=0}^{\infty}\frac{((\omega-x)(t+s))^{\chi}}{\chi!}\sum_{n,\kappa=0}^{\infty}{}_{\mathbb{G}}F_{n+\kappa}^{(c,\upsilon)}(x,y,z)\frac{t^n s^\kappa}{n!\kappa!}. \tag{3.16}$$

Now, making use of identity (3.14) in the right hand side of the above equation then replacing n by $n-l$ and κ by $\kappa-m$ in the right hand side of the resultant equation gives

$$\sum_{n,\kappa=0}^{\infty}{}_{\mathbb{G}}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z)\frac{t^n s^\kappa}{n!\kappa!}=\sum_{n,\kappa=0}^{\infty}\sum_{l,m=0}^{n,\kappa}\frac{(\omega-x)^{l+m}{}_{\mathbb{G}}F_{n+\kappa-l-m}^{(c,\upsilon)}(x,y,z)t^n s^\kappa}{l!m!(n-l)!(\kappa-m)!}. \tag{3.17}$$

Finally, comparing the coefficients of the like powers of t and s in both sides of Eq. (3.17) yields the assertion in Eq. (3.13). □

Corollary 3 *Taking $z=0$ and replacing ω by $\omega+x$ Eq. (3.13), we get*

$$ {}_{\mathbb{G}}F_{n+\kappa}^{(\upsilon)}(\omega+x,y)=\sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}\omega^{l+m}{}_{\mathbb{G}}F_{n+\kappa-l-m}^{(\upsilon)}(x,y). \tag{3.18}$$

Theorem 9 *Let $\upsilon,\sigma\in\mathbb{N}_0$, then the following identities hold true:*

$$ {}_{\mathbb{G}}F_n^{(c,\upsilon+\sigma)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_\kappa^{(\upsilon)}{}_{\mathbb{G}}F_{n-\kappa}^{(c,\sigma)}(x,y,z) \tag{3.19}$$

and

$$ {}_{\mathbb{G}}F_n^{(s,\upsilon+\sigma)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_\kappa^{(\upsilon)}{}_{\mathbb{G}}F_{n-\kappa}^{(s,\sigma)}(x,y,z). \tag{3.20}$$

Proof Rewriting generating relation (2.1) in the following form:

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_n^{(c,\upsilon+\sigma)}(x,y,z)\frac{t^n}{n!}&=\frac{2^{\upsilon+\sigma}}{(2-e^t)^{2(\upsilon+\sigma)}}e^{xt}\varphi(y,t)\cos(zt)\\&=\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}\frac{2^{\sigma}}{(2-e^t)^{2\sigma}}e^{xt}\varphi(y,t)\cos(zt),\end{aligned} \tag{3.21}$$

which, upon using Eqs. (1.2) and (2.1) and then after simplification yields the assertion in Eq. (3.19). The assertion in Eq. (3.20) can be proved similarly. □

Theorem 10 *Let $n\in\mathbb{N}_0$ and $i=\sqrt{-1}\in\mathbb{C}$, then*

$$ {}_{\mathbb{G}}F_n^{(\upsilon)}(x+iz,y)={}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)+i{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z). \tag{3.22}$$

Proof Taking $z = 0$ and replacing x by $x + iz$ in Eq. (2.1), it follows that

$$\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(\upsilon)}(x+iz,y)\frac{t^n}{n!} = \frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{(x+iz)t}\varphi(y,t)$$
$$= \frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\cos(zt) + i\frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\sin(zt), \quad (3.23)$$

which, upon using Eqs. (2.1) and (2.2) in the right hand side and then comparing the coefficients of the like powers of t in both sides of the resultant equation yields the assertion in Eq. (3.22). □

Theorem 11 *For $\upsilon \in \mathbb{N}_0$, the following identities hold true*:

$$x^n\varphi(y,t)\cos(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^\delta\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}\,{}_{\mathbb{G}}F_\kappa^{(c,\upsilon)}(x,y,z) \quad (3.24)$$

and

$$x^n\varphi(y,t)\sin(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^\delta\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}\,{}_{\mathbb{G}}F_\kappa^{(s,\upsilon)}(x,y,z). \quad (3.25)$$

Proof Rewriting generating relation (2.1) in the following form:

$$2^\upsilon e^{xt}\varphi(y,t)\cos(zt) = \left(2-e^t\right)^{2\upsilon}\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}, \quad (3.26)$$

which can be written as

$$2^\upsilon e^{xt}\varphi(y,t)\cos(zt) = \sum_{\delta=0}^{2\upsilon}(-1)^\delta\binom{2\upsilon}{\delta}2^{2\upsilon-\delta}e^{\delta t}\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}. \quad (3.27)$$

Finally, expanding the exponentials in both sides of the above equation and then after simplification, gives the assertion in Eq. (3.24). Similarly, the assertion in Eq. (3.25) can be proved. □

Theorem 12 *For $n \in \mathbb{N}_0$, the following identity holds true*:

$$\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(\upsilon)}(x)\,{}_{\mathbb{G}}F_\kappa^{(s,\sigma)}(x,y,2z) = 2\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_\kappa^{(s,\upsilon)}(x,z)\,{}_{\mathbb{G}}F_{n-\kappa}^{(c,\sigma)}(x,y,z). \quad (3.28)$$

Proof In view of Eqs. (1.1) and (2.2), we have

$$\sum_{n=0}^{\infty}F_n^{(\upsilon)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} {}_{\mathbb{G}}F_n^{(s,\sigma)}(x,y,2z)\frac{t^n}{n!}$$
$$= \frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\frac{2^\sigma}{(2-e^t)^{2\sigma}}e^{xt}\varphi(y,t)\sin(2zt)$$
$$= 2\frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\sin(zt)\frac{2^\sigma}{(2-e^t)^{2\sigma}}e^{xt}\varphi(y,t)\cos(zt), \quad (3.29)$$

which, upon using Eqs. (1.8) and (2.1) and then after simplification and comparing the coefficients of the like powers of t in both sides of the resultant equation yields the assertion in Eq. (3.28). □

4 Partial derivative equations

In this section, we establish various partial derivative equations including the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ by applying partial derivative operator to generating relations (2.1) and (2.2). To achieve this, the following results are proved.

Theorem 13 *Let $m,n \in \mathbb{N}$ with $n \geqq m$. Then the following partial derivative formulas for the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ hold true*:

$$\frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\} = 2^{-\frac{3\delta}{2}}\sum_{\kappa=0}^{n} m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_\kappa^{(\delta)}\left(-\frac{1}{2}\right){}_{\mathbb{G}}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(x,y,z) \tag{4.1}$$

and

$$\frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\} = 2^{-\frac{\delta}{2}}\sum_{\kappa=0}^{n} \frac{m!\kappa!}{(\kappa+\delta)!}\binom{n}{\kappa}\binom{n-\kappa}{m}\mathfrak{B}_{\kappa+\delta}^{(\delta)}\left(\frac{1}{2}\right){}_{\mathbb{G}}F_{n-\kappa-m}^{(s,\upsilon-\frac{\delta}{2})}(x,y,z), \tag{4.2}$$

respectively.

Proof Applying the derivative operator $\frac{\partial^m}{\partial x^m}$ to the generating relation (2.1) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\}\frac{t^n}{n!} \\
&\quad = t^m\left\{\frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\cos(zt)\right\} \\
&\quad = t^m\left\{2^{-\frac{3\delta}{2}}\left(\frac{2}{-\frac{1}{2}e^t+1}\right)^\delta\frac{2^{\upsilon-\frac{\delta}{2}}}{(2-e^t)^{2(\upsilon-\frac{\delta}{2})}}e^{xt}\varphi(y,t)\cos(zt)\right\},
\end{aligned} \tag{4.3}$$

which, in view of Eqs. (1.10) and (2.1) and after simplification, becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\}\frac{t^n}{n!} \\
&\quad = 2^{-\frac{3\delta}{2}}\sum_{n=0}^{\infty}\sum_{\kappa=0}^{n} m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_\kappa^{(\delta)}\left(-\frac{1}{2}\right){}_{\mathbb{G}}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(x,y,z)\frac{t^n}{n!}.
\end{aligned} \tag{4.4}$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we are led at the assertion in Eq. (4.1). The assertion (4.2) can be proved in a similar way. □

Theorem 14 *Let $m, n \in \mathbb{N}$ with $n \geqq m$. Then*

$$\begin{aligned}
&\frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\right\} \\
&\quad = \sum_{\kappa=0}^{n} m!\binom{n}{\kappa}\binom{\kappa}{m}\left\{{}_{\mathbb{G}}F_{\kappa-m}^{(s,\upsilon)}(x,y,z)C_{n-\kappa}(u,\omega) + {}_{\mathbb{G}}F_{\kappa-m}^{(c,\upsilon)}(x,y,z)S_{n-\kappa}(u,\omega)\right\}. \qquad (4.5)
\end{aligned}$$

Proof Replacing x by $x+u$ and z by $z+\omega$ in Eq. (2.2) and then applying the derivative operator $\frac{\partial^m}{\partial x^m}$ to the resultant equation, it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\right\}\frac{t^n}{n!} \\
&\quad = t^m\left\{\frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{(x+u)t}\varphi(y,t)\sin\bigl((z+\omega)t\bigr)\right\} \\
&\quad = t^m\left\{\frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\sin(zt)e^{ut}\cos(\omega t)\right. \\
&\qquad \left. + \frac{2^\upsilon}{(2-e^t)^{2\upsilon}}e^{xt}\varphi(y,t)\cos(zt)e^{ut}\sin(\omega t)\right\}. \qquad (4.6)
\end{aligned}$$

Next, in view of Eqs. (1.3), (1.4), (2.1) and (2.2), Eq. (4.6) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\right\}\frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{\kappa=0}^{n} m!\binom{\kappa}{m}{}_{\mathbb{G}}F_{\kappa-m}^{(s,\upsilon)}(x,y,z)C_{n-\kappa}(u,\omega)\frac{t^n}{\kappa!(n-\kappa)!} \\
&\qquad + \sum_{n=0}^{\infty}\sum_{\kappa=0}^{n} m!\binom{\kappa}{m}{}_{\mathbb{G}}F_{\kappa-m}^{(c,\upsilon)}(x,y,z)S_{n-\kappa}(u,\omega)\frac{t^n}{\kappa!(n-\kappa)!}, \qquad (4.7)
\end{aligned}$$

which, upon simplification, yields the desired result (4.5). □

Theorem 15 *Let $\upsilon, \sigma \in \mathbb{N}_0$ and $m, n \in \mathbb{N}$ with $n \geqq m$. Then*

$$\begin{aligned}
&\frac{\partial^m}{\partial u^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon+\sigma)}(x+u,y,z)\right\} \\
&\quad = \frac{n!}{2^\upsilon(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon} m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathfrak{B}_{\kappa-m}^{(2\upsilon)}\left(x,\frac{1}{2}\right){}_{\mathbb{G}}F_{n+2\upsilon-\kappa}^{(c,\sigma)}(u,y,z) \qquad (4.8)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^m}{\partial u^m}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon+\sigma)}(x+u,y,z)\right\} \\
&\quad = \frac{n!}{2^{3\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon} m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathcal{G}_{\kappa-m}^{(2\upsilon)}\left(x,-\frac{1}{2}\right){}_{\mathbb{G}}F_{n+2\upsilon-\kappa}^{(s,\sigma)}(u,y,z). \qquad (4.9)
\end{aligned}$$

Proof Replacing x by $x+u$ and υ by $\upsilon+\sigma$ in Eq. (2.1) and then applying the derivative operator $\frac{\partial^m}{\partial x^m}$ to the resultant equation, it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon+\sigma)}(x+u,y,z)\right\}\frac{t^n}{n!} \\
&= t^m\left\{\frac{2^{\upsilon+\sigma}}{(2-e^t)^{2(\upsilon+\sigma)}}e^{(x+u)t}\varphi(y,t)\cos(zt)\right\} \\
&= t^m\left\{\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}\frac{2^{\sigma}}{(2-e^t)^{2\sigma}}e^{ut}\varphi(y,t)\cos(zt)\right\} \\
&= t^m\left\{2^{-\upsilon}t^{-2\upsilon}\left(\frac{t}{\frac{1}{2}e^t-1}\right)^{2\upsilon}e^{xt}\frac{2^{\sigma}}{(2-e^t)^{2\sigma}}e^{ut}\varphi(y,t)\cos(zt)\right\},
\end{aligned} \tag{4.10}$$

which, in view of Eqs. (1.9) and (2.1), becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon+\sigma)}(x+u,y,z)\right\}\frac{t^n}{n!} \\
&= t^m\left\{2^{-\upsilon}t^{-2\upsilon}\sum_{n=0}^{\infty}\mathfrak{B}_n^{(2\upsilon)}\left(x,\frac{1}{2}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_{\mathbb{G}}F_n^{(c,\sigma)}(u,y,z)\frac{t^n}{n!}\right\}.
\end{aligned} \tag{4.11}$$

Finally, simplifying and then equating the coefficients of the like powers of t in the resultant equation yields the assertion in Eq. (4.8). Similarly, we can prove the desired result (4.9). □

Theorem 16 *For $n \in \mathbb{N}$, we have*

$$\frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}(n-\kappa)\binom{n}{\kappa}\mathcal{E}_\kappa^{(2\delta)}\left(-\frac{1}{2}\right){}_{\mathbb{G}}F_{n-\kappa-1}^{(c,\upsilon-\delta)}(x,y,z) \tag{4.12}$$

and

$$\frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}\frac{(n-\kappa)\kappa!}{(\kappa+2\delta)!}\binom{n}{\kappa}\mathcal{G}_{\kappa+2\delta}^{(2\delta)}\left(-\frac{1}{2}\right){}_{\mathbb{G}}F_{n-\kappa-1}^{(s,\upsilon-\delta)}(x,y,z). \tag{4.13}$$

Proof Differentiating generating relation (2.1) with respect to x, gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\}\frac{t^n}{n!} \\
&= t\left\{\frac{1}{2^{3\delta}}\left(\frac{2}{-\frac{1}{2}e^t+1}\right)^{2\delta}\left(\frac{2^{\upsilon-\delta}}{(2-e^t)^{2(\upsilon-\delta)}}\right)e^{xt}\varphi(y,t)\cos(zt)\right\},
\end{aligned} \tag{4.14}$$

which, upon using Eqs. (1.10) and (2.1) then simplifying and equating the coefficients of the like powers of t in the resultant equation yields the assertion in Eq. (4.12). The result (4.13) can be derived similarly. □

Remark 1 For $n \in \mathbb{N}$ and in view of Eqs. (2.1) and (2.2), the following results can be obtained:

$$\frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\} = n\,{}_{\mathbb{G}}F_{n-1}^{(c,\upsilon)}(x,y,z), \tag{4.15}$$

$$\frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\} = n\,{}_{\mathbb{G}}F_{n-1}^{(s,\upsilon)}(x,y,z), \tag{4.16}$$

$$\frac{\partial}{\partial z}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\} = -n\,{}_{\mathbb{G}}F_{n-1}^{(s,\upsilon)}(x,y,z), \tag{4.17}$$

and

$$\frac{\partial}{\partial z}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\} = n\,{}_{\mathbb{G}}F_{n-1}^{(c,\upsilon)}(x,y,z). \tag{4.18}$$

Furthermore, we have

$$\frac{\partial}{\partial x}\left\{{}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)\right\} = \frac{\partial}{\partial z}\left\{{}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)\right\}. \tag{4.19}$$

The theory of hybrid special polynomials has been one of the most emerging research topic in mathematical analysis and extensively studied to find useful properties and identities. Applications of various properties of multivariable hybrid special polynomials arise in problems of number theory, combinatorics, theoretical physics and other areas of pure and applied mathematics provide motivation for introducing a new class of generalized Fubini-type polynomials and explore its properties.

The properties and applications of these polynomials lie within the root polynomials. To explore the applications of the hybrid class of generalized Fubini-type polynomials, we have:

1. The hybrid polynomials comprising Fubini type polynomials occurs in many application not only in combinatorial analysis, but also other branches of mathematics, engineering and related areas.
2. The reason of interest for the hybrid polynomials related with truncated exponential polynomials originates from the fact that they appear in the theory of flattened Beam which assumes a role of foremost significance in optics and particularly in super-Gaussian optical resonators and plays an important role to evaluate integrals including products of special functions.
3. The motivation for the hybrid polynomials related with Laguerre polynomials is because of their intrinsic scientific significance and to the way that these polynomials are demonstrated to be natural solutions of a particular set of partial differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings.
4. The hybrid polynomials involving Hermite polynomials occur in probability as the Edgeworth series; in combinatorics, they arise in the umbral calculus as an example of an Appell sequence (play an important role in various problems connected with functional equations, interpolation problems, approximation theory, summation methods); in numerical analysis, they play a role in Gaussian quadrature; and in physics, they appear in quantum mechanical and optical beam transport problems.

In the next section, certain special cases of the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ are considered.

5 Special cases

We note that corresponding to each member belonging to the 2VGP $\mathbb{G}_n(x,y)$, there exists a new special hybrid polynomial belonging to the generalized Fubini-type polynomials of parametric kind ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$. The results related to these special hybrid polynomials can be obtained from the results established in the previous sections.

5.1 Gould–Hopper–Fubini-type polynomials

Since, for $\varphi(y,t) = e^{yt^r}$, the 2VGP $\mathbb{G}_n(x,y)$ reduce to the Gould–Hopper polynomials $\mathcal{H}_n^{(r)}(x,y)$ (Table 1(I)), for the same choices of $\varphi(y,t)$, the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ reduce to the Gould–Hopper–Fubini-type polynomials (GHFTP) ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)$. Thus, by taking $\varphi(y,t) = e^{yt^r}$ in the results established in Sects. 2–4, we can obtain the corresponding results for the GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)$, these results are listed in Tables 2 and 3.

Table 2 Results for ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$

Generating function	$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}} e^{xt+yt^r}\cos(zt) = \sum_{n=0}^{\infty} {}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{GHFc} = x + ryD_x^{r-1} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}} - z\tan(zD_x)$, $\hat{P}_{GHFc} := D_x$
Differential equation	$(xD_x + ryD_x^r + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x - z\tan(zD_x)D_x - n)\, {}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z) = 0$
Identities and relations	${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(c,\upsilon)}(z)\mathcal{H}_\kappa^{(r)}(x,y)$ ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x+u,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}{}_{\mathcal{H}^{(r)}}F_{n-\kappa}^{(c,\upsilon)}(x,y,z)u^\kappa$ ${}_{\mathcal{H}^{(r)}}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z) = \sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}(\omega-x)^{l+m}\, {}_{\mathcal{H}^{(r)}}F_{n+\kappa-l-m}^{(c,\upsilon)}(x,y,z)$ ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon+\sigma)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_\kappa^{(\upsilon)}\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa}^{(c,\sigma)}(x,y,z)$ $x^n e^{yt^r}\cos(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^\delta\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}\, {}_{\mathcal{H}^{(r)}}F_\kappa^{(c,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)\} = 2^{-\frac{3\delta}{2}}\sum_{\kappa=0}^{n} m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_\kappa^{(\delta)}(-\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(x,y,z)$ $\frac{\partial^m}{\partial u^m}\{{}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon+\sigma)}(x+u,y,z)\} = \frac{n!}{2^{\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon} m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathfrak{B}_{\kappa-m}^{(2\upsilon)}(x,\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n+2\upsilon-\kappa}^{(c,\sigma)}(u,y,z)$ $\frac{\partial}{\partial x}\{{}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}(n-\kappa)\binom{n}{\kappa}\mathcal{E}_\kappa^{(2\delta)}(-\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa-1}^{(c,\upsilon-\delta)}(x,y,z)$

Table 3 Results for ${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)$

Generating function	$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}} e^{xt+yt^r}\sin(zt) = \sum_{n=0}^{\infty} {}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{GHFs} = x + ryD_x^{r-1} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}} + z\cot(zD_x)$, $\hat{P}_{GHFs} := D_x$
Differential equation	$(xD_x + ryD_x^r + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x + z\cot(zD_x)D_x - n)\, {}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z) = 0$
Identities and relations	${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(z)\mathcal{H}_\kappa^{(r)}(x,y)$ ${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x+u,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(x,z)\mathcal{H}_\kappa^{(r)}(u,y)$ ${}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon+\sigma)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_\kappa^{(\upsilon)}\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa}^{(s,\sigma)}(x,y,z)$ $x^n e^{yt^r}\sin(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^\delta\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}\, {}_{\mathcal{H}^{(r)}}F_\kappa^{(s,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)\} = 2^{-\frac{\delta}{2}}\sum_{\kappa=0}^{n}\frac{m!\kappa!}{(\kappa+\delta)!}\binom{n}{\kappa}\binom{n-\kappa}{m}\mathfrak{B}_{\kappa+\delta}^{(\delta)}(\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa-m}^{(s,\upsilon-\frac{\delta}{2})}(x,y,z)$ $\frac{\partial^m}{\partial u^m}\{{}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon+\sigma)}(x+u,y,z)\} = \frac{n!}{2^{3\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon} m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathcal{G}_{\kappa-m}^{(2\upsilon)}(x,-\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n+2\upsilon-\kappa}^{(s,\sigma)}(u,y,z)$ $\frac{\partial}{\partial x}\{{}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z)\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}\frac{(n-\kappa)\kappa!}{(\kappa+2\delta)!}\binom{n}{\kappa}\mathcal{G}_{\kappa+2\delta}^{(2\delta)}(-\frac{1}{2})\, {}_{\mathcal{H}^{(r)}}F_{n-\kappa-1}^{(s,\upsilon-\delta)}(x,y,z)$

Furthermore, in view of Eqs. (3.22), (3.28) and (4.5), we have

$$
{}_{\mathcal{H}^{(r)}}F_n^{(\upsilon)}(x+iz,y) = {}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z) + i\,{}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x,y,z), \tag{5.1}
$$

$$
\begin{aligned}
&\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(\upsilon)}(x)\,{}_{\mathcal{H}^{(r)}}F_{\kappa}^{(s,\sigma)}(x,y,2z)\\
&\quad = 2\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(s,\upsilon)}(x,z)\,{}_{\mathcal{H}^{(r)}}F_{n-\kappa}^{(c,\sigma)}(x,y,z)
\end{aligned} \tag{5.2}
$$

and

$$
\begin{aligned}
&\frac{\partial^m}{\partial x^m}\left\{{}_{\mathcal{H}^{(r)}}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\right\}\\
&\quad = \sum_{\kappa=0}^{n} m!\binom{n}{\kappa}\binom{\kappa}{m}\left\{{}_{\mathcal{H}^{(r)}}F_{\kappa-m}^{(s,\upsilon)}(x,y,z)C_{n-\kappa}(u,\omega) + {}_{\mathcal{H}^{(r)}}F_{\kappa-m}^{(c,\upsilon)}(x,y,z)S_{n-\kappa}(u,\omega)\right\}.
\end{aligned} \tag{5.3}
$$

5.2 Laguerre–Fubini-type polynomials

Since, for $\varphi(y,t) = C_0(yt)$, the 2VGP $\mathbb{G}_n(x,y)$ reduce to the Laguerre polynomials $\mathrm{L}_n(y,x)$ (Table 1(II)). Therefore, for the same choice of $\varphi(y,t)$, the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(y,x,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(y,x,z)$ reduce to Laguerre–Fubini-type polynomials ${}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)$ and ${}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)$. Thus, by taking $\varphi(y,t) = C_0(yt)$ in the results established in Sects. 2–4, we can obtain the corresponding results for Laguerre–Fubini-type polynomials ${}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)$ and ${}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)$, these results are listed in Tables 4 and 5.

Furthermore, in view of Eqs. (3.22), (3.28) and (4.5), we have

$$
{}_{\mathrm{L}}F_n^{(\upsilon)}(y,x+iz) = {}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z) + i\,{}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z), \tag{5.4}
$$

$$
\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(\upsilon)}(x)\,{}_{\mathrm{L}}F_{\kappa}^{(s,\sigma)}(y,x,2z) = 2\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(s,\upsilon)}(x,z)\,{}_{\mathrm{L}}F_{n-\kappa}^{(c,\sigma)}(y,x,z) \tag{5.5}
$$

Table 4 Results for ${}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)$

Generating function	$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}C_0(yt)\cos(zt) = \sum_{n=0}^{\infty}{}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{LFc} = x - D_y^{-1} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}} - z\tan(zD_x)$, $\hat{P}_{LFc} := D_x$
Differential equation	$(xD_x - D_y^{-1}D_x + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x - z\tan(zD_x)D_x - n)\,{}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z) = 0$
Identities and relations	${}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(c,\upsilon)}(z)\mathrm{L}_{\kappa}(y,x)$ ${}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x+u,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}{}_{\mathrm{L}}F_{n-\kappa}^{(c,\upsilon)}(y,x,z)u^{\kappa}$ ${}_{\mathrm{L}}F_{n+\kappa}^{(c,\upsilon)}(\omega,x,z) = \sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}(\omega-x)^{l+m}\,{}_{\mathrm{L}}F_{n+\kappa-l-m}^{(c,\upsilon)}(y,x,z)$ ${}_{\mathrm{L}}F_n^{(c,\upsilon+\sigma)}(y,x,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}\,{}_{\mathrm{L}}F_{n-\kappa}^{(c,\sigma)}(y,x,z)$ $x^nC_0(yt)\cos(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}\,{}_{\mathrm{L}}F_{\kappa}^{(c,\upsilon)}(y,x,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)\} = 2^{-\frac{3\delta}{2}}\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_{\kappa}^{(\delta)}(-\frac{1}{2})\,{}_{\mathrm{L}}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(y,x,z)$ $\frac{\partial^m}{\partial u^m}\{{}_{\mathrm{L}}F_n^{(c,\upsilon+\sigma)}(y,x+u,z)\} = \frac{n!}{2^{\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathfrak{B}_{\kappa-m}^{(2\upsilon)}(x,\frac{1}{2})\,{}_{\mathrm{L}}F_{n+2\upsilon-\kappa}^{(c,\sigma)}(u,x,z)$ $\frac{\partial}{\partial x}\{{}_{\mathrm{L}}F_n^{(c,\upsilon)}(y,x,z)\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}(n-\kappa)\binom{n}{\kappa}\mathcal{E}_{\kappa}^{(2\delta)}(-\frac{1}{2})\,{}_{\mathrm{L}}F_{n-\kappa-1}^{(c,\upsilon-\delta)}(y,x,z)$

Table 5 Results for ${}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)$

Generating function	$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}e^{xt}C_0(yt)\sin(zt)=\sum_{n=0}^{\infty}{}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{LFs}=x-D_y^{-1}+\frac{2\upsilon e^{D_x}}{2-e^{D_x}}+z\cot(zD_x)$, $\hat{P}_{LFs}:=D_x$
Differential equation	$(xD_x-D_y^{-1}D_x+\frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x+z\cot(zD_x)D_x-n){}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)=0$
Identities and relations	${}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(z)\mathrm{L}_{\kappa}(y,x)$ ${}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x+u,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(x,z)\mathrm{L}_{\kappa}(y,u)$ ${}_{\mathrm{L}}F_n^{(s,\upsilon+\sigma)}(y,x,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}{}_{\mathrm{L}}F_{n-\kappa}^{(s,\sigma)}(y,x,z)$ $x^nC_0(yt)\sin(zt)=\sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}{}_{\mathrm{L}}F_{\kappa}^{(s,\upsilon)}(y,x,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)\}=2^{-\frac{\delta}{2}}\sum_{\kappa=0}^{n}\frac{m!\kappa!}{(\kappa+\delta)!}\binom{n}{\kappa}\binom{n-\kappa}{m}\mathfrak{B}_{\kappa+\delta}^{(\delta)}(\frac{1}{2}){}_{\mathrm{L}}F_{n-\kappa-m}^{(s,\upsilon-\frac{\delta}{2})}(y,x,z)$ $\frac{\partial^m}{\partial u^m}\{{}_{\mathrm{L}}F_n^{(s,\upsilon+\sigma)}(y,x+u,z)\}=\frac{n!}{2^{3\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathcal{G}_{\kappa-m}^{(2\upsilon)}(x,-\frac{1}{2}){}_{\mathrm{L}}F_{n+2\upsilon-\kappa}^{(s,\sigma)}(y,u,z)$ $\frac{\partial}{\partial x}\{{}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x,z)\}=\frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}\frac{(n-\kappa)\kappa!}{(\kappa+2\delta)!}\binom{n}{\kappa}\mathcal{G}_{\kappa+2\delta}^{(2\delta)}(-\frac{1}{2}){}_{\mathrm{L}}F_{n-\kappa-1}^{(s,\upsilon-\delta)}(y,x,z)$

Table 6 Results for ${}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)$

Generating function	$(\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}})(\frac{e^{xt}}{1-yt^s})\cos(zt)=\sum_{n=0}^{\infty}{}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{TEFc}=x+\frac{syD_x^{s-1}}{1-yD_x^s}+\frac{2\upsilon e^{D_x}}{2-e^{D_x}}-z\tan(zD_x)$, $\hat{P}_{TEFc}:=D_x$
Differential equation	$(xD_x+\frac{syD_x^s}{1-yD_x^s}+\frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x-z\tan(zD_x)D_x-n){}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)=0$
Identities and relations	${}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(c,\upsilon)}(z)e_{\kappa}^{(s)}(x,y)$ ${}_{e^{(s)}}F_n^{(c,\upsilon)}(x+u,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}{}_{e^{(s)}}F_{n-\kappa}^{(c,\upsilon)}(x,y,z)u^{\kappa}$ ${}_{e^{(s)}}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z)=\sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}(\omega-x)^{l+m}{}_{e^{(s)}}F_{n+\kappa-l-m}^{(c,\upsilon)}(x,y,z)$ ${}_{e^{(s)}}F_n^{(c,\upsilon+\sigma)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}{}_{e^{(s)}}F_{n-\kappa}^{(c,\sigma)}(x,y,z)$ $(\frac{x^n}{1-yt^s})\cos(zt)=\sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}{}_{e^{(s)}}F_{\kappa}^{(c,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)\}=2^{-\frac{3\delta}{2}}\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_{\kappa}^{(\delta)}(-\frac{1}{2}){}_{e^{(s)}}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(x,y,z)$ $\frac{\partial^m}{\partial u^m}\{{}_{e^{(s)}}F_n^{(c,\upsilon+\sigma)}(x+u,y,z)\}=\frac{n!}{2^{\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathfrak{B}_{\kappa-m}^{(2\upsilon)}(x,\frac{1}{2}){}_{e^{(s)}}F_{n+2\upsilon-\kappa}^{(c,\sigma)}(u,y,z)$ $\frac{\partial}{\partial x}\{{}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)\}=\frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}(n-\kappa)\binom{n}{\kappa}\mathcal{E}_{\kappa}^{(2\delta)}(-\frac{1}{2}){}_{e^{(s)}}F_{n-\kappa-1}^{(c,\upsilon-\delta)}(x,y,z)$

and

$$\frac{\partial^m}{\partial x^m}\left\{{}_{\mathrm{L}}F_n^{(s,\upsilon)}(y,x+u,z+\omega)\right\}$$
$$=\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{\kappa}{m}\left\{{}_{\mathrm{L}}F_{\kappa-m}^{(s,\upsilon)}(y,x,z)C_{n-\kappa}(u,\omega)+{}_{\mathrm{L}}F_{\kappa-m}^{(c,\upsilon)}(y,x,z)S_{n-\kappa}(u,\omega)\right\}. \tag{5.6}$$

5.3 Truncated exponential–Fubini-type polynomials

Since, for $\varphi(y,t)=\frac{1}{1-yt^s}$, the 2VGP $\mathbb{G}_n(x,y)$ reduce to the truncated exponential polynomials of order s $e_n^{(s)}(x,y)$ (Table 1(III)). Therefore, for the same choice of $\varphi(y,t)$, the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ reduce to truncated exponential–Fubini-type polynomials ${}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)$. Thus, by taking $\varphi(y,t)=\frac{1}{1-yt^s}$ in the results established in Sects. 2–4, we can obtain the corresponding results for the truncated exponential–Fubini-type polynomials ${}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)$, these results are listed in Tables 6 and 7.

Table 7 Results for ${}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)$

Generating function	$\frac{2^{\upsilon}}{(2-e^t)^{2\upsilon}}(\frac{e^{xt}}{1-yt^s})\sin(zt) = \sum_{n=0}^{\infty} {}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\hat{M}_{TEFs} = x + \frac{syD_x^{s-1}}{1-yD_x^s} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}} + z\cot(zD_x)$, $\hat{P}_{TEFs} := D_x$
Differential equation	$(xD_x + \frac{syD_x^s}{1-yD_x^s} + \frac{2\upsilon e^{D_x}}{2-e^{D_x}}D_x + z\cot(zD_x)D_x - n)_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z) = 0$
Identities and relations	${}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(z)e_{\kappa}^{(s)}(x,y)$
	${}_{e^{(s)}}F_n^{(s,\upsilon)}(x+u,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(x,z)e_{\kappa}^{(s)}(u,y)$
	${}_{e^{(s)}}F_n^{(s,\upsilon+\sigma)}(x,y,z) = \sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}{}_{e^{(s)}}F_{n-\kappa}^{(s,\sigma)}(x,y,z)$
	$(\frac{x^n}{1-yt^s})\sin(zt) = \sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}{}_{e^{(s)}}F_{\kappa}^{(s,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^m}{\partial x^m}\{{}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)\} = 2^{-\frac{\delta}{2}}\sum_{\kappa=0}^{n}\frac{m!\kappa!}{(\kappa+\delta)!}\binom{n}{\kappa}\binom{n-\kappa}{m}\mathfrak{B}_{\kappa+\delta}^{(\delta)}(\frac{1}{2}){}_{e^{(s)}}F_{n-\kappa-m}^{(s,\upsilon-\frac{\delta}{2})}(x,y,z)$
	$\frac{\partial^m}{\partial u^m}\{{}_{e^{(s)}}F_n^{(s,\upsilon+\sigma)}(x+u,y,z)\} = \frac{n!}{2^{3\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathcal{G}_{\kappa-m}^{(2\upsilon)}(x,-\frac{1}{2}){}_{e^{(s)}}F_{n+2\upsilon-\kappa}^{(s,\sigma)}(u,y,z)$
	$\frac{\partial}{\partial x}\{{}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z)\} = \frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}\frac{(n-\kappa)\kappa!}{(\kappa+2\delta)!}\binom{n}{\kappa}\mathcal{G}_{\kappa+2\delta}^{(2\delta)}(-\frac{1}{2}){}_{e^{(s)}}F_{n-\kappa-1}^{(s,\upsilon-\delta)}(x,y,z)$

Furthermore, in view of Eqs. (3.22), (3.28) and (4.5), we have

$$ {}_{e^{(s)}}F_n^{(\upsilon)}(x+iz,y) = {}_{e^{(s)}}F_n^{(c,\upsilon)}(x,y,z) + i\,{}_{e^{(s)}}F_n^{(s,\upsilon)}(x,y,z), \tag{5.7}$$

$$\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(\upsilon)}(x)\,{}_{e^{(s)}}F_{\kappa}^{(s,\sigma)}(x,y,2z) = 2\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(s,\upsilon)}(x,z)\,{}_{e^{(s)}}F_{n-\kappa}^{(c,\sigma)}(x,y,z) \tag{5.8}$$

and

$$\begin{aligned}&\frac{\partial^m}{\partial x^m}\bigl\{{}_{e^{(s)}}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\bigr\}\\&\quad=\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{\kappa}{m}\bigl\{{}_{e^{(s)}}F_{\kappa-m}^{(s,\upsilon)}(x,y,z)C_{n-\kappa}(u,\omega)+{}_{e^{(s)}}F_{\kappa-m}^{(c,\upsilon)}(x,y,z)S_{n-\kappa}(u,\omega)\bigr\}.\end{aligned} \tag{5.9}$$

5.4 Hermite–Appell–Fubini-type polynomials

Since, for $\varphi(y,t) = A(t)e^{yt^2}$, the 2VGP $\mathbb{G}_n(x,y)$ reduce to the Hermite–Appell polynomials ${}_{\mathcal{H}}A_n(x,y)$ (Table 1(IV)). Therefore, for the same choice of $\varphi(y,t)$, the generalized Fubini-type polynomials ${}_{\mathbb{G}}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathbb{G}}F_n^{(s,\upsilon)}(x,y,z)$ reduce to Hermite–Appell–Fubini-type polynomials ${}_{\mathcal{H}A}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathcal{H}A}F_n^{(s,\upsilon)}(x,y,z)$. Thus, by taking $\varphi(y,t) = e^{yt^r}$ in the results established in Sects. 2–4, we can obtain the corresponding results for the Hermite–Appell–Fubini-type polynomials ${}_{\mathcal{H}A}F_n^{(c,\upsilon)}(x,y,z)$ and ${}_{\mathcal{H}A}F_n^{(s,\upsilon)}(x,y,z)$, these results are listed in Tables 8 and 9.

Furthermore, in view of Eqs. (3.22), (3.28) and (4.5), we have

$$ {}_{\mathcal{H}A}F_n^{(\upsilon)}(x+iz,y) = {}_{\mathcal{H}A}F_n^{(c,\upsilon)}(x,y,z) + i\,{}_{\mathcal{H}A}F_n^{(s,\upsilon)}(x,y,z), \tag{5.10}$$

$$\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(\upsilon)}(x)\,{}_{\mathcal{H}A}F_{\kappa}^{(s,\sigma)}(x,y,2z) = 2\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(s,\upsilon)}(x,z)\,{}_{\mathcal{H}A}F_{n-\kappa}^{(c,\sigma)}(x,y,z) \tag{5.11}$$

and

$$\begin{aligned}&\frac{\partial^m}{\partial x^m}\bigl\{{}_{\mathcal{H}A}F_n^{(s,\upsilon)}(x+u,y,z+\omega)\bigr\}\\&\quad=\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{\kappa}{m}\bigl\{{}_{\mathcal{H}A}F_{\kappa-m}^{(s,\upsilon)}(x,y,z)C_{n-\kappa}(u,\omega)+{}_{\mathcal{H}A}F_{\kappa-m}^{(c,\upsilon)}(x,y,z)S_{n-\kappa}(u,\omega)\bigr\}.\end{aligned} \tag{5.12}$$

Table 8 Results for ${}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)$

Generating function	$\frac{2^{\upsilon}A(t)}{(2-e^{t})^{2\upsilon}}e^{xt+yt^{2}}\cos(zt)=\sum_{n=0}^{\infty}{}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)\frac{t^{n}}{n!}$
Multiplicative and derivative operators	$\hat{M}_{HAFc}=x+2yD_{x}+\frac{A'(D_{x})}{A(D_{x})}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}-z\tan(zD_{x}),\ \hat{P}_{HAFc}:=D_{x}$
Differential equation	$(xD_{x}+2yD_{x}^{2}+\frac{A'(D_{x})}{A(D_{x})}D_{x}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}D_{x}-z\tan(zD_{x})D_{x}-n){}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)=0$
Identities and relations	${}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(c,\upsilon)}(z){}_{\mathcal{H}}A_{\kappa}(x,y)$ ${}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x+u,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}{}_{\mathcal{H}}{}_{A}F_{n-\kappa}^{(c,\upsilon)}(x,y,z)u^{\kappa}$ ${}_{\mathcal{H}}{}_{A}F_{n+\kappa}^{(c,\upsilon)}(\omega,y,z)=\sum_{l=0}^{n}\sum_{m=0}^{\kappa}\binom{n}{l}\binom{\kappa}{m}(\omega-x)^{l+m}{}_{\mathcal{H}}{}_{A}F_{n+\kappa-l-m}^{(c,\upsilon)}(x,y,z)$ ${}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon+\sigma)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}{}_{\mathcal{H}}{}_{A}F_{n-\kappa}^{(c,\sigma)}(x,y,z)$ $x^{n}A(t)e^{yt^{2}}\cos(zt)=\sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}{}_{\mathcal{H}}{}_{A}F_{\kappa}^{(c,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^{m}}{\partial x^{m}}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)\}=2^{-\frac{3\delta}{2}}\sum_{\kappa=0}^{n}m!\binom{n}{\kappa}\binom{n-\kappa}{m}\mathcal{E}_{\kappa}^{(\delta)}(-\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n-\kappa-m}^{(c,\upsilon-\frac{\delta}{2})}(x,y,z)$ $\frac{\partial^{m}}{\partial u^{m}}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon+\sigma)}(x+u,y,z)\}=\frac{n!}{2^{\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathfrak{B}_{\kappa-m}^{(2\upsilon)}(x,\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n+2\upsilon-\kappa}^{(c,\sigma)}(u,y,z)$ $\frac{\partial}{\partial x}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(c,\upsilon)}(x,y,z)\}=\frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}(n-\kappa)\binom{n}{\kappa}\mathcal{E}_{\kappa}^{(2\delta)}(-\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n-\kappa-1}^{(c,\upsilon-\delta)}(x,y,z)$

Table 9 Results for ${}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)$

Generating function	$\frac{2^{\upsilon}A(t)}{(2-e^{t})^{2\upsilon}}e^{xt+yt^{2}}\sin(zt)=\sum_{n=0}^{\infty}{}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)\frac{t^{n}}{n!}$
Multiplicative and derivative operators	$\hat{M}_{HAFs}=x+2yD_{x}+\frac{A'(D_{x})}{A(D_{x})}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}+z\cot(zD_{x}),\ \hat{P}_{HAFs}:=D_{x}$
Differential equation	$(xD_{x}+2yD_{x}^{2}+\frac{A'(D_{x})}{A(D_{x})}D_{x}+\frac{2\upsilon e^{D_{x}}}{2-e^{D_{x}}}D_{x}+z\cot(zD_{x})D_{x}-n){}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)=0$
Identities and relations	${}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(z){}_{\mathcal{H}}A_{\kappa}(x,y)$ ${}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x+u,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{n-\kappa}^{(s,\upsilon)}(x,z){}_{\mathcal{H}}A_{\kappa}(u,y)$ ${}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon+\sigma)}(x,y,z)=\sum_{\kappa=0}^{n}\binom{n}{\kappa}F_{\kappa}^{(\upsilon)}{}_{\mathcal{H}}{}_{A}F_{n-\kappa}^{(s,\sigma)}(x,y,z)$ $x^{n}A(t)e^{yt^{2}}\sin(zt)=\sum_{\delta=0}^{2\upsilon}\sum_{\kappa=0}^{n}(-1)^{\delta}\binom{2\upsilon}{\delta}\binom{n}{\kappa}2^{\upsilon-\delta}\delta^{n-\kappa}{}_{\mathcal{H}}{}_{A}F_{\kappa}^{(s,\upsilon)}(x,y,z)$
Partial derivatives equations	$\frac{\partial^{m}}{\partial x^{m}}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)\}=2^{-\frac{\delta}{2}}\sum_{\kappa=0}^{n}\frac{m!\kappa!}{(\kappa+\delta)!}\binom{n}{\kappa}\binom{n-\kappa}{m}\mathfrak{B}_{\kappa+\delta}^{(\delta)}(\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n-\kappa-m}^{(s,\upsilon-\frac{\delta}{2})}(x,y,z)$ $\frac{\partial^{m}}{\partial u^{m}}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon+\sigma)}(x+u,y,z)\}=\frac{n!}{2^{3\upsilon}(n+2\upsilon)!}\sum_{\kappa=0}^{n+2\upsilon}m!\binom{n+2\upsilon}{\kappa}\binom{\kappa}{m}\mathcal{G}_{\kappa-m}^{(2\upsilon)}(x,-\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n+2\upsilon-\kappa}^{(s,\sigma)}(u,y,z)$ $\frac{\partial}{\partial x}\{{}_{\mathcal{H}}{}_{A}F_{n}^{(s,\upsilon)}(x,y,z)\}=\frac{1}{2^{3\delta}}\sum_{\kappa=0}^{n}\frac{(n-\kappa)\kappa!}{(\kappa+2\delta)!}\binom{n}{\kappa}\mathcal{G}_{\kappa+2\delta}^{(2\delta)}(-\frac{1}{2}){}_{\mathcal{H}}{}_{A}F_{n-\kappa-1}^{(s,\upsilon-\delta)}(x,y,z)$

6 Conclusions

In our present work, two-hybrid classes of the generalized Fubini-type polynomials of a parametric kind are introduced, and their properties are investigated with the help of operational methods. Several important identities and relations are established for these hybrid type polynomials. Partial derivative formulas including the generalized Fubini-type polynomials are also derived. The Gould–Hopper–Fubini-type polynomials, Laguerre–Fubini-type polynomials, truncated exponential-Fubini-type polynomials, and Hermite–Appell–Fubini-type polynomials are introduced as special cases of the generalized Fubini-type polynomials of a parametric kind. The numerical computations of zeros and graphs related to these special cases can also be done. Furthermore, the symmetry identities, differential and integral equations associated with the generalized Fubini-type polynomials can be established. These aspects may be considered in further investigation.

Appendix

The 3D surface plots are more informative and better for analysis. It helps to visualize the response surface and hence to provide a clearer concept.

In this section, we give some interesting graphical representations for GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ for suitable values of the parameters and indices. The first few GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ for $r=2$ and $\nu=4$ are

$$\begin{aligned}
{}_{\mathcal{H}^{(r)}}F_0^{(c,\upsilon)}(x,y,z) &= 16,\\
{}_{\mathcal{H}^{(r)}}F_1^{(c,\upsilon)}(x,y,z) &= 128+16x,\\
{}_{\mathcal{H}^{(r)}}F_2^{(c,\upsilon)}(x,y,z) &= 1280+256x+16x^2+32y-16z^2,\\
{}_{\mathcal{H}^{(r)}}F_3^{(c,\upsilon)}(x,y,z) &= 15{,}104+3840x+384x^2+16x^3+768y+96xy-384z^2-48xz^2,\\
{}_{\mathcal{H}^{(r)}}F_4^{(c,\upsilon)}(x,y,z) &= 204{,}032+60{,}416x+7680x^2+512x^3+16x^4+15{,}360y+3072xy\\
&\quad+192x^2y+192y^2-7680z^2-1536xz^2-96x^2z^2-192yz^2+16z^4.
\end{aligned}$$

The shapes of the GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ for $r=2$, $\nu=4$, $y=3$, $z=5$ and $-100\le x\le 100$ for $n=1,2,3,\dots,10$ are displayed in Fig. 1.

The surface plots of the GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ for $r=2$, $\nu=4$, $y=3$ and $n=20$ are displayed in Fig. 2.

Numerical results for number of real and complex zeros of the GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$ for $r=2$, $s=4$, $y=3$ and $z=5$ are listed in Table 10.

Approximate solutions satisfying the GHFP ${}_{H^{(r)}}F_n^{(c,\nu)}(x,y,z)=0$ for $r=2$, $\nu=4$, $y=3$ and $z=4$ are given in Table 11.

The zeros of the GHFP ${}_{H^{(r)}}F_n^{(c,\nu)}(x,y,z)=0$ for $r=2$, $y=3$, $z=5$, $n=20$ and $x\in\mathbb{C}$ are plotted in Fig. 3.

In Fig. 3, we choose $\nu=10$ (top-left), $\nu=20$ (top-right), $\nu=30$ (bottom-left) and $\nu=40$ (bottom-right).

For $b,d\in\mathbb{R}$ and $x\in\mathbb{C}$, GHFP ${}_{H^{(r)}}F_n^{(c,\nu)}(x,b,d)$ has $\operatorname{Im}(x)=0$ reflection symmetry which is shown graphically in Fig. 4.

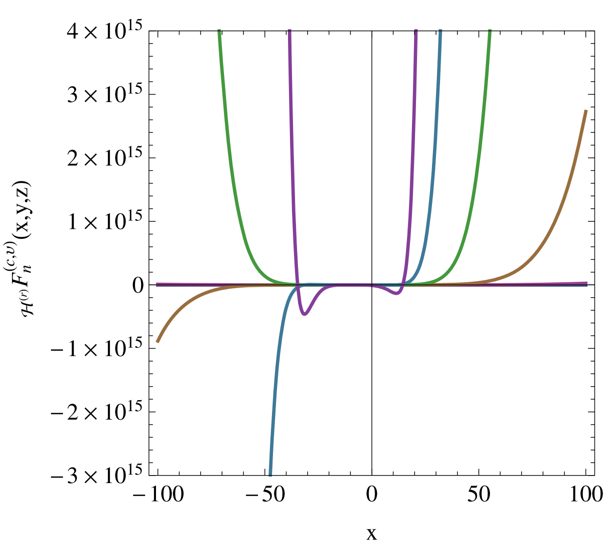

Figure 1 Curve of GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$

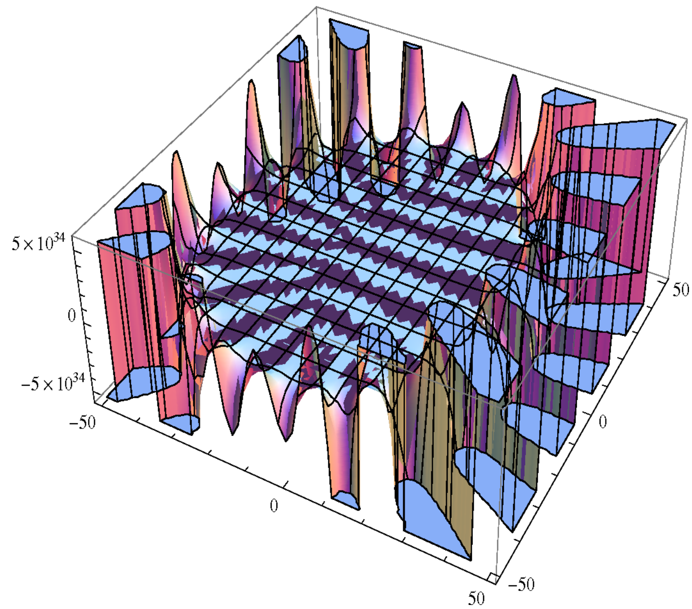

Figure 2 Surface plot of GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$

Table 10 Numbers of real and complex zeros of ${}_{H^{(r)}}F_n^{(c,\nu)}(x,y,z) = 0$

Degree n	Number of real zeros	Number of complex zeros
1	1	0
2	2	0
3	1	2
4	2	2
5	3	2
6	2	4
7	3	4

Table 11 Approximate solutions of ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$

Degree n	Real roots	Complex roots	
1	−8.0	–	
2	−9.7321, −6.2679	–	
3	−12.4486	−5.7757 − 2.4171i,	−5.7757 + 2.4171i
4	−15.7331, −3.2498	−6.5085 − 5.0405i,	−6.5085 + 5.0405i
5	−18.8334, −8.0471, −0.4824	−6.3186 − 7.73086i,	−6.3186 + 7.73086i
6	−21.9822, 2.4281	−8.4121 − 3.3098i,	−8.4121 + 3.3098i,
		−5.8108 − 10.0564i,	−5.81082 + 10.0564i
7	−25.1389, −9.3486, 5.3876	−8.2941 − 5.9519i,	−8.2941 + 5.9519i,
		−5.1559 − 12.1304i,	−5.1559 + 12.1304i

Real zeros of the GHFP ${}_{H^{(r)}}F_n^{(c,\nu)}(x,y,z) = 0$ for $r = 2$, $s = 4$, $y = 3$, $z = -5$, $x \in \mathbb{R}$ and $1 \leq n \leq 20$ are plotted in Fig. 5.

Stacks of zeros of GHFP ${}_{H^{(r)}}F_n^{(c,\nu)}(x,y,z) = 0$ for $r = 2$, $\nu = 4$, $y = 3$, $z = 5$ and $1 \leq n \leq 20$ form a 3-D structure and are presented in Fig. 6.

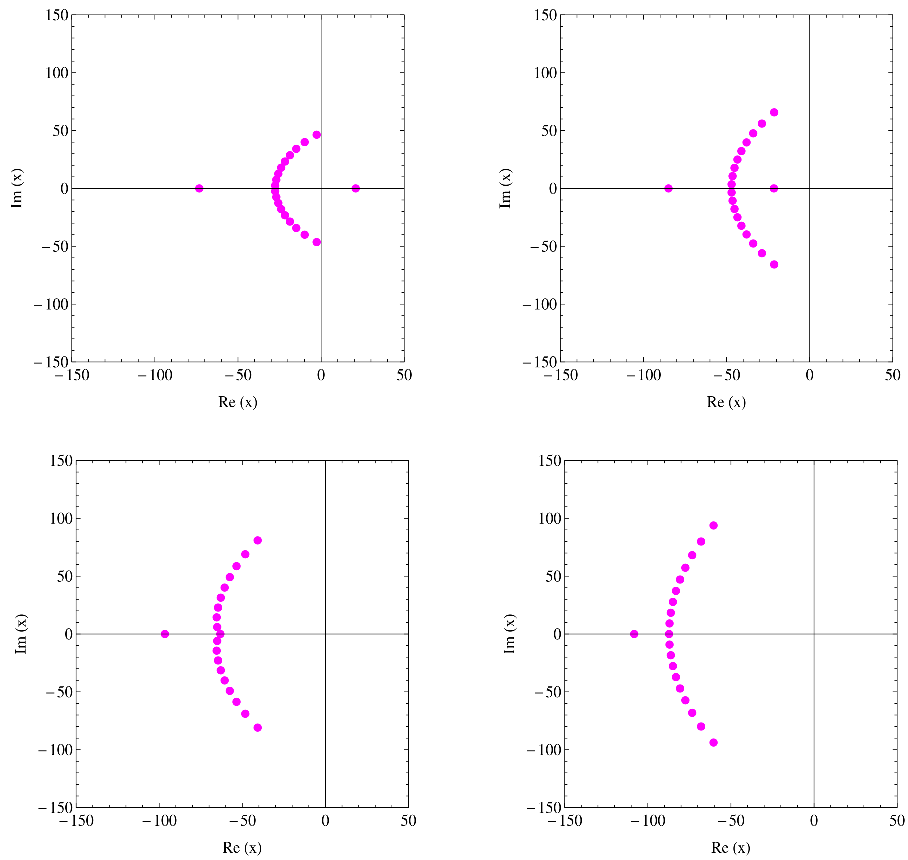

Figure 3 Zeros of GHFTP ${}_{\mathcal{H}^{(r)}}F_n^{(c,\upsilon)}(x,y,z)$

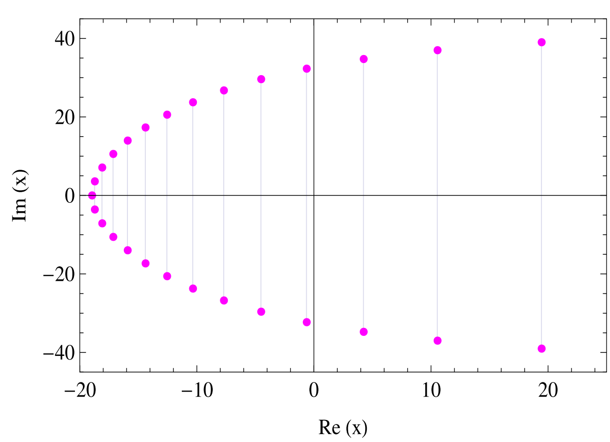

Figure 4 ${}_{H^{(r)}}F_{30}^{(c,v)}(x,b,c)$ has $\mathrm{Im}(x) = 0$ reflection symmetry

Mathematical problems can be investigated more effectively and more thoroughly using computers. The ability to make the figures on the computer screen empowers us to envision and produce numerous problems, analyze the properties of figures and search for new patterns.

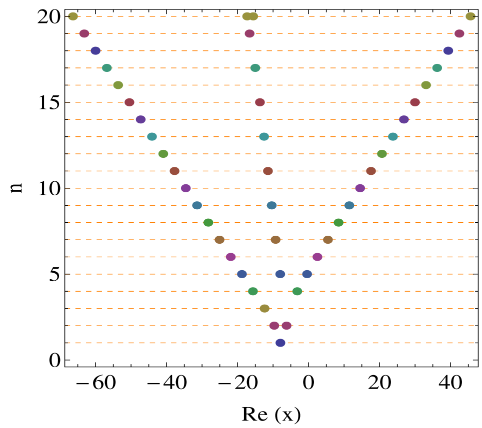

Figure 5 Real zeros of ${}_{H(r)}F_n^{(c,v)}(x,y,z) = 0$

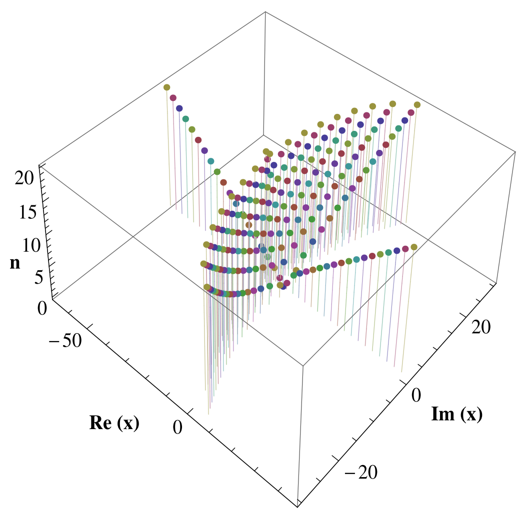

Figure 6 Stacks of zeros of GHFP ${}_{H(r)}F_n^{(c,v)}(x,y,z) = 0$

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