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Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences

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Abstract

In this paper, we consider a generalization of Horadam sequence $\{w_n\}$ which is defined by the recurrence relation $w_n = \chi(n)w_{n-1} + cw_{n-2}$, where $\chi(n) = a$ if n is even, $\chi(n) = b$ if n is odd with arbitrary initial conditions w_0, w_1 and nonzero real numbers a, b and c. As a special case, by taking the initial conditions 0, 1 and 2, b we define the sequences $\{u_n\}$ and $\{v_n\}$, respectively. The main purpose of this study is to derive some basic properties of the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by using a matrix approach.

MSC: 11B39; 05A15

Keywords: Horadam sequence; Bi-periodic Fibonacci sequence; Matrix method

1 Introduction

A generalization of the Horadam sequence $\{w_n\}$ is defined by the recurrence relation

$$w_n = \chi(n)w_{n-1} + cw_{n-2}, \quad n \geq 2, \tag{1.1}$$

where $\chi(n) = a$ if n is even, $\chi(n) = b$ if n is odd with arbitrary initial conditions w_0, w_1 and nonzero real numbers a, b and c. They emerged as a generalization of the best known sequences in the literature, such as the Horadam sequence, Fibonacci–Lucas sequence, k-Fibonacci–k-Lucas sequence, Pell–Pell–Lucas sequence, Jacobsthal–Jacobsthal–Lucas sequence, etc. Here we call the sequence $\{w_n\}$ a *generalized bi-periodic Horadam sequence*. In particular, by taking the initial conditions 0, 1 and 2, b we call these sequences a *generalized bi-periodic Fibonacci sequence* $\{u_n\}$ and *generalized bi-periodic Lucas sequence* $\{v_n\}$, respectively.

Some modified versions of the sequence $\{w_n\}$ have been studied by several authors. For the case of $w_0 = 0$, $w_1 = 1$ and $c = 1$, the sequence $\{w_n\}$ reduces to the bi-periodic Fibonacci sequence, and some basic properties of this sequence can be found in [4, 10, 18]. Its companion sequence, the bi-periodic Lucas sequence, was studied in [2, 6, 14, 15]. For the case of $c = 1$, the sequence $\{w_n\}$ reduces to the bi-periodic Horadam sequence, and several properties of this sequence were given in [4, 13]. For a further generalization of the sequence $\{w_n\}$, we refer to [1, 9].

Table 1 Special cases of the sequence $\{w_n\}$

$\{w_n\}$	$\{w_n(w_0,w_1;a,b,c)\}$	generalized bi-periodic Horadam sequence
$\{u_n\}$	$\{w_n(0,1;a,b,c)\}$	generalized bi-periodic Fibonacci sequence
$\{v_n\}$	$\{w_n(2,b;a,b,c)\}$	generalized bi-periodic Lucas sequence
$\{q_n\}$	$\{w_n(0,1;a,b,1)\}$	bi-periodic Fibonacci sequence [4]
$\{p_n\}$	$\{w_n(2,a;b,a,1)\}$	bi-periodic Lucas sequence [2]
$\{W_n\}$	$\{w_n(w_0,w_1;a,b,1)\}$	bi-periodic Horadam sequence [4]
$\{H_n\}$	$\{w_n(w_0,w_1;p,p,-q)\}$	Horadam sequence [5]
$\{F_n\}$	$\{w_n(0,1;1,1,1)\}$	Fibonacci sequence
$\{L_n\}$	$\{w_n(2,1;1,1,1)\}$	Lucas sequence
$\{F_{k,n}\}$	$\{w_n(0,1;k,k,1)\}$	k-Fibonacci sequence
$\{L_{k,n}\}$	$\{w_n(0,k;k,k,1)\}$	k-Lucas sequence
$\{P_n\}$	$\{w_n(0,1;2,2,1)\}$	Pell sequence
$\{PL_n\}$	$\{w_n(2,2;2,2,1)\}$	Pell–Lucas sequence
$\{J_n\}$	$\{w_n(0,1;1,1,2)\}$	Jacobsthal sequence
$\{JL_n\}$	$\{w_n(2,1;1,1,2)\}$	Jacobsthal–Lucas sequence

On the other hand, the matrix method is extremely useful for obtaining some well-known Fibonacci properties, such as Cassini's identity, d'Ocagne's identity, and the convolution property. For the detailed history of the matrix technique see [3, 7, 8, 11, 16, 17]. The 2×2 matrix representation for the general case of the sequence $\{w_n\}$ was given firstly in [15], and several properties were obtained for the even indices terms of this sequence. Then, in [12], the author defined a new matrix identity for the bi-periodic Fibonacci sequence as follows:

$$S := \begin{bmatrix} ab & ab \\ 1 & 0 \end{bmatrix} \quad\Rightarrow\quad S^n = (ab)^{\lfloor \frac{n}{2}\rfloor} \begin{bmatrix} b^{\zeta(n)}q_{n+1} & a^{\zeta(n)}bq_n \\ a^{-\zeta(n+1)}q_n & b^{\zeta(n)}q_{n-1} \end{bmatrix}, \tag{1.2}$$

where $\{q_n\}$ is the bi-periodic Fibonacci sequence and $\zeta(n)$ is the parity function. By using this matrix identity, simple proofs of several identities of the bi-periodic Fibonacci and Lucas numbers were given. One of the main objectives of this study is to generalize the matrix identity (1.2) for the sequence $\{w_n\}$.

Similar to the notation of the classical Horadam sequence in [5], we can state several sequences in terms of the generalized bi-periodic Horadam sequence $\{w_n\} := \{w_n(w_0,w_1;a,b,c)\}$ in Table 1.

The outline of this paper as follows: in Sect. 2, inspired by the matrix identity (1.2), we give analogous matrix representations for the generalized bi-periodic Fibonacci and the generalized bi-periodic Lucas numbers. Then, we generalize the matrix identity (1.2) to the generalized bi-periodic Horadam numbers. Thus, one can develop many matrix identities by choosing appropriate initial values in our matrix formula. We state several properties of these numbers by using matrix approach which provides a very simple proof. Section 3 is devoted to obtaining more generalized expressions for the generalized bi-periodic Horadam numbers, by using the matrix method in [16].

2 Matrix representations for $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$

First, we define the matrix $U := \left[\begin{smallmatrix} ab & cb \\ a & 0 \end{smallmatrix}\right]$. For any nonnegative integer n, by using induction, we have

$$U^n = (ab)^{\lfloor \frac{n}{2}\rfloor} \begin{bmatrix} b^{\zeta(n)}u_{n+1} & cba^{-\zeta(n+1)}u_n \\ a^{\zeta(n)}u_n & cb^{\zeta(n)}u_{n-1} \end{bmatrix}, \tag{2.1}$$

where u_n is the nth generalized bi-periodic Fibonacci number. Since the matrix U is invertible, then

$$U^{-n} = \frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{(-abc)^n} \begin{bmatrix} cb^{\zeta(n)}u_{n-1} & -cba^{-\zeta(n+1)}u_n \\ -a^{\zeta(n)}u_n & b^{\zeta(n)}u_{n+1} \end{bmatrix}.$$

By using the matrix identity (2.1) and using a similar method to [12, Theorem 1], one can obtain the following results which give some basic properties of $\{u_n\}$. Note that the results (1)–(3) can be found in [18, Theorem 9], but here we obtain these identities by using matrix approach.

Lemma 1 *The sequence $\{u_n\}$ satisfies the following identities*:

(1) $(\frac{a}{b})^{\zeta(n)}u_n^2 - (\frac{a}{b})^{\zeta(n+1)}u_{n-1}u_{n+1} = \frac{a}{b}(-c)^{n-1}$,
(2) $(\frac{b}{a})^{\zeta(mn+n)}u_m u_{n+1} + (\frac{b}{a})^{\zeta(mn+m)}cu_n u_{m-1} = u_{n+m}$,
(3) $(\frac{b}{a})^{\zeta(mn+n)}u_n u_{m+1} - (\frac{b}{a})^{\zeta(mn+m)}u_m u_{n+1} = (-c)^m u_{n-m}$,
(4) $(\frac{b}{a})^{\zeta(mn+n)}u_m u_{n-m+1} + c(\frac{b}{a})^{\zeta(mn)}u_{m-1}u_{n-m} = u_n$.

Now we consider the matrix equality

$$K := \frac{1}{2}\begin{bmatrix} ab & \Delta \\ 1 & ab \end{bmatrix} \quad \Rightarrow \quad K^n = \frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{2}\begin{bmatrix} a^{\zeta(n)}v_n & \Delta a^{\zeta(n)-1}u_n \\ a^{\zeta(n)-1}u_n & a^{\zeta(n)}v_n \end{bmatrix}, \tag{2.2}$$

where $\Delta := a^2b^2 + 4abc \neq 0$. By using the method in [12, Theorem 4], one can obtain the following results which give some relations involving both the generalized bi-periodic Fibonacci and the generalized bi-periodic Lucas numbers.

Lemma 2 *The sequences $\{u_n\}$ and $\{v_n\}$ satisfy the following identities*:

(1) $v_n^2 - \frac{\Delta}{a^2}u_n^2 = 4(\frac{b}{a})^{\zeta(n)}(-c)^n$,
(2) $v_m v_n + \frac{\Delta}{a^2}u_m u_n = 2(\frac{b}{a})^{\zeta(n)\zeta(m)}v_{n+m}$,
(3) $u_m v_n + u_n v_m = 2(\frac{b}{a})^{\zeta(n)\zeta(m)}u_{n+m}$,
(4) $v_m v_n - \frac{\Delta}{a^2}u_m u_n = 2(-c)^m(\frac{a}{b})^{-\zeta(n)\zeta(m)}v_{n-m}$,
(5) $u_n v_m - u_m v_n = 2(-c)^m(\frac{a}{b})^{-\zeta(n)\zeta(m)}u_{n-m}$,
(6) $v_{n+m} + (-c)^m v_{n-m} = (\frac{a}{b})^{\zeta(n)\zeta(m)}v_m v_n$,
(7) $u_{n+m} + (-c)^m u_{n-m} = (\frac{a}{b})^{\zeta(n)\zeta(m)}u_n v_m$.

We define the matrix $H := K + abcK^{-1} = [\begin{smallmatrix} 0 & \Delta \\ 1 & 0 \end{smallmatrix}]$. It is clear that we have the matrix relation

$$K^n = \frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{2}\left(a^{\zeta(n)-1}u_n H + a^{\zeta(n)}v_n I\right). \tag{2.3}$$

Also from the relation

$$w_n = u_n w_1 + c\left(\frac{b}{a}\right)^{\zeta(n)}u_{n-1}w_0, \tag{2.4}$$

we have $v_n = bu_n + 2c(\frac{b}{a})^{\zeta(n)}u_{n-1}$. Then we have

$$K^n = (ab)^{\lfloor \frac{n}{2} \rfloor}\left(a^{\zeta(n)-1}u_n K + cb^{\zeta(n)}u_{n-1}I\right). \tag{2.5}$$

By using the matrix relations (2.3) and (2.5), we give Theorem 1 and Theorem 2, respectively.

Theorem 1 *Let* $D := 1 - a^{\zeta(m)}v_m + (ab)^{\zeta(m)}(-c)^m \neq 0$, *then*

$$\begin{aligned}
&\sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)-1}u_{mj+r}\\
&\quad=\frac{1}{D}\Big((ab)^{\lfloor\frac{r}{2}\rfloor}a^{\zeta(r)-1}\big(u_r-(-c)^m a^{\zeta(m)\zeta(r+1)}b^{\zeta(m)\zeta(r)}u_{r-m}\big)\\
&\qquad-(ab)^{\lfloor\frac{mn+m+r}{2}\rfloor}a^{\zeta(mn+m+r)-1}\\
&\qquad\times\big(u_{mn+m+r}+(-c)^m a^{\zeta(m)\zeta(mn+m+r+1)}b^{\zeta(m)\zeta(mn+m+r)}u_{mn+r}\big)\Big),\\
&\sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)}v_{mj+r}\\
&\quad=\frac{1}{D}\Big((ab)^{\lfloor\frac{r}{2}\rfloor}a^{\zeta(r)}\big(v_r-(-c)^m a^{\zeta(m)\zeta(r+1)}b^{\zeta(m)\zeta(r)}v_{r-m}\big)\\
&\qquad-(ab)^{\lfloor\frac{mn+m+r}{2}\rfloor}a^{\zeta(mn+m+r)}\\
&\qquad\times\big(v_{mn+m+r}+(-c)^m a^{\zeta(m)\zeta(mn+m+r+1)}b^{\zeta(m)\zeta(mn+m+r)}v_{mn+r}\big)\Big).
\end{aligned}$$

Proof It is clear that

$$I-\big(K^m\big)^{n+1}=\big(I-K^m\big)\sum_{j=0}^{n}K^{mj}.$$

Since

$$\det\big(I-K^m\big)=1-a^{\zeta(m)}v_m+(ab)^{\zeta(m)}(-c)^m 0\neq 0,$$

then

$$\begin{aligned}
\big(I-K^m\big)^{-1}&=\frac{1}{D}\begin{bmatrix}1-a^{\zeta(m)}\frac{v_m}{2} & \Delta a^{\zeta(m)-1}\frac{u_m}{2}\\ a^{\zeta(m)-1}\frac{u_m}{2} & 1-a^{\zeta(m)}\frac{v_m}{2}\end{bmatrix}\\
&=\frac{1}{D}\bigg(\bigg(1-a^{\zeta(m)}\frac{v_m}{2}\bigg)I+a^{\zeta(m)-1}\frac{u_m}{2}H\bigg).
\end{aligned}$$

By using the matrix identity (2.2), we have

$$\begin{aligned}
&\big(I-K^m\big)^{-1}\big(I-\big(K^m\big)^{n+1}\big)K^r\\
&\quad=\sum_{j=0}^{n}K^{mj+r}\\
&\quad=\begin{bmatrix}\sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)}\frac{v_{mj+r}}{2} & \Delta\sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)-1}\frac{u_{mj+r}}{2}\\ \sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)-1}\frac{u_{mj+r}}{2} & \sum_{j=0}^{n}(ab)^{\lfloor\frac{mj+r}{2}\rfloor}a^{\zeta(mj+r)}\frac{v_{mj+r}}{2}\end{bmatrix}.
\end{aligned}\tag{2.6}$$

On the other hand, we have

$$\begin{aligned}
&\left(I-K^m\right)^{-1}\left(K^r-K^{mn+m+r}\right)\\
&=\frac{1}{D}\left(\left(1-a^{\zeta(m)}\frac{v_m}{2}\right)I+a^{\zeta(m)-1}\frac{u_m}{2}H\right)\left(K^r-K^{mn+m+r}\right)\\
&=\frac{1}{D}\left(\left(1-a^{\zeta(m)}\frac{v_m}{2}\right)\left(K^r-K^{mn+m+r}\right)+a^{\zeta(m)-1}\frac{u_m}{2}H\left(K^r-K^{mn+m+r}\right)\right)\\
&=\frac{1}{D}\left(\left(1-a^{\zeta(m)}\frac{v_m}{2}\right)\begin{bmatrix} X & \Delta Y\\ Y & X\end{bmatrix}+a^{\zeta(m)-1}\frac{u_m}{2}\begin{bmatrix} \Delta Y & \Delta X\\ X & \Delta Y\end{bmatrix}\right),
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
X &:= (ab)^{\lfloor\frac{r}{2}\rfloor}a^{\zeta(r)}\frac{v_r}{2}-(ab)^{\lfloor\frac{mn+m+r}{2}\rfloor}a^{\zeta(mn+m+r)}\frac{v_{mn+m+r}}{2},\\
Y &:= (ab)^{\lfloor\frac{r}{2}\rfloor}a^{\zeta(r)-1}\frac{u_r}{2}-(ab)^{\lfloor\frac{mn+m+r}{2}\rfloor}a^{\zeta(mn+m+r)-1}\frac{u_{mn+m+r}}{2}.
\end{aligned}$$

By equating the corresponding entries of (2.6) and (2.7), and using the fifth identity of Lemma 2, we get the desired result. The remaining result can be proven similarly by using the fourth identity of Lemma 2. □

Theorem 2 *For any nonnegative integers n, r and m with $m > 1$, we have*

$$\begin{aligned}
u_{mn+r} &= \frac{a^{1-\zeta(mn+r)}}{(ab)^{\lfloor\frac{mn+r}{2}\rfloor}}\sum_{i=0}^{n}\binom{n}{i}c^{n-i}u_m^i u_{m-1}^{n-i}u_{i+r}\delta[m,n,r,i],\\
v_{mn+r} &= \frac{a^{1-\zeta(mn+r)}}{(ab)^{\lfloor\frac{mn+r}{2}\rfloor}}\sum_{i=0}^{n}\binom{n}{i}c^{n-i}u_m^i u_{m-1}^{n-i}v_{i+r}\delta[m,n,r,i],
\end{aligned}$$

where

$$\delta[m,n,r,i] := (ab)^{\lfloor\frac{i+r}{2}\rfloor+n\lfloor\frac{m}{2}\rfloor}a^{-\zeta(m+1)i-1+\zeta(i+r)}b^{\zeta(m)(n-i)}.$$

Proof By considering the matrix identities (2.5) and (2.2), then equating the corresponding entries we obtain the desired results. □

Note that Theorem 1 and Theorem 2 can be seen as a generalization of the results in [11].

Finally, we define the matrix $T := \left[\begin{smallmatrix} abw_1+cbw_0 & cbw_1\\ aw_1 & cbw_0\end{smallmatrix}\right]$. By induction we have

$$TU^n = (ab)^{\lfloor\frac{n+1}{2}\rfloor}\begin{bmatrix} b^{\zeta(n+1)}w_{n+2} & cba^{-\zeta(n)}w_{n+1}\\ a^{\zeta(n+1)}w_{n+1} & cb^{\zeta(n+1)}w_n\end{bmatrix},\tag{2.8}$$

where w_n is the nth generalized bi-periodic Horadam number.

If we take the determinant of both sides of Eq. (2.8) and taking $n \to n-1$, we obtain Cassini's identity for the sequence $\{w_n\}$:

$$\left(\frac{b}{a}\right)^{\zeta(n)}w_{n-1}w_{n+1}-\left(\frac{b}{a}\right)^{\zeta(n+1)}w_n^2 = (-1)^n c^{n-1}\left(w_1^2-bw_0w_1-c\frac{b}{a}w_0^2\right).\tag{2.9}$$

The matrix T can be written as

$$T = cbw_0 I + w_1 U, \tag{2.10}$$

where I is the 2×2 unit matrix. It is easy to see that

$$TU^n = cbw_0 U^n + w_1 U^{n+1}. \tag{2.11}$$

If we equate the corresponding entries of the matrix equality (2.11), we get the identity (2.4). Also, the generalized bi-periodic Horadam numbers for negative subscripts can be defined as

$$w_{-n} = -\frac{a^{\zeta(n+1)} b^{\zeta(n)}}{c} w_{-n+1} + \frac{1}{c} w_{-n+2}, \tag{2.12}$$

so that the matrix identity (2.8) holds for every integer n. From (2.11), we have

$$(-c)^n w_{-n} = \left(\frac{b}{a}\right)^{\zeta(n)} w_0 u_{n+1} - w_1 u_n, \tag{2.13}$$

which reduces to

$$u_{-n} = \frac{(-1)^{n+1}}{c^n} u_n \quad \text{and} \quad v_{-n} = \frac{(-1)^n}{c^n} v_n \tag{2.14}$$

for the generalized bi-periodic Fibonacci and Lucas numbers, respectively.

3 More general results for $\{w_n\}$

Besides the matrix U, the nth power of the matrix $A := \left[\begin{smallmatrix} ab & abc \\ 1 & 0 \end{smallmatrix}\right]$ also has entries involving generalized bi-periodic Fibonacci numbers; that is,

$$A^n = \begin{bmatrix} ab & abc \\ 1 & 0 \end{bmatrix}^n = (ab)^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} b^{\zeta(n)} u_{n+1} & cba^{\zeta(n)} u_n \\ a^{-\zeta(n+1)} u_n & cb^{\zeta(n)} u_{n-1} \end{bmatrix}. \tag{3.1}$$

If n is even, then

$$A^n \begin{bmatrix} w_1 \\ a^{-1} w_0 \end{bmatrix} = (ab)^{\frac{n}{2}} \begin{bmatrix} w_{n+1} \\ a^{-1} w_n \end{bmatrix}, \qquad A^n \begin{bmatrix} cbw_2 \\ cw_1 \end{bmatrix} = (ab)^{\frac{n}{2}} \begin{bmatrix} cbw_{n+2} \\ cw_{n+1} \end{bmatrix}. \tag{3.2}$$

By combining Eqs. (3.1) and (3.2) for even n,

$$\begin{aligned} \begin{bmatrix} w_{n+1} \\ a^{-1} w_n \end{bmatrix} &= \begin{bmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{bmatrix} \begin{bmatrix} w_1 \\ a^{-1} w_0 \end{bmatrix}, \\ \begin{bmatrix} cbw_{n+2} \\ cw_{n+1} \end{bmatrix} &= \begin{bmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{bmatrix} \begin{bmatrix} cbw_2 \\ cw_1 \end{bmatrix}. \end{aligned} \tag{3.3}$$

We get (2.4) by comparing entries in (3.3). We generalize it further as follows.

Theorem 3 *Let n and p be any positive integers. Then*

$$w_{n+p} = \left(\frac{b}{a}\right)^{\zeta(n+1)\zeta(p)} u_n w_{p+1} + c\left(\frac{b}{a}\right)^{\zeta(n)\zeta(p+1)} u_{n-1} w_p. \tag{3.4}$$

Proof Let n and p be even. By (3.2), (3.3),

$$\begin{aligned}(ab)^{\frac{n+p}{2}} \begin{bmatrix} w_{n+p+1} \\ a^{-1} w_{n+p} \end{bmatrix} &= A^{n+p} \begin{bmatrix} w_1 \\ a^{-1} w_0 \end{bmatrix} \\ &= (ab)^{\frac{p}{2}} A^n \begin{bmatrix} w_{p+1} \\ a^{-1} w_p \end{bmatrix} \\ &= (ab)^{\frac{n+p}{2}} \begin{bmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{bmatrix} \begin{bmatrix} w_{p+1} \\ a^{-1} w_p \end{bmatrix}. \end{aligned} \tag{3.5}$$

By comparing both entries of the matrices on both sides of (3.5), we get (3.4). Similarly, we obtain the following equation by (3.2) and (3.3):

$$\begin{bmatrix} cbw_{n+p+2} \\ cw_{n+p+1} \end{bmatrix} = \begin{bmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{bmatrix} \begin{bmatrix} cbw_{p+2} \\ cw_{p+1} \end{bmatrix}. \tag{3.6}$$

By comparing entries of the matrices in (3.6), we get the desired result. □

By Theorem 3, we have the following matrix identities for even n and even p:

$$\begin{bmatrix} w_{n+p+1} \\ a^{-1} w_p \end{bmatrix} = \begin{bmatrix} u_{n+1} & bcu_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_{p+1} \\ a^{-1} w_p \end{bmatrix}, \tag{3.7}$$

$$\begin{bmatrix} cbw_{n+p+2} \\ cw_{p+1} \end{bmatrix} = \begin{bmatrix} u_{n+1} & bcu_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} cbw_{p+2} \\ cw_{p+1} \end{bmatrix}. \tag{3.8}$$

The following theorem is a generalization of Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 4 *Let n, p and q be any positive integers, then we have the following identity*:

$$\begin{aligned}&\left(\frac{b}{a}\right)^{\zeta(n)\zeta(p)\zeta(q)} w_{n+p} w_{n+q} - \left(\frac{b}{a}\right)^{\zeta(n+1)\zeta(p)\zeta(q)} w_n w_{n+p+q} \\ &\quad = \left(\frac{b}{a}\right)^{\zeta(n)\zeta(p+1)\zeta(q+1)} (-c)^n u_p u_q \left(w_1^2 - bw_0 w_1 - \frac{b}{a} c w_0^2\right).\end{aligned}$$

Proof For the case of even n, odd p and odd q, we note that

$$c\left(w_{n+p} w_{n+q} - \frac{b}{a} w_n w_{n+p+q}\right) = \begin{bmatrix} w_{n+q} & a^{-1} w_n \end{bmatrix} \begin{bmatrix} cw_{n+p} \\ -bcw_{n+p+q} \end{bmatrix}. \tag{3.9}$$

By (3.7) and (3.8), we get the following two equations:

$$\begin{aligned} (ab)^{\frac{n}{2}}\begin{bmatrix} w_{n+q} & a^{-1}w_n \end{bmatrix} &= (ab)^{\frac{n}{2}}\begin{bmatrix} w_{n+1} & a^{-1}w_n \end{bmatrix}\begin{bmatrix} u_q & 0 \\ bcu_{q-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} w_1 & a^{-1}w_0 \end{bmatrix}\left(A^T\right)^n\begin{bmatrix} u_q & 0 \\ bcu_{q-1} & 1 \end{bmatrix}, \end{aligned} \tag{3.10}$$

$$(ab)^{\frac{n+p-1}{2}}\begin{bmatrix} cw_{n+p} \\ -bcw_{n+p+q} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -bcu_{q-1} & u_q \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -abc & ab \end{bmatrix}^{n+p-1}\begin{bmatrix} cw_1 \\ -bcw_2 \end{bmatrix}. \tag{3.11}$$

We note that

$$\begin{aligned} &\begin{bmatrix} u_q & 0 \\ bcu_{q-1} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -bcu_{q-1} & u_q \end{bmatrix} = u_q I, \\ &\begin{bmatrix} ab & 1 \\ abc & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -abc & ab \end{bmatrix} = -abcI. \end{aligned} \tag{3.12}$$

We take the product of equations (3.10) and (3.11), and by using (3.9), (3.12), we get

$$\begin{aligned} &(ab)^{n+\frac{p-1}{2}}c\left(w_{n+p}w_{n+q} - \frac{b}{a}w_n w_{n+p+q}\right) \\ &\quad = u_q\begin{bmatrix} w_1 & a^{-1}w_0 \end{bmatrix}\begin{bmatrix} ab & 1 \\ abc & 0 \end{bmatrix}^n\begin{bmatrix} 0 & -1 \\ -abc & ab \end{bmatrix}^{n+p-1}\begin{bmatrix} cw_1 \\ -bcw_2 \end{bmatrix} \\ &\quad = (-abc)^n u_q\begin{bmatrix} w_1 & a^{-1}w_0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -abc & ab \end{bmatrix}^{p-1}\begin{bmatrix} cw_1 \\ -bcw_2 \end{bmatrix} \\ &\quad = (-abc)^n u_q\begin{bmatrix} w_1 & a^{-1}w_0 \end{bmatrix}(ab)^{\frac{p-1}{2}}\begin{bmatrix} cu_{p-2} & -a^{-1}u_{p-1} \\ -bcu_{p-1} & u_p \end{bmatrix}\begin{bmatrix} cw_1 \\ -bcw_2 \end{bmatrix}. \end{aligned} \tag{3.13}$$

We compute the following matrix product:

$$\begin{aligned} &\begin{bmatrix} w_1 & a^{-1}w_0 \end{bmatrix}\begin{bmatrix} cu_{p-2} & -a^{-1}u_{p-1} \\ -bcu_{p-1} & u_p \end{bmatrix}\begin{bmatrix} cw_1 \\ -bcw_2 \end{bmatrix} \\ &\quad = c^2w_1^2u_{p-2} - \frac{b}{a}c^2w_0w_1u_{p-1} + \frac{b}{a}cw_1w_2u_{p-1} - \frac{b}{a}cw_0w_2u_p \\ &\quad = c^2w_1^2u_{p-2} - \frac{b}{a}cw_1u_{p-1}(cw_0 - w_2) - \frac{b}{a}cw_0u_p(aw_1 + cw_0) \\ &\quad = c^2w_1^2u_{p-2} + bcw_1^2u_{p-1} - bcw_0w_1u_p - \frac{b}{a}c^2w_0^2u_p \\ &\quad = cw_1^2(u_{p-2} + bu_{p-1}) - bcw_0w_1u_p - \frac{b}{a}c^2w_0^2u_p \\ &\quad = cw_1^2u_p - bcw_0w_1u_p - \frac{b}{a}c^2w_0^2u_p = cu_p\left(w_1^2 - bw_0w_1 - \frac{b}{a}cw_0^2\right). \end{aligned}$$

We replace it in (3.13) to get the result as desired for the case of even n, odd p and odd q.

For the case of odd n', we take $n' = n + 1$ and then do a similar computation to the one above for

$$(ab)^{n+\frac{p+1}{2}} \begin{bmatrix} bcw_{(n+1)+p} & cw_{(n+1)} \end{bmatrix} \begin{bmatrix} a^{-1}w_{(n+1)+q} \\ -w_{(n+1)+p+q} \end{bmatrix}$$

where n is even, p, q are odd.

For the case of even n, even p and odd q, we do a similar computation to the one above for

$$(ab)^{n+\frac{p}{2}} \begin{bmatrix} w_{n+q} & a^{-1}w_n \end{bmatrix} \begin{bmatrix} a^{-1}w_{n+p} \\ -w_{n+p+q} \end{bmatrix}.$$

For the case of odd n, even p and odd q, we do a similar computation as above for

$$(ab)^{n-1+\frac{p}{2}} \begin{bmatrix} bcw_{n+q} & cw_n \end{bmatrix} \begin{bmatrix} cw_{n+p} \\ -bcw_{n+p+q} \end{bmatrix}.$$

The other cases for even q can be proven similarly. □

We state another two matrix identities for even n:

$$\begin{aligned} \begin{bmatrix} w_1 & bcw_0 \end{bmatrix} A^n &= (ab)^{\frac{n}{2}} \begin{bmatrix} w_{n+1} & bcw_n \end{bmatrix}, \\ \begin{bmatrix} a^{-1}w_2 & cw_1 \end{bmatrix} A^n &= (ab)^{\frac{n}{2}} \begin{bmatrix} a^{-1}w_{n+2} & cw_{n+1} \end{bmatrix}. \end{aligned} \tag{3.14}$$

The following result is a generalization of (2) and (4) of Lemma 1.

Theorem 5 *For positive integers n and m, we have*

$$\left(\frac{b}{a}\right)^{\zeta(mn+n)} w_{n+1}w_m + \left(\frac{b}{a}\right)^{\zeta(mn+m)} cw_nw_{m-1} = w_1w_{m+n} + \left(\frac{b}{a}\right)^{\zeta(m+n)} cw_0w_{m+n-1}.$$

Proof For even n and odd m, by (3.7) and (3.14),

$$\begin{aligned} &(ab)^{\frac{n+m-1}{2}} \left(w_{n+1}w_m + \frac{b}{a} cw_nw_{m-1} \right) \\ &= (ab)^{\frac{n+m-1}{2}} \begin{bmatrix} w_{n+1} & bcw_n \end{bmatrix} \begin{bmatrix} w_m \\ a^{-1}w_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} w_1 & bcw_0 \end{bmatrix} A^n A^{m-1} \begin{bmatrix} w_1 \\ a^{-1}w_0 \end{bmatrix} \\ &= \begin{bmatrix} w_1 & bcw_0 \end{bmatrix} A^{m+n-1} \begin{bmatrix} w_1 \\ a^{-1}w_0 \end{bmatrix} \\ &= (ab)^{\frac{m+n-1}{2}} \begin{bmatrix} w_1 & bcw_0 \end{bmatrix} \begin{bmatrix} w_{m+n} \\ a^{-1}w_{m+n-1} \end{bmatrix}. \end{aligned}$$

Hence the result follows. For the case of odd n and even m, we do a similar computation on

$$(ab)^{\frac{n+m-1}{2}}\begin{bmatrix} a^{-1}w_m & cw_{m-1}\end{bmatrix}\begin{bmatrix} bcw_{n+1} \\ cw_n\end{bmatrix}.$$

For even n and even m, we do a similar computation on

$$(ab)^{\frac{m+n-2}{2}}\begin{bmatrix} a^{-1}w_m & cw_{m-1}\end{bmatrix}\begin{bmatrix} w_{n+1} \\ a^{-1}w_n\end{bmatrix}.$$

For odd n and odd m, we do a similar computation on

$$(ab)^{\frac{m+n-2}{2}}\begin{bmatrix} w_m & bcw_{m-1}\end{bmatrix}\begin{bmatrix} bcw_{n+1} \\ cw_n\end{bmatrix}.$$

□

By substituting $m = n + 1$ in Theorem 5, we get the following corollary, which is a generalization of the classical result $F_{n+1}^2 + F_n^2 = F_{2n+1}$ for Fibonacci numbers.

Corollary 1 *For positive integer n, we have*

$$\left(\frac{b}{a}\right)^{\zeta(n)} w_{n+1}^2 + \left(\frac{b}{a}\right)^{\zeta(n+1)} cw_n^2 = w_1 w_{2n+1} + \left(\frac{b}{a}\right) cw_0 w_{2n}.$$

By Corollary 1, we prove the following result, which is a generalization of the classical result $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ for Fibonacci numbers.

Theorem 6 *For positive integer n, we have*

$$w_{n+1}^2 - c^2 w_{n-1}^2 = a^{\zeta(n)} b^{\zeta(n+1)} (w_1 w_{2n} + cw_0 w_{2n-1}).$$

Proof For even n, by Corollary 1,

$$\begin{aligned} w_{n+1}^2 - c^2 w_{n-1}^2 &= \left(w_{n+1} + \frac{b}{a} cw_n^2\right) - \left(\frac{b}{a} cw_n^2 + c^2 w_{n-1}^2\right) \\ &= \left(w_1 w_{2n+1} + \frac{b}{a} cw_0 w_{2n}\right) - c\left(w_1 w_{2n-1} + \frac{b}{a} cw_0 w_{2n-2}\right) \\ &= w_1 (w_{2n+1} - cw_{2n-1}) + \frac{b}{a} cw_0 (w_{2n} - cw_{2n-2}) \\ &= bw_1 w_{2n} + bcw_0 w_{2n-1}. \end{aligned}$$

For odd n, by Corollary 1,

$$\begin{aligned} \left(\frac{b}{a}\right)^2 \left(w_{n+1}^2 - c^2 w_{n-1}^2\right) &= \frac{b}{a}\left(\frac{b}{a} w_{n+1}^2 + cw_n^2\right) - \frac{b}{a} c\left(w_n^2 + \frac{b}{a} cw_{n-1}^2\right) \\ &= \frac{b}{a}\left(w_1 w_{2n+1} + \frac{b}{a} cw_0 w_{2n}\right) - \frac{b}{a} c\left(w_1 w_{2n-1} + \frac{b}{a} cw_0 w_{2n-2}\right) \end{aligned}$$

$$= \frac{b}{a}\left(w_1(w_{2n+1} - cw_{2n-1}) + \frac{b}{a}cw_0(w_{2n} - cw_{2n-2})\right)$$
$$= \frac{b}{a}(bw_1w_{2n} + bcw_0w_{2n-1}).$$

So the proof is complete after simple algebraic manipulations of both sides of the equation. □

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