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Some generalized Volterra–Fredholm type dynamical integral inequalities in two independent variables on time scale pairs

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Abstract

In this paper, we study some new Volterra–Fredholm type dynamical integral inequalities in two independent variables on time scale pairs, which provide explicit bounds on unknown functions. These inequalities generalize and extend some known inequalities and can be used as effective tools in the qualitative theory of certain classes of partial dynamic equations on time scales. Finally, an example is provided to illustrate the usefulness of our result.

Keywords: Time scale; Dynamical integral inequality; Volterra–Fredholm type; Two independent variables

1 Introduction

Beginning in 1988, a seminal paper by Stefan Hilger [1] initiated a theory capable of containing both continuous and discrete analysis in a consistent way. Since then, the theory has attracted wide attention. As one of the most fundamental objects, dynamic equations on time scales has been extensively investigated in recent years, we refer the reader to the books [2, 3] and to the papers [4–24] and the references therein.

As we all know, inequalities are a powerful tool in the study of qualitative properties of solutions of differential, integral, and difference equations, and so on. During the last few years, a lot of dynamic inequalities have been extended by many authors. See [25–37]. For example, Anderson [28] considered the following nonlinear integral inequality in two independent variables on time scale pairs:

$$u^p(x, y) \leq a(x, y) + b(x, y) \int_{x_0}^x \int_y^\infty [c(s, t)u^q(s, t) + d(s, t)u^r(s, t) + e(s, t)] \nabla t \Delta s.$$

Ferreira and Torres [36] studied the following nonlinear integral inequality in two independent variables on time scale pairs:

$$u^p(x, y) \leq a(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y c(x, y, s, t)u^q(s, t) \Delta t \Delta s.$$

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Volterra–Fredholm-type integral inequality is an important integral inequality, which contains a definite integral of the unknown function, and has been given much attention by many authors, see [38–46] and the references therein. For example, Feng et al. [38] studied the following Volterra–Fredholm-type finite difference inequality:

$$\begin{aligned} u^p(m, n) &\leq a(m, n) \\ &+ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c(s, t, m, n) u^q(s, t) + \sum_{\xi=m_0}^s \sum_{\eta=n_0}^t d(\xi, \eta, m, n) u^r(\xi, \eta) \right] \\ &+ \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f(s, t, m, n) u^l(s, t) + \sum_{\xi=m_0}^s \sum_{\eta=n_0}^t g(\xi, \eta, m, n) u^k(\xi, \eta) \right]. \end{aligned}$$

Meng and Gu [46] considered the following nonlinear Volterra–Fredholm-type dynamic integral inequality on time scales:

$$\begin{aligned} u(x) &\leq k + \int_{x_0}^x f_1(s) w(u(s)) \Delta s + \int_{x_0}^x f_2(s) \int_{x_0}^s f_3(\tau) w(u(\tau)) \Delta \tau \Delta s \\ &+ \int_{x_0}^\alpha f_1(s) w(u(s)) \Delta s + \int_{x_0}^\alpha f_2(s) \int_{x_0}^s f_3(\tau) w(u(\tau)) \Delta \tau \Delta s. \end{aligned}$$

But to our knowledge, Volterra–Fredholm-type dynamic integral inequalities in two independent variables on time scale pairs have been paid little attention in the literature so far. Motivated by the work done in [36, 38, 46], in this paper, we establish some generalized Volterra–Fredholm-type dynamic integral inequalities in two independent variables on time scale pairs, which not only extend some existing results in the literature, unify some known continuous and discrete inequalities, but also may be applied to the analysis of certain classes of partial dynamic equations on time scales.

2 Preliminaries

In what follows, we assume that \mathbf{T}_1 and \mathbf{T}_2 are two time scales with at least two points, $x_0, \alpha \in \mathbf{T}_1$, $y_0, \beta \in \widetilde{\mathbf{T}}_2$, $\alpha > x_0$, $\beta > y_0$, $\widetilde{\mathbf{T}}_1 = [x_0, \infty) \cap \mathbf{T}_1$, $\widetilde{\mathbf{T}}_2 = [y_0, \infty) \cap \mathbf{T}_2$, $I_1 = [x_0, \alpha] \cap \mathbf{T}_1$, $I_2 = [y_0, \beta] \cap \mathbf{T}_2$, $D = \{(x, y, s, t) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2 \times \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2 : x_0 \leq s \leq x, y_0 \leq t \leq y\}$, $E = \{(x, y, s, t) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2 \times I_1 \times I_2\}$. \mathcal{R} denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{P \in \mathcal{R}, 1 + \mu(t)P(t) > 0, t \in \mathbf{T}\}$. \mathbf{R} denotes the set of real numbers, $\mathbf{R}_+ = [0, \infty)$, while \mathbf{Z} denotes the set of integers.

Lemma 2.1 ([33]) *Let $m > 0$, $n > 0$, $p > 0$, $\alpha > 0$ and $\beta > 0$ be given, then for each $x \geq 0$,*

$$mx^\alpha - nx^\beta \leq \frac{m(\beta - \alpha)}{\beta - p} \left(\frac{(\beta - p)n}{(\alpha - p)m} \right)^{(\alpha - p)/(\alpha - \beta)} x^p$$

holds for the cases when $0 < p < \alpha < \beta$ or $0 < \beta < \alpha < p$.

Lemma 2.2 ([47]) *Assume that $x \geq 0$, $p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$,*

$$x^{q/p} \leq \frac{q}{p} K^{(q-p)/p} x + \frac{p-q}{p} K^{q/p}.$$

Lemma 2.3 ([2, Theorem 6.1]) Suppose y and f are rd-continuous functions and $p \in \mathcal{R}^+$. Then

$$y^\Delta(t) \leq p(t)y(t) + f(t), \quad \text{for all } t \in \mathbf{T}$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau, \quad \text{for all } t \in \mathbf{T}.$$

Lemma 2.4 ([36]) Let $u, a, f \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \mathbf{R}_+)$, with a and f nondecreasing in each of the variables and $g \in C(D, \mathbf{R}_+)$ be nondecreasing in x and y . If

$$u(x, y) \leq a(x, y) + f(x, y) \int_{x_0}^x \int_{y_0}^y g(x, y, s, t)u(s, t)\Delta t\Delta s, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$$

then

$$u(x, y) \leq a(x, y)e_{p(x, y, \cdot)}(x, x_0), \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$$

$$\text{where } p(x, y, s) = \int_{y_0}^y f(x, y)g(x, y, s, t)\Delta t.$$

Lemma 2.5 Let $u, c, d \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \mathbf{R}_+)$ and $k \geq 0$ be a constant. If

$$u(x, y) \leq k + \int_{x_0}^x \int_{y_0}^y \left[c(s, t)u(s, \sigma(t)) + d(s, t)u(s, t) \right] \Delta t\Delta s, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (1)$$

then

$$u(x, y) \leq ke_{p(\cdot, y)}(x, x_0), \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (2)$$

where

$$p(x, y) = \int_{y_0}^y h(x, t)\Delta t, \quad \text{and} \quad (3)$$

$$h(x, y) = c(x, y) + d(x, y). \quad (4)$$

Proof For an arbitrary $\varepsilon > 0$, denote

$$z(x, y) = k + \varepsilon + \int_{x_0}^x \int_{y_0}^y \left[c(s, t)u(s, \sigma(t)) + d(s, t)u(s, t) \right] \Delta t\Delta s. \quad (5)$$

From the assumptions, we have z is positive and nondecreasing in each of the variables. By (1) and (5), we have that

$$u(x, y) \leq z(x, y), \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2.$$

Delta differentiating with respect to the first variable and then with respect to the second, we obtain

$$\begin{aligned} \frac{\partial}{\Delta_2 y} \left(\frac{\partial z(x,y)}{\Delta_1 x} \right) &= c(x,y)u(x,\sigma(y)) + d(x,y)u(x,y) \\ &\leq c(x,y)z(x,\sigma(y)) + d(x,y)z(x,y), \quad (x,y) \in \tilde{T}_1 \times \tilde{T}_2. \end{aligned} \quad (6)$$

From (6), we get

$$\begin{aligned} \frac{z(x,y) \frac{\partial}{\Delta_2 y} \left(\frac{\partial z(x,y)}{\Delta_1 x} \right)}{z(x,y)z(x,\sigma(y))} &\leq \left[c(x,y) + d(x,y) \frac{z(x,y)}{z(x,\sigma(y))} \right] \\ &\leq \left[c(x,y) + d(x,y) \right] \\ &= h(x,y), \end{aligned}$$

where $h(x,y)$ is defined as in (4). Hence,

$$\frac{z(x,y) \frac{\partial}{\Delta_2 y} \left(\frac{\partial z(x,y)}{\Delta_1 x} \right)}{z(x,y)z(x,\sigma(y))} - \frac{\frac{\partial z(x,y)}{\Delta_1 x} \frac{\partial z(x,y)}{\Delta_2 y}}{z(x,y)z(x,\sigma(y))} \leq h(x,y), \quad (7)$$

i.e.,

$$\frac{\partial}{\Delta_2 y} \left(\frac{\frac{\partial z(x,y)}{\Delta_1 x}}{z(x,y)} \right) \leq h(x,y).$$

Delta integrating with respect to the second variable from y_0 to y and noting that $\frac{\partial z(x,y)}{\Delta_1 x}|_{(x,y_0)} = 0$, we have

$$\frac{\frac{\partial z(x,y)}{\Delta_1 x}}{z(x,y)} \leq \int_{y_0}^y h(x,t) \Delta t,$$

that is,

$$\frac{\partial z(x,y)}{\Delta_1 x} \leq z(x,y) \int_{y_0}^y h(x,t) \Delta t. \quad (8)$$

By (3) and (8) we get

$$\frac{\partial z(x,y)}{\Delta_1 x} \leq p(x,y)z(x,y).$$

From Lemma 2.3 and $z(x_0, y) = k + \varepsilon$, we obtain

$$z(x,y) \leq (k + \varepsilon) e_{p(\cdot,y)}(x, x_0). \quad (9)$$

Noting that $u(x,y) \leq z(x,y)$ and ε is arbitrary, it follows (2). This completes the proof. \square

Lemma 2.6 Let $u, c, d \in C(\widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2, \mathbf{R}_+)$ and $k \geq 0$ be a constant. If

$$u(x, y) \leq k + \int_{x_0}^x \int_{y_0}^y \left[c(s, t)u(\sigma(s), t) + d(s, t)u(s, t) \right] \Delta t \Delta s, \quad (x, y) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2,$$

then

$$u(x, y) \leq ke_{p(x, \cdot)}(y, y_0), \quad (x, y) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2,$$

where

$$p(x, y) = \int_{x_0}^x h(s, y) \Delta s, \quad \text{and}$$

$$h(x, y) = c(x, y) + d(x, y).$$

The proof of the Lemma is similar to the proof in Lemma 2.5, and therefore is omitted.

3 Main results

Theorem 3.1 Let $u, a, b, h \in C(\widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2, \mathbf{R}_+)$, with b and h nondecreasing in each variable, $c, d \in C(D, R_+)$ and let $f, g \in C(E, R_+)$ be nondecreasing in x and y . Assume p, q, r, m and n are nonnegative constants with $p \geq q, p \geq r, p \geq m, p \geq n, p \neq 0$. Suppose that u satisfies the following inequality:

$$\begin{aligned} u^p(x, y) &\leq a(x, y) \\ &+ b(x, y) \int_{x_0}^x \int_{y_0}^y \left[c(x, y, s, t)u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta)u^r(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[f(x, y, s, t)u^m(s, t) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta)u^n(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s, \\ (x, y) &\in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2. \end{aligned} \tag{10}$$

If there exist positive constants K_1 and K_2 such that

$$\begin{aligned} \lambda &:= \int_{x_0}^\alpha \int_{y_0}^\beta \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) e_{R(s, t, \cdot)}(s, x_0) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) e_{R(\tau, \eta, \cdot)}(\tau, x_0) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &< \frac{1}{h(\alpha, \beta)}, \end{aligned} \tag{11}$$

then for arbitrary positive constants K_3 and K_4 ,

$$u(x, y) \leq \left[a(x, y) + \frac{A(\alpha, \beta)}{1 - \lambda h(\alpha, \beta)} e_{R(x, y, \cdot)}(x, x_0) \right]^{1/p}, \quad (x, y) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2, \tag{12}$$

where

$$\begin{aligned} A(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, t) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} a(\tau, \eta) + \frac{p-r}{p} K_4^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s, \end{aligned} \quad (13)$$

$$R(x, y, s) = \int_{y_0}^y b(x, y) F(x, y, s, t) \Delta t, \quad \text{and} \quad (14)$$

$$F(x, y, s, t) = \frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) \Delta \eta \Delta \tau. \quad (15)$$

Proof Denote

$$\begin{aligned} z(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y \left[c(x, y, s, t) u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) u^r(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[f(x, y, s, t) u^m(s, t) + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s, \\ (x, y) &\in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2. \end{aligned} \quad (16)$$

Then z is nondecreasing in each variable on $\widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2$. From (10) and (16), we get

$$u(x, y) \leq [a(x, y) + z(x, y)]^{1/p}, \quad (x, y) \in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2. \quad (17)$$

By (16) and (17), we obtain

$$\begin{aligned} z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) [a(s, t) + z(s, t)]^{q/p} \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) [a(\tau, \eta) + z(\tau, \eta)]^{r/p} \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) [a(s, t) + z(s, t)]^{m/p} \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) [a(\tau, \eta) + z(\tau, \eta)]^{n/p} \Delta \eta \Delta \tau \right\} \Delta t \Delta s, \\ (x, y) &\in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2. \end{aligned} \quad (18)$$

Then for K_1, K_2 satisfying (11) and arbitrary $K_3, K_4 > 0$, it follows from Lemma 2.2 that

$$[a(x, y) + z(x, y)]^{q/p} \leq \frac{q}{p} K_3^{(q-p)/p} [a(x, y) + z(x, y)] + \frac{p-q}{p} K_3^{q/p}, \quad (19)$$

$$[a(x, y) + z(x, y)]^{r/p} \leq \frac{r}{p} K_4^{(r-p)/p} [a(x, y) + z(x, y)] + \frac{p-r}{p} K_4^{r/p}, \quad (20)$$

$$[a(x, y) + z(x, y)]^{m/p} \leq \frac{m}{p} K_1^{(m-p)/p} [a(x, y) + z(x, y)] + \frac{p-m}{p} K_1^{m/p}, \quad (21)$$

$$[a(x, y) + z(x, y)]^{n/p} \leq \frac{n}{p} K_2^{(n-p)/p} [a(x, y) + z(x, y)] + \frac{p-n}{p} K_2^{n/p}. \quad (22)$$

According to (18)–(22), we have

$$\begin{aligned} z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} [a(s, t) + z(s, t)] + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} [a(\tau, \eta) + z(\tau, \eta)] + \frac{p-r}{p} K_4^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} [a(s, t) + z(s, t)] + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} [a(\tau, \eta) + z(\tau, \eta)] + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, t) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} a(\tau, \eta) + \frac{p-r}{p} K_4^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &= A(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &\leq A(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_1^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\ &\quad \left. + z(s, t) \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \end{aligned}$$

$$= A(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y F(x, y, s, t) z(s, t) \Delta t \Delta s, \\ (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (23)$$

where $A(x, y)$ is defined in (13), and

$$B(x, y) = h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\ \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s.$$

From (23) and Lemma 2.4, we have

$$z(x, y) \leq (A(x, y) + B(x, y)) e_{R(x, y, \cdot)}(x, x_0), \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (24)$$

where $R(x, y)$ is defined in (14). By (24) and since A, B are nondecreasing in each variable on $\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2$, we obtain

$$z(x, y) \leq (A(x, y) + B(x, y)) e_{R(x, y, \cdot)}(x, x_0) \\ \leq (A(\alpha, \beta) + B(\alpha, \beta)) e_{R(x, y, \cdot)}(x, x_0) \\ = C(\alpha, \beta) e_{R(x, y, \cdot)}(x, x_0), \quad (25)$$

where $C(x, y) = A(x, y) + B(x, y)$. From the definitions of B, C, λ and (25), we obtain

$$C(\alpha, \beta) = A(\alpha, \beta) + B(\alpha, \beta) \\ = A(\alpha, \beta) + h(\alpha, \beta) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) z(s, t) \right. \\ \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ \leq A(\alpha, \beta) + h(\alpha, \beta) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) C(\alpha, \beta) e_{R(s, t, \cdot)}(s, x_0) \right. \\ \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) C(\alpha, \beta) e_{R(\tau, \eta, \cdot)}(\tau, x_0) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ = A(\alpha, \beta) + C(\alpha, \beta) h(\alpha, \beta) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) e_{R(s, t, \cdot)}(s, x_0) \right. \\ \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) e_{R(\tau, \eta, \cdot)}(\tau, x_0) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ = A(\alpha, \beta) + \lambda C(\alpha, \beta) h(\alpha, \beta).$$

So we get

$$C(\alpha, \beta) \leq \frac{A(\alpha, \beta)}{1 - \lambda h(\alpha, \beta)}. \quad (26)$$

Noting (17), (25), and (26), we get the desired inequality (12). This completes the proof. \square

Remark 3.1 If we take $T = \mathbf{N}$, $b(x, y) = h(x, y) \equiv 1$, then Theorem 3.1 reduces to [38, Theorem 5]. If we take $T = \mathbf{N}$, $b(x, y) = h(x, y) \equiv 1$, $c(x, y, s, t) = c(s, t)$, $f(x, y, s, t) = c(s, t)$, $d(x, y, s, t) = g(x, y, s, t) \equiv 0$, then Theorem 3.1 reduces to [39, Theorem 2.1].

Theorem 3.2 Assume $l \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \mathbf{R}_+)$ and $b \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, (0, \infty))$ are nondecreasing in each variable, $v \in C(D, (0, \infty))$ is nondecreasing in x and y , $w \in C(D, (0, \infty))$ is nonincreasing in x and y . Assume $u, a, c, d, f, g, h, p, q, r, m$ and n are defined as in Theorem 3.1; while k and θ are nonnegative constants with $0 < p < k < \theta$ or $0 < \theta < k < p$. Suppose that u satisfies the following inequality:

$$\begin{aligned} u^p(x, y) &\leq a(x, y) \\ &+ b(x, y) \int_{x_0}^x \int_{y_0}^y \left[c(x, y, s, t) u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) u^r(\tau, \eta) \Delta\eta \Delta\tau \right] \Delta t \Delta s \\ &+ l(x, y) \int_{x_0}^x \int_{y_0}^y [v(x, y, s, t) u^k(s, t) - w(x, y, s, t) u^\theta(s, t)] \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[f(x, y, s, t) u^m(s, t) + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta\eta \Delta\tau \right] \Delta t \Delta s, \\ (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned} \quad (27)$$

If there exist positive constants K_1 and K_2 such that

$$\begin{aligned} \tilde{\lambda} &:= \int_{x_0}^\alpha \int_{y_0}^\beta \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) e_{\tilde{R}(s,t,\cdot)}(s, x_0) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) e_{\tilde{R}(\tau,\eta,\cdot)}(\tau, x_0) \Delta\eta \Delta\tau \right] \Delta t \Delta s \\ &< \frac{1}{h(\alpha, \beta)}, \end{aligned} \quad (28)$$

then for arbitrary positive constants K_3 and K_4 ,

$$u(x, y) \leq \left[a(x, y) + \frac{\tilde{A}(\alpha, \beta)}{1 - \tilde{\lambda} h(\alpha, \beta)} e_{\tilde{R}(x,y,\cdot)}(x, x_0) \right]^{1/p}, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (29)$$

where

$$\begin{aligned} \tilde{A}(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, t) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} a(\tau, \eta) + \frac{p-r}{p} K_4^{r/p} \right] \Delta\eta \Delta\tau \right\} \Delta t \Delta s \\ &+ l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) a(s, t) \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) a(\tau, \eta) \Delta\eta \Delta\tau \right\} \Delta t \Delta s \end{aligned}$$

$$+ \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \Big\} \Delta t \Delta s, \quad (30)$$

$$\tilde{R}(x, y, s) = \int_{y_0}^y b(x, y) \tilde{F}(x, y, s, t) \Delta t, \quad (31)$$

$$\begin{aligned} \tilde{F}(x, y, s, t) &= \frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) + \frac{l(x, y)}{b(x, y)} \varphi(x, y, s, t) \\ &\quad + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) \Delta \eta \Delta \tau, \quad \text{and} \end{aligned} \quad (32)$$

$$\varphi(x, y, s, t) = \frac{\nu(x, y, s, t)(\theta - k)}{\theta - p} \left(\frac{(\theta - p)w(x, y, s, t)}{(k - p)v(x, y, s, t)} \right)^{(k-p)/(k-\theta)}. \quad (33)$$

Proof From Lemma 2.1 and (27), we have

$$\begin{aligned} u^p(x, y) &\leq a(x, y) \\ &\quad + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[c(x, y, s, t) u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) u^r(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &\quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) u^p(s, t) \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[f(x, y, s, t) u^m(s, t) + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s, \\ (x, y) &\in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned} \quad (34)$$

Denote

$$\begin{aligned} z(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y \left[c(x, y, s, t) u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) u^r(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &\quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) u^p(s, t) \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left[f(x, y, s, t) u^m(s, t) + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s, \\ (x, y) &\in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned} \quad (35)$$

From the assumptions on v and w , we have that φ is nondecreasing in x and y , then z is nondecreasing in each variable on $\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2$. From (34) and (35), we get

$$u(x, y) \leq [a(x, y) + z(x, y)]^{1/p}, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \quad (36)$$

By (35) and (36), we obtain

$$\begin{aligned}
 z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) [a(s, t) + z(s, t)]^{q/p} \right. \\
 &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) [a(\tau, \eta) + z(\tau, \eta)]^{r/p} \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
 &\quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) [a(s, t) + z(s, t)] \Delta t \Delta s \\
 &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) [a(s, t) + z(s, t)]^{m/p} \right. \\
 &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) [a(\tau, \eta) + z(\tau, \eta)]^{n/p} \Delta \eta \Delta \tau \right\} \Delta t \Delta s, \\
 (x, y) &\in \widetilde{\mathbf{T}}_1 \times \widetilde{\mathbf{T}}_2. \tag{37}
 \end{aligned}$$

For K_1, K_2 satisfying (28) and arbitrary $K_3, K_4 > 0$, it follows from Lemma 2.2 that

$$[a(x, y) + z(x, y)]^{q/p} \leq \frac{q}{p} K_3^{(q-p)/p} [a(x, y) + z(x, y)] + \frac{p-q}{p} K_3^{q/p}, \tag{38}$$

$$[a(x, y) + z(x, y)]^{r/p} \leq \frac{r}{p} K_4^{(r-p)/p} [a(x, y) + z(x, y)] + \frac{p-r}{p} K_4^{r/p}, \tag{39}$$

$$[a(x, y) + z(x, y)]^{m/p} \leq \frac{m}{p} K_1^{(m-p)/p} [a(x, y) + z(x, y)] + \frac{p-m}{p} K_1^{q/p}, \tag{40}$$

$$[a(x, y) + z(x, y)]^{n/p} \leq \frac{n}{p} K_2^{(n-p)/p} [a(x, y) + z(x, y)] + \frac{p-n}{p} K_2^{r/p}. \tag{41}$$

According to (37)–(41), we have

$$\begin{aligned}
 z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} [a(s, t) + z(s, t)] + \frac{p-q}{p} K_3^{q/p} \right] \right. \\
 &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} [a(\tau, \eta) + z(\tau, \eta)] + \frac{p-r}{p} K_4^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
 &\quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) [a(s, t) + z(s, t)] \Delta t \Delta s \\
 &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} [a(s, t) + z(s, t)] + \frac{p-m}{p} K_1^{q/p} \right] \right. \\
 &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} [a(\tau, \eta) + z(\tau, \eta)] + \frac{p-n}{p} K_2^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
 &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, t) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\
 &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(x, y, \tau, \eta) \left[\frac{r}{p} K_4^{(r-p)/p} a(\tau, \eta) + \frac{p-r}{p} K_4^{r/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
 &\quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) a(s, t) \Delta t \Delta s
 \end{aligned}$$

$$\begin{aligned}
& + h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
& \quad + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\
& \quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) z(s, t) \Delta t \Delta s \\
& \quad + h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\
& = \tilde{A}(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\
& \quad + l(x, y) \int_{x_0}^x \int_{y_0}^y \varphi(x, y, s, t) z(s, t) \Delta t \Delta s \\
& = \tilde{A}(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\
& \quad + b(x, y) \int_{x_0}^x \int_{y_0}^y \frac{l(x, y)}{b(x, y)} \varphi(x, y, s, t) z(s, t) \Delta t \Delta s \\
& \leq \tilde{A}(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) + \frac{l(x, y)}{b(x, y)} \varphi(x, y, s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{r}{p} K_4^{(r-p)/p} d(x, y, \tau, \eta) \Delta \eta \Delta \tau \right] z(s, t) \Delta t \Delta s \\
& = \tilde{A}(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \tilde{F}(x, y, s, t) z(s, t) \Delta t \Delta s,
\end{aligned}$$

$$(x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$$

where \tilde{A} and \tilde{F} are defined in (30) and (32),

$$\begin{aligned}
B(x, y) &= h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\
&\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s.
\end{aligned}$$

The rest of the argument is similar to that of Theorem 3.1, and therefore is omitted. This completes the proof. \square

Theorem 3.3 Assume that $u, a \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \mathbf{R}_+)$, while $b, c, d, f, g, h, p, q, r, m$ and n are defined as in Theorem 3.1. Suppose that u satisfies the following inequality:

$$\begin{aligned} u^p(x, y) &\leq a(x, y) \\ &+ b(x, y) \int_{x_0}^x \int_{y_0}^y [c(x, y, s, t)u^q(s, \sigma(t)) + d(x, y, s, t)u^r(s, t)] \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta [f(x, y, s, t)u^m(s, t) \\ &+ \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta)u^n(\tau, \eta) \Delta \eta \Delta \tau] \Delta t \Delta s, \\ (x, y) &\in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned} \quad (42)$$

If there exist positive constants K_1 and K_2 such that

$$\begin{aligned} \xi := & \int_{x_0}^\alpha \int_{y_0}^\beta \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) e_{Q(\cdot, t)}(s, x_0) \right. \\ & + \left. \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) e_{Q(\cdot, \eta)}(\tau, x_0) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ & < \frac{1}{h(\alpha, \beta)}, \end{aligned} \quad (43)$$

then for arbitrary positive constants K_3 and K_4 ,

$$u(x, y) \leq \left[a(x, y) + \frac{\tilde{A}(\alpha, \beta)}{1 - \xi h(\alpha, \beta)} e_{Q(\cdot, y)}(x, x_0) \right]^{1/p}, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \quad (44)$$

where

$$\begin{aligned} \tilde{A}(x, y) = & b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, \sigma(t)) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ & + d(x, y, s, t) \left[\frac{r}{p} K_4^{(r-p)/p} a(s, t) + \frac{p-r}{p} K_4^{r/p} \right] \} \Delta t \Delta s \\ & + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ & + \left. \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) \right. \right. \\ & + \left. \left. \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s, \end{aligned} \quad (45)$$

$$Q(x, y) = \int_{y_0}^y h(\alpha, \beta, x, t) \Delta t, \quad \text{and} \quad (46)$$

$$h(\alpha, \beta, s, t) = \frac{q}{p} K_3^{(q-p)/p} b(\alpha, \beta) c(\alpha, \beta, s, t) + \frac{r}{p} K_4^{(r-p)/p} b(\alpha, \beta) d(\alpha, \beta, s, t). \quad (47)$$

Proof Denote

$$\begin{aligned}
z(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y [c(x, y, s, t) u^q(s, \sigma(t)) + d(x, y, s, t) u^r(s, t)] \Delta t \Delta s \\
&\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta [f(x, y, s, t) u^m(s, t) \\
&\quad + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta \eta \Delta \tau] \Delta t \Delta s. \tag{48}
\end{aligned}$$

Then z is nondecreasing in each variable on $\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2$. From (42) and (48), we get

$$u(x, y) \leq [a(x, y) + z(x, y)]^{1/p}, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \tag{49}$$

By (48) and (49), we obtain

$$\begin{aligned}
z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \{c(x, y, s, t)[a(s, \sigma(t)) + z(s, \sigma(t))]^{q/p} \\
&\quad + d(x, y, s, t)[a(s, t) + z(s, t)]^{r/p} \Delta \eta \Delta \tau\} \Delta t \Delta s \\
&\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \{f(x, y, s, t)[a(s, t) + z(s, t)]^{m/p} \\
&\quad + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta)[a(\tau, \eta) + z(\tau, \eta)]^{n/p} \Delta \eta \Delta \tau\} \Delta t \Delta s, \\
(x, y) &\in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \tag{50}
\end{aligned}$$

For K_1, K_2 satisfying (43) and arbitrary $K_3, K_4 > 0$, it follows from Lemma 2.2 that

$$[a(x, \sigma(y)) + z(x, \sigma(y))]^{q/p} \leq \frac{q}{p} K_3^{(q-p)/p} [a(x, \sigma(y)) + z(x, \sigma(y))] + \frac{p-q}{p} K_3^{q/p}, \tag{51}$$

$$[a(x, y) + z(x, y)]^{r/p} \leq \frac{r}{p} K_4^{(r-p)/p} [a(x, y) + z(x, y)] + \frac{p-r}{p} K_4^{r/p}, \tag{52}$$

$$[a(x, y) + z(x, y)]^{m/p} \leq \frac{m}{p} K_1^{(m-p)/p} [a(x, y) + z(x, y)] + \frac{p-m}{p} K_1^{m/p}, \tag{53}$$

$$[a(x, y) + z(x, y)]^{n/p} \leq \frac{n}{p} K_2^{(n-p)/p} [a(x, y) + z(x, y)] + \frac{p-n}{p} K_2^{n/p}. \tag{54}$$

According to (50)–(54), we have

$$\begin{aligned}
z(x, y) &\leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} [a(s, \sigma(t)) + z(s, \sigma(t))] \right. \right. \\
&\quad \left. \left. + \frac{p-q}{p} K_3^{q/p} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + d(x, y, s, t) \left[\frac{r}{p} K_4^{(r-p)/p} [a(s, t) + z(s, t)] + \frac{p-r}{p} K_4^{r/p} \right] \Delta t \Delta s \\
& + h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} [a(s, t) + z(s, t)] + \frac{p-m}{p} K_1^{m/p} \right] \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} [a(\tau, \eta) + z(\tau, \eta)] + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
& \leq b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(s, \sigma(t)) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\
& \quad \left. + d(x, y, s, t) \left[\frac{r}{p} K_4^{(r-p)/p} a(s, t) + \frac{p-r}{p} K_4^{r/p} \right] \right\} \Delta t \Delta s \\
& + h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\
& + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, \sigma(t)) \right. \\
& \quad \left. + \frac{r}{p} K_4^{(r-p)/p} d(x, y, s, t) z(s, t) \right] \Delta t \Delta s \\
& + h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\
& \quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\
& = \tilde{A}(x, y) + B(x, y) + b(x, y) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(x, y, s, t) z(s, \sigma(t)) \right. \\
& \quad \left. + \frac{r}{p} K_4^{(r-p)/p} d(x, y, s, t) z(s, t) \right] \Delta t \Delta s \\
& \leq \tilde{A}(\alpha, \beta) + B(\alpha, \beta) + b(\alpha, \beta) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(\alpha, \beta, s, t) z(s, \sigma(t)) \right. \\
& \quad \left. + \frac{r}{p} K_4^{(r-p)/p} d(\alpha, \beta, s, t) z(s, t) \right] \Delta t \Delta s \\
& = \tilde{C}(\alpha, \beta) + b(\alpha, \beta) \int_{x_0}^x \int_{y_0}^y \left[\frac{q}{p} K_3^{(q-p)/p} c(\alpha, \beta, s, t) z(s, \sigma(t)) \right. \\
& \quad \left. + \frac{r}{p} K_4^{(r-p)/p} d(\alpha, \beta, s, t) z(s, t) \right] \Delta t \Delta s,
\end{aligned}$$

$(x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$

where $\tilde{C}(x, y) = \tilde{A}(x, y) + B(x, y)$, \tilde{A} is defined in (45), and

$$\begin{aligned}
B(x, y) &= h(x, y) \int_{x_0}^{\alpha} \int_{y_0}^{\beta} \left[\frac{m}{p} K_1^{(m-p)/p} f(x, y, s, t) z(s, t) \right. \\
&\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(x, y, \tau, \eta) z(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s.
\end{aligned}$$

From Lemma 2.5, we have

$$z(x, y) \leq \tilde{C}(\alpha, \beta) e_{Q(\cdot, y)}(x, x_0), \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$$

where Q is defined in (46). The rest of the proof is similar to that of Theorem 3.1, and therefore is omitted. This completes the proof. \square

Theorem 3.4 Assume $u, a \in C(\tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2, \mathbf{R}_+)$, while $b, c, d, f, g, h, p, q, r, m$ and n are defined as in Theorem 3.1. Suppose that u satisfies the following inequality:

$$\begin{aligned} u^p(x, y) &\leq a(x, y) \\ &+ b(x, y) \int_{x_0}^x \int_{y_0}^y [c(x, y, s, t) u^q(\sigma(s), t) + d(x, y, s, t) u^r(s, t)] \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta [f(x, y, s, t) u^m(s, t) \\ &+ \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) u^n(\tau, \eta) \Delta \eta \Delta \tau] \Delta t \Delta s, \\ (x, y) &\in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned}$$

If there exist positive constants K_1 and K_2 such that

$$\begin{aligned} \xi &:= \int_{x_0}^\alpha \int_{y_0}^\beta \left[\frac{m}{p} K_1^{(m-p)/p} f(\alpha, \beta, s, t) e_{\tilde{Q}(s, \cdot)}(t, y_0) \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t \frac{n}{p} K_2^{(n-p)/p} g(\alpha, \beta, \tau, \eta) e_{\tilde{Q}(\tau, \cdot)}(\eta, y_0) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &< \frac{1}{h(\alpha, \beta)}, \end{aligned}$$

then for arbitrary positive constants K_3 and K_4 ,

$$u(x, y) \leq \left[a(x, y) + \frac{\tilde{F}(\alpha, \beta)}{1 - \xi h(\alpha, \beta)} e_{\tilde{Q}(x, \cdot)}(y, y_0) \right]^{1/p}, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2,$$

where

$$\begin{aligned} \tilde{F}(x, y) &= b(x, y) \int_{x_0}^x \int_{y_0}^y \left\{ c(x, y, s, t) \left[\frac{q}{p} K_3^{(q-p)/p} a(\sigma(s), t) + \frac{p-q}{p} K_3^{q/p} \right] \right. \\ &\quad \left. + d(x, y, s, t) \left[\frac{r}{p} K_4^{(r-p)/p} a(s, t) + \frac{p-r}{p} K_4^{r/p} \right] \right\} \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta \left\{ f(x, y, s, t) \left[\frac{m}{p} K_1^{(m-p)/p} a(s, t) + \frac{p-m}{p} K_1^{m/p} \right] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t g(x, y, \tau, \eta) \left[\frac{n}{p} K_2^{(n-p)/p} a(\tau, \eta) + \frac{p-n}{p} K_2^{n/p} \right] \Delta \eta \Delta \tau \right\} \Delta t \Delta s, \end{aligned}$$

and

$$\tilde{Q}(x, y) = \int_{x_0}^x h(\alpha, \beta, s, y) \Delta s.$$

The proof of the theorem is similar to that of Theorem 3.3, and therefore is omitted.

4 Application

In this section, we will present an application for our results.

Example 1 Consider the following partial dynamic equation with positive and negative coefficients:

$$\begin{cases} \frac{\partial}{\partial y} \left(\frac{\partial u(x,y)}{\partial_1 x} \right) \\ = c(x,y)u^q(x,y) + v(x,y)u^k(x,y) \\ - w(x,y)u^\theta(x,y) + \int_{x_0}^x \int_{y_0}^y d(\tau, \eta)u^r(\tau, \eta) \Delta \eta \Delta \tau \\ + g(x,y) \int_{x_0}^\alpha \int_{y_0}^\beta f(s,t)u^m(s,t) \Delta t \Delta s, \quad (x,y) \in \tilde{T}_1 \times \tilde{T}_2, \\ u(x, y_0) = \phi(x), \quad u(x_0, y) = \psi(y), \quad u(x_0, y_0) = u_0, \end{cases} \quad (55)$$

where $u, c, d, f, g \in C(\tilde{T}_1 \times \tilde{T}_2, \mathbf{R}_+)$, $v, w \in C(\tilde{T}_1 \times \tilde{T}_2, (0, \infty))$, q, r and m are nonnegative constants with $1 \geq q, 1 \geq r, 1 \geq m$. Assume θ is a quotient of an even integer over odd integer, k is a nonnegative constant with $0 < 1 < k < \theta$ or $0 < \theta < k < 1$.

If there exists a positive constant K_1 such that

$$\begin{aligned} \tilde{\lambda} &:= \int_{x_0}^\alpha \int_{y_0}^\beta [mK_1^{m-1}f(s,t)e_{\tilde{R}(\cdot,t)}(s,x_0)] \Delta t \Delta s \\ &< \frac{1}{h(\alpha, \beta)}, \end{aligned} \quad (56)$$

then for arbitrary positive constants K_3 and K_4 ,

$$u(x, y) \leq a(x, y) + \frac{\tilde{A}(\alpha, \beta)}{1 - \tilde{\lambda}h(\alpha, \beta)} e_{\tilde{R}(\cdot,y)}(x, x_0), \quad (x, y) \in \tilde{T}_1 \times \tilde{T}_2, \quad (57)$$

where

$$a(x, y) = |\phi(x)| + |\psi(y)| + |u_0|, \quad (58)$$

$$\begin{aligned} \tilde{A}(x, y) &= \int_{x_0}^x \int_{y_0}^y \left\{ c(s, t) [qK_3^{q-1}a(s, t) + (1-q)K_3^q] \right. \\ &\quad \left. + \int_{x_0}^s \int_{y_0}^t d(\tau, \eta) [rK_4^{r-1}a(\tau, \eta) + (1-r)K_4^r] \Delta \eta \Delta \tau \right\} \Delta t \Delta s \\ &\quad + \int_{x_0}^x \int_{y_0}^y \varphi(s, t)a(s, t) \Delta t \Delta s \\ &\quad + h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta f(s, t) [mK_1^{m-1}a(s, t) + (1-m)K_1^m] \Delta t \Delta s, \end{aligned} \quad (59)$$

$$h(x, y) = \int_{x_0}^x \int_{y_0}^y g(s, t) \Delta t \Delta s, \quad (60)$$

$$\tilde{R}(x, y) = \int_{y_0}^y \tilde{F}(x, t) \Delta t, \quad (61)$$

$$\tilde{F}(x, y) = qK_3^{q-1}c(x, y) + \varphi(x, y) + \int_{x_0}^x \int_{y_0}^y rK_4^{r-1}d(\tau, \eta) \Delta \eta \Delta \tau, \quad \text{and} \quad (62)$$

$$\varphi(x, y) = \frac{(\theta - k)v(x, y)}{\theta - 1} \left(\frac{(\theta - 1)w(x, y)}{(k - 1)v(x, y)} \right)^{(k-1)/(k-\theta)}. \quad (63)$$

Proof Let $u(x, y)$ be a solution of (55). Then, it satisfies the following dynamical integral equation:

$$\begin{aligned} u(x, y) &= \phi(x) + \psi(y) - u_0 \\ &+ \int_{x_0}^x \int_{y_0}^y \left[c(s, t)u^q(s, t) + \int_{x_0}^s \int_{y_0}^t d(\tau, \eta)u^r(\tau, \eta) \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &+ \int_{x_0}^x \int_{y_0}^y [v(s, t)u^k(s, t) - w(s, t)u^\theta(s, t)] \Delta t \Delta s \\ &+ \int_{x_0}^x \int_{y_0}^y g(s, t) \Delta t \Delta s \int_{x_0}^\alpha \int_{y_0}^\beta f(s, t)u^m(s, t) \Delta t \Delta s. \end{aligned} \quad (64)$$

Then from (58), (60), and (63), we have

$$\begin{aligned} |u(x, y)| &\leq a(x, y) \\ &+ \int_{x_0}^x \int_{y_0}^y \left[c(s, t)|u(s, t)|^q + \int_{x_0}^s \int_{y_0}^t d(\tau, \eta)|u(\tau, \eta)|^r \Delta \eta \Delta \tau \right] \Delta t \Delta s \\ &+ \int_{x_0}^x \int_{y_0}^y [v(s, t)|u(s, t)|^k - w(s, t)|u(s, t)|^\theta] \Delta t \Delta s \\ &+ h(x, y) \int_{x_0}^\alpha \int_{y_0}^\beta f(s, t)|u(s, t)|^m \Delta t \Delta s, \quad (x, y) \in \tilde{\mathbf{T}}_1 \times \tilde{\mathbf{T}}_2. \end{aligned} \quad (65)$$

An application of Theorem 3.2 with $p = 1$, $b(x, y) = l(x, y) \equiv 1$, $c(x, y, s, t) = c(s, t)$, $d(x, y, s, t) = d(s, t)$, $v(x, y, s, t) = v(s, t)$, $w(x, y, s, t) = w(s, t)$, $f(x, y, s, t) = f(s, t)$ and $g(x, y, s, t) \equiv 0$ yields (57). \square

5 Conclusions

We have established several generalized Volterra–Fredholm-type dynamical integral inequalities in two independent variables on time scale pairs using an inequality introduced in [33]. As one can see, Theorems 3.1–3.4 generalize many known results in the literature. Theorem 3.2 can be applied to deal with the bounds of solutions of certain partial dynamic equation with positive and negative coefficients. Moreover, unlike some existing results in the literature (e.g., [28, 36, 37]), the integral inequalities considered in this paper involve the forward jump operator $\sigma(x)$ on a time scale, which results in difficulties in the estimation on the explicit bounds of the unknown function $u(x, y)$.

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Authors' contributions

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