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Existence and uniqueness of nontrivial solution for nonlinear fractional multi-point boundary value problem with a parameter

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Abstract

In this paper, a class of fractional boundary value problems with a parameter are discussed. We give some sufficient conditions to guarantee that above problems have a unique solution and construct the corresponding iterative sequences for approximating the unique solution.

MSC: 34B15; 34B18

Keywords: Nonhomogeneous boundary condition; Mixed monotone operator; Iterative solution; Partial order structure

1 Introduction

In this paper, we consider the existence and uniqueness of the solution of the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t),u(t))+g(t,u(t),u(t))-b=0, \quad t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ D_{0^+}^{\beta}u(1)=\sum_{i=1}^{m-2}\xi_i D_{0^+}^{\beta}u(\eta_i)+\lambda, \end{cases} \tag{1.1}$$

where $b>0$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives with $n-1<\alpha\le n$, $n-2<\beta\le n-1$, $n\ge 2$ $(n\in\mathbb{N})$, $\alpha-\beta-1>0$, $0<\xi_i,\eta_i<1$, $i=1,2,3,\ldots,m-2$, $m\ge 3$, $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}<1$. $f,g:(0,1)\times(-\infty,+\infty)\times(-\infty,+\infty)\to(-\infty,+\infty)$ are continuous, and f,g may be singular at $t=0,1$, λ is a parameter.

The problem (1.1) with $\lambda=0$ has been investigated by many authors [1–8]. Li et al. [1] considered the following fractional three-point boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,\\ u(0)=0, \qquad D_{0^+}^{\beta}u(1)=aD_{0^+}^{\beta}u(\xi), \end{cases} \tag{1.2}$$

where $1<\alpha\le 2$, $0\le\beta\le 1$, $0\le a\le 1$ and $\xi\in(0,1)$. The authors firstly derived the corresponding Green's function of the problem (1.2). Based on the above result, the problem

(1.2) is reduced to an equivalent integral equation. By using the Banach contraction mapping principle and a nonlinear alternative of Leray–Schauder type, the authors obtained the existence and multiplicity theorems of positive solutions for the problem (1.2). Subsequently, Peng and Zhou [2] studied the existence of positive solutions for the problem (1.2), the main tools adopted in [2] are topological degree theory and bifurcation techniques. In fact, in [3], Kaufmann and Mboumi have considered the fractional two-point boundary value problem (1.2) when $a = 0$ and $\beta = 1$. Furthermore, Lv [4] considered the positive solutions of the following m-point boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in [0,1], \\ u(0) = 0, \qquad D_{0^+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i), \end{cases} \tag{1.3}$$

where $1 < \alpha \le 2$, $0 \le \beta \le 1$, $\alpha - \beta - 1 \ge 0$, $0 < \xi_i, \eta_i < 1$, $i = 1, 2, 3, \ldots, m-2$, $m \ge 3$, and $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$. Lv studied the existence of minimal and maximal positive solutions for the problem (1.3). Moreover, Lv [5] used the fixed point theorem to study m-point fractional problem with the p-Laplacian operator.

In [9], Sang and Ren studied the following fractional boundary value problem:

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = f(t, u(t), u(t)) + g(t, u(t), u(t)) - b, \quad t \in (0,1), \ n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, \quad i = 0, \ldots, n-2, \\ D_{0^+}^{\beta} u(1) = 0, \quad 1 \le \beta \le n-2, \end{cases} \tag{1.4}$$

where $n \ge 3$, $b > 0$ is a constant, $f, g : [0,1] \times (-\infty, +\infty) \times (-\infty, +\infty) \longrightarrow (-\infty, +\infty)$ are continuous functions. The problem (1.4) includes the well-known elastic beam equation and fractional problems considered in [10–16].

Very recently, Wang et al. [17] discussed the following higher-order three-point fractional problem:

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = f(t, u(t), u(t)) + g(t, u(t)), \quad t \in (0,1), \ n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, \quad i = 0, \ldots, n-2, \\ D_{0^+}^{\nu} u(1) = b D_{0^+}^{\nu} u(\xi), \quad n-2 \le \nu \le n-1, \end{cases} \tag{1.5}$$

where $n \in \mathbb{N}$, $n \ge 2$, $0 \le b \le 1$, and $0 < \xi < 1$. $f(t, u, v)$ may be singular at $t = 0, 1$ and $v = 0$, $g(t, u)$ may be singular at $t = 0, 1$. By the properties of the Green function and two fixed point theorems for sum-type operator, the authors derived sufficient conditions for the existence and uniqueness of positive solutions to the problem (1.5).

On the other hand, fractional boundary value problems with parameters have received considerable attention [18–27]. Tan, Tan and Zhou [18] considered the existence of positive solutions for fractional differential equations with a parameter as follows:

$$\begin{cases} -D_{0^+}^{\alpha} x(t) = f_1(t, x(t), x(t)) + f_2(t, x(t)), \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(k)}(0) = 0, \quad 0 \le k \le n-2, \\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) + \lambda, \quad m \ge 3, \end{cases} \tag{1.6}$$

where $n-1 < \alpha \le n, n \ge 2, f_1 : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty), f_2 : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\beta-1} < 1$, and λ is a parameter. In

[19, 20], the authors studied nonlinear boundary value problem with boundary conditions $u(0) - \sum_{i=1}^{m} a_i u(t_i) = \lambda_1$ and $u(1) - \sum_{i=1}^{m} b_i u(t_i) = \lambda_2$. In addition, Graef and Kong [21] considered the boundary value problem with fractional q-derivatives, and studied the existence of positive solutions according to different ranges of parameter. Moreover, Li et al. [22] considered infinite point boundary value problem for fractional differential equations with perturbed parameter. In [24], Lee and Park considered non-local problems with the boundary value condition $u(1) - \int_0^1 g(s)u(s)\,ds = b$. In [25], Wang and Guo studied fractional differential equations with boundary condition $x(1) = \int_0^1 k(s)g(x(s))\,ds + \mu$. Jia and Liu [26] discussed the effect of the mixed boundary condition $m_2 u(1) + n_2 u'(1) = \int_0^1 g(s)u(s)\,ds + a$.

In this paper, we first consider the Green function of the m-point boundary value problem (1.1) with a parameter. Then we define a new set, which is not a subset of a cone. So we extend the results of the cone mapping established in [17, 18] to the non-cone cases. Finally, we will consider the singularity of f, g and provide some sufficient conditions to guarantee that the problem (1.1) has a unique solution and construct two iterative sequences of solutions.

The rest of this paper is structured as follows. In Sect. 2, we will give some definitions and related lemmas to prove the main result. In Sect. 3, the existence and uniqueness of the solution to the problem (1.1) is proved, and an example supporting conclusion is given.

2 Preliminaries and related lemmas

In this section, we will provide some necessary basic definitions and lemmas to prove our main theorem, which can be found in [28–32].

Throughout our article, we define its base space as a Banach space. Let E be a Banach space, and θ be the zero element of E. If there are (1) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$ and (2) $x \in P$, $-x \in P \Rightarrow x = \theta$, then we call that a nonempty closed convex set $P \subset E$ is a cone. Define an ordered relation in E: $x \leq y$ if and only if $y - x \in P$. If there exists a positive constant N such that, for all $x, y \in E$, $\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|$, then P is called a normal cone. Given $h > \theta$, we denote P_h by

$$P_h = \{x \in E \mid \text{ there exist } \lambda > 0, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}.$$

Let $e \in P$ with $\theta \leq e \leq h$, denote

$$P_{h,e} = \{x \in E | x + e \in P_h\}.$$

Definition 2.1 ([28, 29]) If $B(x, y)$ is increasing in x, and decreasing in y, then $B : P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator. i.e., for every $u_i, v_i \in P_{h,e}$ $(i = 1, 2)$, with $u_1 \geq v_1$, $u_2 \leq v_2$, implies $B(u_1, u_2) \geq B(v_1, v_2)$.

Definition 2.2 ([31, 32]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $h \in C[0, 1]$ is defined by

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t h(s)(t-s)^{n-\alpha-1}\,ds,$$

where $n = [\alpha] + 1$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds.$$

Definition 2.3 ([32]) Let $\beta > -1$, $\alpha > 0$ and $t > 0$. Then

$$D_{0^+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$$

Lemma 2.1 ([7]) *Let* $u \in C[0,1] \cap L^1[0,1]$, $\alpha > 0$, *then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n = [\alpha] + 1$.

Lemma 2.2 *Let* $h(t) \in C(0,1) \cap L^1(0,1)$, *then the following fractional boundary value problem*:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) h(s)\,ds + \lambda \frac{\Gamma(\alpha-\beta) t^{\alpha-1}}{A\Gamma(\alpha)},$$

where $n-1 < \alpha \le n$, $n-2 < \beta \le n-1$, $n \ge 2$, $m \ge 3$,

$$G(t,s) = G_1(t,s) + G_2(t,s),$$

in which

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{A\Gamma(\alpha)} \begin{cases} t^{\alpha-1} \sum_{0 \le s \le \eta_i} \xi_i [\eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (\eta_i - s)^{\alpha-\beta-1}], & 0 \le t,s \le 1, \\ t^{\alpha-1} \sum_{\eta_i \le s \le 1} \xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & 0 \le t,s \le 1, \end{cases}$$

with

$$A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}.$$

Proof Using Lemma 2.1, we get

$$u(t) = -I_{0^+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

From condition $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we obtain $c_n = c_{n-1} = \cdots = c_2 = 0$. Thus

$$u(t) = -I_{0^+}^{\alpha} h(t) + c_1 t^{\alpha-1}.$$

By Definition 2.3, we deduce that

$$\begin{aligned} D_{0^+}^{\beta} u(t) &= -I_{0^+}^{\alpha-\beta} h(t) + D_{0^+}^{\beta} t^{\alpha-1} c_1 \\ &= -I_{0^+}^{\alpha-\beta} h(t) + \frac{\Gamma(\alpha)c_1}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\beta-1}h(s)}{\Gamma(\alpha-\beta)}\,ds + \frac{\Gamma(\alpha)c_1}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned}$$

From the boundary value condition $D_{0^+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda$, we have

$$\begin{aligned} D_{0^+}^{\beta} u(1) &= -\int_0^1 \frac{(1-s)^{\alpha-\beta-1}h(s)}{\Gamma(\alpha-\beta)}\,ds + \frac{\Gamma(\alpha)c_1}{\Gamma(\alpha-\beta)} \\ &= \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda, \end{aligned}$$

which yields

$$c_1 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \left(\sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda + \int_0^1 \frac{(1-s)^{\alpha-\beta-1}h(s)}{\Gamma(\alpha-\beta)}\,ds \right).$$

Thus

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha} h(t) + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \left(\sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda \right) t^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} h(s) t^{\alpha-1}\,ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \left(\sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda \right) t^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} h(s) t^{\alpha-1}\,ds. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) &= \sum_{i=1}^{m-2} \xi_i \left(-I_{0^+}^{\alpha-\beta} h(\eta_i) + D_{0^+}^{\beta} \eta_i^{\alpha-1} c_1 \right) \\ &= -\sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-\beta-1}h(s)}{\Gamma(\alpha-\beta)}\,ds + \sum_{i=1}^{m-2} \xi_i \frac{\Gamma(\alpha)\eta_i^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} c_1 \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-\beta-1} h(s)}{\Gamma(\alpha-\beta)}\,ds \\
&\quad + \sum_{i=1}^{m-2} \xi_i \left(\sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) + \lambda + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} h(s)}{\Gamma(\alpha-\beta)}\,ds \right) \eta_i^{\alpha-\beta-1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{i=1}^{m-2} \xi_i D_{0^+}^{\beta} u(\eta_i) &= -\frac{1}{A} \sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-\beta-1} h(s)}{\Gamma(\alpha-\beta)}\,ds + \frac{\lambda}{A} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} \\
&\quad + \frac{1}{A} \int_0^1 \frac{(1-s)^{\alpha-\beta-1} h(s) \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\,ds,
\end{aligned}
$$

where

$$
A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}.
$$

Therefore, the boundary value problem (2.1) has the unique solution

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} t^{\alpha-1} h(s)\,ds \\
&\quad - \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta-1} t^{\alpha-1} h(s)\,ds \\
&\quad + \frac{\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}}{A\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} t^{\alpha-1} h(s)\,ds + \frac{\Gamma(\alpha-\beta) t^{\alpha-1} \lambda}{A\Gamma(\alpha)} \\
&= \int_0^1 G_1(t,s) h(s)\,ds + \int_0^1 G_2(t,s) h(s)\,ds + \frac{\Gamma(\alpha-\beta) t^{\alpha-1} \lambda}{A\Gamma(\alpha)} \\
&= \int_0^1 G(t,s) h(s)\,ds + \frac{\Gamma(\alpha-\beta) t^{\alpha-1} \lambda}{A\Gamma(\alpha)}.
\end{aligned}
$$

The proof is complete. □

Lemma 2.3 *Let*

$$
C(s) = \frac{1}{A} \sum_{0 \le s \le \eta_i} \xi_i \bigl[\eta_i^{\alpha-\beta-1} (1-s)^{\alpha-\beta-1} - (\eta_i - s)^{\alpha-\beta-1} \bigr] + \sum_{s \ge \eta_i} \xi_i \eta_i^{\alpha-\beta-1} (1-s)^{\alpha-\beta-1}
$$

and

$$
D = \frac{1}{A} \left(1 + \sum_{i=1}^{m-2} \xi_i \bigl(1 - \eta_i^{\alpha-\beta-1} \bigr) \right).
$$

Then the function $G(t,s)$ defined in Lemma 2.2 satisfies

$$
C(s) t^{\alpha-1} \le \Gamma(\alpha) G(t,s) \le D t^{\alpha-1},
$$

where $t, s \in [0,1]$.

Lemma 2.4 ([9]) *Let $P \subset E$ be a normal cone, and let $M, N : P_{h,e} \times P_{h,e} \longrightarrow E$ be two mixed monotone operators. Suppose that*

(L1) *for all $t \in [0,1]$ and $x, y \in P_{h,e}$, there exists $\psi(t) \in (t,1)$ such that*

$$M\bigl(tx + (t-1)e, t^{-1}y + \bigl(t^{-1} - 1\bigr)e\bigr) \geq \psi(t)M(x,y) + \bigl(\psi(t) - 1\bigr)e;$$

(L2) *for all $t \in [0,1]$ and $x, y \in P_{h,e}$,*

$$N\bigl(tx + (t-1)e, t^{-1}y + \bigl(t^{-1} - 1\bigr)e\bigr) \geq tN(x,y) + (t-1)e;$$

(L3) *$M(h,h) \in P_{h,e}$ and $N(h,h) \in P_{h,e}$;*

(L4) *for all $x, y \in P_{h,e}$, there exists a constant $\delta > 0$ such that*

$$M(x,y) \geq \delta N(x,y) + (\delta - 1)e.$$

Then the operator equation $M(x,x) + N(x,x) + e = x$ has a unique solution x^* in $P_{h,e}$, for any initial values $x_0, y_0 \in P_{h,e}$, we can get the following iterative sequences:

$$\begin{aligned} x_n &= M(x_{n-1}, y_{n-1}) + N(x_{n-1}, y_{n-1}) + e, \\ y_n &= M(y_{n-1}, x_{n-1}) + N(y_{n-1}, x_{n-1}) + e, \quad n = 1, 2, \ldots, \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

3 Main result

In this section, we will consider the existence and uniqueness of the solution to the boundary value problem (1.1).

For convenience in the proof, we work in a Banach space $E = C[0,1]$. Let $P \subset E$ be defined by $P = \{u \in E | u(t) \geq 0, t \in [0,1]\}$, it is clear that P is a normal cone. Let

$$e(t) = \frac{bt^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{bt^{\alpha-1}}{\alpha\Gamma(\alpha)} + \frac{bt^{\alpha-1}(\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} - \sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta})}{A(\alpha-\beta)\Gamma(\alpha)}.$$

Theorem 3.1 *Assume that*

(C1) *$f, g : (0,1) \times [-e^*, +\infty) \times [-e^*, +\infty) \to (-\infty, +\infty)$ are continuous and f, g may be singular at $t = 0, 1$, where $e^* = \max\{e(t) : t \in [0,1]\}$. For $t \in [0,1]$, $g(t, 0, H) \geq 0$ with $g(t, 0, H) \not\equiv 0$ where $H \geq \frac{b}{A\Gamma(\alpha)(\alpha-\beta)}$;*

(C2) *for fixed $t \in [0,1]$ and $y \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ are increasing in $x \in [-e^*, +\infty)$; for fixed $t \in [0,1]$ and $x \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ are decreasing in $y \in [-e^*, +\infty)$;*

(C3) *for $\mu \in (0,1)$ and $t \in [0,1]$, there exists $\psi(\mu) \in (\mu, 1)$ such that*
  (a) *$f(t, \mu x + (\mu-1)\rho, \mu^{-1}y + (\mu^{-1} - 1)\rho) \geq \psi(\mu)f(t,x,y)$,*
  (b) *$g(t, \mu x + (\mu-1)\rho, \mu^{-1}y + (\mu^{-1} - 1)\rho) \geq \mu g(t,x,y)$,*
  where $x, y \in [-e^*, +\infty)$, $\rho \in [0, e^*]$;

(C4) *for all $t \in [0,1]$, $x, y \in [-e^*, +\infty)$, there exists $\delta > 0$ such that*

$$f(t,x,y) \geq \delta g(t,x,y) + \frac{\delta^2\Gamma(\alpha-\beta)}{C(s)A};$$

(C5) $\int_0^1 f(s,H,0)\,ds < \infty$ *and* $\int_0^1 g(s,H,0)\,ds < \infty$. *Then, for every* $\lambda \in (0,\delta]$, *the problem* (1.1) *has a unique nontrivial solution* u^* *in* $P_{h,e}$, *where* $h(t) = Ht^{\alpha-1}$, *for all* $t \in [0,1]$. *We can construct two iterative sequences*:

$$\begin{aligned}
\omega_n(t) &= \int_0^1 G(t,s)\bigl(f\bigl(s,\omega_{n-1}(s),\tau_{n-1}(s)\bigr) + g\bigl(s,\omega_{n-1}(s),\tau_{n-1}(s)\bigr)\bigr)\,ds - e(t) \\
&\quad + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)}, \quad n = 1,2,\ldots, \\
\tau_n(t) &= \int_0^1 G(t,s)\bigl(f\bigl(s,\tau_{n-1}(s),\omega_{n-1}(s)\bigr) + g\bigl(s,\tau_{n-1}(s),\omega_{n-1}(s)\bigr)\bigr)\,ds - e(t) \\
&\quad + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)}, \quad n = 1,2,\ldots,
\end{aligned}$$

for any initial values $\omega_0, \tau_0 \in P_{h,e}$, *the sequences* $\{\omega_n(t)\}$, $\{\tau_n(t)\}$ *approximate* u^*, *that is*, $\omega_n \to u^*$ *and* $\tau_n \to u^*$ *as* $n \to \infty$.

Proof By Lemma 2.2, we obtain

$$\begin{aligned}
\int_0^1 G(t,s)\,ds &= \int_0^1 G_1(t,s)\,ds + \int_0^1 G_2(t,s)\,ds \\
&= \frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{t^{\alpha-1}}{\alpha\Gamma(\alpha)} + \frac{t^{\alpha-1}(\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} - \sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta})}{A(\alpha-\beta)\Gamma(\alpha)}.
\end{aligned}$$

For all $t \in [0,1]$,

$$\begin{aligned}
0 < e(t) &= \frac{bt^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{bt^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{bt^{\alpha-1}(\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} - \sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta})}{A(\alpha-\beta)\Gamma(\alpha)} \\
&\leq \frac{bt^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} + \frac{bt^{\alpha-1}\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}}{A(\alpha-\beta)\Gamma(\alpha)} \\
&= \frac{bt^{\alpha-1}}{A(\alpha-\beta)\Gamma(\alpha)} \leq Ht^{\alpha-1} = h(t),
\end{aligned}$$

where $H \geq \frac{b}{A(\alpha-\beta)\Gamma(\alpha)}$. Hence, $0 < e(t) \leq h(t)$ and $P_{h,e} = \{u \in E | u + e \in P_h\}$. By Lemma 2.3, the solution to problem (1.1) has the following expression:

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\bigl(f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s),u(s)\bigr) - b\bigr)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),u(s)\bigr)\,ds \\
&\quad - b\int_0^1 G(t,s)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),u(s)\bigr)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&\quad - \left(\frac{bt^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{bt^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{bt^{\alpha-1}(\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} - \sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta})}{A(\alpha-\beta)\Gamma(\alpha)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)f\big(s,u(s),u(s)\big)\,ds + \int_0^1 G(t,s)g\big(s,u(s),u(s)\big)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} - e(t) \\
&= \int_0^1 G(t,s)f\big(s,u(s),u(s)\big)\,ds - e(t) + \int_0^1 G(t,s)g\big(s,u(s),u(s)\big)\,ds - e(t) \\
&\quad + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} + e(t).
\end{aligned}
$$

For every $t \in [0,1]$, $u, v \in P_{h,e}$, we define the following operators:

$$
\begin{aligned}
&M(u,v)(t) = \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds - e(t), \\
&N(u,v)(t) = \int_0^1 G(t,s)g\big(s,u(s),v(s)\big)\,ds - e(t) + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)}.
\end{aligned}
$$

Clearly, $u(t)$ is the solution to problem (1.1), if and only if $u(t)$ is the fixed point of the operator $M(u,v)(t) + N(u,v)(t) + e(t)$. Therefore, if it can be proved that the operators M, N satisfy all the conditions of the Lemma 2.4, then the conclusion of Theorem 3.1 holds.

(1) By (C3), for $t \in [0,1]$, $\mu \in (0,1)$, $x, y \in P_{h,e}$, and $\rho \in [0, e^*]$, we have

$$
\begin{aligned}
&f\big(t, \mu^{-1}x + \big(\mu^{-1}-1\big)\rho, \mu y + (\mu-1)\rho\big) \le \psi(\mu)^{-1} f(t,x,y), \\
&g\big(t, \mu^{-1}x + \big(\mu^{-1}-1\big)\rho, \mu y + (\mu-1)\rho\big) \le \mu^{-1} g(t,x,y).
\end{aligned}
$$

For all $u, v \in P_{h,e}$, there exists $0 < m < 1$ such that $mh - e \le u, v \le \frac{1}{m}h - e$, where $h(t) = Ht^{\alpha-1}$. From $h(t) \ge e(t)$, we get
$(m-1)e \le mh - e \le u, v \le \frac{1}{m}h - e \le \frac{1}{m}h + (\frac{1}{m}-1)e$. Thus

$$
\begin{aligned}
f\big(t,u(t),v(t)\big) &\le f\left(t, \frac{1}{m}h(t) + \left(\frac{1}{m}-1\right)e, (m-1)e\right) \le \psi(m)^{-1} f\big(t,h(t),0\big) \\
&= \psi(m)^{-1} f\big(t, Ht^{\alpha-1}, 0\big) \le \psi(m)^{-1} f(t,H,0), \\
g\big(t,u(t),v(t)\big) &\le g\left(t, \frac{1}{m}h(t) + \left(\frac{1}{m}-1\right)e, (m-1)e\right) \le \frac{1}{m} g\big(t,h(t),0\big) \\
&= \frac{1}{m} g\big(t, Ht^{\alpha-1}, 0\big) \le \frac{1}{m} g(t,H,0).
\end{aligned}
$$

In view of (C5), we get

$$
\begin{aligned}
\int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds &\le \int_0^1 G(t,s)\psi(m)^{-1} f(s,H,0)\,ds \\
&\le \frac{Dt^{\alpha-1}\psi(m)^{-1}}{\Gamma(\alpha)} \int_0^1 f(s,H,0)\,ds < \infty, \\
\int_0^1 G(t,s)g\big(s,u(s),v(s)\big)\,ds &\le \int_0^1 G(t,s)\frac{1}{m} g(s,H,0)\,ds \\
&\le \frac{Dt^{\alpha-1}}{m\Gamma(\alpha)} \int_0^1 g(s,H,0)\,ds < \infty.
\end{aligned}
$$

Therefore

$$\begin{aligned} M(u,v)(t) &= \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds - e(t) < \infty, \\ N(u,v)(t) &= \int_0^1 G(t,s)g\big(s,u(s),v(s)\big)\,ds - e(t) + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} < \infty, \end{aligned}$$

that is, M, N are well-defined.

(2) From (C2), for every $u_i, v_i \in P_{h,e}$ $(i = 1,2)$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we have

$$\begin{aligned} M(u_1,v_1)(t) &= \int_0^1 G(t,s)f\big(s,u_1(s),v_1(s)\big)\,ds - e(t) \\ &\geq \int_0^1 G(t,s)f\big(s,u_2(s),v_2(s)\big)\,ds - e(t) = M(u_2,v_2)(t), \\ N(u_1,v_1)(t) &= \int_0^1 G(t,s)g\big(s,u_1(s),v_1(s)\big)\,ds - e(t) + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\ &\geq \int_0^1 G(t,s)g\big(s,u_2(s),v_2(s)\big)\,ds - e(t) + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\ &= N(u_2,v_2)(t). \end{aligned}$$

Hence, M and N are two mixed monotone operators.

(3) By (C3), for $\mu \in (0,1)$, $t \in [0,1]$, there exists $\psi(\mu) \in (\mu,1)$ such that

$$\begin{aligned} &M\big(\mu u + (\mu-1)e, \mu^{-1}v + \big(\mu^{-1}-1\big)e\big)(t) \\ &\quad= \int_0^1 G(t,s)f\big(s,\mu u(s) + (\mu-1)e, \mu^{-1}v(s) + \big(\mu^{-1}-1\big)e\big)\,ds - e(t) \\ &\quad\geq \psi(\mu)\int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds - e(t) \\ &\quad= \psi(\mu)\int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds - e(t) + \psi(\mu)e(t) - \psi(\mu)e(t) \\ &\quad= \psi(\mu)M(u,v)(t) + \big(\psi(\mu)-1\big)e(t) \end{aligned}$$

and

$$\begin{aligned} &N\big(\mu u + (\mu-1)e, \mu^{-1}v + \big(\mu^{-1}-1\big)e\big)(t) \\ &\quad= \int_0^1 G(t,s)g\big(s,\mu u(s) + (\mu-1)e, \mu^{-1}v(s) + \big(\mu^{-1}-1\big)e\big)\,ds \\ &\qquad - e(t) + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\ &\quad\geq \mu\int_0^1 G(t,s)g\big(s,u(s),v(s)\big)\,ds - e(t) + \frac{\mu\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} + \mu e(t) - \mu e(t) \\ &\quad= \mu N(u,v)(t) + (\mu-1)e(t). \end{aligned}$$

(4) For all $t \in [0,1]$, combining with (C1) and (C2), we have

$$\begin{aligned} M(h,h)(t) + e(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds \\ &= \int_0^1 G(t,s)f\bigl(s,Hs^{\alpha-1},Hs^{\alpha-1}\bigr)\,ds \\ &\le \int_0^1 \frac{Dt^{\alpha-1}}{\Gamma(\alpha)}f(s,H,0)\,ds \\ &= \frac{D}{\Gamma(\alpha)}\int_0^1 f(s,H,0)\,ds\cdot t^{\alpha-1} \\ &= \frac{D}{H\Gamma(\alpha)}\int_0^1 f(s,H,0)\,ds\cdot h(t) \end{aligned}$$

and

$$\begin{aligned} M(h,h)(t) + e(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds \\ &= \int_0^1 G(t,s)f\bigl(s,Hs^{\alpha-1},Hs^{\alpha-1}\bigr)\,ds \\ &\ge \int_0^1 \frac{C(s)t^{\alpha-1}}{\Gamma(\alpha)}f(s,0,H)\,ds \\ &= \frac{1}{H\Gamma(\alpha)}\int_0^1 C(s)f(s,0,H)\,ds\cdot h(t). \end{aligned}$$

From (C1), (C2) and (C4), for $s \in [0,1]$, we derive that

$$f(s,H,0) \ge f(s,0,H) \ge \delta g(s,H,0) + \frac{\Gamma(\alpha-\beta)\delta^2}{AC(s)} \ge 0.$$

Thus

$$\int_0^1 f(s,H,0)\,ds \ge \int_0^1 f(s,0,H)\,ds \ge \int_0^1 \left(\delta g(s,H,0) + \frac{\Gamma(\alpha-\beta)\delta^2}{AC(s)}\right)ds \ge 0.$$

Let

$$\begin{aligned} l_1 &= \frac{D}{H\Gamma(\alpha)}\int_0^1 f(s,H,0)\,ds, \\ l_2 &= \frac{1}{H\Gamma(\alpha)}\int_0^1 C(s)f(s,0,H)\,ds. \end{aligned}$$

Therefore $l_2h(t) \le M(h,h)(t) + e(t) \le l_1h(t)$, that is, $M(h,h) \in P_{h,e}$. Similarly, we obtain

$$\begin{aligned} N(h,h)(t) + e(t) &= \int_0^1 G(t,s)g\bigl(s,h(s),h(s)\bigr)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\ &= \int_0^1 G(t,s)g\bigl(s,Hs^{\alpha-1},Hs^{\alpha-1}\bigr)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \frac{Dt^{\alpha-1}}{\Gamma(\alpha)} g(s,H,0)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&= \left(\frac{D}{\Gamma(\alpha)} \int_0^1 g(s,H,0)\,ds + \frac{\Gamma(\alpha-\beta)\lambda}{A\Gamma(\alpha)}\right) t^{\alpha-1} \\
&= \left(\frac{D}{H\Gamma(\alpha)} \int_0^1 g(s,H,0)\,ds + \frac{\Gamma(\alpha-\beta)\lambda}{HA\Gamma(\alpha)}\right) h(t)
\end{aligned}
$$

and

$$
\begin{aligned}
N(h,h)(t) + e(t) &= \int_0^1 G(t,s) g\big(s,h(s),h(s)\big)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&= \int_0^1 G(t,s) g\big(s,Hs^{\alpha-1},Hs^{\alpha-1}\big)\,ds + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} \\
&\ge \int_0^1 \frac{C(s)t^{\alpha-1}}{\Gamma(\alpha)} g(s,0,H)\,ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 C(s) g(s,0,H)\,ds \cdot t^{\alpha-1} \\
&= \frac{1}{H\Gamma(\alpha)} \int_0^1 C(s) g(s,0,H)\,ds \cdot h(t).
\end{aligned}
$$

Let

$$
\begin{aligned}
l_3 &= \frac{1}{H\Gamma(\alpha)} \int_0^1 C(s) g(s,0,H)\,ds, \\
l_4 &= \frac{D}{H\Gamma(\alpha)} \int_0^1 g(s,H,0)\,ds + \frac{\Gamma(\alpha-\beta)\lambda}{HA\Gamma(\alpha)}.
\end{aligned}
$$

Thus $l_3 h(t) \le N(h,h)(t) + e(t) \le l_4 h(t)$, that is, $N(h,h) \in P_{h,e}$.

(5) For all $u, v \in P_{h,e}$, $t \in [0,1]$ and $\lambda \in (0,\delta]$, by (C4), we have

$$
\begin{aligned}
M(u,v)(t) &= \int_0^1 G(t,s) f\big(s,u(s),v(s)\big)\,ds - e(t) \\
&\ge \int_0^1 G(t,s)\left(\delta g\big(t,u(s),v(s)\big) + \frac{\delta^2\Gamma(\alpha-\beta)}{C(s)A}\right) ds - e(t) \\
&= \delta \int_0^1 G(t,s) g\big(t,u(s),v(s)\big)\,ds + \int_0^1 G(t,s) \frac{\delta^2\Gamma(\alpha-\beta)}{C(s)A}\,ds - e(t) \\
&\ge \delta \int_0^1 G(t,s) g\big(t,u(s),v(s)\big)\,ds + \delta \int_0^1 \frac{C(s)t^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\delta\Gamma(\alpha-\beta)}{C(s)A}\,ds - e(t) \\
&\ge \delta \int_0^1 G(t,s) g\big(t,u(s),v(s)\big)\,ds + \delta \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\lambda}{A\Gamma(\alpha)} - e(t) + \delta e(t) - \delta e(t) \\
&= \delta N(u,v)(t) + (\delta-1)e(t).
\end{aligned}
$$

Thus $M(u,v)(t) \ge \delta N(u,v)(t) + (\delta-1)e(t)$. Consequently, all the conditions of Lemma 2.4 are satisfied, and the conclusion of Theorem 3.1 holds. □

Next, we will use an example to illustrate our main result.

Example 3.1 Consider the following boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t)+\frac{2}{\sqrt{1-t^2}}+(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1)^{\frac{1}{3}}+(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1)^{\frac{1}{2}}+\frac{40\Gamma(\frac{5}{4})}{3} \\ \quad +(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1)^{-\frac{1}{5}}+(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1)^{-1}-10=0, \\ u(0)=0, \\ D_{0^+}^{\frac{1}{2}}u(1)=\frac{1}{10}D_{0^+}^{\frac{1}{2}}u(\frac{1}{4})+\frac{1}{10}D_{0^+}^{\frac{1}{2}}u(\frac{1}{2})+\frac{1}{10}D_{0^+}^{\frac{1}{2}}u(\frac{3}{4})+\lambda, \end{cases} \tag{3.1}$$

where $\lambda\in(0,\frac{1}{2}]$ is a positive parameter. Then the problem (3.1) has a unique solution.

Proof The problem (3.1) can be viewed as the problem (1.1) when $n=2$, $\alpha=\frac{3}{2}$, $\beta=\frac{1}{4}$, $b=10$, $\eta_1=\frac{1}{4}$, $\eta_2=\frac{1}{2}$, $\eta_3=\frac{3}{4}$, $\xi_1=\xi_2=\xi_3=\frac{1}{10}$. Then we have

$$A\approx 0.7521>0$$

and

$$\begin{aligned} C(s) &\geq \frac{1}{A}\left[\frac{1}{10}\cdot\frac{1}{4}^{\frac{1}{4}}(1-s)^{\frac{1}{4}}+\frac{1}{10}\cdot\frac{1}{2}^{\frac{1}{4}}(1-s)^{\frac{1}{4}}+\frac{1}{10}\cdot\frac{3}{4}^{\frac{1}{4}}(1-s)^{\frac{1}{4}}-\frac{1}{10}\left(\frac{1}{4}-s\right)^{\frac{1}{4}}\right. \\ &\quad \left.-\frac{1}{10}\left(\frac{1}{2}-s\right)^{\frac{1}{4}}-\frac{1}{10}\left(\frac{3}{4}-s\right)^{\frac{1}{4}}\right] \\ &\geq \frac{1}{40}. \end{aligned}$$

A direct calculation yields

$$H\geq\frac{32}{2\Gamma(\frac{3}{2})}$$

and

$$e(t)=\frac{6}{25\Gamma(\frac{3}{2})}t^{\alpha-1}\leq Ht^{\alpha-1}=h(t).$$

Thus

$$e^*=\frac{6}{25\Gamma(\frac{3}{2})}.$$

Let

$$\begin{aligned} f(t,u,v)&=\frac{1}{\sqrt{1-t^2}}+\left(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}}+\left(v(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-\frac{1}{5}}+\frac{40\Gamma(\frac{5}{4})}{3}, \\ g(t,u,v)&=\frac{1}{\sqrt{1-t^2}}+\left(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}}+\left(v(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1}. \end{aligned}$$

It is easy to check that $f,g:(0,1)\times[-\frac{6}{25\Gamma(\frac{3}{2})},+\infty)\times[-\frac{6}{25\Gamma(\frac{3}{2})},+\infty)\to(-\infty,+\infty)$ are continuous, $f(t,u,v)$, $g(t,u,v)$ are increasing in u and decreasing in v, and f, g are singular at

$t = 1$. For $t \in [0,1]$, $g(t,0,H) = \frac{1}{\sqrt{1-t^2}} + (\frac{6}{25\Gamma(\frac{3}{2})} + 1)^{\frac{1}{3}} + (H + \frac{6}{25\Gamma(\frac{3}{2})} + 1)^{-1} > 0$. Thus, (C1) and (C2) are satisfied.

For $\mu \in (0,1)$, $u, v \in P_{h,e}$, $\rho \in [0, \frac{6}{25\Gamma(\frac{3}{2})}]$, there exists $\psi(\mu) \in (\mu, 1)$ such that

$$\begin{aligned}
&f\bigl(t, \mu u + (\mu - 1)\rho, \mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\rho\bigr) \\
&\quad = \frac{1}{\sqrt{1-t^2}} + \left(\mu u + (\mu - 1)\rho + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{2}} \\
&\qquad + \left(\mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\rho + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\quad \geq \frac{1}{\sqrt{1-t^2}} + \left(\mu u + (\mu - 1)\frac{6}{25\Gamma(\frac{3}{2})} + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{2}} \\
&\qquad + \left(\mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\frac{6}{25\Gamma(\frac{3}{2})} + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\quad = \frac{1}{\sqrt{1-t^2}} + \left(\mu u + \mu\frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{2}} + \left(\mu^{-1}v + \mu^{-1}\frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\quad \geq \frac{\mu^{\frac{1}{2}}}{\sqrt{1-t^2}} + \mu^{\frac{1}{2}}\left(u + \frac{6}{25\Gamma(\frac{3}{2})} + \frac{1}{\mu}\right)^{\frac{1}{2}} + \mu^{\frac{1}{5}}\left(v + \frac{6}{25\Gamma(\frac{3}{2})} + \mu\right)^{-\frac{1}{5}} + \frac{40\mu^{\frac{1}{2}}\Gamma(\frac{5}{4})}{3} \\
&\quad \geq \frac{\mu^{\frac{1}{2}}}{\sqrt{1-t^2}} + \mu^{\frac{1}{2}}\left(u + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{2}} + \mu^{\frac{1}{2}}\left(v + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} + \frac{40\mu^{\frac{1}{2}}\Gamma(\frac{5}{4})}{3} \\
&\quad = \psi(\mu)f(t,u,v),
\end{aligned}$$

where $\psi(\mu) = \mu^{\frac{1}{2}}$. Moreover, we deduce that

$$\begin{aligned}
&g\bigl(t, \mu u + (\mu - 1)\rho, \mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\rho\bigr) \\
&\quad = \frac{1}{\sqrt{1-t^2}} + \left(\mu u + (\mu - 1)\rho + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{3}} \\
&\qquad + \left(\mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\rho + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-1} \\
&\quad \geq \frac{1}{\sqrt{1-t^2}} + \left(\mu u + (\mu - 1)\frac{6}{25\Gamma(\frac{3}{2})} + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{3}} \\
&\qquad + \left(\mu^{-1}v + \bigl(\mu^{-1} - 1\bigr)\frac{6}{25\Gamma(\frac{3}{2})} + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-1} \\
&\quad = \frac{1}{\sqrt{1-t^2}} + \mu^{\frac{1}{3}}\left(u + \frac{6}{25\Gamma(\frac{3}{2})} + \frac{1}{\mu}\right)^{\frac{1}{3}} + \mu\left(v + \frac{6}{25\Gamma(\frac{3}{2})} + \mu\right)^{-1} \\
&\quad \geq \frac{\mu}{\sqrt{1-t^2}} + \mu\left(u + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{\frac{1}{3}} + \mu\left(v + \frac{6}{25\Gamma(\frac{3}{2})} + 1\right)^{-1} \\
&\quad = \mu g(t,u,v).
\end{aligned}$$

Thus, (C3) is satisfied. Furthermore, for $u,v\in P_{h,e}$, letting $\delta=\frac{1}{2}$, we have

$$\begin{aligned}
f(t,u,v) &= \frac{1}{\sqrt{1-t^2}} + \left(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}} + \left(v(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\geq \frac{1}{\sqrt{1-t^2}} + \left(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}} + \left(v(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\geq \frac{1}{2}\Bigg[\frac{1}{\sqrt{1-t^2}} + \left(u(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}} + \left(v(t)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1} \\
&\quad + \frac{1}{2}\cdot\frac{160\Gamma(\frac{5}{4})}{3}\Bigg] \\
&\geq \frac{1}{2}\left(g(t,u,v)+\frac{1}{2}\cdot\frac{4\Gamma(\frac{5}{4})}{3C(s)}\right) \\
&= \delta g(t,u,v)+\delta^2\frac{4\Gamma(\frac{5}{4})}{3C(s)}.
\end{aligned}$$

Therefore, (C4) holds. In addition, we get

$$\begin{aligned}
&\int_0^1 f(s,H,0)\,ds \\
&\quad= \int_0^1\left(\frac{1}{\sqrt{1-s^2}} + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3}\right)ds \\
&\quad= \int_0^1\frac{1}{\sqrt{1-s^2}}\,ds + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} \\
&\quad= \frac{\pi}{2} + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-\frac{1}{5}} + \frac{40\Gamma(\frac{5}{4})}{3} < \infty,
\end{aligned}$$

similarly,

$$\begin{aligned}
\int_0^1 g(s,H,0)\,ds &= \int_0^1\left(\frac{1}{\sqrt{1-s^2}} + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1}\right)ds \\
&= \int_0^1\frac{1}{\sqrt{1-s^2}}\,ds + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1} \\
&= \frac{\pi}{2} + \left(H+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}} + \left(\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1} < \infty.
\end{aligned}$$

Thus, (C5) is satisfied. Therefore, the application of Theorem 3.1 ensures that the problem (3.1) has a unique solution u^* for $\lambda\in(0,\frac{1}{2}]$, and we can construct the following iterative sequences:

$$\begin{aligned}
\omega_n(t) &= \int_0^1 G(t,s)\left(\frac{1}{\sqrt{1-s^2}} + \left(\omega_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}} + \left(\tau_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{-1}{5}}\right. \\
&\quad \left.+\frac{40\Gamma(\frac{5}{4})}{3}\right)ds + \int_0^1 G(t,s)\left(\frac{1}{\sqrt{1-s^2}} + \left(\omega_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left(\tau_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1}\right)ds \\
&-\frac{6}{25\Gamma(\frac{3}{2})}t^{\alpha-1}+\frac{3\Gamma(\frac{5}{4})t^{\frac{1}{2}}\lambda}{4\Gamma(\frac{3}{2})}, \quad n=1,2,\ldots, \\
\tau_n(t)=&\int_0^1 G(t,s)\left(\frac{1}{\sqrt{1-s^2}}+\left(\tau_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{2}}+\left(\omega_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{-1}{5}}\right. \\
&\left.+\frac{40\Gamma(\frac{5}{4})}{3}\right)ds+\int_0^1 G(t,s)\left(\frac{1}{\sqrt{1-s^2}}+\left(\tau_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{\frac{1}{3}}\right. \\
&\left.+\left(\omega_{n-1}(s)+\frac{6}{25\Gamma(\frac{3}{2})}+1\right)^{-1}\right)ds \\
&-\frac{6}{25\Gamma(\frac{3}{2})}t^{\alpha-1}+\frac{3\Gamma(\frac{5}{4})t^{\frac{1}{2}}\lambda}{4\Gamma(\frac{3}{2})}, \quad n=1,2,\ldots,
\end{aligned}
$$

for any initial values $\omega_0, \tau_0 \in P_{h,e}$, we have $\omega_n \to u^*$ and $\tau_n \to u^*$ as $n \to \infty$. □

Remark 3.1 For problem (3.1), we can take some negative values in nonlinear term $f + g - 10$. However, the authors of [18] required that the nonlinear term is non-negative. Therefore, Theorem 3.1 in [18] cannot be applied to dealing with the problem (3.1).

4 Conclusions

In this paper, we establish the existence and uniqueness theorem of the solution for fractional m-point boundary value problem. Our tool is mixed monotone fixed point theorem involving non-cone mapping. Furthermore, two iterative sequences to approximate the unique solution are also given.

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