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Estimation of f-divergence and Shannon entropy by Levinson type inequalities via new Green's functions and Lidstone polynomial

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Abstract

By using new Green's functions and Lidstone interpolating polynomial, some new generalizations of Levinson type inequalities for $(2p+1)$-convex functions are obtained. In seek of applications of our results to information theory, new generalizations based on f-divergence estimates are also proven. Moreover, some inequalities for Shannon entropies are deduced as well.

Keywords: Information theory; Levinson's inequality; Lidstone interpolating polynomial; Green's functions

1 Introduction and preliminaries

The idea of Shannon entropy is the central job of information speculation, now and again implied as a measure of uncertainty. The entropy of a random variable is described with respect to a probability distribution, and it can be shown that it is a decent measure of random. The assignment of Shannon entropy is to assess the typical least number of bits expected to encode a progression of pictures subject to the letters, including the size and the repetition of the symbols.

Divergences between probability distributions can be interpreted as measures of distance between them. An assortment of sorts of divergences exist, for example the $\mathfrak{f}$-divergences (especially, Kullback–Leibler divergences, Hellinger distance, and total variation distance), Rényi divergences, Jensen–Shannon divergences, etc. (see [1, 2]). There are a lot of papers dealing with the subject of inequalities and entropies, see, e.g., [3–7] and the references therein. Jensen's inequality deals with one kind of data points, Levinson's inequality deals with two types of data points.

1.1 Csiszár divergence

In [8, 9] Csiszár gave the following definition:

Definition 1 Let f be a convex function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}^n_+$ be such that $\sum_{\rho=1}^n r_\rho = 1$ and $\sum_{\rho=1}^n k_\rho = 1$. Then f-divergence functional is defined by

$$\mathbb{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^n k_\rho f\left(\frac{r_\rho}{k_\rho}\right).$$

By defining the following:

$$f(0) := \lim_{x\to 0^+} f(x); \qquad 0f\left(\frac{0}{0}\right) := 0; \qquad 0f\left(\frac{a}{0}\right) := \lim_{x\to 0^+} xf\left(\frac{a}{0}\right), \quad a > 0,$$

he stated that nonnegative probability distributions can also be used.

Using the definition of f-divergence functional, Horv́ath *et al.* [10] gave the following functional:

Definition 2 Let $\mathbb{I}$ be an interval contained in $\mathbb{R}$ and $f : \mathbb{I} \to \mathbb{R}$ be a function. Also let $\tilde{\mathbf{r}} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ and $\tilde{\mathbf{k}} = (k_1, \ldots, k_n) \in (0, \infty)^n$ be such that

$$\frac{r_\rho}{k_\rho} \in \mathbb{I}, \quad \rho = 1, \ldots, n.$$

Then

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^n k_\rho f\left(\frac{r_\rho}{k_\rho}\right). \tag{1}$$

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: firstly, applications of convex functions are directly involved in modern analysis; secondly, many important inequalities are results of applications of convex functions, and convex functions are closely related to inequalities (see [11]).

Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [11, p. 14].

Levinson generalized Ky Fan's inequality for 3-convex in [12] (see also [13, p. 32, Theorem 1]) as follows:

Theorem 1 *Let $f : \mathbb{I} = (0, 2\lambda) \to \mathbb{R}$ be such that f is 3-convex. Also let $0 < x_\rho < \lambda$ and $p_\rho > 0$, then*

$$\frac{1}{P_n}\sum_{\rho=1}^n p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^n p_\rho x_\rho\right) \le \frac{1}{P_n}\sum_{\rho=1}^n p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^n p_\rho (2\lambda - x_\rho)\right). \tag{2}$$

Inequality (2) gives us the following functional:

$$J_1(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho (2\lambda - x_\rho)\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \geq 0. \tag{3}$$

In [14], Popoviciu noticed that Levinson's inequality (2) is substantial on $(0, 2\lambda)$ for 3-convex functions, while in [15] (see additionally [13, p. 32, Theorem 2]) Bullen gave distinctive confirmation of Popoviciu's result and furthermore the converse of (2).

Theorem 2

(a) *Let* $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a* 3*-convex function and* $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ *for* $\rho = 1, 2, \ldots, n$ *be such that*

$$\max\{x_1 \cdots x_n\} \leq \min\{y_1 \cdots y_n\}, \quad x_1 + y_1 = \cdots = x_n + y_n \tag{4}$$

and $p_\rho > 0$*, then*

$$\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \leq \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right). \tag{5}$$

(b) *If* $p_\rho > 0$*, inequality* (5) *is valid for all* x_ρ*,* y_ρ *satisfying condition* (4)*, and function* f *is continuous, then* f *is* 3*-convex.*

The following functional arises from inequality (5):

$$J_2(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \geq 0. \tag{6}$$

Remark 1 In the above results, if the function f is 3-convex, then $J_k(f(\cdot)) \geq 0$ for $k = 1, 2$ and $J_k(f(\cdot)) = 0$ for $f(x) = x$ or $f(x) = x^2$ or a constant function f.

In the following result, Pečarić [16] (see also [13, p. 32, Theorem 4]) proved inequality (5) by weakening condition (4).

Theorem 3 *Let* $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $f^3(t) \geq 0$*,* $p_\rho > 0$*. Also let* $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ *be such that* $x_\rho + y_\rho = 2\check{c}$ *for* $\rho = 1, \ldots, n$*,* $x_\rho + x_{n-\rho+1} \leq 2\check{c}$ *and* $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$*. Then inequality* (5) *holds.*

In [17], Mercer proved that inequality (5) still holds after replacing the symmetry condition with symmetric variances of points. His result is given in the following theorem.

Theorem 4 *Let* $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $f^3(t) \geq 0$, p_ρ *are positive such that* $\sum_{\rho=1}^{n} p_\rho = 1$. *Also, let* x_ρ, y_ρ *satisfy* $\max\{x_1 \cdots x_\rho\} \leq \min\{y_1 \cdots y_\rho\}$ *and*

$$\sum_{\rho=1}^{n} p_\rho \left(x_\rho - \sum_{\rho=1}^{n} p_\rho x_\rho \right)^2 = \sum_{\rho=1}^{n} p_\rho \left(y_\rho - \sum_{\rho=1}^{n} p_\rho y_\rho \right)^2, \tag{7}$$

then (5) *holds.*

Lidstone polynomials are useful in literature to generalize a number of novel inequalities including Jensen, Ostrowski, Chebysev, and Hermite–Hadamard type inequalities. In the literature, several extensions and generalizations of the said inequalities are found via Lidstone interpolation. However, all these results involve only one type of data points and are for the class of convex functions along with generalization for $(2p)$-convex functions.

The following result was proved by Wider in [18]:

Lemma 1.1 *If* $f \in C^\infty[0,1]$, *then*

$$f(t) = \sum_{l=0}^{p-1} \left[f^{(2l)}(0)\Theta_l(1-t) + f^{(2l)}(0)\Theta_l(t) \right] + \int_0^1 G_p(t,s) f^{(2p)}(t)\, dt,$$

where Θ_l *is a polynomial of degree* $(2l+1)$ *defined by the relations*

$$\Theta_0(t) = t, \qquad \Theta_p''(t) = \Theta_{p-1}(t), \qquad \Theta_p(0) = \Theta_p(1) = 0, \quad p \geq 1,$$

and

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & s \leq t; \\ (s-1)t, & t \leq s, \end{cases} \tag{8}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ *on* $[0,1]$, *and with the successive iterates of* $G(t,s)$

$$G_p(t,s) = \int_0^1 G_1(t,k) G_{p-1}(k,s)\, dk, \quad p \geq 2. \tag{9}$$

The Lidstone polynomial can be expressed in terms of $G_p(t,s)$ as

$$\Theta_p(t) = \int_0^1 G_p(t,s) s\, ds. \tag{10}$$

The Lidstone series representation of $f \in C^{2p}[\zeta_1, \zeta_2]$ is given in [19] as follows:

$$\begin{aligned} f(x) = {} & \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - x}{\zeta_2 - \zeta_1} \right) + \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \Theta_l\left(\frac{x - \zeta_1}{\zeta_2 - \zeta_1} \right) \\ & + (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{x - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) f^{(2p)}(t)\, dt. \end{aligned} \tag{11}$$

The error function $e_{\mathcal{F}}(t)$ can be represented in terms of Green's function $G_{\mathcal{F},n}(t,s)$ of the boundary value problem

$$\begin{aligned}
&z^{(n)}(t) = 0,\\
&z^{(i)}(\zeta_1) = 0, \quad 0 \le i \le p,\\
&z^{(i)}(\zeta_2) = 0, \quad p+1 \le i \le n-1,\\
&e_F(t) = \int_{\zeta_1}^{\zeta_2} G_{F,n}(t,s) f^{(n)}(s)\,ds, \quad t \in [\zeta_1, \zeta_2],
\end{aligned}$$

where

$$G_{F,n}(t,s) = \frac{1}{(n-1)!}\begin{cases} \sum_{i=0}^{p}\binom{n-1}{i}(t-\zeta_1)^i(\zeta_1-s)^{n-i-1}, & \zeta_1 \le s \le t;\\ -\sum_{i=p+1}^{n-1}\binom{n-1}{i}(t-\zeta_1)^i(\zeta_1-s)^{n-i-1}, & t \le s \le \zeta_2. \end{cases} \tag{12}$$

In [20] Aras Gazić *et al.* proved the following result:

Theorem 5 *Let $f \in C^n[\zeta_1,\zeta_2]$ and P_F be its 'two-point right focal' interpolating polynomial. Then, for $\zeta_1 \le a_1 < a_2 \le \zeta_2$ and $0 \le p \le n-2$,*

$$\begin{aligned}
f(t) &= P_F(t) + e_F(t)\\
&= \sum_{i=0}^{p}\frac{(t-a_1)^i}{i!}f^{(i)}(a_1)\\
&\quad + \sum_{j=0}^{n-p-2}\left(\sum_{i=0}^{j}\frac{(t-a_1)^{p+1+i}(a_1-a_2)^{j-i}}{(p+1+i)!(j-i)!}\right)f^{(p+1+j)}(a_2)\\
&\quad + \int_{a_1}^{a_2} G_{F,n}(t,s) f^{(n)}(s)\,ds,
\end{aligned} \tag{13}$$

where $G_{F,n}(t,s)$ is the Green's function defined by (12).

We have the following two cases from (13).

(*Case*-1) *For $n = 3$ and $p = 0$*

$$\begin{aligned}
f(t) ={}& f(a_1) + (t-a_1)f^{(1)}(a_2) + (t-a_1)(a_1-a_2)f^{(2)}(a_2) + \frac{(t-a_1)^2}{2}f^{(2)}(a_2)\\
&+ \int_{a_1}^{a_2} G_1(t,s) f^{(3)}(s)\,ds,
\end{aligned} \tag{14}$$

where

$$G_1(t,s) = \begin{cases} (a_1-s)^2, & a_1 \le s \le t;\\ -(t-a_1)(a_1-s) + \frac{1}{2}(t-a_1)^2, & t \le s \le a_2. \end{cases} \tag{15}$$

(*Case*-2) *For $n = 3$ and $p = 1$*

$$f(t) = f(a_1) + (t-a_1)f^{(1)}(a_2) + \frac{(t-a_1)^2}{2}f^{(2)}(a_2) + \int_{a_1}^{a_2} G_2(t,s) f^{(3)}(s)\,ds, \tag{16}$$

where

$$G_2(t,s) = \begin{cases} \frac{1}{2}(a_1 - s)^2 + (t - a_1)(a_1 - s), & a_1 \leq s \leq t; \\ -\frac{1}{2}(t - a_1)^2, & t \leq s \leq a_2. \end{cases} \tag{17}$$

In [21], Pečarić *et al.* gave a probabilistic version of inequality (2) under condition (7). In [22] an operator version of probabilistic Levinson's inequality was discussed. In [20], Gazić *et al.* considered the class of $2p$-convex functions and generalized Jensen's inequality and converses of Jensen's inequality by using Lidstone's interpolating polynomials. All generalizations that exist in literature refer only to one type of data points. But in this paper, motivated by the above discussion, Levinson type inequalities are generalized for $(2p+1)$-convex function via Lidstone interpolating polynomial involving two types of data points for higher order convex functions.

2 Main results

Motivated by functional (6), we generalize the following new results with the help of Lidstone interpolating polynomial given by (11).

2.1 Generalization of Bullen type inequalities for (2p + 1)-convex functions

First we define the following functional:

$\mathcal{F}$: Let $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function, $x_1, \dots, x_n$ and $y_1, \dots, y_m \in \mathbb{I}_1$ be such that

$$\max\{x_1 \cdots x_n\} \leq \min\{y_1 \cdots y_m\}, \quad x_1 + y_1 = \cdots = x_n + y_m. \tag{18}$$

Also let $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $(q_1, \dots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^n p_\rho = 1$ $\sum_{\varrho=1}^m q_\varrho = 1$ and $x_\rho, y_\varrho, \sum_{\rho=1}^n p_\rho x_\rho, \sum_{\varrho=1}^m q_\varrho y_\varrho \in \mathbb{I}_1$. Then

$$\breve{J}(f(\cdot)) = \sum_{\varrho=1}^m q_\varrho f(y_\varrho) - f\left(\sum_{\varrho=1}^m q_\varrho y_\varrho\right) - \sum_{\rho=1}^n p_\rho f(x_\rho) + f\left(\sum_{\rho=1}^n p_\rho x_\rho\right). \tag{19}$$

Theorem 6 *Assume $\mathcal{F}$ with $f \in C^{2p+1}[\zeta_1, \zeta_2]$, and let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then*

$$\begin{aligned} \breve{J}(f(\cdot)) = {} & \sum_{l=0}^{p-2} (\zeta_2 - \zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} \breve{J}(G_k(\cdot, s)) \Theta_l\left(\frac{\zeta_2 - s}{\zeta_2 - \zeta_1}\right) ds \right. \\ & \left. + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} \breve{J}(G_k(\cdot, s)) \Theta_l\left(\frac{s - \zeta_1}{\zeta_2 - \zeta_1}\right) ds \right] \\ & + (\zeta_2 - \zeta_1)^{2p-3} \int_{\zeta_1}^{\zeta_2} f^{(2p+1)}(\nu) \\ & \times \left(\int_{\zeta_1}^{\zeta_2} \breve{J}(G_k(\cdot, s)) G_p\left(\frac{s - \zeta_1}{\zeta_2 - \zeta_1}, \frac{\nu - \zeta_1}{\zeta_2 - \zeta_1}\right) ds \right) d\nu, \end{aligned} \tag{20}$$

where

$$\breve{J}\big(G_k(\cdot,s)\big) = \sum_{\varrho=1}^{m} q_\varrho G_k(y_\varrho,s) - G_k\left(\sum_{\varrho=1}^{m} q_\varrho(y_\varrho,s)\right) - \sum_{\rho=1}^{n} p_\rho G_k(x_\rho,s) + G_k\left(\sum_{\rho=1}^{n} p_\rho x_\rho,s\right) \tag{21}$$

and $G_k(\cdot,s)$ $(k = 1,2)$ *are defined in* (15) *and* (17) *respectively.*

Proof Applying (19) to identities (14) and (16) respectively along with there defined new Green's functions, by means of simple calculations and following the properties of $\breve{J}(f(\cdot))$, we get

$$\breve{J}\big(f(\cdot)\big) = \int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big) f^{(3)}(s)\,ds. \tag{22}$$

Using Lidstone series representation (11) on the function $f^{(3)}(s)$, we have

$$f^{(3)}(s) = \sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l+3)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right) + \sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l+3)}(\zeta_2) \times \Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right) + (\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1},\frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\right) f^{(2p+3)}(\nu)\,d\nu.$$

Replacing p by $p-1$, we get

$$f^{(3)}(s) = \sum_{l=0}^{p-2}(\zeta_2-\zeta_1)^{2l}\left(f^{(2l+3)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right) + f^{(2l+3)}(\zeta_2)\Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right)\right) + (\zeta_2-\zeta_1)^{2p-3}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1},\frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\right) f^{(2p+1)}(\nu)\,d\nu. \tag{23}$$

Now, using (23) in (22) yields

$$\breve{J}\big(f(\cdot)\big) = \int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big)\left[\sum_{l=0}^{p-2}(\zeta_2-\zeta_1)^{2l}\left(f^{(2l+3)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right) + f^{(2l+3)}(\zeta_2)\Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right)\right) + (\zeta_2-\zeta_1)^{2p-3} \times \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1},\frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\right) f^{(2p+1)}(\nu)\,d\nu\right] ds. \tag{24}$$

After rearranging the terms in (24), we have

$$\breve{J}\big(f(\cdot)\big) = \sum_{l=0}^{p-2}(\zeta_2-\zeta_1)^{2l}\left[f^{(2l+3)}(\zeta_1)\int_{\zeta_1}^{\zeta_2}\breve{J}\big(G_k(\cdot,s)\big)\Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right)ds + f^{(2l+3)}(\zeta_2)\int_{\zeta_1}^{\zeta_2}\breve{J}\big(G_k(\cdot,s)\big)\Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right)ds\right]$$

$$+ (\zeta_2 - \zeta_1)^{2p-3} \int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big) \times \left(\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1} \right) f^{(2p+1)}(\nu)\, d\nu \right) ds. \tag{25}$$

Executing Fubini's theorem in the last term of (25) yields (20). □

As an application we obtain Bullen type inequality for $(2p+1)$-convex functions.

Theorem 7 *Assume that all the conditions of Theorem 6 hold, and let f be a $(2p+1)$-convex function. Then, for $k = 1, 2$, we have the following result:*

If

$$\int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big) G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1} \right) ds \ge 0, \tag{26}$$

then

$$\breve{J}\big(f(\cdot)\big) \ge \sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{\zeta_2 - s}{\zeta_2-\zeta_1} \right) ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} \breve{J}\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1} \right) ds \right]. \tag{27}$$

Proof Since the function f is $(2p+1)$-convex and is $(2p+1)$ times differentiable, we have

$$f^{(2p+1)}(x) \ge 0 \quad \forall x \in \mathbb{I}_1,$$

therefore, by using (26) in (27), we get the required results respectively. □

Remark 2

(i) In Theorem 7, inequality (26) holds in reverse direction if the inequality in (27) is reversed.

(ii) Inequality in (27) is also reversed if f is $(2p+1)$-concave.

If we put $m = n$, $p_\rho = q_\varrho$ and use positive weights in (19), then $\breve{J}(\cdot)$ converts to the functional $J_2(\cdot)$ defined in (6), also in this case (20), (21), (26), and (27) become

$$J_2\big(f(\cdot)\big) = \sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{\zeta_2 - s}{\zeta_2-\zeta_1} \right) ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1} \right) ds \right] + (\zeta_2-\zeta_1)^{2p-3} \int_{\zeta_1}^{\zeta_2} f^{(2p+1)}(\nu) \times \left(\int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1} \right) ds \right) d\nu, \tag{28}$$

$$J_2\big(G_k(\cdot,s)\big) = \sum_{\rho=1}^{n} p_\rho G_k(y_\rho,s) - G_k\left(\sum_{\rho=1}^{n} p_\rho (y_\rho,s)\right) - \sum_{\rho=1}^{n} p_\rho G_k(x_\rho,s) + G_k\left(\sum_{\rho=1}^{n} p_\rho x_\rho, s\right), \tag{29}$$

$$\int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\right) ds \geq 0, \tag{30}$$

and

$$J_2\big(f(\cdot)\big) \geq \sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right) ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} J_2\big(G_k(\cdot,s)\big) \Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right) ds \right]. \tag{31}$$

Theorem 8 *Let $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a $(2p+1)$-convex function. Also let $(p_1, \dots, p_n)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$. Then, for the functional $J_2(\cdot)$ defined in (6), and using $\Theta_p(t)$ defined in Lemma 1.1, we have the following:*

(i) *For $k = 1, 2$, inequality (31) holds provided that p is odd.*

(ii) *For fixed $k = 1, 2$, let inequality (31) be satisfied and*

$$\sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \Theta_l\left(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\right) + f^{(2l+3)}(\zeta_2) \Theta_l\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\right) \right] \geq 0. \tag{32}$$

Then

$$J_2\big(f(\cdot)\big) \geq 0. \tag{33}$$

Proof It is clear that Green's functions $G_k(\cdot,s)$ defined in (15) and (17) are 3-convex and the weights are assumed to be positive. Therefore, applying Theorem 2 and using Remark 1, we have $J_2(G_k(\cdot,s)) \geq 0$ for fixed $k = 1, 2$.

(i) $G_p(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1}) \geq 0$ for odd p, therefore (30) holds. As f is $(2p+1)$-convex, hence, by following Theorem 7, we get (31).

(ii) Using (32) in (31), we get (33) for fixed $k = 1, 2$. □

In the next result we give a generalization of the Levinson type inequality given in [16] (see also [13]).

Theorem 9 *Let $f \in C^{2p+1}[\zeta_1, \zeta_2]$, $(p_1, \dots, p_n) \in \mathbb{R}^n$, $(q_1, \dots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\varrho=1}^{m} q_\varrho = 1$ and $\sum_{\varrho=1}^{m} q_\varrho y_\varrho$ and $\sum_{\rho=1}^{n} p_\rho x_\rho \in \mathbb{I}_1$. Also let $x_1, \dots, x_n$ and $y_1, \dots, y_m \in \mathbb{I}_1$ be such that $x_\rho + y_\varrho = 2\breve{c}$ and $x_\rho + x_{n-\rho+1} \leq 2\breve{c}$, $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \breve{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma 1.1, then (20) holds.*

Proof Proof is similar to Theorem 6 by assuming conditions given in the statement. □

As an application, we obtain generalizations of Levinson type functional for $(2p+1)$-convex functions $(p > 1)$.

Theorem 10 *Let $f \in C^{2p+1}[\zeta_1,\zeta_2]$ $(p > 1)$, $(p_1,\dots,p_n) \in \mathbb{R}^n$, $(q_1,\dots,q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\varrho=1}^{m} q_\varrho = 1$ and $\sum_{\varrho=1}^{m} q_\varrho y_\varrho$ and $\sum_{\rho=1}^{n} p_\rho x_\rho \in \mathbb{I}_1$. Also let $x_1,\dots,x_n$ and $y_1,\dots,y_m \in \mathbb{I}_1$ be such that $x_\rho + y_\varrho = 2\check{c}$ and $x_\rho + x_{n-\rho+1} \le 2\check{c}$, $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \le \check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma 1.1. If (26) is valid, then (27) is also valid.*

Proof Proof is similar to Theorem 7. □

Theorem 11 *Let $f \in C^{2p+1}[\zeta_1,\zeta_2]$ $(p > 1)$, $(p_1,\dots,p_n)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$. Also let $x_1,\dots,x_n$ and $y_1,\dots,y_n \in \mathbb{I}_1$ be such that $x_\rho + y_\varrho = 2\check{c}$ and $x_\rho + x_{n-\rho+1}$, $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \le \check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then*
(i) *For $k = 1, 2$, inequality (31) holds provided that p is odd.*
(ii) *For fixed $k = 1, 2$, let inequality (31) be satisfied, then (33) holds.*

Proof Proof is similar to Theorem 10. □

In the next result, a Levinson type inequality is given (for positive weights) under Mercer's condition (7).

Corollary 1 *Let $f : \mathbb{I}_1 = [\zeta_1,\zeta_2] \to \mathbb{R}$ be a $(2p + 1)$-convex function, x_ρ, y_ρ satisfy (7) and* $\max\{x_1 \cdots x_n\} \le \min\{y_1 \cdots y_n\}$. *Also let p_ρ be such that $\sum_{\rho=1}^{n} p_\rho = 1$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then (28) holds.*

Proof We get (28) after using the conditions given in the statement and following similar steps as in the proof of Theorem 6. □

2.2 Generalization of Levinson type inequality for $(2p + 1)$-convex functions

Motivated by functional (3), we generalize the following new results with the help of Lidstone interpolating polynomial given by (11).

First we defined the following functional:

$\mathcal{H}$: Let $f : \mathbb{I}_2 = [0, 2a] \to \mathbb{R}$ be a function, $x_1,\dots,x_n \in (0,a)$, $(p_1,\dots,p_n) \in \mathbb{R}^n$, $(q_1,\dots,q_m) \in \mathbb{R}^m$ are real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$ and $\sum_{\varrho=1}^{m} q_\varrho = 1$. Also let x_ρ, $\sum_{\varrho=1}^{m} q_\varrho(2a - x_\varrho)$ and $\sum_{\rho=1}^{n} p_\rho \in \mathbb{I}_2$. Then

$$\tilde{J}(f(\cdot)) = \sum_{\varrho=1}^{m} q_\varrho f(2a - x_\varrho) - f\left(\sum_{\varrho=1}^{m} q_\varrho (2a - x_\varrho)\right) - \sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{34}$$

Theorem 12 *Assume $\mathcal{H}$ with $f \in C^{2p+1}[0,2a]$, and let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then, for $0 \le \zeta_1 < \zeta_2 \le 2a$, we have*

$$\tilde{J}(f(\cdot)) = \sum_{l=0}^{p-2} (\zeta_2 - \zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_k(\cdot,s))\Theta_l\left(\frac{\zeta_2 - s}{\zeta_2 - \zeta_1}\right) ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_k(\cdot,s))\Theta_l\left(\frac{s - \zeta_1}{\zeta_2 - \zeta_1}\right) ds \right]$$

$$+ (\zeta_2 - \zeta_1)^{2p-3} \int_{\zeta_1}^{\zeta_2} f^{(2p+1)}(\nu) \times \left(\int_{\zeta_1}^{\zeta_2} \check{J}\big(G_k(\cdot, s)\big) G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1} \right) ds \right) d\nu, \tag{35}$$

where

$$\tilde{J}\big(G_k(\cdot, s)\big) = \sum_{\varrho=1}^{m} q_\varrho G_k(2a - x_\varrho, s) - G_k\left(\sum_{\varrho=1}^{m} q_\varrho (2a - x_\varrho, s) \right) - \sum_{\rho=1}^{n} p_\rho G_k(x_\rho, s) + G_k\left(\sum_{\rho=1}^{n} p_\rho x_\rho, s \right) \tag{36}$$

and $G_k(\cdot, s)$ $(k = 1, 2)$ *are defined in* (15) *and* (17) *respectively.*

Proof Replace $\mathbb{I}_1$ with $\mathbb{I}_2$ and y_ϱ with $2a - x_\varrho$ in Theorem 6, we get the required result. □

In the following theorem we obtain generalizations of Levinson's inequality (for real weights) for $(2p + 1)$-convex functions.

Theorem 13 *Assume that all the conditions of Theorem 12 hold, and let f be a* $(2p + 1)$-*convex function. Then, for* $k = 1, 2$ *and* $0 \leq \zeta_1 < \zeta_2 \leq 2a$, *we have the following inequalities*:

If

$$\int_{\zeta_1}^{\zeta_2} \tilde{J}\big(G_k(\cdot, s)\big) G_p\left(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1} \right) ds \geq 0, \tag{37}$$

then

$$\tilde{J}\big(f(\cdot)\big) \geq \sum_{l=0}^{p-2} (\zeta_2 - \zeta_1)^{2l} \left[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} \tilde{J}\big(G_k(\cdot, s)\big) \Theta_l\left(\frac{\zeta_2 - s}{\zeta_2 - \zeta_1} \right) ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} \tilde{J}\big(G_k(\cdot, s)\big) \Theta_l\left(\frac{s - \zeta_1}{\zeta_2 - \zeta_1} \right) ds \right]. \tag{38}$$

Proof Similar to Theorem 7. □

Remark 3 In Theorem 13, inequality in (38) holds in reverse direction if the inequality in (37) is reversed.

If we put $m = n$, $p_\rho = q_\varrho$ and use positive weights in (34), then $\tilde{J}(\cdot)$ converts to the functional $J_1(\cdot)$ defined in (2), also in this case (35), (36), (37), and (38) become, for

$0 \le \zeta_1 < \zeta_2 \le 2a$,

$$J_1(f(\cdot)) = \sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \biggl[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) \Theta_l\biggl(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\biggr)\,ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) \Theta_l\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\biggr)\,ds \biggr] + (\zeta_2-\zeta_1)^{2p-3} \int_{\zeta_1}^{\zeta_2} f^{(2p+1)}(\nu) \times \biggl(\int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) G_p\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\biggr)\,ds \biggr)\,d\nu, \tag{39}$$

$$J_1(G_k(\cdot,s)) = \sum_{\rho=1}^{n} p_\rho G_k(2a-x_\varrho,s) - G_k\Biggl(\sum_{\rho=1}^{n} p_\rho(2a-x_\varrho,s)\Biggr) - \sum_{\rho=1}^{n} p_\rho G_k(x_\rho,s) + G_k\Biggl(\sum_{\rho=1}^{n} p_\rho x_\rho, s\Biggr), \tag{40}$$

$$\int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) G_p\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}, \frac{\nu-\zeta_1}{\zeta_2-\zeta_1}\biggr)\,ds \ge 0, \tag{41}$$

and

$$J_1(f(\cdot)) \ge \sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \biggl[f^{(2l+3)}(\zeta_1) \int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) \Theta_l\biggl(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\biggr)\,ds + f^{(2l+3)}(\zeta_2) \int_{\zeta_1}^{\zeta_2} J_1(G_k(\cdot,s)) \Theta_l\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\biggr)\,ds \biggr]. \tag{42}$$

Theorem 14 *Let $f : \mathbb{I}_2 = [0, 2a] \to \mathbb{R}$ be a $(2p+1)$-convex function. Also let $(p_1, \dots, p_n)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$. Then, for the functional $J_1(\cdot)$ defined in (3), and using $\Theta_p(t)$ defined in Lemma 1.1, we have the following*:

(i) *For $k = 1, 2$, inequality (42) holds provided that p is odd.*

(ii) *For fixed $k = 1, 2$, let inequality (42) be satisfied and*

$$\sum_{l=0}^{p-2} (\zeta_2-\zeta_1)^{2l} \biggl[f^{(2l+3)}(\zeta_1) \Theta_l\biggl(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\biggr) + f^{(2l+3)}(\zeta_2) \Theta_l\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\biggr) \biggr] \ge 0. \tag{43}$$

Then

$$J_1(f(\cdot)) \ge 0. \tag{44}$$

Proof By using Theorem 13 and Remark 1. □

Remark 4 Čebyšev, Grüss, and Ostrowski type new bounds related to the obtained generalizations can also be discussed. Moreover, we can also give related mean value theorems by using non-negative functionals (20) and (35) to construct the new families of n-exponentially convex functions and Cauchy means related to these functionals as given in Sect. 4 of [23].

3 Estimation of *f*-divergence and Shannon entropy

In this section we obtain applications of information theory. We apply Theorem 7 for $(2p+1)$-convex functions to $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}},\tilde{\mathbf{k}})$.

Theorem 15 *Let* $\tilde{\mathbf{r}}=(r_1,\ldots,r_n)\in\mathbb{R}^n$, $\tilde{\mathbf{w}}=(w_1,\ldots,w_m)\in\mathbb{R}^m$, $\tilde{\mathbf{k}}=(k_1,\ldots,k_n)\in(0,\infty)^n$, *and* $\tilde{\mathbf{t}}=(t_1,\ldots,t_m)\in(0,\infty)^m$ *be such that*

$$\frac{r_\rho}{k_\rho}\in\mathbb{I},\quad \rho=1,\ldots,n,$$

and

$$\frac{w_\varrho}{t_\varrho}\in\mathbb{I},\quad \varrho=1,\ldots,m.$$

Also let $f\in C^{2p+1}[\zeta_1,\zeta_2]$ *be such that* f *is* $(2p+1)$*-convex (for odd* p*) function, then*

$$\begin{aligned} J_{cis}\bigl(f(\cdot)\bigr) \geq \sum_{l=0}^{p-2}(\zeta_2-\zeta_1)^{2l}\biggl[f^{(2l+3)}(\zeta_1)\int_{\zeta_1}^{\zeta_2}J\bigl(G_k(\cdot,s)\bigr)\Theta_l\biggl(\frac{\zeta_2-s}{\zeta_2-\zeta_1}\biggr)\,ds \\ +f^{(2l+3)}(\zeta_2)\int_{\zeta_1}^{\zeta_2}J\bigl(G_k(\cdot,s)\bigr)\Theta_l\biggl(\frac{s-\zeta_1}{\zeta_2-\zeta_1}\biggr)\,ds\biggr] \end{aligned} \tag{45}$$

for $k=1,2$

$$\begin{aligned} J\bigl(G_k(\cdot,s)\bigr) = \sum_{\varrho=1}^{m}\frac{t_\varrho}{\sum_{\varrho=1}^{m}t_\varrho}G_k\biggl(\frac{w_\varrho}{t_\varrho},s\biggr)-G_k\Biggl(\sum_{\varrho=1}^{m}\frac{w_\varrho}{\sum_{\varrho=1}^{m}t_\varrho},s\Biggr)-\sum_{\rho=1}^{n}\frac{k_\rho}{\sum_{\rho=1}^{n}k_\rho}G_k\biggl(\frac{r_\rho}{k_\rho},s\biggr) \\ +G_k\Biggl(\sum_{\rho=1}^{n}\frac{r_\rho}{\sum_{\rho=1}^{n}k_\rho},s\Biggr). \end{aligned} \tag{46}$$

Proof It is clear that Green's functions $G_k(\cdot,s)$ defined in (15) and (17) are 3-convex, therefore $J(G_k(\cdot,s))\geq 0$ for fixed $k=1,2$. Also $G_p(\frac{s-\zeta_1}{\zeta_2-\zeta_1},\frac{\nu-\zeta_1}{\zeta_2-\zeta_1})\geq 0$ for odd p, therefore (26) holds. Hence, using $p_\rho=\frac{k_\rho}{\sum_{\rho=1}^{n}k_\rho}$, $x_\rho=\frac{r_\rho}{k_\rho}$, $q_\varrho=\frac{t_\varrho}{\sum_{\varrho=1}^{m}t_\varrho}$, $y_\varrho=\frac{w_\varrho}{t_\varrho}$ in Theorem 7, (27) becomes (45), where $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}},\tilde{\mathbf{k}})$ is defined in (1) and

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{w}},\tilde{\mathbf{t}}):=\sum_{\varrho=1}^{m}t_\varrho f\biggl(\frac{w_\varrho}{t_\varrho}\biggr). \tag{47}$$

□

3.1 Shannon entropy

Definition 3 (see [10]) The $\mathcal{S}$hannon entropy of positive probability distribution $\tilde{\mathbf{k}}=(k_1,\ldots,k_n)$ is defined by

$$\mathcal{S}:=-\sum_{\rho=1}^{n}k_\rho\log(k_\rho). \tag{48}$$

Corollary 2 *Let* $\tilde{\mathbf{k}}=(k_1,\ldots,k_n)$ *and* $\tilde{\mathbf{t}}=(t_1,\ldots,t_m)$ *be positive probability distributions. Also let* $\tilde{\mathbf{r}}=(r_1,\ldots,r_n)\in(0,\infty)^n$ *and* $\tilde{\mathbf{w}}=(w_1,\ldots,w_m)\in(0,\infty)^m$.

If base of log *is greater than* 1 *and* $p = \text{odd}$ $(p > 2)$, *then*

$$J_s(\cdot) \geq \sum_{l=0}^{p-2} (\zeta_2 - \zeta_1)^{2l} \left[\frac{(-1)^{2l+2}(2l+2)!}{(\zeta_1)^{2l+3}} \int_{\zeta_1}^{\zeta_2} J\big(G_k(\cdot,s)\big) \Theta_l \left(\frac{\zeta_2 - s}{\zeta_2 - \zeta_1} \right) ds + \frac{(-1)^{2l+2}(2l+2)!}{(\zeta_1)^{2l+3}} \int_{\zeta_1}^{\zeta_2} J\big(G_k(\cdot,s)\big) \Theta_l \left(\frac{s - \zeta_1}{\zeta_2 - \zeta_1} \right) ds \right], \tag{49}$$

where

$$J_s(\cdot) = \sum_{\varrho=1}^{m} t_\varrho \log(w_\varrho) + \tilde{\mathcal{S}} - \log\left(\sum_{\varrho=1}^{m} w_\varrho \right) - \sum_{\rho=1}^{n} k_\rho \log(r_\rho) + \mathcal{S} + \log\left(\sum_{\rho=1}^{n} r_\rho \right), \tag{50}$$

and for fixed $k = 1, 2$, $J(G_k(\cdot,s))$ *is the same as defined in* (46).

Proof The function $f : x \to \log(x)$ is $(2p+1)$-convex for odd p $(p > 1)$ and base of log is greater than 1. Therefore we use $f = \log(x)$ in (45) to get (49), where $\mathcal{S}$ is defined in (48) and

$$\tilde{\mathcal{S}} = -\sum_{\varrho=1}^{m} t_\varrho \log(t_\varrho).$$

□

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