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On zeros and growth of solutions of complex difference equations

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Abstract

Let f be an entire function of finite order, let $n \geq 1$, $m \geq 1$, $L(z,f) \not\equiv 0$ be a linear difference polynomial of f with small meromorphic coefficients, and $P_d(z,f) \not\equiv 0$ be a difference polynomial in f of degree $d \leq n-1$ with small meromorphic coefficients. We consider the growth and zeros of $f^n(z)L^m(z,f) + P_d(z,f)$. And some counterexamples are given to show that Theorem 3.1 proved by I. Laine (J. Math. Anal. Appl. 469:808–826, 2019) is not valid. In addition, we study meromorphic solutions to the difference equation of type $f^n(z) + P_d(z,f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$, where $n \geq 2$, $P_d(z,f) \not\equiv 0$ is a difference polynomial in f of degree $d \leq n-2$ with small mromorphic coefficients, p_i, α_i $(i = 1,2)$ are nonzero constants such that $\alpha_1 \neq \alpha_2$. Our results are improvements and complements of Laine 2019, Latreuch 2017, Liu and Mao 2018.

MSC: 39B32; 39A10; 30D35

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1 Introduction and main results

In this paper, we assume familiarity with the basic results and standard notations of Nevanlinna theory [7, 9, 21]. In addition, we use $\rho(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros' sequence of f. For simplicity, we denote by $S(r,f)$ any quantify satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to f.

Nowadays, there has been substantial interest in Nevanlinna theory for differences, as well as meromorphic solutions of difference and functional equations; see, e.g., [1–6, 10–14, 18, 19, 22, 23]. With the establishment of difference analogue of Nevanlinna theory, many outstanding achievements on the complex difference theory are accomplished.

Halburd and Korhonen [5] in 2006 and Chiang and Feng [3] in 2008 presented a difference analogue of the lemma on the logarithmic derivative as follows.

Lemma A (See [3, Corollary 2.5]) *Let f be a meromorphic function of finite order ρ and let η be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\rho-1+\varepsilon}\right).$$

Halburd and Korhonen [5] also established the difference analogue of Clunie lemma.

Lemma B (See [5, Corollary 3.3]) *Let f be a nonconstant finite-order meromorphic solution of*

$$f^n(z)P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are difference polynomials in f with small meromorphic coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z,f)$, as a polynomial in f and its shifts, is $\leq n$, then

$$m\bigl(r, P(z,f)\bigr) = o\biggl(\frac{T(r+|c|,f)}{r^\delta}\biggr) + o\bigl(T(r,f)\bigr)$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

Using the same methods as in the proof of [9, Lemma 2.4.2] and Lemma B, we have a similar conclusion as follows.

Lemma C *Let f be a nonconstant finite-order meromorphic solution of*

$$f^n(z)P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are differential-difference polynomials in f with small meromorphic coefficients. If the total degree of $Q(z,f)$, as a polynomial in f, its derivatives, and its shifts, is $\leq n$, then

$$m\bigl(r, P(z,f)\bigr) = S(r,f)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

This paper is organized as follows. In Sect. 2, we will consider the growth and zeros of certain types of complex difference polynomials and complex difference equations. In Sect. 3, we will study meromorphic solutions of certain type of nonlinear difference equations.

2 Zero distribution of some complex difference polynomials

We now recall the following result proved in [12]; see Theorem 2 therein:

Theorem D (See [12]) *Let f be a transcendental entire function of finite order, and c be a nonzero complex constant. Then for $n \geq 2$, $f^n f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

This paper prompted many related investigations during the last 12 years, such as [10, 14, 22]. In what follows, we use $L(z,f) := \sum_{i=1}^{l} a_i(z) f(z+\lambda_i)$ with small meromorphic coefficients, without being otherwise specified.

In 2019, Laine [10, Theorem 2.1] presented an extension to Theorem D as follows:

Theorem E (See [10]) *Let f be a transcendental entire function of finite order ρ, b_0 be a nonvanishing small meromorphic function of f, $L(z,f)$ be nonvanishing and $n \geq 2$, $m \geq 1$. Then $F := f^n L^m(z,f) - b_0$ has sufficiently many zeros to satisfy $\lambda(F) = \rho$.*

Remark 2.1 In Theorem E, a meromorphic function α is said to be small, relative to a given meromorphic function f of finite order ρ, if for any $\varepsilon > 0$, and for some $\lambda < \rho$, $T(r,\alpha) = O(r^{\lambda+\varepsilon}) + S(r,f)$ outside of a possible exceptional set of finite logarithmic measure.

In this section, our purpose is to improve and extend the results in [10] for an entire function f by considering the zero distribution of $f^n L^m(z,f) + P_d(z,f)$, where $n \geq 2$, $m \geq 1$, $P_d(z,f)$ is a difference polynomial in f of degree $d \leq n-2$, with small meromorphic coefficients. We obtain

Theorem 2.1 *Let f be a transcendental entire function of finite order ρ, $L(z,f)$, $P_d(z,f)$ be nonvanishing and $n \geq 2$, $m \geq 1$, $d \leq n-2$. Then $F := f^n L^m(z,f) + P_d(z,f)$ has sufficiently many zeros and satisfies $\lambda(F) = \rho(F) = \rho$.*

Remark 2.2 The following examples show that our estimates in Theorem 2.1 are accurate, and the condition $d \leq n-2$ is necessary in Theorem 2.1.

Example 2.1 If $f(z) = 1 + e^z$, $P_1(z,f) = -4f(z + \log 2)$, $L(z,f) = f(z) + f(z + i\pi)$, then $F := f^2(z)L^2(z,f) + P_1(z,f) = 4e^{2z}$ has no zeros, and $0 = \lambda(F) < \rho(F) = \rho = 1$.

Example 2.2 If $f(z) = z + e^z$, $P_0(z,f) = -z(z - \log 2)^m$, $L(z,f) = 2f(z) - f(z + \log 2)$, $m \geq 1$, then $F := f(z)L^m(z,f) + P_0(z,f) = (z - \log 2)^m e^z$ has finitely many zeros only, and $0 = \lambda(F) < \rho(F) = \rho = 1$.

Remark 2.3 Example 2.2 shows that the following result due to Laine [10] may be invalid. Recently, I. Laine and Z. Latreuch [11] gave an extension and a complete version of [10, Theorem 3.1].

Theorem F (See [10]) *Let f be a transcendental entire function of finite order ρ, b_0 be a nonvanishing small function of f, and $L(z,f)$ be nonvanishing. If $m \geq 2$, then $F := fL^m(z,f) - b_0$ satisfies $\lambda(F) = \rho$.*

Examples 2.1 and 2.2 imply that $F := f^n L^m(z,f) + P_d(z,f)$ may have no zeros or finitely many zeros only under the condition $d \leq n-1$, where $n \geq 2$, $m \geq 1$. Enlightened by Examples 2.1–2.2 and Theorem F, we consider the growth and zeros of entire solutions of the following equation:

$$f^n(z)L^m(z,f) + P_d(z,f) = \gamma(z)e^{h(z)}, \tag{2.1}$$

where $n \geq 1$, $m \geq 1$, $P_d(z,f)$ is a difference polynomial in f of degree $d \leq n-1$, with small meromorphic coefficients, $\gamma(z)$ is a small function relative to f, and $h(z)$ is a polynomial. Now we state our result as follows.

Theorem 2.2 *Let f be a transcendental entire function of Eq. (2.1) with finite order ρ, where $L(z,f)$, $P_d(z,f)$ are nonvanishing. Then $\lambda(f) = \deg h(z) = \rho$.*

Before proving Theorems 2.1 and 2.2, we need the following lemmas.

Lemma 2.1 (See [20, Theorem 1.51]) *Suppose that $f_1, f_2, \dots, f_n$ ($n \geq 2$) are meromorphic functions and $g_1, g_2, \dots, g_n$ are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$.

(2) *$g_j - g_k$ are not constants for $1 \leq j < k \leq n$.*

(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\bigl(T\bigl(r, e^{g_h - g_k}\bigr)\bigr) \quad (r \to \infty, r \notin E),$$

where $E \subset [1, \infty)$ has finite linear measure $\int_E dr < \infty$ or finite logarithmic measure $\int_E \frac{dr}{r} < \infty$. Then $f_j \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.2 *Let f be a transcendental entire function of finite order ρ, $P_d(z,f)$ be difference polynomial in f of degree $d \leq n-1$, $L(z,f) := \sum_{i=1}^{l} a_i(z) f(z + \lambda_i)$, with small meromorphic coefficients. Let $L(z,f)$, $P_d(z,f)$ be nonvanishing and $n \geq 1$, $m \geq 1$. Then $F := f^n L^m(z,f) + P_d(z,f)$ satisfies $\rho(F) = \rho$.*

Proof Set

$$P_d(z,f) = \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{t_\mu} f(z + \delta_{\mu j})^{l_{\mu j}}, \tag{2.2}$$

where I is a finite set of the index μ, t_μ, $l_{\mu j}$ ($\mu \in I, j = 1, \dots, t_\mu$) are natural numbers, $\delta_{\mu j}$ ($\mu \in I, j = 1, \dots, t_\mu$) are distinct complex constants. Denoting $g_{\mu j}(z) := \frac{f(z+\delta_{\mu j})}{f(z)}$ and substituting this equality into (2.2) yields

$$P_d(z,f) = \sum_{\mu \in I} \left(b_\mu(z) \prod_{j=1}^{t_\mu} g_{\mu j}^{l_{\mu j}}(z) \right) f^{l_\mu}(z) = \sum_{k=0}^{d} c_k(z) f^k(z), \tag{2.3}$$

where $l_\mu = \sum_{j=1}^{t_\mu} l_{\mu j}$, $d = \max_{\mu \in I}\{l_\mu\}$, $c_k(z) = \sum_{l_\mu = k} (b_\mu(z) \prod_{j=1}^{t_\mu} g_{\mu j}^{l_{\mu j}}(z))$ ($k = 0, \dots, d$). By applying Lemma A, we have $m(r, c_k(z)) = S(r,f)$ ($k = 0, \dots, d$), which gives

$$m\bigl(r, P_d(z,f)\bigr) \leq d m(r,f) + S(r,f) \leq (n-1) m(r,f) + S(r,f). \tag{2.4}$$

From (2.4) and Lemma A, we obtain

$$\begin{aligned} T(r,F) &= T\bigl(r, f^n L^m(z,f) + P_d(z,f)\bigr) \\ &= m\bigl(r, f^n L^m(z,f) + P_d(z,f)\bigr) + S(r,f) \\ &\leq m\bigl(r, f^n\bigr) + m\bigl(r, L^m(z,f)/f^m\bigr) + m\bigl(r, f^m\bigr) + m\bigl(r, P_d(z,f)\bigr) + S(r,f) \\ &\leq (2n + m - 1) T(r,f) + S(r,f), \end{aligned}$$

namely, $T(r,F) \leq (2n+m-1)T(r,f) + S(r,f)$, and then $\rho(F) \leq \rho$. If $\rho(F) < \rho$, then $T(r,F) = S(r,f)$. Rewrite $F = f^n L^m(z,f) + P_d(z,f)$ as

$$f^n L^m(z,f) = F - P_d(z,f).$$

By applying Lemma B, we see that

$$m\big(r, L^m(z,f)\big) = S(r,f), \qquad m\big(r, fL^m(z,f)\big) = S(r,f).$$

Note that $L(z,f) \not\equiv 0$, which implies

$$\begin{aligned} T(r,f) = m(r,f) &\le m\big(r, fL^m(z,f)\big) + m\big(r, 1/L^m(z,f)\big) \\ &\le T\big(r, L^m(z,f)\big) + S(r,f) \\ &= m\big(r, L^m(z,f)\big) + S(r,f) = S(r,f), \end{aligned}$$

a contradiction. Thus $\rho(F) = \rho$. □

Proof of Theorem 2.1 Suppose that $\lambda(F) = \lambda < \rho$. By the Hadamard representation, we assume that

$$F = f^n L^m(z,f) + P_d(z,f) = \gamma(z)e^{h(z)}, \tag{2.5}$$

where $h(z)$ is a polynomial of degree $\le \rho$ and $T(r,\gamma(z)) = O(r^{\lambda+\varepsilon}) + S(r,f)$, $\gamma(z) \not\equiv 0$, or else, by using the same reasoning as in the proof of Lemma 2.1, we obtain $T(r,f) = S(r,f)$, a contradiction. If $\deg h(z) \le \mu < \rho$, then

$$T(r,F) = O\big(r^{\lambda+\varepsilon}\big) + O\big(r^{\mu+\varepsilon}\big) + S(r,f),$$

resulting in a contradiction $\rho \le \max\{\lambda, \mu\} < \rho$ by Lemma 2.2. Thus $\deg h(z) = \rho$. Denote $L(z,f) := L$ and $P_d(z,f) = P_d$. Differentiating (2.5) yields

$$nf^{n-1}f'L^m + mf^nL^{m-1}L' + P_d' = \big(\gamma' + \gamma h'\big)e^h. \tag{2.6}$$

Eliminating e^h from (2.5) and (2.6), we have

$$f^{n-1}\psi = Q_d(z,f), \tag{2.7}$$

where

$$\psi = nf'L^m + mfL^{m-1}L' - \left(\frac{\gamma'}{\gamma} + h'\right)fL^m$$

and

$$Q_d(z,f) = \left(\frac{\gamma'}{\gamma} + h'\right)P_d - P_d',$$

while $Q_d(z,f)$ is a differential-difference polynomial in f of degree $\le n-2$. By applying Lemma C, we get

$$m(r,\psi) = S(r,f), \qquad m(r,f\psi) = S(r,f).$$

If $\psi \not\equiv 0$, we conclude that

$$\begin{aligned} T(r,f) &= m(r,f) \le m(r,f\psi)) + m(r,1/\psi) \\ &\le T(r,\psi) + S(r,f) \\ &= m(r,\psi) + S(r,f) = S(r,f), \end{aligned}$$

which is impossible. If $\psi \equiv 0$, then $Q_d(z,f) \equiv 0$. Thus $Q_d(z,f) = (\frac{\gamma'}{\gamma} + h')P_d - P_d' \equiv 0$, and we get $P_d = C\gamma(z)e^{h(z)}$, where $C \in \mathbb{C} \setminus \{0\}$. If $C = 1$, then $P_d = \gamma(z)e^{h(z)}$. Substituting this equality into (2.5) yields $f^n L^m \equiv 0$, a contradiction. If $C \neq 1$, then $\gamma(z)e^{h(z)} = \frac{1}{C}P_d$. Putting this equality into (2.5), we have

$$f^n L^m(z,f) = \frac{1-C}{C} P_d(z,f).$$

Recalling that $d \le n-2$ and by applying Lemma B, we have

$$m\bigl(r, L^m(z,f)\bigr) = S(r,f), \qquad m\bigl(r, fL^m(z,f)\bigr) = S(r,f).$$

Making using of the above two equalities and noting that $L(z,f) \not\equiv 0$, we get

$$\begin{aligned} T(r,f) &= m(r,f) \le m\bigl(r, fL^m(z,f)\bigr) + m\bigl(r, 1/L^m(z,f)\bigr) \\ &\le T\bigl(r, L^m(z,f)\bigr) + S(r,f) \\ &= m\bigl(r, L^m(z,f)\bigr) + S(r,f) = S(r,f), \end{aligned}$$

resulting in a contradiction $T(r,f) = S(r,f)$.

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 Let $F := f^n L^m(z,f) + P_d(z,f)$. It follows from Lemma 2.2 that $\rho(F) = \rho$. On the other hand, f is a transcendental entire function to Eq. (2.1). So as in the beginning of the proof of Theorem 2.1, we obtain $\deg h(z) = \rho$. We now deduce that $\lambda(f) = \rho$. On the contrary, suppose that $\lambda(f) < \rho$. By the Hadamard factorization theorem [20, Theorem 2.5], we assume that

$$f(z) = P(z)e^{Q(z)}, \tag{2.8}$$

where $P(z)$ is the canonical product of f formed with the zeros of f, and satisfies $\lambda(P) = \rho(P) < \rho$, $Q(z)$ is a polynomial of degree ρ. Substituting (2.8) into $P_d(z,f)$ yields

$$\begin{aligned} P_d(z,f) &= \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{t_\mu} f(z+\delta_{\mu j})^{l_{\mu j}} \\ &= \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{t_\mu} \bigl(P(z+\delta_{\mu j})e^{Q(z+\delta_{\mu j})}\bigr)^{l_{\mu j}} \\ &= \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{t_\mu} \bigl(P(z+\delta_{\mu j})e^{Q(z+\delta_{\mu j})-Q(z)}\bigr)^{l_{\mu j}} e^{l_{\mu j}Q(z)} \end{aligned}$$

$$= \sum_{\mu\in I}\left(b_\mu(z)\prod_{j=1}^{t_\mu}\left(P(z+\delta_{\mu j})e^{Q(z+\delta_{\mu j})-Q(z)}\right)^{l_{\mu j}}\right)e^{l_\mu Q(z)}$$
$$= \sum_{k=0}^{d} c_k(z)e^{kQ(z)},$$

where $l_\mu = \sum_{j=1}^{t_\mu} l_{\mu j}$, $d = \max_{\mu\in I}\{l_\mu\}$, $c_k(z) = \sum_{l_\mu=k}(b_\mu(z)\prod_{j=1}^{t_\mu}(P(z+\delta_{\mu j})e^{Q(z+\delta_{\mu j})-Q(z)})^{l_{\mu j}})$ $(k = 0,\dots,d)$. Noting that $T(r,P(z+\delta_{\mu j})) = S(r,f)$, $\deg(Q(z+\delta_{\mu j})-Q(z)) = \deg Q(z)-1$, we see that $T(r,c_k(z)) = S(r,f)(k = 0,\dots,d)$. Substituting (2.8) into $L^m(z,f)$, we have

$$L^m(z,f) = \left(\sum_{i=1}^{l} a_i(z)f(z+\lambda_i)\right)^m$$
$$= \left(\sum_{i=1}^{l} a_i(z)P(z+\lambda_i)e^{Q(z+\lambda_i)}\right)^m$$
$$= (\sum_{i=1}^{l}\left(a_i(z)P(z+\lambda_i)e^{Q(z+\lambda_i)-Q(z)}e^{Q(z)}\right)^m$$
$$= (\sum_{i=1}^{l}\left(a_i(z)P(z+\lambda_i)e^{Q(z+\lambda_i)-Q(z)}\right)^m e^{mQ(z)}$$
$$= C(z)e^{mQ(z)}.$$

Recalling that $T(r,P(z+\lambda_i)) = S(r,f)$, $\deg(Q(z+\lambda_i)-Q(z)) = \deg Q(z)-1$, we see that $T(r,C(z)) = S(r,f)$. Since $L(z,f)\not\equiv 0$, one has $C(z)\not\equiv 0$. Rewrite (2.1) as

$$C(z)P^n(z)e^{(m+n)Q(z)} + \sum_{k=0}^{d} c_k(z)e^{kQ(z)} = \gamma(z)e^{h(z)}. \tag{2.9}$$

Noting that $\deg h(z) = \rho = \deg Q(z)$, we consider two cases below:

(*i*) If $\deg(h(z)-jQ(z)) = \rho, j = 1,2,\dots,d(\le n-1), n+m$, by applying Lemma 2.1, we obtain

$$C(z)P^n \equiv \gamma(z) \equiv 0,$$

then $L(z,f)\equiv 0$ or $f(z)\equiv 0$, which is impossible.

(*ii*) If $\deg(h(z)-jQ(z)) < \rho, j = 1,2,\dots,d(\le n-1), n+m$. We consider two subcases below:

Subcase (*ii*1). If $\deg(h(z)-(m+n)Q(z)) < \rho$, then $\deg(h(z)-jQ(z)) = \rho$, $j = 1,2,\dots,d(\le n-1)$, so rewriting (2.9) as

$$\left(C(z)P^n(z)-\gamma(z)e^{h(z)-(m+n)Q(z)}\right)e^{(m+n)Q(z)} + \sum_{k=0}^{d} c_k(z)e^{kQ(z)} = 0 \tag{2.10}$$

and by applying Lemma 2.1, we obtain

$$c_k(z)\equiv 0 \quad (k = 0,1,\dots,d\le n-1),$$

so then $P_d(z,f) = \sum_{k=0}^{d} c_k(z)e^{kQ(z)}\equiv 0$, which is impossible.

Subcase (*ii*2). If there exists a $j_0, j_0 \in \{1, 2, \ldots, d(\leq n-1)\}$ such that $\deg(h(z) - jQ(z)) < \rho$, then $\deg(h(z) - (m+n)Q(z)) = \rho$, so, by Lemma 2.1 and (2.9), we get $C(z)P^n(z) \equiv 0$, and then $L(z,f) \equiv 0$ or $f(z) \equiv 0$, a contradiction.

This completes the proof of Theorem 2.2. □

3 Meromorphic solutions of certain difference equations

Recently, there has been a renewed interest in the existence of entire or meromorphic solutions for nonlinear differential or difference equations, or differential-difference equations in the complex plane, see [1, 13, 15–19, 23].

In 2011, Li proved the following result, see [16], Theorem 2.

Theorem G (See [16]) *Let $n \geq 2$ be an integer, $Q_d(z,f)$ be a differential polynomial in f of degree $d \leq n-2$, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants and $\alpha_1 \neq \alpha_2$. If f is a transcendental meromorphic solution of the following equation*:

$$f^n(z) + Q_d(z,f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{3.1}$$

and satisfying $N(r,f) = S(r,f)$, then one of the following holds:

(*i*) $f(z) = c_0 + c_1 e^{\alpha_1 z/n}$;

(*ii*) $f(z) = c_0 + c_2 e^{\alpha_2 z/n}$;

(*iii*) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$ *and* $\alpha_1 + \alpha_2 = 0$,

where c_0 is a small function of f and c_1, c_2 are constants satisfying $c_i^2 = p_i$, $i = 1, 2$.

Replacing the differential polynomial $Q_d(z,f)$ in Eq. (3.1) by a difference, or differential-difference polynomial $P_d(z,f)$, many scholars considered the existence of solutions of the following equation:

$$f^n(z) + P_d(z,f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{3.2}$$

where $n \geq 2$, $P_d(z,f)$ is a difference or differential-difference polynomial in f of degree $d \leq n-1$, with small meromorphic coefficients.

In 2018, Liu and Mao [18] investigated the entire solutions of finite order of difference Eq. (3.2) and obtained the following result corresponding to Theorem G.

Theorem H (See [18]) *Let $n \geq 2$ be an integer, $P_d(z,f)$ be a difference polynomial in f of degree $d \leq n-2$ such that $P_d(z,0) \not\equiv 0$, p_1, p_2 be nonzero small functions of e^z, and let α_1, α_2 be nonzero constants. If $\frac{\alpha_1}{\alpha_2} < 0$, and Eq. (3.2) has an entire solution f of finite order, then $\alpha_1 + \alpha_2 = 0$ and $f(z) = \gamma_1 e^{\alpha_1 z/n} + \gamma_2 e^{\alpha_2 z/n}$, where γ_1, γ_2 are constants satisfying $\gamma_i^2 = p_i$, $i = 1, 2$.*

It is natural to ask whether the conditions $P_d(z,0) \not\equiv 0$ and $\frac{\alpha_1}{\alpha_2} < 0$ in Theorem H can be omitted or not. In this section, we give an affirmative answer to this question by proving the following result.

Theorem 3.1 *Let $n \geq 2$ be an integer, $P_d(z,f)$ be a difference polynomial in f of degree $d \leq n-2$, and p_1, p_2, α_1, α_2 be nonzero constants and $\alpha_1 \neq \alpha_2$. If f is a transcendental meromorphic solution of Eq. (3.2), and satisfying $N(r,f) = S(r,f)$, then $\rho(f) = 1$, and one of the following holds*:

(i) $f(z) = \gamma_0 + \gamma_1 e^{\alpha_1 z/n}$;
(ii) $f(z) = \gamma_0 + \gamma_2 e^{\alpha_2 z/n}$;
(iii) $f(z) = \gamma_1 e^{\alpha_1 z/n} + \gamma_2 e^{\alpha_2 z/n}$ *and* $\alpha_1 + \alpha_2 = 0$;
where $\gamma_0, \gamma_1, \gamma_2$ *are constants satisfying* $\gamma_i^2 = p_i$, $i = 1, 2$.

Remark 3.1 The following examples show that the condition $P_d(z, 0) \not\equiv 0$ is not necessary in Theorem 3.1.

Example 3.1 The difference equation

$$f^3(z) + f(z + \log 2) - f(z) = e^{3z} + e^z.$$

has an entire solution $f(z) = e^z$, where $P_1(z,f) = f(z + \log 2) - f(z)$ and $P_1(z, 0) = 0$.

Example 3.2 The difference equation

$$f^3(z) - 3f(z) = e^{3z} + e^{-3z}.$$

has an entire solution $f(z) = e^z + e^{-z}$, where $P_1(z,f) = -3f(z)$ and $P_1(z, 0) = 0$.

The following lemmas will be used in the proof of Theorem 3.1.

Lemma 3.1 (See [16, Lemma 6]) *Suppose that f is a transcendental meromorphic function, a, b, c, d are small functions with respect to f and* $acd \not\equiv 0$. *If*

$$af^2 + bff' + c(f')^2 = d,$$

then

$$c(b^2 - 4ac)\frac{d'}{d} + b(b^2 - 4ac) - c(b^2 - 4ac)' + (b^2 - 4ac)c' = 0.$$

In particular, if a, b, c, d are constants and $b^2 - 4ac \neq 0$, *then* $b = 0$ *and*

$$f(z) = \gamma_1 e^{\lambda z} + \gamma_2 e^{-\lambda z},$$

where γ_1, γ_2 *and* λ *are nonzero constants.*

Remark 3.2 The condition $acd \not\equiv 0$ in Lemma 3.1 is not necessary and it can be replaced with $cd \not\equiv 0$.

Lemma 3.2 (See [8, p. 247]) *Suppose that* $f(z)$ *is a transcendental meromorphic function and* $K > 1$. *Then there exists a set* $M(K)$ *of upper logarithmic density at most* $\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K - 1))\exp(e(1 - K))\}$ *such that for every positive integer k, we have*

$$\limsup_{\substack{r\to\infty \\ r\notin M(K)}} \frac{T(r,f)}{T(r,f^{(k)})} \leq 3eK.$$

Remark 3.3 By Lemma 3.2, we see that if f is a transcendental meromorphic function, and if φ satisfies $T(r,\varphi^{(k)}) = S_1(r,f)$, then $T(r,\varphi) = S_1(r,f)$, where $S_1(r,f)$ is defined to be any quantity such that for any positive number ε there exists a set $E(\varepsilon)$ whose upper logarithmic density is less than ε, and $S_1(r,f) = o(T(r,f))$ as $r \to \infty$, $r \notin E(\varepsilon)$.

The proof given below for Theorem 3.1 is different from the proof previously given for the preceding result in [18].

Proof of Theorem 3.1 Clearly, $\rho(p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) = 1$, where $\alpha_1 \neq \alpha_2$. Similar as in the beginning of the proof of Lemma 2.2, we have

$$\begin{aligned} m\bigl(r,P_d(z,f)\bigr) &\le dm(r,f) + S(r,f) \\ &\le (n-2)m(r,f) + S(r,f) \le (n-2)T(r,f) + S(r,f). \end{aligned}$$

Noting that $N(r,f) = S(r,f)$, by (3.2) and the above inequality, we have

$$\begin{aligned} T\bigl(r,p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}\bigr) &= T\bigl(r,f^n + P_d(z,f)\bigr) \\ &= m\bigl(r,f^n + P_d(z,f)\bigr) + S(r,f) \\ &\le m\bigl(r,f^n\bigr) + m\bigl(r,P_d(z,f)\bigr) + S(r,f) \\ &\le 2(n-1)T(r,f) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} T\bigl(r,p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}\bigr) &= T\bigl(r,f^n + P_d(z,f)\bigr) \\ &= m\bigl(r,f^n + P_d(z,f)\bigr) + S(r,f) \\ &\ge m\bigl(r,f^n\bigr) - m\bigl(r,P_d(z,f)\bigr) + S(r,f) \\ &\ge 2T(r,f) + S(r,f). \end{aligned}$$

From the above two inequalities, we derive

$$2T(r,f) + S(r,f) \le T\bigl(r,p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}\bigr) \le 2(n-1)T(r,f) + S(r,f),$$

thus $\rho(f) = 1$. Denote $P_d := P_d(z,f)$. Suppose that f is a transcendental meromorphic solution of Eq. (3.2) which satisfies $N(r,f) = S(r,f)$. By differentiating (3.2), we have

$$nf^{n-1}f' + P_d' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{3.3}$$

Eliminating $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ from (3.2) and (3.3), respectively, we have

$$\alpha_1 f^n - nf^{n-1}f' + \alpha_1 P_d - P_d' = (\alpha_1 - \alpha_2)p_2e^{\alpha_2 z}, \tag{3.4}$$

$$\alpha_2 f^n - nf^{n-1}f' + \alpha_2 P_d - P_d' = (\alpha_2 - \alpha_1)p_1e^{\alpha_1 z}. \tag{3.5}$$

Differentiating (3.5) yields

$$n\alpha_2 f^{n-1}f' - n(n-1)f^{n-2}\big(f'\big)^2 - nf^{n-1}f'' + \alpha_2 P_d' - P_d'' = \alpha_1(\alpha_2-\alpha_1)p_1e^{\alpha_1 z}. \tag{3.6}$$

Eliminating $e^{\alpha_1 z}$ from (3.5) and (3.6), we get

$$f^{n-2}\varphi(z) = -Q_d(z,f), \tag{3.7}$$

where

$$\varphi(z) = \alpha_1\alpha_2 f^2 - n(\alpha_1+\alpha_2)ff' + n(n-1)\big(f'\big)^2 + nff'' \tag{3.8}$$

and

$$Q_d(z,f) = \alpha_1\alpha_2 P_d - (\alpha_1+\alpha_2)P_d' + P_d'', \tag{3.9}$$

while $Q_d(z,f)$ is a differential-difference polynomial in f of degree $\leq n-2$. By (3.7) and Lemma C, we have $m(r,\varphi) = S(r,f)$. Note that $N(r,f) = S(r,f)$, thus $T(r,\varphi) = S(r,f)$. We distinguish two cases below:

Case 1. If $\varphi \equiv 0$, then $Q_d(z,f) \equiv 0$. Since $\alpha_1 \neq \alpha_2$, we see that $\alpha_1 P_d - P_d' \equiv 0$ and $\alpha_2 P_d - P_d' \equiv 0$ cannot hold simultaneously. Suppose that $\alpha_2 P_d - P_d' \not\equiv 0$. By (3.9), we have

$$\alpha_2 P_d - P_d' = Ae^{\alpha_1 z}, \tag{3.10}$$

where A is a nonzero constant. Substituting (3.10) into (3.5), we have

$$f^{n-1}\big(\alpha_2 f - nf'\big) = \frac{[(\alpha_2-\alpha_1)p_1 - A]\alpha_2}{A}P_d - \frac{(\alpha_2-\alpha_1)p_1 - A}{A}P_d'. \tag{3.11}$$

Since the right-hand side of (3.11) is a differential-difference polynomial in f of degree $\leq n-2$, by Lemma C, we have $m(r,\alpha_2 f - nf') = S(r,f)$ and $m(r,f(\alpha_2 f - nf')) = S(r,f)$, and then $\alpha_2 f - nf' \equiv 0$. Otherwise, $\alpha_2 f - nf' \not\equiv 0$, thus we have

$$\begin{aligned} T(r,f) &= m(r,f) + S(r,f) \leq m\big(r,f\big(\alpha_2 f - nf'\big)\big) + m\big(r,1/\big(\alpha_2 f - nf'\big)\big) + S(r,f) \\ &\leq T\big(r,\alpha_2 f - nf'\big) + S(r,f) \\ &= m\big(r,\alpha_2 f - nf'\big) + S(r,f) = S(r,f), \end{aligned}$$

resulting in a contradiction $T(r,f) \leq S(r,f)$. Since $\alpha_2 f - nf' \equiv 0$, one gets $f^n = \widetilde{p}_2 e^{\alpha_2 z}$, $\widetilde{p}_2 \in \mathbb{C}\setminus\{0\}$. Substituting this equality and (3.10) into (3.2) yields

$$\left(1 - \frac{p_2}{\widetilde{p}_2}\right)f^n = \frac{\alpha_2 p_1 - A}{A}P_d - \frac{p_1}{A}P_d'.$$

If $p_2 \neq \widetilde{p}_2$, by Lemma C, we have $T(r,f) = m(r,f) + N(r,f) = S(r,f)$, a contradiction. Therefore $p_2 = \widetilde{p}_2$, $f(z) = \gamma_2 e^{\frac{\alpha_2}{n}z}$, $\gamma_2^n = p_2$. Similarly, if $\alpha_1 P_d - P_d' \not\equiv 0$, then we obtain $f(z) = \gamma_1 e^{\frac{\alpha_1}{n}z}$, $\gamma_1^n = p_1$.

Case 2. If $\varphi \not\equiv 0$, by applying $T(r,\varphi) = S(r,f)$ and the lemma on the logarithmic derivative, we have

$$\begin{aligned} 2m\left(r,\frac{1}{f}\right) &= m\left(r,\frac{1}{f^2}\right) \le m\left(r,\frac{\varphi}{f^2}\right) + m\left(r,\frac{1}{\varphi}\right) \\ &\le T(r,\varphi) + S(r,f) = S(r,f). \end{aligned} \tag{3.12}$$

By (3.8), if z_0 is a multiple zero of f, then z_0 must be a zero of φ. Hence $N_{(2}(r,\frac{1}{f}) = S(r,f)$, where $N_{(2}(r,\frac{1}{f})$ denotes the counting function of multiple zeros of f, which implies

$$T(r,f) = N_{1)}\left(r,\frac{1}{f}\right) + S(r,f), \tag{3.13}$$

where $N_{1)}(r,\frac{1}{f})$ denotes the counting function of simple zeros of f, and we deduce that f has infinitely many simple zeros. Differentiating (3.8) gives

$$\varphi' = 2\alpha_1\alpha_2 ff' - n(\alpha_1+\alpha_2)ff'' - n(\alpha_1+\alpha_2)\big(f'\big)^2 + n(2n-1)f'f'' + nff'''. \tag{3.14}$$

If z_0 is a simple zero of f, it follows from (3.8) and (3.14) that z_0 is a zero of $(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]f'$. Define

$$\alpha := \frac{(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]f'}{f}, \tag{3.15}$$

then we have $T(r,\alpha) = S(r,f)$. It follows that

$$f'' = \frac{1}{2n-1}\left[(n-1)\frac{\varphi'}{\varphi} + (\alpha_1+\alpha_2)\right]f' + \frac{\alpha}{(2n-1)\varphi}f. \tag{3.16}$$

Substituting (3.16) into (3.8) yields

$$af^2 + bff' + c\big(f'\big)^2 = \varphi, \tag{3.17}$$

where $a = \alpha_1\alpha_2 + \frac{n\alpha}{(2n-1)\varphi}$, $b = \frac{n(n-1)}{2n-1}[\frac{\varphi'}{\varphi} - 2(\alpha_1+\alpha_2)]$, $c = n(n-1)$. By Lemma 3.1, we have

$$c\big(4ac-b^2\big)\frac{\varphi'}{\varphi} = c\big(4ac-b^2\big)' - b\big(4ac-b^2\big). \tag{3.18}$$

We consider two subcases as follows.

Subcase 2.1. Suppose that $4ac - b^2 \not\equiv 0$. It follows from (3.18) that

$$2n\frac{\varphi'}{\varphi} = (2n-1)\frac{(4ac-b^2)'}{4ac-b^2} + 2(\alpha_1+\alpha_2). \tag{3.19}$$

By integration, we see that there exists a $C \in \mathbb{C} \setminus \{0\}$ such that

$$e^{2(\alpha_1+\alpha_2)z} = C\varphi^{2n}\big(4ac-b^2\big)^{-(2n-1)}, \tag{3.20}$$

which implies $e^{2(\alpha_1+\alpha_2)z} \in S(f)$, then $\alpha_1 + \alpha_2 = 0$. It follows from (3.4) and (3.5) that

$$f^{2n-2}\psi + R(z,f) = -(\alpha_1 - \alpha_2)^2 p_1 p_2, \tag{3.21}$$

where $\psi = \alpha_1\alpha_2 f^2 + n^2(f')^2$, $R(z,f)$ is a differential-difference polynomial of degree $\leq 2n-2$. By applying Lemma C, we have $m(r,\psi) = S(r,f)$, then $T(r,\psi) = S(r,f)$. We deduce that $\psi \not\equiv 0$. Otherwise, we assume that $\psi = \alpha_1\alpha_2 f^2 + n^2(f')^2 \equiv 0$. Then $(\frac{f'}{f})^2 \equiv -\frac{\alpha_1\alpha_2}{n^2}$, which implies that

$$\begin{aligned} 0 &= N\left(r, -\frac{\alpha_1\alpha_2}{n^2}\right) = 2N\left(r, \frac{f'}{f}\right) \\ &= 2\left[\overline{N}(r,f) + \overline{N}(r,1/f)\right] = 2N_{1)}(r,1/f) + S(r,f), \end{aligned}$$

combining with (3.13), we have $T(r,f) = S(r,f)$, a contradiction. By Lemma 3.1 and (3.2), we see that ψ must be constant and $f(z) = \gamma_1 e^{\alpha_1 z/n} + \gamma_2 e^{\alpha_2 z/n}$, $\alpha_1 + \alpha_2 = 0$, where γ_1, γ_2 are constants satisfying $\gamma_i^2 = p_i$, $i = 1, 2$.

Subcase 2.2. Suppose that $4ac - b^2 \equiv 0$. Differentiating (3.17) yields

$$\varphi' = a'f^2 + \left(2a + b'\right)ff' + b\left(f'\right)^2 + bff'' + 2cf'f''. \tag{3.22}$$

Suppose z_0 is a simple zero of f which is not a zero of a, b. It follows from (3.17) and (3.22) that z_0 is a zero of $2\varphi f'' - (\varphi' - \frac{b}{c}\varphi)f'$. Denote

$$\beta := \frac{2\varphi f'' - (\varphi' - \frac{b}{c}\varphi)f'}{f}, \tag{3.23}$$

then we have $T(r,\beta) = S(r,f)$. It follows that

$$f'' = \left(\frac{1}{2}\frac{\varphi'}{\varphi} - \frac{b}{2c}\right)f' + \frac{\beta}{2\varphi}f. \tag{3.24}$$

Substituting (3.24) into (3.22) yields

$$\varphi' = df^2 + hff' + c\frac{\varphi'}{\varphi}\left(f'\right)^2, \tag{3.25}$$

where $d = a' + \frac{b\beta}{2\varphi}$, $h = 2a + b' + \frac{b}{2}\frac{\varphi'}{\varphi} - \frac{b^2}{2c} + \frac{c\beta}{\varphi}$. Eliminating $(f')^2$ from (3.17) and (3.25), we have

$$Af + Bf' \equiv 0, \tag{3.26}$$

where

$$\begin{aligned} A &= d - a\frac{\varphi'}{\varphi} = a' + \frac{b\beta}{2\varphi} - a\frac{\varphi'}{\varphi}, \\ B &= h - b\frac{\varphi'}{\varphi} = 2a + b' - \frac{b}{2}\frac{\varphi'}{\varphi} - \frac{b^2}{2c} + \frac{c\beta}{\varphi}. \end{aligned}$$

Note that A and B are small functions of f. If z_0 is a simple zero of f and not a zero of A, B, it follows from (3.26) that $A = B \equiv 0$. By (3.24), we have

$$f'' = \left(\frac{1}{2}\frac{\varphi'}{\varphi} - \frac{b}{2c}\right)f' - \frac{1}{b}\left(a' - a\frac{\varphi'}{\varphi}\right)f, \tag{3.27}$$

where $b = \frac{n(n-1)}{2n-1}[\frac{\varphi'}{\varphi} - 2(\alpha_1 + \alpha_2)] \not\equiv 0$. Otherwise, $b \equiv 0$, by $4ac - b^2 \equiv 0$, and then $a \equiv 0$. It follows from (3.17) that $c(f')^2 \equiv \varphi$, then $T(r,f') = S(r,f)$. By Lemma 3.2 and Remark 3.3, we obtain that $T(r,f) = S_1(r,f)$, a contradiction. Substituting $4ac - b^2 \equiv 0$ into (3.27) yields

$$\begin{aligned} f'' &= \frac{1}{2n-1}\left[(n-1)\frac{\varphi'}{\varphi} + (\alpha_1 + \alpha_2)\right]f' \\ &\quad - \frac{1}{2(2n-1)}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_1 + \alpha_2)\frac{\varphi'}{\varphi}\right]f. \end{aligned} \tag{3.28}$$

It follows from (3.16) and (3.28) that

$$\frac{\alpha}{\varphi} = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_1 + \alpha_2)\frac{\varphi'}{\varphi}\right]. \tag{3.29}$$

If $\varphi' \equiv 0$, then $\frac{\alpha}{\varphi} \equiv 0$. Substituting this identity into $4ac - b^2 \equiv 0$ yields

$$n(n-1)\left(\frac{\alpha_1}{\alpha_2}\right)^2 - \left[n^2 + (n-1)^2\right]\frac{\alpha_1}{\alpha_2} + n(n-1) = 0,$$

which implies that $\frac{\alpha_1}{\alpha_2} = \frac{n-1}{n}$ or $\frac{\alpha_1}{\alpha_2} = \frac{n}{n-1}$. By substituting $\varphi' \equiv 0$ into (3.28), we obtain

$$f'' = \frac{1}{2n-1}(\alpha_1 + \alpha_2)f',$$

then

$$f = \frac{(2n-1)C_1}{\alpha_1 + \alpha_2}e^{\frac{(\alpha_1+\alpha_2)z}{2n-1}} + C_2,$$

where $C_1, C_2 \in \mathbb{C}\setminus\{0\}$. Otherwise, one of C_1 and C_2 is equal to zero, then $N(r, 1/f) = S(r,f)$. It follows from (3.13) that $T(r,f) = S(r,f)$, a contradiction. As $\frac{\alpha_1}{\alpha_2} = \frac{n-1}{n}$, then

$$f = \frac{nC_1}{\alpha_2}e^{\frac{\alpha_2}{n}z} + C_2.$$

As $\frac{\alpha_1}{\alpha_2} = \frac{n}{n-1}$, then

$$f = \frac{nC_1}{\alpha_1}e^{\frac{\alpha_1}{n}z} + C_2.$$

If $\varphi' \not\equiv 0$, differentiating (3.29) gives

$$\left(\frac{\alpha}{\varphi}\right)' = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)'' - \left(\frac{\varphi'}{\varphi}\right)'\frac{\varphi'}{\varphi} + (\alpha_1 + \alpha_2)\left(\frac{\varphi'}{\varphi}\right)'\right]. \tag{3.30}$$

It follows from $4ac - b^2 \equiv 0$ that $bb' = 2ca'$, namely,

$$\left(\frac{\alpha}{\varphi}\right)' = \frac{n-1}{2(2n-1)}\left(\frac{\varphi'}{\varphi}\right)'\left[\frac{\varphi'}{\varphi} - 2(\alpha_1 + \alpha_2)\right]. \tag{3.31}$$

Denoting $\gamma := \frac{\varphi'}{\varphi}$ and combining (3.30) with (3.31) yields

$$(\alpha_1 + \alpha_2)\gamma' = n\gamma\gamma' - (2n-1)\gamma''. \tag{3.32}$$

If $\gamma' \equiv 0$, then $\varphi = C_3 e^{C_4 z}$, $C_3, C_4 \in \mathbb{C}$. It follows from $\varphi' \not\equiv 0$ that $C_3 C_4 \neq 0$, which implies that $\varphi \notin S(f)$, a contradiction. If $\gamma' \not\equiv 0$, it follows from (3.32) that

$$e^{(\alpha_1+\alpha_2)z} = C_5 \varphi^n \left(\left(\frac{\varphi'}{\varphi}\right)'\right)^{-(2n-1)}, \quad C_5 \in \mathbb{C} \setminus \{0\},$$

which implies that $e^{(\alpha_1+\alpha_2)z} \in S(f)$, and then $\alpha_1 + \alpha_2 = 0$. Using similar reasoning as in Subcase 2.1, we obtain $f(z) = \gamma_1 e^{\alpha_1 z/n} + \gamma_2 e^{\alpha_2 z/n}$, $\alpha_1 + \alpha_2 = 0$, where γ_1, γ_2 are constants satisfying $\gamma_i^2 = p_i$, $i = 1, 2$.

This completes the proof of Theorem 3.1. □

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