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Inverse spectral problem of a class of fourth-order eigenparameter-dependent boundary value problems

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Abstract

This paper deals with a class of inverse spectral problems of fourth-order boundary value problems with eigenparameter-dependent boundary conditions. Under the equivalent conditions of the problem and a certain type of matrix eigenvalue problem some coefficient functions are reconstructed from the given three sets of interlacing real numbers and several additional conditions. The key technique is the method of inverse matrix eigenvalue problems of a two-banded matrix.

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1 Introduction

This paper is a generalization of [26] and [5]. In [26], a class of inverse problems of Sturm–Liouville problems with eigenparameter-dependent boundary conditions has been considered by the authors. And in [5], the authors studied the inverse fourth-order boundary value problems with finite spectrum under separated boundary conditions. Since the higher-order boundary value problems with eigenparameter-dependent boundary conditions have their own theoretical and application background, we will investigate an inverse spectral problem of the fourth-order boundary value problems with eigenparameter-dependent boundary conditions. The reader may find the historical and research background on boundary value problems with eigenparameter-dependent boundary conditions in [20, 26] and the references therein.

It is well known that the inverse spectral problems play an important role in many scientific fields which is motivated by recovering operators from their spectral data. Such problems have been widely studied by mathematicians, physicists, and engineers. Especially the inverse matrix eigenvalue problems and inverse Sturm–Liouville problems have been deeply studied in the last several decades [7, 8, 10, 11, 13, 22–25]. Such problems are always connected with the problems such as the vibrating systems [12], the classical moment problems [19], and quantum mechanics [18].

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In recent years, boundary value problems (BVPs) of Atkinson type which have a finite spectrum have been considered by some scholars [1–4, 6, 14, 15, 17]. These problems are connected with some physical problems such as frequencies of vibrating strings and diffusion operators [21]. Besides the finite spectrum results and matrix representations of these problems, the corresponding inverse spectral problems of such a type of problems have also been investigated most recently [5, 9, 16, 26, 27]. In 2012, Kong and Zettl considered the inverse Sturm–Liouville problems with finite spectrum of Atkinson type by using the so-called matrix representations of these problems [16]. In 2018, Cai and Zheng [9] generated the results in [16] and investigated a class of inverse Sturm–Liouville problems with discontinuous boundary conditions. The inverse spectral problems of Sturm–Liouville problems with eigenparameter- dependent boundary conditions and fourth-order boundary value problems were studied by Zhang and Ao in [26, 27] and [5], respectively. Since the fourth-order boundary value problems and the eigenparameter-dependent boundary value problems have their own significant background, we will discuss an inverse spectral problem on it.

Let us consider the fourth-order boundary value problem generated by the equation

$$(py'')'' + qy = \lambda wy, \quad \text{on } I = [a, b], -\infty < a < b < +\infty, \tag{1.1}$$

where the coefficients satisfy the condition

$$r = 1/p, q, w \in L(I, \mathbb{R}),$$

here $L(I, \mathbb{R})$ denotes the real valued functions which are Lebesgue integrable on I , and the eigenparameter-dependent boundary conditions (BCs)

$$A_\lambda Y(a) + B_\lambda Y(b) = 0, \quad Y = [y, y', py'', (py'')']^T, \tag{1.2}$$

where

$$A_\lambda = \begin{bmatrix} \lambda\alpha'_1 + \alpha_1 & \lambda\alpha'_2 + \alpha_2 & \lambda\alpha'_3 + \alpha_3 & \lambda\alpha'_4 + \alpha_4 \\ \lambda\beta'_1 + \beta_1 & \lambda\beta'_2 + \beta_2 & \lambda\beta'_3 + \beta_3 & \lambda\beta'_4 + \beta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda\gamma'_1 + \gamma_1 & \lambda\gamma'_2 + \gamma_2 & \lambda\gamma'_3 + \gamma_3 & \lambda\gamma'_4 + \gamma_4 \\ \lambda\delta'_1 + \delta_1 & \lambda\delta'_2 + \delta_2 & \lambda\delta'_3 + \delta_3 & \lambda\delta'_4 + \delta_4 \end{bmatrix}$$

with $\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i, \gamma'_i, \delta_i, \delta'_i \in \mathbb{R}, i = 1, 2, 3, 4$, and satisfying

$$\begin{aligned}
 \text{rank} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \end{pmatrix} = 2, & \quad \text{rank} \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \end{pmatrix} = 2, \\
 \text{rank} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma'_1 & \gamma'_2 & \gamma'_3 & \gamma'_4 \end{pmatrix} = 2, & \quad \text{rank} \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \delta'_1 & \delta'_2 & \delta'_3 & \delta'_4 \end{pmatrix} = 2, \\
 \text{rank} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} = 2, & \quad \text{rank} \begin{pmatrix} \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \end{pmatrix} = 2, \\
 \text{rank} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix} = 2, & \quad \text{rank} \begin{pmatrix} \gamma'_1 & \gamma'_2 & \gamma'_3 & \gamma'_4 \\ \delta'_1 & \delta'_2 & \delta'_3 & \delta'_4 \end{pmatrix} = 2.
 \end{aligned} \tag{1.3}$$

Here λ is the spectral parameter.

The fourth-order eigenparameter-dependent BVPs (1.1), (1.2) are of Atkinson type if for some positive integer $n \geq 1$, there exists a partition of the interval $[a, b]$

$$a = a_0 < b_0 < a_1 < b_1 < \dots < a_n < b_n = b, \tag{1.4}$$

such that

$$r(t) = \frac{1}{p(t)} = 0 \quad \text{on } [a_i, b_i], i = 0, 1, \dots, n, \quad \int_{b_{i-1}}^{a_i} r(t) dt \neq 0, \tag{1.5}$$

$$\begin{aligned}
 \int_{b_{i-1}}^{a_i} r(t)t dt \neq 0, \quad \int_{b_{i-1}}^{a_i} r(t)t^2 dt \neq 0, \quad i = 1, 2, \dots, n; \\
 q(t) = 0 \quad \text{on } [b_{i-1}, a_i], i = 1, 2, \dots, n,
 \end{aligned} \tag{1.6}$$

and

$$\begin{aligned}
 w(t) = 0 \quad \text{on } [b_{i-1}, a_i], i = 1, 2, \dots, n, \quad \int_{a_i}^{b_i} w(t) dt > 0, \\
 \int_{a_i}^{b_i} w(t)t dt > 0, \quad \int_{a_i}^{b_i} w(t)t^2 dt > 0, \quad i = 0, 1, \dots, n,
 \end{aligned} \tag{1.7}$$

and BC (1.2) satisfies the conditions (1.3).

To present our discussions in this paper we need to introduce the following notations. Let

$$\begin{aligned}
 r_i &= \int_{b_{i-1}}^{a_i} r(t) dt, & \hat{r}_i &= \int_{b_{i-1}}^{a_i} r(t)t dt, & \check{r}_i &= \int_{b_{i-1}}^{a_i} r(t)t^2 dt, & i = 1, 2, \dots, n, \\
 q_i &= \int_{a_i}^{b_i} q(t) dt, & \hat{q}_i &= \int_{a_i}^{b_i} q(t)t dt, & \check{q}_i &= \int_{a_i}^{b_i} q(t)t^2 dt, & i = 0, 1, \dots, n, \\
 w_i &= \int_{a_i}^{b_i} w(t) dt, & \hat{w}_i &= \int_{a_i}^{b_i} w(t)t dt, & \check{w}_i &= \int_{a_i}^{b_i} w(t)t^2 dt, & i = 0, 1, \dots, n,
 \end{aligned} \tag{1.8}$$

and assume that

$$\theta_i = \hat{r}_i^2 - r_i \check{r}_i \neq 0, \quad i = 1, 2, \dots, n.$$

For a given equation (1.1) with coefficients satisfying (1.5)–(1.7), let us set

$$\begin{aligned} \vec{\alpha} &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4), & \vec{\alpha}' &= (\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4), & \vec{\beta} &= (\beta_1, \beta_2, \beta_3, \beta_4), \\ \vec{\beta}' &= (\beta'_1, \beta'_2, \beta'_3, \beta'_4), & \vec{\gamma} &= (\gamma_1, \gamma_2, \gamma_3, \gamma_4), & \vec{\gamma}' &= (\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4), \\ \vec{\delta} &= (\delta_1, \delta_2, \delta_3, \delta_4), & \vec{\delta}' &= (\delta'_1, \delta'_2, \delta'_3, \delta'_4), & \vec{0} &= (0, 0, 0, 0). \end{aligned}$$

Then we let $\sigma(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}', \vec{\gamma}, \vec{\gamma}', \vec{\delta}, \vec{\delta}')$ be the spectrum of the fourth-order eigenparameter-dependent BVPs (1.1), (1.2).

Assume $k \in \mathbb{N}^+$ such that $k > 2$, and let \mathbb{M}_{2k} be the set of $2k \times 2k$ matrices over the reals. For any $Y \in \mathbb{M}_{2k}$, we denote by $\sigma(Y)$ the set of eigenvalues of Y . Furthermore, let Y_1 be the principal submatrix obtained from Y by removing its first row and column, and Y^1 its submatrix obtained from Y by removing the last row and column. Let Y_{12} be the principal submatrix obtained from Y by removing its first two rows and first two columns, and Y^{12} its submatrix obtained from Y by removing the last two rows and the last two columns. For any 2×2 matrix F , we denote by $Y_{12}[F]$ the matrix of Y_{12} plus F in the (1, 1), (1, 2), (2, 1), (2, 2) entries. We denote by $Y^{12}[F]$ the matrix of Y^{12} plus F in the $(2k - 3, 2k - 3), (2k - 3, 2k - 2), (2k - 2, 2k - 3), (2k - 2, 2k - 2)$ entries.

For any $Y, Z \in \mathbb{M}_{2k}$, we say that λ^* is an eigenvalue of the matrix-pair (Y, Z) if there exists a nontrivial vector $u \in \mathbb{R}^{2k}$ such that $(Y - \lambda^*Z)u = 0$. We denote by $\sigma(Y, Z)$ the set of eigenvalues of (Y, Z) . Clearly, $\lambda^* \in \sigma(Y)$ if and only if $\lambda^* \in \sigma(Y, I_{2k})$, where I_{2k} is the identity matrix in \mathbb{M}_{2k} .

2 Equivalent matrix representations of fourth-order eigenparameter-dependent BVPs

To illustrate our main result on inverse fourth-order eigenparameter-dependent BVPs, we will show the following lemmas, which have been given in [1].

Lemma 2.1 ([1]) *Let $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$. Then, for any $\eta_0, \eta_1, \eta_2 \in \mathbb{R}$, there exists a unique polynomial function $P(t) = \tau_2 t^2 + \tau_1 t + \tau_0$, such that*

$$\int_{x_1}^{x_2} P(t) dt = \eta_0, \quad \int_{x_1}^{x_2} P(t)t dt = \eta_1, \quad \int_{x_1}^{x_2} P(t)t^2 dt = \eta_2. \tag{2.1}$$

Denote the polynomial constructed in Lemma 2.1 by $\chi(x_1, x_2, \eta_0, \eta_1, \eta_2)$ and define the piecewise polynomial functions $\bar{r}(t) = \frac{1}{\bar{p}(t)}, \bar{q}(t)$ and $\bar{w}(t)$ on $[a, b]$ by

$$\begin{aligned} \bar{r}(t) &= \frac{1}{\bar{p}(t)} = \begin{cases} \chi(b_{i-1}, a_i, r_i, \hat{r}_i, \check{r}_i), & t \in [b_{i-1}, a_i], i = 1, 2, \dots, n; \\ 0, & t \in [a_i, b_i], i = 0, 1, \dots, n; \end{cases} \\ \bar{q}(t) &= \begin{cases} \chi(a_i, b_i, q_i, \hat{q}_i, \check{q}_i), & t \in [a_i, b_i], i = 0, 1, \dots, n; \\ 0, & t \in [b_{i-1}, a_i], i = 1, 2, \dots, n; \end{cases} \\ \bar{w}(t) &= \begin{cases} \chi(a_i, b_i, w_i, \hat{w}_i, \check{w}_i), & t \in [a_i, b_i], i = 0, 1, \dots, n; \\ 0, & t \in [b_{i-1}, a_i], i = 1, 2, \dots, n. \end{cases} \end{aligned}$$

Then we can state the following lemma.

Lemma 3.1 ([22]) *Let $\{\lambda_i : i = 1, \dots, 2k\}$, $\{\mu_i : i = 1, \dots, 2k - 1\}$ and $\{\xi_i : i = 1, \dots, 2k - 2\}$ be three sets of real numbers satisfying the interlacing property*

$$\begin{aligned} \lambda_1 &\leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \lambda_{2k-1} \leq \mu_{2k-1} \leq \lambda_{2k}, \\ \mu_1 &\leq \xi_1 \leq \mu_2 \leq \xi_2 \leq \dots \leq \mu_{2k-2} \leq \xi_{2k-2} \leq \mu_{2k-1}. \end{aligned} \tag{3.2}$$

Then there exists a two-banded matrix $\mathbf{J}_b \in \mathbb{M}_{2k}$ such that

$$\begin{aligned} \sigma(\mathbf{J}_b) &= \{\lambda_i : i = 1, \dots, 2k\}, \quad \sigma((\mathbf{J}_b)_1) = \{\mu_i : i = 1, \dots, 2k - 1\}, \\ \sigma((\mathbf{J}_b)_{12}) &= \{\xi_i : i = 1, \dots, 2k - 2\}. \end{aligned}$$

Next, we consider the block matrices in \mathbb{M}_{2k} of the form

$$\begin{bmatrix} C_1 & D_1 & & & & & & & \\ D'_1 & C_2 & D_2 & & & & & & \\ & & \dots & \dots & \dots & & & & \\ & & & & & D_{k-2}^T & C_{k-1} & D'_{k-1} & \\ & & & & & & D_{k-1} & C_k & \end{bmatrix}, \tag{3.3}$$

which is block symmetric except for the $D_1, D'_1, D_{k-1}, D'_{k-1}$ entries.

Definition 3.2 A matrix $\mathbf{M} \in \mathbb{M}_{2k}$ in the form of (3.3) is called a 2×2 block pseudo-Jacobi matrix if $C_i = \begin{bmatrix} c_i & \hat{c}_i \\ \hat{c}_i & c_i \end{bmatrix}$, $c_i, \hat{c}_i, \check{c}_i \neq 0$, $i = 2, \dots, k - 1$; $D_i = \begin{bmatrix} d_i & \hat{d}_i \\ \hat{d}_i & d_i \end{bmatrix}$, $d_i, \hat{d}_i, \check{d}_i \neq 0$, $i = 2, \dots, k - 2$, and $C_1 = \begin{bmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{bmatrix}$, $D_1 = \begin{bmatrix} d_{11}^{(1)} & d_{12}^{(1)} \\ d_{21}^{(1)} & d_{22}^{(1)} \end{bmatrix}$, $D'_1 = \begin{bmatrix} d_{11}^{(1)'} & d_{12}^{(1)'} \\ d_{21}^{(1)'} & d_{22}^{(1)'} \end{bmatrix}$, $D_{k-1} = \begin{bmatrix} d_{11}^{(k-1)} & d_{12}^{(k-1)} \\ d_{21}^{(k-1)} & d_{22}^{(k-1)} \end{bmatrix}$, $D'_{k-1} = \begin{bmatrix} d_{11}^{(k-1)'} & d_{12}^{(k-1)'} \\ d_{21}^{(k-1)'} & d_{22}^{(k-1)'} \end{bmatrix}$, $C_k = \begin{bmatrix} c_{11}^{(k)} & c_{12}^{(k)} \\ c_{21}^{(k)} & c_{22}^{(k)} \end{bmatrix}$ with each $D_i, i = 1, \dots, k - 1$, and C_1, C_k, D'_1, D'_{k-1} are 2×2 nonsingular matrices.

From Lemma 3.1, we can deduce the following theorem which includes the key technique to solve our main result.

Theorem 3.1 *Let $\{\lambda_i : i = 1, \dots, 2k\}$, $\{\mu_i : i = 1, \dots, 2k - 1\}$ and $\{\xi_i : i = 1, \dots, 2k - 2\}$ be three sets of real numbers satisfying the interlacing property (3.2).*

If

$$\mathbf{W} = \begin{bmatrix} -B'_1 & -B'_2 & & & & & & & \\ & W_0 & & & & & & & \\ & & W_1 & & & & & & \\ & & & \dots & & & & & \\ & & & & W_{k-4} & & & & \\ & & & & & W_{k-3} & & & \\ & & & & & & -B'_3 & -B'_4 & \end{bmatrix}$$

is an 'almost' block diagonal matrix with $W_j = \begin{bmatrix} w_j & \hat{w}_j \\ \hat{w}_j & w_j \end{bmatrix}$, $j = 0, 1, \dots, k - 3$, $B'_i = \begin{bmatrix} b_{11}^{(i)} & b_{12}^{(i)} \\ b_{21}^{(i)} & b_{22}^{(i)} \end{bmatrix}$, $i = 1, 2, 3, 4$ satisfying $w_i, \hat{w}_i, \check{w}_i > 0$, $\det(W_i) > 0$, $i = 0, 1, \dots, k - 3$, $\frac{w_i \hat{w}_{i+1} - \hat{w}_i w_{i+1}}{w_{i+1} \det(W_i)} = \frac{\hat{d}_{i+2}}{d_{i+2}}$, $i = 0, 1, \dots, k - 4$, where $\hat{d}_{i+2}, \check{d}_{i+2}, i = 0, 1, \dots, k - 4$ are entries of \mathbf{J}_b as mentioned in Lemma 3.1,

and $\det(B'_i) \neq 0, i = 1, 2, 3, 4$. Then there exists a block pseudo-Jacobi matrix $\mathbf{M} \in \mathbb{M}_{2k}$ such that

$$\begin{aligned} \sigma(\mathbf{M}, \mathbf{W}) &= \{\lambda_i : i = 1, \dots, 2k\}, \\ \sigma(\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2], \mathbf{W}_{12}) &= \{\xi_i : i = 1, \dots, 2k - 2\}. \end{aligned} \tag{3.4}$$

Proof Since $\{\lambda_i : i = 1, \dots, 2k\}, \{\mu_i : i = 1, \dots, 2k - 1\}$ and $\{\xi_i : i = 1, \dots, 2k - 2\}$ are three sets of real numbers satisfying the interlacing property (3.2), hence by Lemma 3.1, there exists a two-banded matrix $\mathbf{J}_b \in \mathbb{M}_{2k}$ such that

$$\begin{aligned} \sigma(\mathbf{J}_b) &= \{\lambda_i : i = 1, \dots, k\}, \quad \sigma((\mathbf{J}_b)_1) = \{\mu_i : i = 1, \dots, k - 1\}, \\ \sigma((\mathbf{J}_b)_{12}) &= \{\xi_i : i = 1, \dots, k - 2\}. \end{aligned}$$

Now for each $\lambda = \lambda_i, i = 1, \dots, 2k$, there exists a nontrivial $u \in \mathbb{R}^{2k}$ such that

$$(\mathbf{J}_b - \lambda I_{2k})u = 0. \tag{3.5}$$

Set

$$\mathbf{H} = \begin{bmatrix} I_2 & -(B'_1)^{-1}B'_2 & & & \\ & I_2 & & & \\ & & \dots & & \\ & & & I_2 & \\ & & & -(B'_4)^{-1}B'_3 & I_2 \end{bmatrix},$$

then one has $\mathbf{W}\mathbf{H} = \mathbf{L}\mathbf{U}$, with $\mathbf{L} = \text{diag}(L_1, R_0^T, R_1^T, \dots, R_{k-3}^T, L_2)$, $\mathbf{U} = \text{diag}(U_1, R_0, R_1, \dots, R_{k-3}, U_2)$, where $U_1 = (R_0^T)^{-1}(\mathcal{D}_1^T)^{-1}E_2, U_2 = (R_{k-3}^T)^{-1}(\mathcal{D}_{k-1})^{-1}(-E_2), L_1 = -B'_1(U_1)^{-1}, L_2 = -B'_4(U_2)^{-1}, R_i = \begin{bmatrix} \sqrt{w_i} & \frac{\hat{w}_i}{\sqrt{w_i}} \\ 0 & \frac{\sqrt{\hat{w}_i w_i - \hat{w}_i^2}}{\sqrt{w_i}} \end{bmatrix}, i = 0, 1, \dots, k - 3, \mathcal{D}_1^T, \mathcal{D}_{k-1} \in \mathbf{J}_b$.

Let $u = \mathbf{U}\mathbf{H}^{-1}\tilde{u}$ and left multiplying Eq. (3.5) by \mathbf{L} we obtain

$$(\mathbf{L}\mathbf{J}_b\mathbf{U}\mathbf{H}^{-1} - \lambda\mathbf{L}\mathbf{U}\mathbf{H}^{-1})\tilde{u} = 0, \quad \text{i.e.,} \quad (\mathbf{M} - \lambda\mathbf{W})\tilde{u} = 0,$$

where $\mathbf{M} = \mathbf{L}\mathbf{J}_b\mathbf{U}\mathbf{H}^{-1}$. Clearly, $\lambda \in \sigma(\mathbf{M}, \mathbf{W})$ and $\mathbf{M} \in \mathbb{M}_{2k}$ is a block pseudo-Jacobi matrix. The same argument applies to each $\xi = \xi_i, i = 1, \dots, 2k - 2$, then we obtain the conclusion that $\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2] = \mathbf{L}_{12}(\mathbf{J}_b)_{12}\mathbf{U}_{12}(\mathbf{H}^{-1})_{12}, \mathbf{W}_{12} = \mathbf{L}_{12}\mathbf{U}_{12}(\mathbf{H}^{-1})_{12}$ and $\xi \in \sigma(\mathbf{M}_{12}, \mathbf{W}_{12})$. Thus,

$$\sigma(\mathbf{J}_b) \subset \sigma(\mathbf{M}, \mathbf{W}) \quad \text{and} \quad \sigma((\mathbf{J}_b)_{12}) \subset \sigma(\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2], \mathbf{W}_{12}).$$

Similarly by reversing the steps above we can get

$$\sigma(\mathbf{J}_b) \supset \sigma(\mathbf{M}, \mathbf{W}) \quad \text{and} \quad \sigma((\mathbf{J}_b)_{12}) \supset \sigma(\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2], \mathbf{W}_{12}).$$

Consequently,

$$\sigma(\mathbf{J}_b) = \sigma(\mathbf{M}, \mathbf{W}) \quad \text{and} \quad \sigma((\mathbf{J}_b)_{12}) = \sigma(\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2], \mathbf{W}_{12}).$$

This completes the proof. □

Corollary 3.1 When $M_{12}[-E_2(B'_1)^{-1}B'_2]$ and W_{12} are replaced by $M^{12}[E_2(B'_4)^{-1}B'_3]$ and W^{12} , respectively, then the similar conclusion of Theorem 3.1 still holds.

Corollary 3.2 When $w_i, \hat{w}_i, \check{w}_i > 0$ are replaced by $w_i, \hat{w}_i, \check{w}_i < 0$ for $i = 1, \dots, k$, then the similar conclusion of Theorem 3.1 still holds.

4 Main result

We now state our main result on the inverse spectral problem of the fourth-order eigenparameter-dependent BVPs (1.1), (1.2).

Theorem 4.1 Let $\{\lambda_i : i = 1, \dots, 2k\}$, $\{\mu_i : i = 1, \dots, 2k-1\}$ and $\{\xi_i : i = 1, \dots, 2k-2\}$ be three sets of real numbers satisfying the interlacing property (3.2). Let $n = k-3$. Then, for any $-\infty < a < b < +\infty$, any partition (1.4), any $w \in L(I, \mathbb{R})$ satisfying (1.7), any real numbers $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i, i = 1, 2, 3, 4$ satisfying $\alpha'_1\beta'_2 - \alpha'_2\beta'_1 \neq 0, \alpha'_3\beta'_4 - \alpha'_4\beta'_3 \neq 0, \gamma'_1\delta'_2 - \gamma'_2\delta'_1 \neq 0, \gamma'_3\delta'_4 - \gamma'_4\delta'_3 \neq 0$, we have the following:

(a) There exist $r, q \in L(I, \mathbb{R})$ satisfying (1.5) and (1.6), the values of eigenparameter-dependent BCs $\alpha_i, \beta_i, \gamma_i, \delta_i, i = 1, 2, 3, 4$, such that the associated equivalent family of the fourth-order eigenparameter-dependent BVPs (1.1), (1.2) has the spectrum

$$\begin{aligned} \sigma(\bar{\alpha}, \bar{\alpha}', \bar{\beta}, \bar{\beta}', \bar{\gamma}, \bar{\gamma}', \bar{\delta}, \bar{\delta}') &= \{\lambda_i : i = 1, \dots, 2k\}, \\ \sigma(\bar{0}, \bar{\alpha}', \bar{0}, \bar{\beta}', \bar{\gamma}, \bar{\gamma}', \bar{0}, \bar{\delta}') &= \{\xi_i : i = 1, \dots, 2k-2\}. \end{aligned}$$

(b) There exist $r, q \in L(I, \mathbb{R})$ satisfying (1.5) and (1.6), the values of eigenparameter-dependent BCs $\alpha_i, \beta_i, \gamma_i, \delta_i, i = 1, 2, 3, 4$, such that the associated equivalent family of the fourth-order eigenparameter-dependent BVPs (1.1), (1.2) has the spectrum

$$\begin{aligned} \sigma(\bar{\alpha}, \bar{\alpha}', \bar{\beta}, \bar{\beta}', \bar{\gamma}, \bar{\gamma}', \bar{\delta}, \bar{\delta}') &= \{\lambda_i : i = 1, \dots, 2k\}, \\ \sigma(\bar{\alpha}, \bar{\alpha}', \bar{\beta}, \bar{\beta}', \bar{0}, \bar{\gamma}', \bar{0}, \bar{\delta}') &= \{\xi_i : i = 1, \dots, 2k-2\}. \end{aligned}$$

Proof (a) For any $-\infty < a < b < +\infty$ and a given partition (1.4) of $[a, b]$, define $w_i = \int_a^{b_i} w(t) dt, \hat{w}_i = \int_a^{b_i} w(t)t dt, \check{w}_i = \int_a^{b_i} w(t)t^2 dt, i = 0, 1, \dots, n, W_i = \begin{bmatrix} w_i & \hat{w}_i \\ \hat{w}_i & \check{w}_i \end{bmatrix}, i = 0, 1, \dots, n$, satisfying the conditions $\frac{w_i\hat{w}_{i+1}-\hat{w}_i w_{i+1}}{w_{i+1}\det(W_i)} = \frac{d_{i+2}}{d_{i+2}}, i = 0, 1, \dots, n-1$. For any real numbers $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i, i = 1, 2, 3, 4$, satisfying $\alpha'_1\beta'_2 - \alpha'_2\beta'_1 \neq 0, \alpha'_3\beta'_4 - \alpha'_4\beta'_3 \neq 0, \gamma'_1\delta'_2 - \gamma'_2\delta'_1 \neq 0, \gamma'_3\delta'_4 - \gamma'_4\delta'_3 \neq 0$, define $B'_1 = \begin{bmatrix} \alpha'_3 & a\alpha'_3+\alpha'_4 \\ \beta'_3 & a\beta'_3+\beta'_4 \end{bmatrix}, B'_2 = \begin{bmatrix} \alpha'_1 & a\alpha'_1+\alpha'_2 \\ \beta'_1 & a\beta'_1+\beta'_2 \end{bmatrix}, B'_3 = \begin{bmatrix} \gamma'_3 & b\gamma'_3+\gamma'_4 \\ \delta'_3 & b\delta'_3+\delta'_4 \end{bmatrix}, B'_4 = \begin{bmatrix} \gamma'_1 & b\gamma'_1+\gamma'_2 \\ \delta'_1 & b\delta'_1+\delta'_2 \end{bmatrix}$. Then define

$$W = \begin{bmatrix} -B'_1 & -B'_2 & & & & & & & & \\ & W_0 & & & & & & & & \\ & & W_1 & & & & & & & \\ & & & \dots & & & & & & \\ & & & & W_{n-1} & & & & & \\ & & & & & W_n & & & & \\ & & & & & -B'_3 & -B'_4 & & & \end{bmatrix}.$$

By (1.7), $w_i > 0, \hat{w}_i > 0, \check{w}_i > 0, i = 0, 1, \dots, n$, and $\det(B'_i) \neq 0, i = 1, 2, 3, 4$. Since $k = n + 3$, by Theorem 3.1, there exists a block pseudo-Jacobi matrix $M \in M_{2n+6}$ in the form of (3.3)

satisfying $D'_1 = -D'_{k-1} = E_2$ such that

$$\begin{aligned} \sigma(\mathbf{M}, \mathbf{W}) &= \{\lambda_i : i = 1, \dots, 2n + 6\}, \\ \sigma(\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2], \mathbf{W}_{12}) &= \{\xi_i : i = 1, \dots, 2n + 4\}. \end{aligned}$$

Let

$$\begin{aligned} \theta_i &= \frac{1}{\check{d}_i^2 - \check{d}_i d_i}, \quad i = 1, \dots, n; \\ r_i &= \theta_i d_i, \quad \hat{r}_i = \theta_i \hat{d}_i, \quad \check{r}_i = \theta_i \check{d}_i, \quad i = 1, \dots, n; \\ q_i &= c_{i+2} + \frac{r_i}{\theta_i} + \frac{r_{i+1}}{\theta_{i+1}}, \quad \hat{q}_i = \hat{c}_{i+2} + \frac{\hat{r}_i}{\theta_i} + \frac{\hat{r}_{i+1}}{\theta_{i+1}}, \\ \check{q}_i &= \check{c}_{i+2} + \frac{\check{r}_i}{\theta_i} + \frac{\check{r}_{i+1}}{\theta_{i+1}}, \quad i = 1, \dots, n - 1, \\ q_0 &= c_2 + \frac{r_1}{\theta_1}, \quad \hat{q}_0 = \hat{c}_2 + \frac{\hat{r}_1}{\theta_1}, \quad \check{q}_0 = \check{c}_2 + \frac{\check{r}_1}{\theta_1}, \\ q_n &= c_{n+2} + \frac{r_n}{\theta_n}, \quad \hat{q}_n = \hat{c}_{n+2} + \frac{\hat{r}_n}{\theta_n}, \quad \check{q}_n = \check{c}_{n+2} + \frac{\check{r}_n}{\theta_n}; \\ \alpha_1 &= d_{11}^{(1)}, \quad \alpha_2 = d_{12}^{(1)} - a * d_{11}^{(1)}, \quad \beta_1 = d_{21}^{(1)}, \quad \beta_2 = d_{22}^{(1)} - a * d_{21}^{(1)}, \\ \alpha_3 &= c_{11}^{(1)}, \quad \alpha_4 = c_{12}^{(1)} - a * c_{11}^{(1)}, \quad \beta_3 = c_{21}^{(1)}, \quad \beta_4 = c_{22}^{(1)} - a * c_{21}^{(1)}, \\ \gamma_1 &= d_{11}^{(k-1)}, \quad \gamma_2 = d_{12}^{(k-1)} - b * d_{11}^{(k-1)}, \quad \delta_1 = d_{21}^{(k-1)}, \quad \delta_2 = d_{22}^{(k-1)} - b * d_{21}^{(k-1)}, \\ \gamma_3 &= c_{11}^{(k)}, \quad \gamma_4 = c_{12}^{(k)} - b * c_{11}^{(k)}, \quad \delta_3 = c_{21}^{(k)}, \quad \delta_4 = c_{22}^{(k)} - b * c_{21}^{(k)}, \end{aligned}$$

and define $\mathbf{P}, \mathbf{Q}, \mathbf{P}_{12}[-E_2(B'_1)^{-1}B'_2]$ and \mathbf{Q}_{12} by (2.3), (2.4), (2.6) and (2.7), respectively. Clearly, $r_i, \hat{r}_i, \check{r}_i \neq 0, i = 1, \dots, n$. It is easy to see that $\mathbf{M} = \mathbf{P} + \mathbf{Q}$ and $\mathbf{M}_{12}[-E_2(B'_1)^{-1}B'_2] = \mathbf{P}_{12}[-E_2(B'_1)^{-1}B'_2] + \mathbf{Q}_{12}$. With the notation in (2.5) and (2.8), we also have $(\mathbf{W})_{12} = \mathbf{W}_{12}$. Therefore,

$$\begin{aligned} \sigma(\mathbf{P} + \mathbf{Q}, \mathbf{W}) &= \{\lambda_i : i = 1, \dots, 2n + 6\}, \\ \sigma(\mathbf{P}_{12}[-E_2(B'_1)^{-1}B'_2] + \mathbf{Q}_{12}, \mathbf{W}_{12}) &= \{\xi_i : i = 1, \dots, 2n + 4\}. \end{aligned}$$

By Lemmas 2.3 and 2.4 we see that for the fourth-order eigenparameter-dependent BVPs (1.1), (1.2)

$$\begin{aligned} \sigma(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}', \vec{\gamma}, \vec{\gamma}', \vec{\delta}, \vec{\delta}') &= \{\lambda_i : i = 1, \dots, 2n + 6\}, \\ \sigma(\vec{0}, \vec{\alpha}', \vec{0}, \vec{\beta}', \vec{\gamma}, \vec{\gamma}', \vec{\delta}, \vec{\delta}') &= \{\xi_i : i = 1, \dots, 2n + 4\}. \end{aligned}$$

We observe that the choice of $r_i, \hat{r}_i, \check{r}_i, i = 1, \dots, n$ and $q_i, \hat{q}_i, \check{q}_i, i = 0, \dots, n$ and all $r, q \in L(I, \mathbb{R})$ by this choice form an equivalent family of the fourth-order eigenparameter-dependent BVPs. This completes the proof.

(b) The proof is similar using Corollary 3.1, and Lemmas 2.3, 2.5, hence the details are omitted here. □

To illustrate our main result we show the following algorithm and example.

Algorithm 1

- Step 1. Input three sets of real numbers $\lambda_1, \lambda_2, \dots, \lambda_{2k}, \mu_1, \mu_2, \dots, \mu_{2k-1}$ and $\xi_1, \xi_2, \dots, \xi_{2k-2}$.
- Step 2. Construct the matrix \mathbf{J}_b by the proof of Problem 2.6 in [22].
- Step 3. Set a, b and the partition of $[a, b]$. Input $\bar{w}(t)$ and $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i \in \mathbb{R}, i = 1, 2, 3, 4$. Set \mathbf{W} and \mathbf{H} to obtain \mathbf{L} and \mathbf{U} . Compute $\mathbf{M} = \mathbf{LJ}_b\mathbf{UH}^{-1}$.
- Step 4. Obtain $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}, i = 1, 2, 3, 4$, and $r_i, \hat{r}_i, \check{r}_i, i = 1, \dots, n, q_i, \hat{q}_i, \check{q}_i, i = 0, \dots, n$ by Theorem 4.1.

Example 1 Given three sets of real numbers satisfying the interlacing property (3.2): $\lambda_1 = -3.8397, \lambda_2 = -3.0377, \lambda_3 = -0.0540, \lambda_4 = 2.5830, \lambda_5 = 5.5791, \lambda_6 = 7.0055, \lambda_7 = 11.3818, \lambda_8 = 15.3820; \mu_1 = -3.0842, \mu_2 = -1.9058, \mu_3 = 2.3327, \mu_4 = 3.2462, \mu_5 = 6.2319, \mu_6 = 9.4212, \mu_7 = 13.7579; \xi_1 = -2.7411, \xi_2 = -1.3502, \xi_3 = 3.1316, \xi_4 = 5.7604, \xi_5 = 8.0532, \xi_6 = 13.1461$. Let $\alpha'_1 = -10, \alpha'_2 = -12, \alpha'_3 = -5, \alpha'_4 = -14, \beta'_1 = -4, \beta'_2 = -2, \beta'_3 = -2, \beta'_4 = -7, \gamma'_1 = 9, \gamma'_2 = -33, \gamma'_3 = -3, \gamma'_4 = 7, \delta'_1 = 3, \delta'_2 = -15, \delta'_3 = -1, \delta'_4 = 1, n = 1$. Given an interval $I = [-2, 3]$ and a partition of it: $-2 < 0 < 1 < 3$. Define the piecewise polynomial function \bar{w} on I by

$$\bar{w}(t) = \begin{cases} 0.7624t^2 - 0.1236t - 0.1401, & t \in [-2, 0), \\ 0, & t \in [0, 1), \\ -3.2811t^2 + 12.3742t - 9.0306, & t \in [1, 3]. \end{cases}$$

Then we can obtain the fourth-order eigenparameter-dependent boundary value problem:

$$\left\{ \begin{array}{l} (\bar{p}y'')'' + \bar{q}y = \lambda \bar{w}y, \quad \text{on } I = [-2, 3], \\ (-10\lambda + 483.2857)y(-2) + (-12\lambda + 1587.3571)y'(-2) \\ \quad + (-5\lambda + 36.1429)(py')(-2) + (-14\lambda + 104.1429)(py'')'(-2) = 0, \\ (-4\lambda + 29.7143)y(-2) + (-2\lambda + 81.1429)y'(-2) \\ \quad + (-2\lambda + 6.8571)(py')(-2) + (-7\lambda + 22.8571)(py'')'(-2) = 0, \\ (9\lambda - 522.0000)y(3) + (-33\lambda + 1380.9000)y'(3) \\ \quad + (-3\lambda + 76.5000)(py'')(3) + (7\lambda - 96.4500)(py'')'(3) = 0, \\ (3\lambda - 14.0000)y(3) + (-15\lambda + 74.9667)y'(3) \\ \quad + (-\lambda + 5.5000)(py'')(3) + (\lambda - 6.1500)(py'')'(3) = 0, \end{array} \right.$$

where the piecewise polynomial functions $\bar{r} = \frac{1}{\bar{p}}$ and \bar{q} on $I = [-2, 3]$ are as follows:

$$\bar{r}(t) = \begin{cases} 0, & t \in [-2, 0), \\ 211.2683t^2 - 201.4703t + 30.9248, & t \in [0, 1), \\ 0, & t \in [1, 3]. \end{cases}$$

$$\bar{q}(t) = \begin{cases} 8.5514t^2 - 9.2826t - 12.0102, & t \in [-2, 0), \\ 0, & t \in [0, 1), \\ -80.7883t^2 + 325.0663t - 291.8755, & t \in [1, 3]. \end{cases}$$

Hence the spectrum of the reconstructed fourth-order eigenparameter-dependent BVPs (1.1), (1.2) is

$$\begin{cases} \sigma(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}', \vec{\gamma}, \vec{\gamma}', \vec{\delta}, \vec{\delta}') \\ \quad = \{-3.8397, -3.0377, -0.0540, 2.5830, 5.5791, 7.0055, 11.3818, 15.3820\}, \\ \sigma(0, \vec{\alpha}', 0, \vec{\beta}', \vec{\gamma}, \vec{\gamma}', \vec{\delta}, \vec{\delta}') \\ \quad = \{-2.7411, -1.3502, 3.1316, 5.7604, 8.0532, 13.1461\}. \end{cases}$$

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Conflicts of interest

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Authors' contributions

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