Soontharanon and Sitthiwirattham *Advances in Difference Equations* (2020) 2020:35
https://doi.org/10.1186/s13662-020-2512-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

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On fractional (p,q)-calculus

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Abstract

In this paper, the new concepts of (p,q)-difference operators are introduced. The properties of fractional (p,q)-calculus in the sense of a (p,q)-difference operator are introduced and developed.

MSC: 39A10; 39A13; 39A70

Keywords: Fractional (p,q)-integral; Fractional (p,q)-difference; (p,q)-calculus

1 Introduction

The q-difference operator was first studied by Jackson [1] and was considered by many researchers (see more information in [2–5]. There are recent works related to q-calculus as seen in [6–8]. The knowledge of q-calculus and difference equations can be applied to physical problems such as molecular problems [9], elementary particle physics, and chemical physics [10–13]. Then, the q-field theory was presented in 1995 [14]. In 1996, the q-Coulomb problem and q-hydrogen atom were investigated by [15–18]. Moreover, Yang–Mills theories and Yang–Mills equation were presented as seen in [19–21]. The theory of quantum group was applied to vibration and rotation molecules with q-algebra and q-Heisenberg algebra technique [22–24]. In 1988, Siegel [25] presented the string theory involving q-calculus.

In the last three decades, applications of q-calculus have been studied and investigated intensively. Inspired and motivated by these applications, many researchers have developed the theory of quantum calculus based on two-parameter (p,q)-integer which is used efficiently in many fields such as difference equations, Lie group, hypergeometric series, physical sciences, and so on. The (p,q)-calculus was first studied in quantum algebras by Chakrabarti and Jagannathan [26]. For some results on the study of (p,q)-calculus, we refer to [27–38]. For example, Sadjang [30] investigated the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas; the boundary value problems for (p,q)-difference equations were initiated in [35, 36]; and we see the concept of (p,q)-Beta and (p,q)-gamma functions in [37, 38]. We can see the applications of (p,q)-calculus in [39–44]. For example, Mursaleen et al. [39–41] investigated some approximation results by using the (p,q)-analogue of Bernstein–Stancu operators, Bleimann–Butzer–Hahn operators, and Lorentz polynomials on a compact disk. Convergence of iterates of (p,q)-Bernstein operator and convergence of Lupaş (p,q)-Bernstein operator are found in [42, 43]. Recently, Nasiruzzaman et al. [44] studied some Opial-type integral inequalities via (p,q)-calculus.

The study of fractional calculus in discrete settings was initiated in [45–47]. Agarwal [45] and Al-Salam [46] introduced fractional q-difference calculus, while Díaz and Osler [47] studied fractional difference calculus. Recently, Brikshavana and Sitthiwirattham [48] introduced fractional Hahn difference calculus. In addition, Patanarapeelert and Sitthiwirattham [49] discussed fractional symmetric Hahn difference calculus. Although many interesting results related to discrete analogues of some topics of continuous fractional calculus have been studied, the theory of discrete fractional calculus remains much less developed than that of continuous fractional calculus.

In particular, the fractional (p,q)-difference equations have not been studied. The gap mentioned above is the motivation for this research. The aim of this paper is to introduce new concepts of (p,q)-integral, and some fundamental properties are studied in the first part of this note. Then, we consider the fractional (p,q)-difference operators of the Riemann–Liouville and Caputo types. A study of this fractional (p,q)-calculus is expected to be of great importance in the development of the (p,q)-function theory, which plays an important role in analysis and applications.

2 Preliminary definitions and properties

In this section, we provide basic definitions, notations, and lemmas that will be used in this study. Letting $0 < q < p \le 1$, we define

$$[k]_{p,q} := \begin{cases} \frac{p^k-q^k}{p-q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N}, \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{k} \frac{p^i-q^i}{p-q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The (p,q)-forward jump operator and the (p,q)-backward jump operator are defined as follows:

$$\sigma_{p,q}^k(t) := \left(\frac{q}{p}\right)^k t \quad \text{and} \quad \rho_{p,q}^k(t) := \left(\frac{p}{q}\right)^k t \quad \text{for } k \in \mathbb{N}, \text{ respectively.}$$

The q-analogue of the power function $(a-b)_q^{\underline{n}}$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is given by

$$(a-b)_q^{\underline{0}} := 1, \qquad (a-b)_q^{\underline{n}} := \prod_{i=0}^{n-1}\left(a - bq^i\right), \quad a,b \in \mathbb{R}.$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^{\underline{n}}$ with $n \in \mathbb{N}_0$ is given by

$$(a-b)_{p,q}^{\underline{0}} := 1, \qquad (a-b)_{p,q}^{\underline{n}} := \prod_{k=0}^{n-1}\left(ap^k - bq^k\right), \quad a,b \in \mathbb{R}.$$

If $\alpha \in \mathbb{R}$, we have a general form:

$$(a-b)_q^{\underline{\alpha}} = a^{\alpha}\prod_{i=0}^{\infty}\frac{1-(\frac{b}{a})q^i}{1-(\frac{b}{a})q^{\alpha+i}}, \quad a \neq 0.$$

Since

$$(a-b)_{p,q}^{\underline{n}} = \prod_{i=0}^{n-1}\left(ap^i - bq^i\right) = \left(\prod_{i=0}^{n-1}p^i\right)\left(\prod_{i=0}^{n-1}\left[a - b\left(\frac{q}{p}\right)^i\right]\right) = p^{\binom{n}{2}}(a-b)_{\frac{q}{p}}^{\underline{n}},$$

where $\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$, and

$$\begin{aligned}(a-b)_{p,q}^{\underline{n}} &= \prod_{i=0}^{n-1}\left(ap^i - bq^i\right) = a^n\prod_{i=0}^{n-1}p^i\left[1 - \frac{b}{a}\left(\frac{q}{p}\right)^i\right] \\ &= a^n\prod_{i=0}^{n-1}p^i\left[1 - \frac{b}{a}\left(\frac{q}{p}\right)^i\right]\cdot\frac{\prod_{j=n}^{\infty}p^j[1-\frac{b}{a}(\frac{q}{p})^j]}{\prod_{j=n}^{\infty}p^j[1-\frac{b}{a}(\frac{q}{p})^j]} \\ &= a^n\frac{\prod_{i=0}^{\infty}p^i[1-\frac{b}{a}(\frac{q}{p})^i]}{\prod_{i=0}^{\infty}p^{i+n}[1-\frac{b}{a}(\frac{q}{p})^{i+n}]} = a^n\prod_{i=0}^{\infty}\frac{1}{p^n}\left[\frac{1-\frac{b}{a}(\frac{q}{p})^i}{1-\frac{b}{a}(\frac{q}{p})^{i+n}}\right],\end{aligned}$$

we obtain

$$(a-b)_{p,q}^{\underline{\alpha}} = p^{\binom{\alpha}{2}}(a-b)_{\frac{q}{p}}^{\underline{\alpha}} = a^\alpha\prod_{i=0}^{\infty}\frac{1}{p^\alpha}\left[\frac{1-\frac{b}{a}(\frac{q}{p})^i}{1-\frac{b}{a}(\frac{q}{p})^{i+\alpha}}\right], \quad a \neq 0.$$

Note that $a_q^{\underline{\alpha}} = a_{p,q}^{\underline{\alpha}} = a^\alpha$ and $(0)_q^{\underline{\alpha}} = (0)_{p,q}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

Lemma 2.1 *For $\alpha,\beta,\gamma,\lambda \in \mathbb{R}$ and $I_{p,q}^T := \{\frac{q^k}{p^{k+1}}T : k \in \mathbb{N}_0\} \cup \{0\}$,*
(a) $(\gamma\beta - \gamma\lambda)_{p,q}^{\underline{\alpha}} = \gamma^\alpha(\beta-\lambda)_{p,q}^{\underline{\alpha}}$,
(b) $(\beta-\gamma)_{p,q}^{\underline{\alpha+\gamma}} = \frac{1}{p^{\alpha\gamma}}(\beta-\gamma)_{p,q}^{\underline{\alpha}}(p^\alpha\beta - q^\alpha\lambda)_{p,q}^{\underline{\gamma}}$,
(c) $(t-s)_{p,q}^{\underline{\alpha}} = 0, \alpha \notin \mathbb{N}_0, t \geq s$, *and* $t,s \in I_{p,q}^T$.

Proof For $\alpha,\beta,\gamma,\lambda \in \mathbb{R}$ and $0 < q < p \leq 1$, we have

$$(\gamma\beta - \gamma\lambda)_{p,q}^{\underline{\alpha}} = (\gamma\beta)^\alpha\prod_{i=0}^{\infty}\frac{1}{p^\alpha}\left[\frac{1-\frac{\lambda}{\beta}(\frac{q}{p})^i}{1-\frac{\lambda}{\beta}(\frac{q}{p})^{i+\alpha}}\right] = \gamma^\alpha(\beta-\lambda)_{p,q}^{\underline{\alpha}},$$

and

$$\begin{aligned}(\beta-\gamma)_{p,q}^{\underline{\alpha+\gamma}} &= \beta^{\alpha+\gamma}\prod_{i=0}^{\infty}\frac{1}{p^{\alpha+\gamma}}\left[\frac{1-\frac{\gamma}{\beta}(\frac{q}{p})^i}{1-\frac{\gamma}{\beta}(\frac{q}{p})^{i+\alpha+\gamma}}\right] \\ &= \beta^{\alpha+\gamma}\frac{\prod_{i=0}^{\infty}\frac{1}{p^{\alpha+\gamma}}[1-\frac{\gamma}{\beta}(\frac{q}{p})^i]}{\prod_{i=0}^{\infty}[1-\frac{\gamma}{\beta}(\frac{q}{p})^{i+\alpha+\gamma}]}\cdot\frac{\prod_{i=0}^{\infty}[1-\frac{\gamma}{\beta}(\frac{q}{p})^{i+\alpha}]}{\prod_{i=0}^{\infty}[1-\frac{\gamma}{\beta}(\frac{q}{p})^{i+\alpha}]} \\ &= \frac{1}{p^{\alpha\gamma}}\cdot\beta^\alpha\prod_{i=0}^{\infty}\frac{1}{p^\alpha}\frac{[1-\frac{\gamma}{\beta}(\frac{q}{p})^i]}{[1-\frac{\gamma}{\beta}(\frac{q}{p})^{i+\alpha}]}\cdot\left(p^\alpha\beta\right)^\gamma\prod_{i=0}^{\infty}\frac{1}{p^\gamma}\frac{[1-\frac{q^\alpha\gamma}{p^\alpha\beta}(\frac{q}{p})^i]}{[1-\frac{q^\alpha\gamma}{p^\alpha\beta}(\frac{q}{p})^{i+\gamma}]} \\ &= \frac{1}{p^{\alpha\gamma}}(\beta-\gamma)_{p,q}^{\underline{\alpha}}\left(p^\alpha\beta - q^\alpha\lambda\right)_{p,q}^{\underline{\gamma}}.\end{aligned}$$

So, (a) and (b) hold.

Letting $t, s \in I_{p,q}^T$, there exist $m, n \in \mathbb{N}_0$ such that $t = \frac{q^m}{p^{m+1}} T$, $s = \frac{q^n}{p^{n+1}} T$, where $t \geq s$, $m \leq n$, and

$$
\begin{aligned}
(t-s)_{p,q}^{\underline{\alpha}} &= \left(\frac{q^m}{p^{m+1}} T - \frac{q^n}{p^{n+1}} T \right)_{p,q}^{\underline{\alpha}} \\
&= \left(\frac{q^m}{p^{m+1}} T \right)^{\alpha} \prod_{i=0}^{\infty} \frac{1}{p^{\alpha}} \left[\frac{1 - (\frac{q}{p})^{i+m-n}}{1 - (\frac{q}{p})^{i+m-n+\alpha}} \right] = 0.
\end{aligned}
$$

Hence, (c) holds. □

Lemma 2.2 *For $m, n \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, and $0 < q < p \leq 1$,*
(a) $(t - \sigma_{p,q}^n(t))_{p,q}^{\underline{\alpha}} = t^{\alpha}(1 - (\frac{q}{p})^n)_{p,q}^{\underline{\alpha}}$,
(b) $(\sigma_{p,q}^m(t) - \sigma_{p,q}^n(t))_{p,q}^{\underline{\alpha}} = (\frac{q}{p})^{m\alpha} t^{\alpha}(1 - (\frac{q}{p})^{n-m})_{p,q}^{\underline{\alpha}}$.

Proof For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$, we have

$$
\left(t - \sigma_{p,q}^n(t)\right)_{p,q}^{\underline{\alpha}} = t^{\alpha} \prod_{i=0}^{\infty} \frac{1}{p^{\alpha}} \left[\frac{1 - (\frac{q}{p})^n (\frac{q}{p})^i}{1 - (\frac{q}{p})^n (\frac{q}{p})^{i+\alpha}} \right] = t^{\alpha} \left(1 - \left(\frac{q}{p} \right)^n \right)_{p,q}^{\underline{\alpha}}
$$

and

$$
\begin{aligned}
\left(\sigma_{p,q}^m(t) - \sigma_{p,q}^n(t)\right)_{p,q}^{\underline{\alpha}} &= \left(\sigma_{p,q}^m(t)\right)^{\alpha} \prod_{i=0}^{\infty} \frac{1}{p^{\alpha}} \left[\frac{1 - (\frac{q}{p})^{n-m} (\frac{q}{p})^i}{1 - (\frac{q}{p})^{n-m} (\frac{q}{p})^{i+\alpha}} \right] \\
&= \left(\frac{q}{p} \right)^{m\alpha} t^{\alpha} \left(1 - \left(\frac{q}{p} \right)^{n-m} \right)_{p,q}^{\underline{\alpha}}.
\end{aligned}
$$

So, (a) and (b) hold. □

The (p,q)-gamma and (p,q)-beta functions are defined by

$$
\begin{aligned}
\Gamma_{p,q}(x) &:= \begin{cases} \frac{(p-q)_{p,q}^{\underline{x-1}}}{(p-q)^{x-1}} = \frac{(1-\frac{q}{p})_{p,q}^{\underline{x-1}}}{(1-\frac{q}{p})^{x-1}}, & x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}, \\ [x-1]_{p,q}!, & x \in \mathbb{N}, \end{cases} \\
B_{p,q}(x,y) &:= \int_0^1 t^{x-1}(1-qt)_{p,q}^{\underline{y-1}} \, d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_q p, q(y)}{\Gamma_{p,q}(x+y)},
\end{aligned}
$$

respectively.

Definition 2.1 For $0 < q < p \leq 1$ and $f : [0, T] \to \mathbb{R}$, we define the (p,q)-difference of f as

$$
D_{p,q} f(t) := \frac{f(pt) - f(qt)}{(p-q)(t)} \quad \text{for } t \neq 0,
$$

where $D_{p,q} f(0) = f'(0)$, provided that f is differentiable at 0. We say that f is (p,q)-differentiable on $I_{p,q}^T$ if $D_{p,q} f(t)$ exists for all $t \in I_{p,q}^T$.

Lemma 2.3 ([30]) *Let f, g be (p,q)-differentiable on $I_{p,q}^T$. The properties of (p,q)-difference operator are as follows:*

(a) $D_{p,q}[f(t)+g(t)] = D_{p,q}f(t) + D_{p,q}g(t)$,
(b) $D_{p,q}[\alpha f(t)] = \alpha D_{p,q}f(t)$ *for* $\alpha \in \mathbb{R}$,
(c) $D_{p,q}[f(t)g(t)] = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t)$,
(d) $D_{p,q}[\frac{f(t)}{g(t)}] = \frac{g(qt)D_{p,q}f(t)-f(qt)D_{p,q}g(t)}{g(pt)g(qt)} = \frac{g(pt)D_{p,q}f(t)-f(pt)D_{p,q}g(t)}{g(pt)g(qt)}$ *for* $g(pt)g(qt) \neq 0$.

Lemma 2.4 *Let* $t \in I^T_{p,q}$, $0 < q < p \leq 1$, $\alpha \geq 1$, *and* $a \in \mathbb{R}$. *Then*
(a) $D_{p,q}(t-a)^{\underline{\alpha}}_{p,q} = [\alpha]_{p,q}(pt-a)^{\underline{\alpha-1}}_{p,q}$,
(b) $D_{p,q}(a-t)^{\underline{\alpha}}_{p,q} = -[\alpha]_{p,q}(a-qt)^{\underline{\alpha-1}}_{p,q}$.

Proof Since $D_{p,q}f(\frac{t}{q}) = \frac{f(\frac{p}{q}t)-f(t)}{(p-q)\frac{t}{q}}$, we have

$$\begin{aligned}
&D_{p,q}\left(\frac{t}{q}-a\right)^{\underline{\alpha}}_{p,q} \\
&= \frac{1}{(p-q)\frac{t}{q}}\left\{\left(\frac{p}{q}t-a\right)^{\underline{\alpha}}_{p,q} - (t-a)^{\underline{\alpha}}_{p,q}\right\} \\
&= \frac{1}{(p-q)\frac{t}{q}}\left\{\left(\frac{p}{q}t\right)^{\alpha} p^{\binom{\alpha}{2}}\prod_{i=0}^{\infty}\left[\frac{1-\frac{a}{t}(\frac{q}{p})^{i+1}}{1-\frac{a}{t}(\frac{q}{p})^{i+\alpha+1}}\right] - t^{\alpha}p^{\binom{\alpha}{2}}\prod_{i=0}^{\infty}\left[\frac{1-\frac{a}{t}(\frac{q}{p})^{i}}{1-\frac{a}{t}(\frac{q}{p})^{i+\alpha}}\right]\right\} \\
&= \frac{p^{\binom{\alpha}{2}}}{(p-q)\frac{t}{q}}\left\{\left(\frac{p}{q}t\right)^{\alpha}\frac{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i+1}]}{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i+\alpha+1}]}\cdot\left[\frac{1-\frac{a}{t}(\frac{q}{p})^{\alpha}}{1-\frac{a}{t}(\frac{q}{p})^{\alpha}}\right] - t^{\alpha}\frac{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i}]}{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i+\alpha}]}\right\} \\
&= p^{\binom{\alpha}{2}+1}\left(\frac{p}{q}t\right)^{\alpha-1}\frac{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i+1}]}{\prod_{i=0}^{\infty}[1-\frac{a}{t}(\frac{q}{p})^{i+\alpha}]}\left\{\frac{[1-\frac{a}{t}(\frac{q}{p})^{\alpha}]-(\frac{p}{q}t)^{\alpha}[1-\frac{a}{t}]}{p-q}\right\} \\
&= p^{\binom{\alpha}{2}+1}\left(\frac{p}{q}t\right)^{\alpha-1}\prod_{i=0}^{\infty}\left[\frac{1-\frac{a}{t}(\frac{q}{p})^{i+1}}{1-\frac{a}{t}(\frac{q}{p})^{i+\alpha}}\right]\left\{\frac{[\alpha]_{p,q}}{p^{\alpha}}\right\} \\
&= [\alpha]_{p,q}p^{\binom{\alpha-1}{2}}\left(\frac{p}{q}t\right)^{\alpha-1}\prod_{i=0}^{\infty}\left[\frac{1-\frac{qa}{pt}(\frac{q}{p})^{i}}{1-\frac{qa}{pt}(\frac{q}{p})^{i+\alpha-1}}\right] \\
&= [\alpha]_{p,q}(pt-a)^{\underline{\alpha-1}}_{p,q}.
\end{aligned}$$

So, (a) holds. Proceeding similarly as above, we obtain

$$D_{p,q}\left(a-\frac{t}{q}\right)^{\underline{\alpha}}_{p,q} = -[\alpha]_{p,q}(a-t)^{\underline{\alpha-1}}_{p,q}.$$

Hence, (b) holds. □

Definition 2.2 Let I be any closed interval of $\mathbb{R}$ containing a, b, and 0. Assuming that $f : I \to \mathbb{R}$ is a given function, we define (p,q)-integral of f from a to b by

$$\int_a^b f(t)\,d_{p,q}t := \int_0^b f(t)\,d_{p,q}t - \int_0^a f(t)\,d_{p,q}t,$$

where

$$\mathcal{I}_{p,q}f(x) = \int_0^x f(t)\,d_{p,q}t = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}x\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. f is called (p,q)-integrable on $[a,b]$. We say that f is (p,q)-integrable on I if it is (p,q)-integrable on $[a,b]$ for all $a,b \in I$.

Next, we define an operator $\mathcal{I}_{p,q}^N$ as

$$\mathcal{I}_{p,q}^0 f(x) = f(x) \quad \text{and} \quad \mathcal{I}_{p,q}^N f(x) = \mathcal{I}_{p,q}\mathcal{I}_{p,q}^{N-1} f(x), \quad N \in \mathbb{N}.$$

The following are the properties of (p,q)-difference and (p,q)-integral operators:

$$D_{p,q}\mathcal{I}_{p,q} f(x) = f(x) \quad \text{and} \quad \mathcal{I}_{p,q} D_{p,q} f(x) = f(x) - f(0).$$

Lemma 2.5 ([30]) *Let* $0 < q < p \leq 1$, $a, b \in I_{p,q}^T$, *and* f, g *be* (p,q)*-integrable on* $I_{p,q}^T$. *Then the following formulas hold*:

(a) $\int_a^a f(t)\, d_{p,q}t = 0$,
(b) $\int_a^b \alpha f(t)\, d_{p,q}t = \alpha \int_a^b f(t)\, d_{p,q}t, \alpha \in \mathbb{R}$,
(c) $\int_a^b f(t)\, d_{p,q}t = -\int_b^a f(t)\, d_{p,q}t$,
(d) $\int_a^b f(t)\, d_{p,q}t = \int_c^b f(t)\, d_{p,q}t + \int_a^c f(t)\, d_{p,q}t, c \in I_{p,q}^T, a < c < b$,
(e) $\int_a^b [f(t) + g(t)]\, d_{p,q}t = \int_a^b f(t)\, d_{p,q}t + \int_a^b g(t)\, d_{p,q}t$,
(f) $\int_a^b [f(pt) D_{p,q} g(t)]\, d_{p,q}t = [f(t)g(t)]_a^b - \int_a^b [g(qt) D_{p,q} f(t)]\, d_{p,q}t$.

We next introduce the fundamental theorem and Leibniz formula of (p,q)-calculus.

Lemma 2.6 ([30] Fundamental theorem of (p,q)-calculus) *Let* $f : I \to \mathbb{R}$ *be continuous at* 0. *Define*

$$F(x) := \int_0^x f(t)\, d_{p,q}t, \quad x \in I.$$

Then F *is continuous at* 0. *Furthermore,* $D_{p,q}F(x)$ *exists for every* $x \in I$ *where*

$$D_{p,q}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{p,q} f(t)\, d_{p,q}t = f(b) - f(a) \quad \textit{for all } a, b \in I.$$

Lemma 2.7 (Leibniz formula of (p,q)-calculus) *Let* $f : I_{p,q}^T \times I_{p,q}^T \to \mathbb{R}$. *Then*

$$D_{p,q}\left[\int_0^t f(t,s)\, d_{p,q}s\right] = \int_0^{qt} {}_tD_{p,q} f(t,s)\, d_{p,q}s + f(pt,t),$$

where ${}_tD_{p,q}$ *is* (p,q)*-difference with respect to* t.

Proof For $t \in I_{p,q}^T$, we have

$$D_{p,q}\left[\int_0^t f(t,s)\, d_{p,q}s\right]$$

$$
\begin{aligned}
&= \frac{1}{(p-q)t}\left\{\int_0^{pt} f(pt,s)\,d_{p,q}s - \int_0^{qt} f(qt,s)\,d_{p,q}s\right\} \\
&= \frac{1}{(p-q)t}\left\{\left[\int_0^{qt} f(pt,s)\,d_{p,q}s - \int_0^{qt} f(qt,s)\,d_{p,q}s\right]\right. \\
&\quad \left. + \left[\int_0^{pt} f(pt,s)\,d_{p,q}s - \int_0^{qt} f(pt,s)\,d_{p,q}s\right]\right\} \\
&= \int_0^{qt} {}_tD_{p,q}f(t,s)\,d_{p,q}s + \frac{1}{(p-q)t}\left[(p-q)pt\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(pt,\left(\frac{q}{p}\right)^k t\right)\right. \\
&\quad \left. -(p-q)qt\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(pt,\left(\frac{q}{p}\right)^{k+1} t\right)\right] \\
&= \int_0^{qt} {}_tD_{p,q}f(t,s)\,d_{p,q}s + f(pt,t).
\end{aligned}
$$

The proof is completed. □

The following lemmas are provided as tools for simplifying our calculations.

Lemma 2.8 *Let* $\alpha,\beta > 0, 0 < q < p \le 1$. *Then*

$$
\begin{aligned}
&\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} s^\beta \,d_{p,q}s = t^{\alpha+\beta}B_{p,q}(\beta+1,\alpha), \\
&\int_0^t \int_0^s (t-qs)_{p,q}^{\underline{\alpha-1}} (s-qx)_{p,q}^{\underline{\beta-1}} \,d_{p,q}x\,d_{p,q}s = \frac{B_{p,q}(\beta+1,\alpha)}{[\beta]_{p,q}} t^{\alpha+\beta}.
\end{aligned}
$$

Proof From the definition of (p,q)-analogue of the power function, (p,q)-beta function, and Definition 3.1, we obtain

$$
\begin{aligned}
\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} s^\beta \,d_{p,q}s &= (p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-\left(\frac{q}{p}\right)^{k+1}t\right)_{p,q}^{\underline{\alpha-1}}\left(\frac{q^k}{p^{k+1}}t\right)^\beta \\
&= (p-q)t^{\alpha+\beta}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(\frac{q^k}{p^{k+1}}\right)^\beta\left(1-q\cdot\frac{q^k}{p^{k+1}}\right)_{p,q}^{\underline{\alpha-1}} \\
&= t^{\alpha+\beta}\int_0^1 s^\beta (1-qs)_{p,q}^{\underline{\alpha-1}}\,d_{p,q}s \\
&= t^{\alpha+\beta}B_{p,q}(\beta+1,\alpha).
\end{aligned}
$$

By Lemma 2.4(b), we have $\int_0^s (s-qx)_{p,q}^{\underline{\beta-1}}\,d_{p,q}x = [-\frac{1}{[\beta]_{p,q}}(s-x)_{p,q}^{\underline{\beta}}]_{x=0}^s = \frac{s^\beta}{[\beta]_{p,q}}$. Hence, we find that

$$
\begin{aligned}
\int_0^t \int_0^s (t-qs)_{p,q}^{\underline{\alpha-1}} (s-qx)_{p,q}^{\underline{\beta-1}} \,d_{p,q}x\,d_{p,q}s &= \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}}\left[\int_0^s (s-qx)_{p,q}^{\underline{\beta-1}}\,d_{p,q}x\right]d_{p,q}s \\
&= \frac{1}{[\beta]_{p,q}}\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} s^\beta\,d_{p,q}s \\
&= \frac{t^{\alpha+\beta}}{[\beta]_{p,q}}B_{p,q}(\beta+1,\alpha).
\end{aligned}
$$

The proof is completed. □

Lemma 2.9 *Let $0 < q < p \leq 1$ and $f : I^T_{p,q} \to \mathbb{R}$ be continuous at 0. Then*

$$\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q}s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q}s\, d_{p,q}\tau.$$

Proof Using the definitions of (p,q)-integral, we find that

$$\begin{aligned}
\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q}s &= (p-q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \int_0^{\frac{q^k}{p^{k+1}}x} f(\tau)\, d_{p,q}\tau \\
&= (p-q)^2 x^2 \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{q^{2k+h}}{p^{2k+h+3}} f\left(\frac{q^{k+h}}{p^{k+h+2}}x\right) \\
&= (p-q)^2 x^2 \sum_{k=0}^{\infty} \sum_{h=k}^{\infty} \frac{q^{k+h}}{p^{k+h+3}} f\left(\frac{q^h}{p^{h+2}}x\right) \\
&= (p-q)^2 x^2 \sum_{h=0}^{\infty} \sum_{k=0}^{h} \frac{q^{k+h}}{p^{k+h+3}} f\left(\frac{q^h}{p^{h+2}}x\right) \\
&= (p-q)x^2 \sum_{h=0}^{\infty} \left(\frac{p^{h+1}-q^{h+1}}{p^{h+1}}\right) \frac{q^h}{p^{h+2}} f\left(\frac{q^h}{p^{h+2}}x\right) \\
&= (p-q)\frac{x}{p} \sum_{h=0}^{\infty} \left[x - pq\left(\frac{q^h}{p^{h+1}} \cdot \frac{x}{p}\right)\right] \frac{q^h}{p^{h+1}} f\left(\frac{q^h}{p^{h+1}} \cdot \frac{x}{p}\right) \\
&= \int_0^{\frac{x}{p}} (x - pq\tau) f(\tau)\, d_{p,q}\tau \\
&= \int_0^{\frac{x}{p}} \left[\int_0^x d_{p,q}s - \int_0^{pq\tau} d_{p,q}s\right] f(\tau)\, d_{p,q}\tau \\
&= \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q}s\, d_{p,q}\tau.
\end{aligned}$$

The proof is completed. □

Next, we introduce the multiple (p,q)-integration as follows.

Theorem 2.1 *For $f : I^T_{p,q} \to \mathbb{R}$ and $n \in \mathbb{N}$, the multiple (p,q)-integration is given by*

$$\begin{aligned}
\mathcal{I}^n_{p,q} f(x) &= \int_0^x \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} f(\tau_n)\, d_{p,q}\tau_n \cdots d_{p,q}\tau_2\, d_{p,q}\tau_1 \\
&= \frac{1}{p^{\binom{n}{2}} [n-1]_{p,q}!} \int_0^x (x - q\tau)^{\underline{n-1}}_{p,q} f\left(\frac{\tau}{p^{n-1}}\right) d_{p,q}\tau. \qquad (2.1)
\end{aligned}$$

Proof We prove by using mathematical induction.

If $n = 1$, then $\mathcal{I}_{p,q} f(x) = \int_0^x f(\tau)\, d_{p,q}\tau$.

If $n = 2$, by Lemma 2.9 we have

$$\begin{aligned}\mathcal{I}_{p,q}^{2}f(x) &= \int_0^x \int_0^s f(\tau)\,d_{p,q}\tau\,d_{p,q}s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\,d_{p,q}s\,d_{p,q}\tau \\ &= \frac{1}{p}\int_0^x (x-q\tau)f\left(\frac{\tau}{p}\right)d_{p,q}\tau \\ &= \frac{1}{p^{\binom{2}{2}}[1]_{p,q}!}\int_0^x (x-\tau)_{p,q}^{\underline{1}} f\left(\frac{\tau}{p}\right)d_{p,q}\tau.\end{aligned}$$

Next, we suppose that Theorem 2.1 holds for $n = k$. For the case $n = k+1$, we have

$$\begin{aligned}&\mathcal{I}_{p,q}^{k+1}f(x) \\ &= \mathcal{I}_{p,q}\left[\frac{1}{p^{\binom{k}{2}}[k-1]_{p,q}!}\int_0^x (x-q\tau)_{p,q}^{\underline{k-1}} f\left(\frac{\tau}{p^{k-1}}\right)d_{p,q}\tau\right] \\ &= \frac{1}{p^{\binom{k}{2}}[k-1]_{p,q}!}\int_0^x\int_0^s (s-q\tau)_{p,q}^{\underline{k-1}} f\left(\frac{\tau}{p^{k-1}}\right)d_{p,q}\tau\,d_{p,q}s \\ &= \frac{(p-q)^2x^2}{p^{\binom{k}{2}}[k-1]_{p,q}!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{2m+n}}{p^{2m+n+3}}\left(\frac{q^m}{p^{m+1}}x - \frac{q^{m+n+1}}{p^{m+n+2}}x\right)_{p,q}^{\underline{k-1}} f\left(\frac{q^{m+n}}{p^{m+n+k+1}}x\right) \\ &= \frac{(p-q)^2x^2}{p^{\binom{k}{2}}[k-1]_{p,q}!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{q^{m+n}}{p^{m+n+3}}\left(\frac{q^m}{p^{m+1}}x\right)^{k-1}\left(1-\left(\frac{q}{p}\right)^{n-m+1}\right)_{p,q}^{\underline{k-1}} f\left(\frac{q^n}{p^{n+k+1}}x\right) \\ &= \frac{(p-q)x^{k+1}}{p^{\binom{k+1}{2}}[k]_{p,q}!}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \\ &\quad\times\left[(p-q)[k]_{p,q}\sum_{m=0}^{n}\frac{q^{mk}}{p^{mk+1}}\left(1-\left(\frac{q}{p}\right)^{n-m+1}\right)_{p,q}^{\underline{k-1}}\right] f\left(\frac{q^n}{p^{n+k+1}}x\right).\end{aligned} \tag{2.2}$$

On the other hand, we have

$$\begin{aligned}\mathcal{I}_{p,q}^{k+1}f(x) &= \frac{1}{p^{\binom{k+1}{2}}[k]_{p,q}!}\int_0^x (x-q\tau)_{p,q}^{\underline{k}} f\left(\frac{\tau}{p^k}\right)d_{p,q}\tau \\ &= \frac{(p-q)x^{k+1}}{p^{\binom{k+1}{2}}[k]_{p,q}!}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\left(\frac{q}{p}\right)^{n+1}\right)_{p,q}^{\underline{k}} f\left(\frac{q^n}{p^{n+k+1}}x\right).\end{aligned} \tag{2.3}$$

To show that (2.2) is equal to (2.3), we consider

$$\begin{aligned}&(p-q)[k]_{p,q}\sum_{m=0}^{n}\frac{q^{mk}}{p^{mk+1}}\left(1-\left(\frac{q}{p}\right)^{n-m+1}\right)_{p,q}^{\underline{k-1}} \\ &= \left(p^k-q^k\right)\left\{\frac{1}{p}\prod_{i=0}^{k-2}\left[p^i - q^i\left(\frac{q}{p}\right)^{n+1}\right] + \cdots + \frac{q^{(n-2)k}}{p^{(n-2)k+1}}\prod_{i=0}^{k-2}\left[p^i - q^i\left(\frac{q}{p}\right)^3\right]\right. \\ &\quad\left. + \frac{q^{(n-1)k}}{p^{(n-1)k+1}}\prod_{i=0}^{k-2}\left[p^i - q^i\left(\frac{q}{p}\right)^2\right] + \frac{q^{nk}}{p^{nk+1}}\prod_{i=0}^{k-2}\left[p^i - q^i\left(\frac{q}{p}\right)\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&= (p^k - q^k)\Bigg\{\frac{1}{p}\prod_{i=0}^{k-2}\left[\frac{p^{i+n+1}-q^{i+n+1}}{p^{n+1}}\right] + \cdots + \frac{q^{(n-2)k}}{p^{(n-2)k+1}}\prod_{i=0}^{k-2}\left[\frac{p^{i+3}-q^{i+3}}{p^3}\right] \\
&\quad + \frac{q^{(n-1)k}}{p^{(n-1)k+1}}\cdot\frac{p^{k+1}-q^{k+1}}{p^3}\prod_{i=0}^{k-3}\left[\frac{p^{i+2}-q^{i+2}}{p^2}\right]\Bigg\} \\
&= (p^k - q^k)\Bigg\{\frac{1}{p}\prod_{i=0}^{k-2}\left[\frac{p^{i+n+1}-q^{i+n+1}}{p^{n+1}}\right] + \cdots + \frac{q^{(n-3)k}}{p^{(n-3)k+1}}\prod_{i=0}^{k-2}\left[\frac{p^{i+4}-q^{i+4}}{p^4}\right] \\
&\quad + \frac{q^{(n-2)k}}{p^{(n-2)k+1}}\left(\frac{p^{k+1}-q^{k+1}}{p^3}\right)\left(\frac{p^{k+2}-q^{k+2}}{p^5}\right)\prod_{i=0}^{k-4}\left[\frac{p^{i+3}-q^{i+3}}{p^3}\right]\Bigg\} \\
&\vdots \\
&= (p^k - q^k)\Bigg\{\frac{1}{p}\prod_{i=0}^{k-2}\left[\frac{p^{i+n+1}-q^{i+n+1}}{p^{n+1}}\right] + \left[\frac{q^k}{p^{k+1}}\left(\frac{p^{k+1}-q^{k+1}}{p^3}\right)\left(\frac{p^{k+2}-q^{k+2}}{p^5}\right)\cdots \right.\\
&\quad \left.\times\left(\frac{p^{k+n-1}-q^{k+n-1}}{p^{2(n-1)+1}}\right)\right]\prod_{i=0}^{k-n-1}\left[\frac{p^{i+n}-q^{i+n}}{p^n}\right]\Bigg\} \\
&= \frac{p^k-q^k}{p}\left(\frac{p^{k+n-1}-q^{k+n-1}}{p^{n+1}}\right)\left(\frac{p^{k+n-2}-q^{k+n-2}}{p^{n+1}}\right)\cdots\left(\frac{p^{k+1}-q^{k+1}}{p^{n+1}}\right) \\
&\quad \times\left\{\frac{p^{k+n}-q^{k+n}}{p^{2n+1}}\right\}\prod_{i=0}^{k-n-2}\left[\frac{p^{i+n+1}-q^{i+n+1}}{p^{n+1}}\right] \\
&= \frac{1}{p^{(n+1)k}}\left\{(p^{k+n}-q^{k+n})\cdots(p^{k+1}-q^{k+1})(p^k-q^k)(p^{k-1}-q^{k-1})\cdots(p^{n+2}-q^{n+2})\right. \\
&\quad \left.\times(p^{n+1}-q^{n+1})\right\} \\
&= \prod_{i=0}^{k-1}\left[\frac{p^{i+n+1}-q^{i+n+1}}{p^{n+1}}\right] = \prod_{i=0}^{k-1}\left[p^i - q^i\left(\frac{q}{p}\right)^{n+1}\right] \\
&= \left(1-\left(\frac{q}{p}\right)^{n+1}\right)_{p,q}^{\underline{k}}.
\end{aligned}
$$

We see that (2.1) holds when $n = k+1$. □

3 Fractional (p,q)-integral

In this section, we introduce fractional (p,q)-integral.

Definition 3.1 For $\alpha > 0, 0 < q < p \le 1$, and f defined on $I_{p,q}^T$, the fractional (p,q)-integral is defined by

$$
\begin{aligned}
\mathcal{I}_{p,q}^{\alpha} f(t) &:= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
&= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-\left(\frac{q}{p}\right)^{k+1}t\right)_{p,q}^{\underline{\alpha-1}} f\left(\frac{q^k}{p^{k+\alpha}}t\right),
\end{aligned}
$$

and $(\mathcal{I}_{p,q}^0 f)(t) = f(t)$.

By Lemma 2.2(a), we have

$$\mathcal{I}_{p,q}^{\alpha}f(t) = \frac{(p-q)t^{\alpha}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{\alpha-1}}f\left(\frac{q^k}{p^{k+\alpha}}t\right). \tag{3.1}$$

Next, we introduce the properties of fractional (p,q)-integral.

Theorem 3.1 *For $f : I_{p,q}^{T} \to \mathbb{R}$, $\alpha > 0$, $0 < q < p \le 1$,*

$$\mathcal{I}_{p,q}^{\alpha}f(t) = \mathcal{I}_{p,q}^{\alpha+1}\left[D_{p,q}f(t)\right] + \frac{f(0)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)}t^{\alpha}.$$

Proof Using Lemma 2.4(b) and Lemma 2.5(f), we obtain

$$\begin{aligned}
\mathcal{I}_{p,q}^{\alpha}f(t) &= -\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)[\alpha]_{p,q}}\int_0^t D_{p,q}(t-s)_{p,q}^{\underline{\alpha}}f\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s \\
&= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)}\left[f(0)t^{\alpha} + \frac{1}{p^{\alpha}}\int_0^t (t-qs)_{p,q}^{\underline{\alpha}}D_{q,\omega}f\left(\frac{s}{p^{\alpha}}\right)d_{p,q}s\right] \\
&= \mathcal{I}_{p,q}^{\alpha+1}\left[D_{p,q}f(t)\right] + \frac{f(0)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)}t^{\alpha}.
\end{aligned}$$

The proof is completed. □

Theorem 3.2 *For $f : I_{p,q}^{T} \to \mathbb{R}$, $\alpha, \beta > 0$, $0 < q < p \le 1$, and $a \in I_{p,q}^{T}$,*

$$\int_0^a (t-qs)_{p,q}^{\underline{\beta-1}}\mathcal{I}_{p,q}^{\alpha}f(ps)\,d_{p,q}s = 0.$$

Proof For $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{I}_{p,q}^{\alpha}f\left(\sigma_{p,q}^{n}(a)\right) &= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\sigma_{p,q}^{n}(a)}\left(\sigma_{p,q}^{n}(a)-qs\right)_{p,q}^{\underline{\alpha-1}}f\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s \\
&= \frac{(p-q)\sigma_{p,q}^{n}(a)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(\sigma_{p,q}^{n}(a)-\sigma_{p,q}^{n+k+1}(a)\right)_{p,q}^{\underline{\alpha-1}}f\left(\frac{q^{k+n}}{p^{k+n+\alpha}}a\right).
\end{aligned}$$

By using Lemma 2.1(c), it implies that $(\sigma_{p,q}^{n}(a)-\sigma_{p,q}^{n+k+1}(a))_{p,q}^{\underline{\alpha-1}} = 0$. Thus,

$$\mathcal{I}_{p,q}^{\alpha}f\left(\sigma_{p,q}^{n}(a)\right) = 0. \tag{3.2}$$

Finally, by Definition 3.1, we find that

$$\begin{aligned}
&\int_0^a (t-qs)_{p,q}^{\underline{\beta-1}}\mathcal{I}_{p,q}^{\alpha}f(ps)\,d_{p,q}s \\
&\quad = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-\left(\frac{q}{p}\right)^{k+1}a\right)_{p,q}^{\underline{\beta-1}}\left[\mathcal{I}_{p,q}^{\alpha}f\left(\sigma_{p,q}^{k}(a)\right)\right] = 0.
\end{aligned}$$

The proof is completed. □

Lemma 3.1 ([50]) *For* $\mu, \alpha, \beta \in \mathbb{R}^+$,

$$\sum_{k=0}^{\infty} q^{\alpha k} \frac{(1-\mu q^{1-k})_q^{\underline{\alpha-1}}(1-\mu q^{1+k})_q^{\underline{\beta-1}}}{(1-q)_q^{\underline{\alpha-1}}(1-q)_q^{\underline{\beta-1}}} = \frac{(1-\mu q)_q^{\underline{\alpha+\beta-1}}}{(1-q)_q^{\underline{\alpha+\beta-1}}}.$$

Theorem 3.3 *For* $f: I_{p,q}^T \to \mathbb{R}$,$\alpha, \beta > 0$, *and* $0 < q < p \leq 1$,

$$\mathcal{I}_{p,q}^{\alpha}\big[\mathcal{I}_{p,q}^{\beta} f(t)\big] = \mathcal{I}_{p,q}^{\beta}\big[\mathcal{I}_{p,q}^{\alpha} f(t)\big] = \mathcal{I}_{p,q}^{\alpha+\beta} f(t).$$

Proof For $t \in I_{p,q}^T$,

$$\begin{aligned}
&\mathcal{I}_{q,\omega}^{\alpha}\mathcal{I}_{q,\omega}^{\beta} f(t) \\
&\quad= \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \int_0^t \int_0^{\frac{x}{p^{\alpha-1}}} (t-qx)_{p,q}^{\underline{\alpha-1}} \left(\frac{x}{p^{\alpha-1}} - qs\right)_{p,q}^{\underline{\beta-1}} f\left(\frac{s}{p^{\beta-1}}\right) d_{p,q}s\, d_{p,q}x \\
&\quad= \frac{(p-q)^2 t^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}+2}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \left(\frac{q}{p}\right)^{k+h+k\beta} \left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{\alpha-1}} \left(1-\left(\frac{q}{p}\right)^{h+1}\right)_{p,q}^{\underline{\beta-1}} \\
&\qquad\times f\left(\frac{q^{k+h}}{p^{k+h+\alpha+\beta}}\right) \\
&\quad= \frac{(p-q)t^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}} \sum_{h=0}^{\infty} \frac{q^h}{p^{h+1}} \left[\frac{(p-q)}{p\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \sum_{k=0}^{h} \left(\frac{q}{p}\right)^{k\beta} \left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{\alpha-1}} \right. \\
&\qquad\left.\times \left(1-\left(\frac{q}{p}\right)^{h-k+1}\right)_{p,q}^{\underline{\beta-1}}\right] f\left(\frac{q^h}{p^{h+\alpha+\beta}} t\right).
\end{aligned}$$

Since $\Gamma_{p,q}(\alpha+\beta) = \frac{(p-q)_{p,q}^{\underline{\alpha+\beta-1}}}{(p-q)^{\alpha+\beta-1}} = \frac{(1-\frac{q}{p})_{p,q}^{\underline{\alpha+\beta-1}}}{(1-\frac{q}{p})^{\alpha+\beta-1}}$ and by Lemma 3.1, we obtain

$$\begin{aligned}
&\sum_{k=0}^{h} \left(\frac{q}{p}\right)^{k\beta} \left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{\alpha-1}} \left(1-\left(\frac{q}{p}\right)^{h-k+1}\right)_{p,q}^{\underline{\beta-1}} \\
&\quad= p^{\binom{\alpha-1}{2}+\binom{\alpha-1}{2}} \sum_{k=0}^{\infty} \left(\frac{q}{p}\right)^{k\beta} \left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{\frac{q}{p}}^{\underline{\alpha-1}} \left(1-\left(\frac{q}{p}\right)^{h-k+1}\right)_{\frac{q}{p}}^{\underline{\beta-1}} \\
&\quad= \Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta) \cdot \frac{(1-\frac{q}{p})^{\alpha+\beta-1}}{(1-\frac{q}{p})} \cdot \frac{(1-(\frac{q}{p})^{h+1})_{p,q}^{\underline{\alpha+\beta-1}}}{(1-\frac{q}{p})_{p,q}^{\underline{\alpha+\beta-1}}} \\
&\quad= \frac{p\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}{p-q} \cdot \frac{(1-(\frac{q}{p})^{h+1})_{p,q}^{\underline{\alpha+\beta-1}}}{\Gamma_{p,q}(\alpha+\beta)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{I}_{q,\omega}^{\alpha}\mathcal{I}_{q,\omega}^{\beta} f(t) &= \frac{(p-q)t^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} \sum_{h=0}^{\infty} \frac{q^h}{p^{h+1}} \left(1-\left(\frac{q}{p}\right)^{h+1}\right)_{p,q}^{\underline{\alpha+\beta-1}} f\left(\frac{q^h}{p^{h+\alpha+\beta}} t\right) \\
&= \frac{(p-q)t}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} \sum_{h=0}^{\infty} \frac{q^h}{p^{h+1}} \left(t-\left(\frac{q}{p}\right)^{h+1} t\right)_{p,q}^{\underline{\alpha+\beta-1}} f\left(\frac{q^h}{p^{h+\alpha+\beta}} t\right)
\end{aligned}$$

$$= \frac{1}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}\int_0^t (t-qx)_{p,q}^{\underline{\alpha+\beta-1}} f\left(\frac{x}{p^{\alpha+\beta-1}}\right) d_{p,q}x$$
$$= \mathcal{I}_{p,q}^{\alpha+\beta} f(t).$$

Similarly, we find that $\mathcal{I}_{p,q}^{\beta}\mathcal{I}_{p,q}^{\alpha} f(t) = \mathcal{I}_{p,q}^{\alpha+\beta} f(t)$. □

4 Fractional (p,q)-difference operator of Riemann–Liouville type

Next, we present the fractional (p,q)-difference operator of Riemann–Liouville.

Definition 4.1 For $\alpha > 0, 0 < q < p \le 1$, and f defined on $I_{p,q}^T$, the fractional (p,q)-difference operator of Riemann–Liouville type of order α is defined by

$$D_{p,q}^{\alpha} f(t) := D_{p,q}^{N}\mathcal{I}_{p,q}^{N-\alpha} f(t),$$

and $D_{p,q}^{0} f(t) = f(t)$, where $N-1 < \alpha < N, N \in \mathbb{N}$.

In the following theorem, we introduce the properties of fractional (p,q)-difference operator of Riemann–Liouville type.

Theorem 4.1 *For* $\alpha > 0, 0 < q < p \le 1$, *and* $f : I_{p,q}^T \to \mathbb{R}$,

$$D_{p,q}^{\alpha}\mathcal{I}_{p,q}^{\alpha} f(t) = f(t).$$

Proof For some $N-1 < \alpha < N, N \in \mathbb{N}$,

$$D_{p,q}^{\alpha}\mathcal{I}_{p,q}^{\alpha} f(t) = D_{p,q}^{N}\mathcal{I}_{p,q}^{N-\alpha}\mathcal{I}_{p,q}^{\alpha} f(t) = D_{p,q}^{N}\mathcal{I}_{p,q}^{N} f(t) = f(t).$$

The proof is completed. □

Theorem 4.2 *For* $\alpha \in (0,1), 0 < q < p \le 1$, *and* $f : I_{p,q}^T \to \mathbb{R}$,

$$\mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) = f(t) + Ct^{\alpha-1}, \quad C \in \mathbb{R}.$$

Proof Let $C(t) = \mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) - f(t)$. Taking $D_{p,q}^{\alpha}$ to the both sides of the above expression and using Theorem 4.1, we have

$$D_{p,q}^{\alpha} C(t) = D_{p,q}^{\alpha}\mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) - D_{p,q}^{\alpha} f(t) = D_{p,q}^{\alpha} f(t) - D_{p,q}^{\alpha} f(t) = 0.$$

On the other hand,

$$\int_0^t (t-qs)_{p,q}^{\underline{-\alpha}}\left(p^{\alpha}s\right)^{\alpha-1} d_{p,q}s = (p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-\left(\frac{q}{p}\right)^{k+1}t\right)_{p,q}^{\underline{-\alpha}}\left(\frac{q^k}{p^{k+1-\alpha}}t\right)^{\alpha-1}$$
$$= (p-q)\sum_{k=0}^{\infty}\frac{q^{k\alpha}}{p^{(k+1)\alpha-\alpha(\alpha-1)}}\left(1-\left(\frac{q}{p}\right)^{k+1}\right).$$

Using the above form, according to Definitions 3.1 and 4.1, we have

$$\begin{aligned}D_{p,q}^{\alpha}t^{\alpha-1} &= D_{p,q}\mathcal{I}_{p,q}^{1-\alpha}t^{\alpha-1}\\ &= D_{p,q}\left[\frac{1}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)}\int_0^t (t-qs)_{p,q}^{\underline{-\alpha}}\left(p^{\alpha}s\right)^{\alpha-1}\,d_{p,q}s\right]\\ &= D_{p,q}\left[(p-q)\sum_{k=0}^{\infty}\frac{q^{k\alpha}}{p^{(k+1)\alpha-\alpha(\alpha-1)}}\left(1-\left(\frac{q}{p}\right)^{k+1}\right)\right]\\ &= 0.\end{aligned}$$

Hence, $C(t) = Ct^{\alpha-1}$. □

Theorem 4.3 *Letting* $\alpha \in (N-1,N), N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f : I_{p,q}^{T} \to \mathbb{R}$, *we get*

$$\mathcal{I}_{p,q}^{\alpha}D_{p,q}^{\alpha}f(t) = f(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}$$

for some $C_i \in \mathbb{R}, i = 1,2,\ldots,N$.

Proof Using Theorems 3.1 and 4.2, we obtain

$$\begin{aligned}\mathcal{I}_{p,q}^{\alpha}D_{p,q}^{\alpha}f(t) &= \mathcal{I}_{p,q}^{\alpha}D_{p,q}^{N}\mathcal{I}_{p,q}^{N-\alpha}f(t)\\ &= \mathcal{I}_{p,q}^{\alpha-1}D_{p,q}^{N-1}\mathcal{I}_{p,q}^{N-\alpha}f(t) - \frac{t^{\alpha-1}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)}D_{p,q}^{N-1}\mathcal{I}_{p,q}^{N-\alpha}f(0)\\ &= \mathcal{I}_{p,q}^{\alpha-2}D_{p,q}^{N-2}\mathcal{I}_{p,q}^{N-\alpha}f(t) - \frac{t^{\alpha-2}}{p^{\binom{\alpha-2}{2}}\Gamma_{p,q}(\alpha-1)}D_{p,q}^{N-2}\mathcal{I}_{p,q}^{N-\alpha}f(0)\\ &\quad - \frac{t^{\alpha-1}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)}D_{p,q}^{N-1}\mathcal{I}_{p,q}^{N-\alpha}f(0)\\ &\quad\vdots\\ &= \mathcal{I}_{p,q}^{\alpha-N+1}D_{p,q}^{\alpha-N+1}f(t) - \frac{t^{\alpha-N+1}}{p^{\binom{\alpha-N+1}{2}}\Gamma_{p,q}(\alpha-N+2)}D_{p,q}\mathcal{I}_{p,q}^{N-\alpha}f(0)\\ &\quad - \cdots - \frac{t^{\alpha-2}}{p^{\binom{\alpha-2}{2}}\Gamma_{p,q}(\alpha-1)}D_{p,q}^{N-2}\mathcal{I}_{p,q}^{N-\alpha}f(0)\\ &\quad - \frac{t^{\alpha-1}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)}D_{p,q}^{N-1}\mathcal{I}_{p,q}^{N-\alpha}f(0).\end{aligned}$$

Using Theorem 4.2, we obtain

$$\mathcal{I}_{p,q}^{\alpha}D_{p,q}^{\alpha}f(t) = f(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}.$$

The proof is completed. □

Corollary 4.1 *Let* $\alpha \in (N-1,N), N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f : I_{p,q}^{T} \to \mathbb{R}$,

$$\mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{t^{\alpha-N+k}}{p^{\binom{\alpha-N+k}{2}} \Gamma_{p,q}(\alpha - N + k + 1)} \big[D_{p,q}^{\alpha-N+k} f(0)\big].$$

5 Fractional (p,q)-difference operator of Caputo type

Now, we introduce fractional (p,q)-difference operator of Caputo type.

Definition 5.1 For $\alpha > 0, 0 < q < p \leq 1$, and f defined on $I_{p,q}^{T}$, the fractional (p,q)-difference operator of Caputo type of order α is defined by

$$\begin{aligned} {}^{C}D_{p,q}^{\alpha} f(t) &:= \mathcal{I}_{p,q}^{N-\alpha} D_{p,q}^{N} f(t) \\ &= \frac{1}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \int_{0}^{t} (t-qs)_{p,q}^{\underline{N-\alpha-1}} D_{p,q}^{N} f\left(\frac{s}{p^{N-\alpha-1}}\right) d_{q,\omega}s, \end{aligned}$$

and ${}^{C}D_{p,q}^{0} f(t) = f(t)$, where $N-1 < \alpha < N, N \in \mathbb{N}$.

Theorem 5.1 *Letting* $\alpha \in (N-1,N), N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f : I_{p,q}^{T} \to \mathbb{R}$ *leads to*

$${}^{C}D_{p,q}^{\alpha} f(t) = \frac{(p-q)t^{N-\alpha}}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - \left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{N-\alpha-1}} D_{p,q}^{N} f\left(\frac{q^k}{p^{k+N-\alpha}} t\right).$$

Proof For $t \in I_{q,\omega}^{T}$, by Definition 5.1, we have

$$\begin{aligned} {}^{C}D_{q,\omega}^{\alpha} f(t) &= \frac{1}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \int_{0}^{t} (t-qs)_{p,q}^{\underline{N-\alpha-1}} D_{p,q}^{N} f\left(\frac{s}{p^{N-\alpha-1}}\right) d_{q,\omega}s \\ &= \frac{(p-q)t}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t - \left(\frac{q}{p}\right)^{k+1} t\right)_{p,q}^{\underline{N-\alpha-1}} D_{p,q}^{N} f\left(\frac{q^k}{p^{k+N-\alpha}} t\right) \\ &= \frac{(p-q)t^{N-\alpha}}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - \left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{\underline{N-\alpha-1}} D_{p,q}^{N} f\left(\frac{q^k}{p^{k+N-\alpha}} t\right). \end{aligned}$$ □

The following theorem presents the properties of fractional Hahn difference operator of Caputo type.

Theorem 5.2 *Let* $\alpha \in (N-1,N), N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f : I_{p,q}^{T} \to \mathbb{R}$. *Then*

$${}^{C}D_{p,q}^{\alpha} \mathcal{I}_{p,q}^{\alpha} f(t) = f(t).$$

Proof For some $N-1 < \alpha < N, N \in \mathbb{N}$, by Definition 5.1 and Theorem 4.3, we have

$$\begin{aligned} {}^{C}D_{p,q}^{\alpha} \mathcal{I}_{p,q}^{\alpha} f(t) &= \mathcal{I}_{p,q}^{N-\alpha} D_{p,q}^{N} \mathcal{I}_{p,q}^{\alpha} f(t) = \mathcal{I}_{p,q}^{N-\alpha} D_{p,q}^{N-\alpha} f(t) \\ &= f(t) - \sum_{k=0}^{N-1} \frac{t^{k-\alpha}}{p^{\binom{k-\alpha}{2}} \Gamma_{p,q}(k-\alpha+1)} \big[D_{p,q}^{k} \mathcal{I}_{p,q}^{\alpha} f(0)\big]. \end{aligned}$$

From (3.2), we have

$$\sum_{k=0}^{N-1} \frac{t^{k-\alpha}}{p^{\binom{k-\alpha}{2}} \Gamma_{p,q}(k-\alpha+1)} \big[D^k_{p,q} \mathcal{I}^\alpha_{p,q} f(0) \big] = 0.$$

It implies that

$${}^C D^\alpha_{p,q} \mathcal{I}^\alpha_{p,q} f(t) = f(t).$$

The proof is completed. □

Theorem 5.3 *Let $\alpha \in (N-1, N), N \in \mathbb{N}, 0 < q < p \leq 1$, and $f : I^T_{p,q} \to \mathbb{R}$. Then*

$$\mathcal{I}^\alpha_{p,q} {}^C D^\alpha_{p,q} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{t^k}{p^{\binom{k}{2}} [k]_{p,q}!} \big[D^k_{p,q} f(0) \big].$$

Proof From Definition 5.1, Theorem 4.3, and Corollary 4.1, we have

$$\begin{aligned} \mathcal{I}^\alpha_{q,\omega} {}^C D^\alpha_{q,\omega} f(t) &= \mathcal{I}^\alpha_{q,\omega} \mathcal{I}^{N-\alpha}_{q,\omega} D^N_{q,\omega} f(t) = \mathcal{I}^N_{q,\omega} D^N_{q,\omega} f(t) \\ &= f(t) - \sum_{k=0}^{N-1} \frac{t^k}{p^{\binom{k}{2}} \Gamma_{p,q}(k+1)} \big[D^k_{p,q} f(0) \big] \\ &= f(t) - \sum_{k=0}^{N-1} \frac{t^k}{p^{\binom{k}{2}} [k]_{p,q}!} \big[D^k_{p,q} f(0) \big]. \end{aligned}$$

The proof is completed. □

Corollary 5.1 *Let $\alpha \in (N-1, N), N \in \mathbb{N}, 0 < q < p \leq 1$, and $f : I^T_{p,q} \to \mathbb{R}$. Then*

$$\mathcal{I}^\alpha_{p,q} {}^C D^\alpha_{p,q} f(t) = f(t) + C_1 t^{N-1} + C_2 t^{N-2} + \cdots + C_N$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, N$.

6 Conclusions

In this paper, we introduced fractional (p,q)-integral, Riemann–Liouville, and Caputo fractional (p,q)-difference operators. Many properties of these fractional (p,q)-operators were proved. This work would be a starting point of many other works. For example, in the future we may define the Laplace transform for fractional (p,q)-calculus, find the fractional (p,q)-convolution product, and compute its fractional (p,q)-Laplace transform. In addition, we could solve many fractional (p,q)-difference equations by using this transform.

Funding
This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-61-KNOW-026.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that they carried out all the work in this manuscript and read and approved the final manuscript.

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Publisher's Note

Received: 6 November 2019 Accepted: 12 January 2020 Published online: 16 January 2020

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