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Existence and global exponential stability of anti-periodic solutions for quaternion-valued cellular neural networks with time-varying delays

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Abstract

In this paper, we are concerned with a class of quaternion-valued cellular neural networks with time-varying transmission delays and leakage delays. By applying a continuation theorem of coincidence degree theory and the Wirtinger inequality as well as constructing a suitable Lyapunov functional, sufficient conditions are derived to ensure the existence and global exponential stability of anti-periodic solutions via direct approaches. Our results are completely new. Finally, numerical examples are also provided to show the effectiveness of our results.

Keywords: Quaternion-valued neural network; Anti-periodic solution; Coincidence degree; Wirtinger inequality; Lyapunov functional

1 Introduction

A quaternion, which was invented by Hamilton in 1843 [1], consists of a real and three imaginary parts. The skew field of quaternion is denoted by

$$\mathbb{H} := \{q = q^R + iq^I + jq^J + kq^K\},$$

where q^R, q^I, q^J, q^K are real numbers and the three imaginary units i, j and k obey Hamilton's multiplication rules:

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad i^2 = j^2 = k^2 = -1$$

and the norm $\|q\| = \sqrt{\bar{q}q} = \sqrt{q\bar{q}} = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2}$, where $\bar{q} = q^R - iq^I - jq^J - kq^K$.

Due to the non-commutativity of quaternion multiplication, the investigation on quaternion is much harder than that on plurality. Fortunately, over the past 20 years, especially in algebra area, quaternion has been a topic for the effective applications in the real world. Also, a new class of differential equations named quaternion differential equations has been already applied successfully to the fields, such as quantum mechanics [2, 3], robotic manipulation [4], fluid mechanics [5], differential geometry [6], communication problems and signal processing [7–9], and neural networks [10–13]. Many scholars tried to shed some light on the information about solutions of quaternion differential equations. For example, the authors of [14] first started the research on the existence of periodic solutions of one-dimensional first order periodic quaternion differential equation by using the coincidence degree theory approach. Subsequently, the author of [15, 16] studied the existence of periodic solution of the quaternion Riccati equation with two-sided coefficients. For more works related the problem of the existence of periodic solutions of the quaternion differential equations, we refer to [17, 18] and the references cited therein. As we know, anti-periodic functions as a special class of the quasi-periodic functions are periodic functions, but not all periodic functions are anti-periodic ones. However, up to date, very few papers have been published on the existence of anti-periodic solutions of the quaternion differential equations [19–21].

On the other hands, complex-valued neural networks (CVNNs) can be seen as an extension of real-valued neural networks (RVNNs). Naturally, CVNNs can be also generalized to quaternion-valued neural networks (QVNNs). In fact, CVNNs employing multi-state activation functions can deal with multi-level information, and have often been applied to the storage of image data [22–27]. QVCNNs can deal with multi-level information, and require only half the connection weight parameters of CVNNs [28]. Moreover, compared with RVNNs and CVNNs, QVNNs perform more prominently when it comes to geometrical transformations, like 2D affine transformations or 3D affine transformations. 3D geometric affine transformations can be represented efficiently and compactly based on QVNNs, especially spatial rotation [29]. It is well known that in the design and implementation of neural networks, the dynamics of neural networks plays a very important role. Recently, the study of QVNNs has received much attention of many scholars and some results about dynamical behaviors of QVNNs have been obtained. However, it is well known that quaternion multiplication does not meet the commutative law, so the research on quaternion is much difficult than that on plurality. Besides, the methods and techniques for analyzing CVNNs or RVNNs cannot be directly applied to study QVNNs. In order to avoid the non-commutativity of quaternion multiplication, two usually feasible methods are to decompose the QVNN into four real-valued or two complex-valued systems based on Hamilton's multiplication rules or the plural decomposition property of quaternion. For example, by decomposing QVNNs into four real-valued systems, in [30], the author dealt with the problem of robust stability for QVNNs with leakage delay, discrete delay and parameter uncertainties; in [31], the global exponential stability for recurrent neural networks with asynchronous time delays is investigated in the quaternion field; in [32], by using Mawhin's continuation theorem of coincidence degree theory and constructing a suitable Lyapunov function, the existence and global exponential stability of periodic solutions for quaternion-valued cellular neural networks with time-varying delays was established; in [33], the existence and global exponential stability of pseudo almost periodic solutions for neutral type quaternion-valued neural networks with delays in the leakage term on time scales was studied by using the exponential dichotomy method and Lyapunov function method; by decompose QVNNs into two complex-valued systems, in [10], some sufficient conditions on the global μ-stability of the QVNNs with unbounded time-varying delays was obtained, in [34], some sufficient conditions on the existence, uniqueness, and global asymptotical stability of the equilibrium point are derived for the continuous-time QVNNs and their discrete-time analogs, respectively.

Moreover, as far as we know, in all known results about the dynamics of quaternion-valued neural networks, the coefficients of the leakage terms in the quaternion-valued neural networks are assumed to be real numbers.

Besides, among all dynamical behaviors of neural networks, the existence and stability of anti-periodic solutions play a key role in designing and implementation of neural networks and it has been attracting the interest of many researchers, we refer to [35–38] and references therein. However, there are only few papers that consider the problems of anti-periodic solutions for QVNNs [19–21]. Thus, it is worth investigating the existence and stability of anti-periodic solutions of QVNNs.

Motivated by the above discussions and considering that various time delays may change the dynamics of a system, in this paper, we are concern with the following quaternion-valued neural network with time-varying transmission delays and leakage delays:

$$\begin{aligned} \dot{x}_p(t) &= -a_p(t)x_p\big(t-\eta_p(t)\big) + \sum_{q=1}^{n} b_{pq}(t)f_q\big(x_q(t)\big) \\ &\quad + \sum_{q=1}^{n} c_{pq}(t)g_q\big(x_q\big(t-\tau_{pq}(t)\big)\big) + Q_p(t), \end{aligned} \tag{1}$$

where $p = 1,2,\dots,n$, $x_p(t) \in \mathbb{H}$ corresponds to the state of the pth unit at time t, $f_q(x_q(t))$, $g_q(x_q(t-\tau_{pq}(t)) \in \mathbb{H}$ denotes the output of the qth unit at time t and $t-\tau_{pq}(t))$, $b_{pq}(t), c_{pq}(t) \in \mathbb{H}$ denote the strength of qth unit on pth unit at time t, respectively, $Q_p(t) \in \mathbb{H}$ is external input on the pth at time t, $\tau_{pq}(t) \geq 0$ corresponds to the transmission delay along the axon of the qth unit on the pth unit at time t, $a_p(t) \in \mathbb{Q}$ represents the coefficient of the leakage terms, $\eta_p(t) \geq 0$ is the delay in the leakage terms.

The initial value of system (1) is given by

$$x_p(s) = \varphi_p(s) \in \mathbb{H}, \quad s \in [-\tau, 0],$$

where $\tau = \max_{1\leq p,q\leq n} \sup_{t\in[0,T]}\{\tau_{pq}(t), \eta_p(t)\}$.

Our main purpose of this paper is by applying a continuation theorem of coincidence degree theory and the Wirtinger inequality as well as constructing a suitable Lyapunov functional to study the existence and global exponential stability of anti-periodic solutions via a direct method. That is, we do not decompose system (1) into real value systems or complex value systems, but study quaternion-valued system (1) directly. Our results are completely new and our methods are different from the previous ones, and can be used to study other types of QVNNs.

This paper is organized as follows. In Sect. 2, we recall some basic definitions and lemmas. In Sect. 3, the existence of anti-periodic solutions of (1) is discussed based on the coincidence degree and the Wirtinger inequality. In Sect. 4, the global exponential stability of anti-periodic solutions of (1) is discussed by constructing a suitable Lyapunov functional. In Sect. 5, two numerical examples are given to demonstrate the obtained results. In Sect. 6, a brief conclusion is given.

2 Preliminaries and lemmas

In this section, we introduce some definitions and recall some lemmas.

Definition 2.1 A function $f:\mathbb{R}\to\mathbb{H}$ is called a T-periodic function if for all $t\in\mathbb{R}$, $f(t+T)=f(t)$.

Definition 2.2 A function $f:\mathbb{R}\to\mathbb{H}$ is called a T-anti-periodic function if for all $t\in\mathbb{R}$, $f(t+T)=-f(t)$.

Remark 2.1 From the definitions above, if $f=f^R+if^I+jf^J+kf^K:\mathbb{R}\to\mathbb{H}$, where $f^R,f^I,f^J,f^K:\mathbb{R}\to\mathbb{R}$, then we see that if f is a T-periodic function, then for every $l=R,I,J,K$, f^l is a T-periodic function, and that if f is a T-anti-periodic function, then for every $l=R,I,J,K$, f^l is a T-anti-periodic function.

Lemma 2.1 ([39]) *Let $\mathbb{X},\mathbb{Y}$ be two Bananch spaces, and let $L:D(L)\subset\mathbb{X}\to\mathbb{Y}$ be a linear operator, $N:\mathbb{X}\to\mathbb{Y}$ is continuous. Assume that L is one-to-one and $\Delta:=L^{-1}N$ is compact. Furthermore, assume there exists a bounded and open subset $\Omega\in\mathbb{X}$ with $0\in\Omega$ such that the equation $Lx=\lambda Nx$ has no solutions in $\partial\Omega\cap D(L)$ for any $\lambda\in(0,1)$. Then the problem $Lx=Nx$ has at least one solution in $\bar{\Omega}$.*

Lemma 2.2 ([39] (Wirtinger inequality)) *If u is a C^1 function such that $u(0)=u(T)$, then*

$$\|u-\bar{u}\|_{L_2}\leq\frac{T}{2\pi}\|\dot{u}\|_{L_2},$$

where $\|u\|_{L_2}:=(\int_0^T|u(t)|^2\,dt)^{\frac{1}{2}}$ and $\bar{u}=\frac{1}{T}\int_0^T u(t)\,dt$.

Lemma 2.3 *For all $a,b\in\mathbb{H}$, $\bar{a}b+\bar{b}a\leq\bar{a}a+\bar{b}b$.*

For convenience, we introduce the following notation:

$$a_p^+=\sup_{t\in[0,2T]}\|a_p(t)\|,\qquad \eta_p^+=\sup_{t\in[0,2T]}\eta_p(t),\qquad \dot{\eta}_p^+=\sup_{t\in[0,2T]}\dot{\eta}_p(t),$$
$$\tau_{pq}^+=\sup_{t\in[0,2T]}\tau_{pq}(t),\qquad \dot{\tau}_{pq}^+=\sup_{t\in[0,2T]}\dot{\tau}_{pq}(t),\qquad b_{pq}^+=\sup_{t\in[0,2T]}\|b_{pq}(t)\|,$$
$$c_{pq}^+=\sup_{t\in[0,2T]}\|c_{pq}(t)\|,\qquad Q_p^+=\sup_{t\in[0,2T]}\|Q_p(t)\|.$$

In order to obtain our results, we introduce the following assumptions.

- (H_1) For $p,q=1,2,\dots,n$, $\tau_{pq},\eta_p\in C^1(\mathbb{R},\mathbb{R}^+)$, $a_p,b_{pq},c_{pq},Q_p\in C(\mathbb{R},\mathbb{H})$ are all T-periodic functions, and τ_{pq} and η_p satisfy $\min_{1\leq p,q\leq n}\{1-\dot{\tau}_{pq}^+\}>0$ and $\min_{1\leq p,q\leq n}\{1-\dot{\eta}_{pq}^+\}>0$, respectively.
- (H_2) For $q=1,2,\dots,n$, $f_q,g_q\in C(\mathbb{H},\mathbb{H})$ satisfy $f_q(-x)=-f_q(x),g_q(-x)=-g_q(x)$ for all $x\in\mathbb{H}$.
- (H_3) For $q=1,2,\dots,n$, there exist constants $F,G>0$ such that $\|f_q(x)\|\leq F$, $\|g_q(x)\|\leq G$ for all $x\in\mathbb{H}$.
- (H_4) For $q=1,2,\dots,n$, there exist constants $L_q^f,L_q^g>0$ such that $\|f_q(x)-f_q(y)\|\leq L_q^f\|x-y\|$, $\|g_q(x)-g_q(y)\|\leq L_q^g\|x-y\|$ for all $x,y\in\mathbb{H}$.

3 Existence of anti-periodic solutions

In this section, we will study the existence of anti-periodic solutions of (1) by applying Lemma 2.1 and the Wirtinger inequality.

Theorem 3.1 *Assume that (H_1)–(H_3) hold. Suppose that*

(H_5) For $p = 1, 2, \dots, n$, $a_p^+ T < \pi\sqrt{1 - \dot{\eta}_p^+}$.

Then system (1) has at least one T-anti-periodic solution.

Proof Let

$$\mathbb{X} = \mathbb{Y} = \bigl\{x = (x_1, x_2, \dots, x_n)^T \in C\bigl(\mathbb{R}, \mathbb{H}^n\bigr), x(t+T) = -x(t), t \in [0, T]\bigr\}$$

be two Bananch spaces equipped with the norms:

$$\|x\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} = \sum_{p=1}^{n} \|x_p\|_0,$$

where $\|x_p\|_0 = \sup_{t\in[0,2T]} \|x_p(t)\|, p = 1, 2, \dots, n$.

Define operators $L : D(L) \cap \mathbb{X} \to \mathbb{Y}$ by $Lx = \dot{x}$, where $D(L) = \{x | x \in \mathbb{X}, \dot{x} \in \mathbb{X}\} \subset \mathbb{X}$, and $N : \mathbb{X} \to \mathbb{Y}$ by

$$Nx = (N_1 x, N_2 x, \dots, N_n x)^T,$$

where

$$\begin{aligned}(N_p x)(t) = {}& -a_p(t) x_p\bigl(t - \eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t) f_q\bigl(x_q(t)\bigr) \\ & + \sum_{q=1}^{n} c_{pq}(t) g_q\bigl(x_q\bigl(t - \tau_{pq}(t)\bigr)\bigr) + Q_p(t), \quad p = 1, 2, \dots, n.\end{aligned}$$

It is easy to see that $\mathrm{Ker} L = \{\mathbf{0}\}$ and $L(D(L)) = \{y \in \mathbb{Y}, \int_0^{2T} y(t)\,dt = 0\} = \mathbb{Y}$. Hence, $L : D(L) \to \mathbb{Y}$ is one-to-one. Denote by L^{-1} the inverse of L and take $\Delta := L^{-1}N$, then by using Arzela–Ascoli theorem, we can verify that Δ is compact. Assume that $x \in D(L)$ is an arbitrary anti-periodic solution of the equation $Lx = \lambda Nx$, for some $\lambda \in (0, 1)$. Then, for $p = 1, 2, \dots, n$, we have

$$\begin{aligned}\dot{x}_p(t) = \lambda \Biggl\{ & -a_p(t) x_p\bigl(t - \eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t) f_q\bigl(x_q(t)\bigr) \\ & + \sum_{q=1}^{n} c_{pq}(t) g_q\bigl(x_q\bigl(t - \tau_{pq}(t)\bigr)\bigr) + Q_p(t) \Biggr\}.\end{aligned} \tag{2}$$

Multiplying by $\bar{x}_p(t)$ from the left on both sides of the system (2), we have

$$\bar{x}_p(t)\dot{x}_p(t) = \lambda \bar{x}_p(t) \Biggl\{ -a_p(t) x_p\bigl(t - \eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t) f_q\bigl(x_q(t)\bigr)$$

$$+\sum_{q=1}^{n} c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr)+Q_p(t)\Biggr\},\quad p=1,2,\ldots,n. \tag{3}$$

Integrating both sides of (3) from 0 to $2T$ and noticing that

$$\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\dot{x}_p(t)\bigr\|\,dt=\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt=\int_0^{2T}\bigl\|\dot{x}_p(t)\bigr\|^2\,dt,\quad p=1,2,\ldots,n,$$

we obtain

$$\begin{aligned}
&\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\\
&\quad=\lambda\int_0^{2T}\Biggl\|\bar{\dot{x}}_p(t)\Biggl\{-a_p(t)x_p\bigl(t-\eta_p(t)\bigr)+\sum_{q=1}^{n}b_{pq}(t)f_q\bigl(x_q(t)\bigr)\\
&\qquad+\sum_{q=1}^{n}c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr)+Q_p(t)\Biggr\}\Biggr\|\,dt\\
&\quad\le a_p^+\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)x_p\bigl(t-\eta_p(t)\bigr)\bigr\|\,dt+\sum_{q=1}^{n}b_{pq}^+F\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|\,dt\\
&\qquad+\sum_{q=1}^{n}c_{pq}^+G\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|\,dt+Q_p^+\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|\,dt\\
&\quad\le\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}\biggl(\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}\biggl(\int_0^{2T}\bigl\|x_p(t)\bigr\|^2\,dt\biggr)^{\frac12}\\
&\qquad+\sum_{q=1}^{n}b_{pq}^+F\sqrt{2T}\biggl(\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}\\
&\qquad+\sum_{q=1}^{n}c_{pq}^+G\sqrt{2T}\biggl(\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}\\
&\qquad+Q_p^+\sqrt{2T}\biggl(\int_0^{2T}\bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac12},
\end{aligned}$$

that is, for $p=1,2,\ldots,n$,

$$\begin{aligned}
\biggl(\int_0^{2T}\bigl\|\dot{x}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}&\le\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}\biggl(\int_0^{2T}\bigl\|x_p(t)\bigr\|^2\,dt\biggr)^{\frac12}+\sum_{q=1}^{n}b_{pq}^+F\sqrt{2T}\\
&\quad+\sum_{q=1}^{n}c_{pq}^+G\sqrt{2T}+Q_p^+\sqrt{2T}.
\end{aligned}\tag{4}$$

Since $x_p\in C^1$ and x_p is a T-anti-periodic function, x_p is a $2T$-periodic function, by Lemma 2.2, we have

$$\biggl(\int_0^{2T}\bigl\|\dot{x}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}\le\frac{a_p^+T}{\pi\sqrt{1-\dot{\eta}_p^+}}\biggl(\int_0^{2T}\bigl\|\dot{x}_p(t)\bigr\|^2\,dt\biggr)^{\frac12}+\sum_{q=1}^{n}b_{pq}^+F\sqrt{2T}$$

$$+\sum_{q=1}^{n} c_{pq}^{+} G\sqrt{2T} + Q_p^{+}\sqrt{2T},$$

hence

$$\left(\int_0^{2T} \left\|\dot{x}_p(t)\right\|^2 dt\right)^{\frac{1}{2}} \le \frac{\sum_{q=1}^{n} b_{pq}^{+} F\sqrt{2T} + \sum_{q=1}^{n} c_{pq}^{+} G\sqrt{2T} + Q_p^{+}\sqrt{2T}}{1 - \frac{a_p^{+}T}{\pi\sqrt{1-\dot{\eta}_p^{+}}}}, \tag{5}$$

where $p = 1,2,\ldots,n$. Since $x_p(t)$ is a T-anti-periodic function, there must exist constants $\xi_p^l \in [0,2T]$ such that $x_p^l(\xi_p^l) = 0, l = R,I,J,K$. Hence, we have

$$\left|x_p^l(t)\right| = \left|x_p^l(\xi_p^l) + \int_{\xi_p^l}^{t} \dot{x}_p^l(s)\,ds\right| \le \int_0^{2T} \left|\dot{x}_p^l(t)\right| dt, \quad p = 1,2,\ldots,n, l = R,I,J,K.$$

Moreover, obviously, $|x_p^l(t)| \le \|x_p(t)\|$, for $l = R,I,J,K$, so we get

$$\begin{aligned}
\left\|x_p(t)\right\| &= \left(\sum_l \left|x_p^l(t)\right|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_l \left(\int_0^{2T} \left|\dot{x}_p^l(t)\right| dt\right)^2\right)^{\frac{1}{2}} \\
&\le 2\int_0^{2T} \left\|\dot{x}_p(t)\right\| dt \\
&\le 2\sqrt{2T}\left(\int_0^{2T} \left\|\dot{x}_p(t)\right\|^2 dt\right)^{\frac{1}{2}}, \quad p = 1,2,\ldots,n.
\end{aligned}$$

From (5), we obtain

$$\left\|x_p(t)\right\| \le \frac{2\sqrt{2T}(\sum_{q=1}^{n} b_{pq}^{+} F\sqrt{2T} + \sum_{q=1}^{n} c_{pq}^{+} G\sqrt{2T} + Q_p^{+}\sqrt{2T})}{1 - \frac{a_p^{+}T}{\pi\sqrt{1-\dot{\eta}_p^{+}}}},$$

thus

$$\|x_p\|_0 \le \frac{2\sqrt{2T}(\sum_{q=1}^{n} b_{pq}^{+} F\sqrt{2T} + \sum_{q=1}^{n} c_{pq}^{+} G\sqrt{2T} + Q_p^{+}\sqrt{2T})}{1 - \frac{a_p^{+}T}{\pi\sqrt{1-\dot{\eta}_p^{+}}}} := M_p,$$

where $p = 1,2,\ldots,n$. Therefore,

$$\|x\|_{\mathbb{X}} = \sum_{p=1}^{n} \|x_p\|_0 \le \sum_{p=1}^{n} M_p := W.$$

Take $\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < W + 1\}$, then it is clear that Ω satisfies all requirements of Lemma 2.1. In view of Lemma 2.1, system (1) has at least one T-anti-periodic solution. The proof is complete. □

Theorem 3.2 *Assume that* (H_1), (H_2) *and* (H_4) *hold. Suppose that*

(H_6) $\Lambda := 1 - \frac{T}{\pi}\sum_{p=1}^{n}[\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}} + \sum_{q=1}^{n}(b_{qp}^+ L_p^f + \frac{c_{qp}^+ L_p^g}{\sqrt{1-\dot{\tau}_{qp}^+}})] > 0.$

Then system (1) has at least one T-anti-periodic solution.

Proof Similar to the proof of Theorem 3.1, suppose that $x \in D(L)$ is an arbitrary anti-periodic solution of the equation $Lx = \lambda Nx$, for some $\lambda \in (0,1)$. Then, for $p = 1,2,\ldots,n$, we have

$$\begin{aligned} \dot{x}_p(t) = \lambda \Biggl\{ & -a_p(t)x_p\bigl(t-\eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t)f_q\bigl(x_q(t)\bigr) \\ & + \sum_{q=1}^{n} c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr) + Q_p(t)\Biggr\}. \end{aligned} \tag{6}$$

Multiplying by $\bar{\dot{x}}_p(t)$ from the left on both sides of the system (2), we have

$$\begin{aligned} \bar{\dot{x}}_p(t)\dot{x}_p(t) = \lambda \bar{\dot{x}}_p(t)\Biggl\{ & -a_p(t)x_p\bigl(t-\eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t)f_q\bigl(x_q(t)\bigr) \\ & + \sum_{q=1}^{n} c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr) + Q_p(t)\Biggr\}, \quad p = 1,2,\ldots,n. \end{aligned} \tag{7}$$

Integrating both sides of (7) from 0 to $2T$ and noticing that

$$\int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\dot{x}_p(t)\bigr\|\,dt = \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt = \int_0^{2T} \bigl\|\dot{x}_p(t)\bigr\|^2\,dt, \quad p = 1,2,\ldots,n,$$

we obtain

$$\begin{aligned} & \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt \\ &\quad = \lambda \int_0^{2T} \Biggl\| \bar{\dot{x}}_p(t)\Biggl\{ -a_p(t)x_p\bigl(t-\eta_p(t)\bigr) + \sum_{q=1}^{n} b_{pq}(t)f_q\bigl(x_q(t)\bigr) \\ &\qquad + \sum_{q=1}^{n} c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr) + Q_p(t)\Biggr\}\Biggr\|\,dt \\ &\quad \le a_p^+ \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|\bigl\|x_p\bigl(t-\eta_p(t)\bigr)\bigr\|\,dt + \sum_{q=1}^{n} b_{pq}^+ L_q^f \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|\bigl\|x_q(t)\bigr\|\,dt \\ &\qquad + \sum_{q=1}^{n} c_{pq}^+ L_q^g \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|\bigl\|x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr\|\,dt + Q_p^+ \int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|\,dt \\ &\quad \le a_p^+ \biggl(\int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \biggl(\int_0^{2T} \bigl\|x_p\bigl(t-\eta_p(t)\bigr)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \\ &\qquad + \sum_{q=1}^{n} b_{pq}^+ L_q^f \biggl(\int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \biggl(\int_0^{2T} \bigl\|x_q(t)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \\ &\qquad + \sum_{q=1}^{n} c_{pq}^+ L_q^g \biggl(\int_0^{2T} \bigl\|\bar{\dot{x}}_p(t)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \biggl(\int_0^{2T} \bigl\|x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr\|^2\,dt\biggr)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&+ Q_p^+\sqrt{2T}\left(\int_0^{2T}\left\|\bar{\dot{x}}_p(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&\le \frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}\left(\int_0^{2T}\left\|\bar{\dot{x}}_p(t)\right\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2T}\left\|x_p(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&+ \sum_{q=1}^n b_{pq}^+ L_q^f\left(\int_0^{2T}\left\|\bar{\dot{x}}_p(t)\right\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2T}\left\|x_q(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&+ \sum_{q=1}^n \frac{c_{pq}^+ L_q^g}{\sqrt{1-\dot{\tau}_{pq}^+}}\left(\int_0^{2T}\left\|\bar{\dot{x}}_p(t)\right\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2T}\left\|x_q(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&+ Q_p^+\sqrt{2T}\left(\int_0^{2T}\left\|\bar{\dot{x}}_p(t)\right\|^2 dt\right)^{\frac{1}{2}},
\end{aligned}
$$

that is, for $p = 1, 2, \dots, n$,

$$
\begin{aligned}
&\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&\le \frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}\left(\int_0^{2T}\left\|x_p(t)\right\|^2 dt\right)^{\frac{1}{2}} + \sum_{q=1}^n\left(b_{pq}^+ L_q^f + \frac{c_{pq}^+ L_q^g}{\sqrt{1-\dot{\tau}_{pq}^+}}\right) \\
&\times\left(\int_0^{2T}\left\|x_q(t)\right\|^2 dt\right)^{\frac{1}{2}} + Q_p^+\sqrt{2T}.
\end{aligned}
\tag{8}
$$

Hence, for $p = 1, 2, \dots, n$,

$$
\begin{aligned}
&\sum_{p=1}^n\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2 dt\right)^{\frac{1}{2}} \\
&\le \sum_{p=1}^n\left[\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}\left(\int_0^{2T}\left\|x_p(t)\right\|^2 dt\right)^{\frac{1}{2}} + \sum_{q=1}^n\left(b_{pq}^+ L_q^f + \frac{c_{pq}^+ L_q^g}{\sqrt{1-\dot{\tau}_{pq}^+}}\right)\right. \\
&\left.\times\left(\int_0^{2T}\left\|x_q(t)\right\|^2 dt\right)^{\frac{1}{2}}\right] + \sqrt{2T}\sum_{p=1}^n Q_p^+ \\
&\le \sum_{p=1}^n\left[\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}} + \sum_{q=1}^n\left(b_{qp}^+ L_p^f + \frac{c_{qp}^+ L_p^g}{\sqrt{1-\dot{\tau}_{pq}^+}}\right)\right] \\
&\times\left(\int_0^{2T}\left\|x_p(t)\right\|^2 dt\right)^{\frac{1}{2}} + \sqrt{2T}\sum_{p=1}^n Q_p^+.
\end{aligned}
\tag{9}
$$

Since $x_p \in D(L)$ and x_p is a T-anti periodic function, x_p is a $2T$-periodic function, by Lemma 2.2, we have

$$
\sum_{p=1}^n\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2 dt\right)^{\frac{1}{2}}
$$

$$\begin{aligned}&\le \frac{T}{\pi}\sum_{p=1}^{n}\left[\frac{a_p^+}{\sqrt{1-\dot{\eta}_p^+}}+\sum_{q=1}^{n}\left(b_{qp}^+L_p^f+\frac{c_{qp}^+L_p^g}{\sqrt{1-\dot{\tau}_{pq}^+}}\right)\right]\\&\quad\times\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2\,dt\right)^{\frac{1}{2}}+\sqrt{2T}\sum_{p=1}^{n}Q_p^+,\end{aligned}$$

hence

$$\sum_{p=1}^{n}\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2\,dt\right)^{\frac{1}{2}}\le\frac{\sqrt{2T}\sum_{p=1}^{n}Q_p^+}{\Lambda},\quad p=1,2,\ldots,n. \tag{10}$$

Since $x_p(t)$ is a T-anti-periodic function, there must exist constants $\xi_p^l\in[0,2T]$ such that $x_p^l(\xi_p^l)=0, p=1,2,\ldots,n, l=R,I,J,K$. Hence, we have

$$\left|x_p^l(t)\right|=\left|x_p^l(\xi_p^l)+\int_{\xi_p^l}^{t}\dot{x}_p^l(s)\,ds\right|\le\int_0^{2T}\left|\dot{x}_p^l(t)\right|dt,\quad p=1,2,\ldots,n, l=R,I,J,K.$$

Moreover, obviously, $|x_p^l(t)|\le\|x_p(t)\|$, for $l=R,I,J,K$, so we get

$$\begin{aligned}\sum_{p=1}^{n}\left\|x_p(t)\right\|&=\sum_{p=1}^{n}\left(\sum_{l}\left|x_p^l(t)\right|^2\right)^{\frac{1}{2}}\\&\le\sum_{p=1}^{n}\left(\sum_{l}\left(\int_0^{2T}\left|\dot{x}_p^l(t)\right|dt\right)^2\right)^{\frac{1}{2}}\\&\le 2\sum_{p=1}^{n}\int_0^{2T}\left\|\dot{x}_p(t)\right\|dt\\&\le 2\sqrt{2T}\sum_{p=1}^{n}\left(\int_0^{2T}\left\|\dot{x}_p(t)\right\|^2\,dt\right)^{\frac{1}{2}}.\end{aligned}$$

From (10), we obtain

$$\sum_{p=1}^{n}\left\|x_p(t)\right\|\le\frac{2T\sum_{p=1}^{n}Q_p^+}{\Lambda}.$$

Therefore,

$$\|x\|_{\mathbb{X}}=\sum_{p=1}^{n}\|x_p\|_0\le\frac{2T\sum_{p=1}^{n}Q_p^+}{\Lambda}:=W.$$

Take $\Omega=\{x\in\mathbb{X}:\|x\|_{\mathbb{X}}<W+1\}$, then system (1) has at least one T-anti periodic solution. The proof is complete. □

4 Global exponential stability

In this section, we study the global exponential stability of anti-periodic solutions of (1) by constructing a suitable Lyapunov functional.

Definition 4.1 Let $x = (x_1, x_2, \dots, x_n)^T$ be an anti periodic solution of system (1) with the initial value $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in C([-\tau, 0], \mathbb{H}^n)$ and $y = (y_1, y_2, \dots, y_n)^T$ be an arbitrary solution of system (1) with the initial value $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T \in C([-\tau, 0], \mathbb{H}^n)$, respectively. If there exist positive constants λ and M such that

$$\|x(t) - y(t)\| \le M\|\varphi - \psi\|_\tau e^{-\lambda t}, \quad t > 0,$$

where

$$\|\varphi - \psi\|_\tau = \sum_{p=1}^{n} \sup_{t \in [-\tau, 0]} \|\varphi_p(t) - \psi_p(t)\|,$$

then the anti-periodic solution of system (1) is said to be globally exponentially stable.

Theorem 4.1 *In system* (1), *let* $\eta_p(t) \equiv 0, a_p \in C(\mathbb{R}, \mathbb{R}^+)$ *with* $a_p^- = \inf_{t\in[0,2T]} a_p(t) > 0, p = 1, 2, \dots, n$. *Assume that* (H_1)–(H_5) *hold, and there exists a positive constant* λ *such that*

(H_7) $\Gamma = \max_{1\le p\le n}\{2\lambda + 2 - 2a_p^- + \sum_{q=1}^n {b_{qp}^+}^2 (L_p^f)^2 + \sum_{q=1}^n {c_{qp}^+}^2 (L_p^g)^2 \frac{e^{2\lambda\tau_{qp}^+}}{1-\dot{\tau}_{qp}^+}\} < 0.$

Then system (1) *has a unique* T*-anti periodic solution that is globally exponentially stable.*

Proof By Theorem 3.1, system (1) has an anti-periodic solution, let x be an anti-periodic solution with the initial value φ and y be an arbitrary anti-periodic solution with the initial value ψ. Taking $z = x - y$, where $z_p = x_p - y_p, p = 1, 2, \dots, n$, we have

$$\begin{aligned}\dot{z}_p(t) = {} & -a_p(t)z_p(t) + \sum_{q=1}^{n} b_{pq}(t)\tilde{f}_q\bigl(z_q(t)\bigr) \\ & + \sum_{q=1}^{n} c_{pq}(t)\tilde{g}_q\bigl(z_q\bigl(t - \tau_{pq}(t)\bigr)\bigr), \quad p = 1, 2, \dots, n, \end{aligned} \tag{11}$$

where $\tilde{f}_q(z_q(t)) = f_q(x_q(t)) - f_q(y_q(t))$, $\tilde{g}_q(z_q(t - \tau_{pq}(t))) = g_q(x_q(t - \tau_{pq}(t))) - g_q(y_q(t - \tau_{pq}(t)))$.

Define a Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{p=1}^{n} e^{2\lambda t}\bigl(\bar{z}_p(t) \cdot z_p(t)\bigr),$$

$$V_2(t) = \sum_{p=1}^{n}\sum_{q=1}^{n} \|c_{pq}\|^2 \bigl(L_q^g\bigr)^2 \frac{e^{2\lambda\tau_{pq}^+}}{1 - \dot{\tau}_{pq}^+} \int_{t-\tau_{pq}(t)}^{t} e^{2\lambda s}\bar{z}_q(s) \cdot z_q(s)\, ds.$$

Calculating the right derivatives $D^+V_1(t)$ of $V_1(t)$ and $D^+V_2(t)$ of $V_2(t)$ along with the solutions of (11), respectively, and by using Lemma 2.3, we have

$$\begin{aligned} & D^+V_1(t) \\ & \quad = \sum_{p=1}^{n} 2\lambda e^{2\lambda t}\bigl(\bar{z}_p(t) \cdot z_p(t)\bigr) + \sum_{p=1}^{n} e^{2\lambda t}\bigl(\dot{\bar{z}}_p(t) \cdot z_p(t) + \bar{z}_p(t) \cdot \dot{z}_p(t)\bigr) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{n} 2\lambda e^{2\lambda t}\big(\bar{z}_p(t)\cdot z_p(t)\big) + \sum_{p=1}^{n} e^{2\lambda t}\Bigg\{\overline{-a_p(t)z_p(t)} + \sum_{q=1}^{n}\overline{b_{pq}(t)\tilde{f}_q\big(z_q(t)\big)} \\
&\quad + \sum_{q=1}^{n}\overline{c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big)}\Bigg\}\cdot z_p(t) + \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)\cdot\Bigg\{-a_p(t)z_p(t) \\
&\quad + \sum_{q=1}^{n} b_{pq}(t)\tilde{f}_q\big(z_q(t)\big) + \sum_{q=1}^{n} c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big)\Bigg\} \\
&= \sum_{p=1}^{n} 2\lambda e^{2\lambda t}\bar{z}_p(t)z_p(t) + \sum_{p=1}^{n} e^{2\lambda t}\Bigg\{-a_p(t)\bar{z}_p(t)z_p(t) \\
&\quad + \sum_{q=1}^{n}\overline{b_{pq}(t)\tilde{f}_q\big(z_q(t)\big)}z_p(t) + \sum_{q=1}^{n}\overline{c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big)}z_p(t)\Bigg\} \\
&\quad + \sum_{p=1}^{n} e^{2\lambda t}\Bigg\{-a_p(t)\bar{z}_p(t)z_p(t) + \sum_{q=1}^{n}\bar{z}_p(t)b_{pq}(t)\tilde{f}_q\big(z_q(t)\big) \\
&\quad + \sum_{q=1}^{n}\bar{z}_p(t)c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big)\Bigg\} \\
&\le \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t)\big(2\lambda - 2a_p^-\big) + \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t) + \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t) \\
&\quad + \sum_{q=1}^{n} e^{2\lambda t}\overline{b_{pq}(t)\tilde{f}_q\big(z_q(t)\big)}b_{pq}(t)\tilde{f}_q\big(z_q(t)\big) \\
&\quad + \sum_{q=1}^{n} e^{2\lambda(t-\tau_{pq}(t))}e^{2\lambda\tau_{pq}^+}\overline{c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big)}c_{pq}(t)\tilde{g}_q\big(z_q\big(t-\tau_{pq}(t)\big)\big) \\
&\le \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t)\big(2\lambda + 2 - 2a_p^-\big) + \sum_{p=1}^{n}\sum_{q=1}^{n} e^{2\lambda t}{b_{pq}^+}^2\big(L_q^f\big)^2\bar{z}_q(t)z_q(t) \\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{n} e^{2\lambda(t-\tau_{pq}(t))}e^{2\lambda\tau_{pq}^+}{c_{pq}^+}^2\big(L_q^g\big)^2\bar{z}_q\big(t-\tau_{pq}(t)\big)z_q\big(t-\tau_{pq}(t)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
D^+V_2(t) &= \sum_{p=1}^{n}\sum_{q=1}^{n}{c_{pq}^+}^2\big(L_q^g\big)^2\frac{e^{2\lambda t}e^{2\lambda\tau_{pq}^+}}{1-\dot{\tau}_{pq}^+}\bar{z}_q(t)z_q(t) \\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{n}{c_{pq}^+}^2\big(L_q^g\big)^2\frac{e^{2\lambda(t-\tau_{pq}(t))}e^{2\lambda\tau_{pq}^+}(1-\dot{\tau}_{pq}(t))}{1-\dot{\tau}_{pq}^+} \\
&\quad \times\bar{z}_q\big(t-\tau_{pq}(t)\big)z_q\big(t-\tau_{pq}(t)\big) \\
&\le \sum_{p=1}^{n}\sum_{q=1}^{n}{c_{pq}^+}^2\big(L_q^g\big)^2\frac{e^{2\lambda t}e^{2\lambda\tau_{pq}^+}}{1-\dot{\tau}_{pq}^+}\bar{z}_q(t)z_q(t) \\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{n}{c_{pq}^+}^2\big(L_q^g\big)^2 e^{2\lambda(t-\tau_{pq}(t))}e^{2\lambda\tau_{pq}^+}\bar{z}_q\big(t-\tau_{pq}(t)\big)z_q\big(t-\tau_{pq}(t)\big).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
D^+V(t) &= D^+V_1(t) + D^+V_2(t) \\
&\leq \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t)\bigl(2\lambda + 2 - 2a_p^-\bigr) \\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{n} e^{2\lambda t}{b_{qp}^+}^2\bigl(L_p^f\bigr)^2\bar{z}_p(t)z_p(t) \\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{n} e^{2\lambda t}\bar{z}_q(t)z_q(t){c_{pq}^+}^2\bigl(L_q^g\bigr)^2\frac{e^{2\lambda\tau_{pq}^+}}{1-\dot{\tau}_{pq}^+} \\
&= \sum_{p=1}^{n} e^{2\lambda t}\bar{z}_p(t)z_p(t)\left(2\lambda + 2 - 2a_p^- + \sum_{q=1}^{n}{b_{qp}^+}^2\bigl(L_p^f\bigr)^2\right. \\
&\quad \left. + \sum_{q=1}^{n}{c_{qp}^+}^2\bigl(L_p^g\bigr)^2\frac{e^{2\lambda\tau_{qp}^+}}{1-\dot{\tau}_{qp}^+}\right) \\
&= \Gamma V_1(t) \leq 0.
\end{aligned}$$

That is, for $t \geq 0$, $V(t) \leq V(0)$. From the definition of $V(t)$, we have

$$V(t) \geq \sum_{p=1}^{n} e^{2\lambda t}\bigl(\bar{z}_p(t)\cdot z_p(t)\bigr) = \sum_{p=1}^{n} e^{2\lambda t}\bigl\|z_p(t)\bigr\|^2$$

and

$$\begin{aligned}
V(0) &= \sum_{p=1}^{n}\bigl(\bar{z}_p(0)\cdot z_p(0)\bigr) + \sum_{p=1}^{n}\sum_{q=1}^{n}{c_{pq}^+}^2\bigl(L_q^g\bigr)^2\frac{e^{2\lambda\tau_{pq}^+}}{1-\dot{\tau}_{pq}^+} \\
&\quad \times \int_{-\tau_{pq}(0)}^{0} e^{2\lambda s}\bar{z}_q(s)\cdot z_q(s)\,ds \\
&\leq \sum_{p=1}^{n}\left\{\sup_{s\in[-\tau,0]}\bigl|z_p(s)\bigr|^2 + \sum_{q=1}^{n}{c_{pq}^+}^2\bigl(L_q^g\bigr)^2\frac{e^{2\lambda\tau_{pq}^+}(1-e^{-2\lambda\tau_{pq}(0)})}{2\lambda(1-\dot{\tau}_{pq}^+)}\right. \\
&\quad \left. \times \sup_{s\in[-\tau,0]}\bigl\|z_q(s)\bigr\|^2\right\} \\
&\leq \sum_{p=1}^{n}\left\{1 + \frac{\sum_{q=1}^{n}{c_{qp}^+}^2(L_p^g)^2(e^{2\lambda\tau_{qp}^+}-1)}{2\lambda(1-\dot{\tau}_{qp}^+)}\right\}\sup_{s\in[-\tau,0]}\bigl\|\varphi_p(s)-\psi_p(s)\bigr\|^2,
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{p=1}^{n} e^{2\lambda t}\bigl\|z_p(t)\bigr\|^2 &\leq \sum_{p=1}^{n}\left\{1 + \frac{\sum_{q=1}^{n}{c_{qp}^+}^2(L_p^g)^2(e^{2\lambda\tau_{qp}^+}-1)}{2\lambda(1-\dot{\tau}_{qp}^+)}\right\} \\
&\quad \times \sup_{s\in[-\tau,0]}\bigl\|\varphi_p(s)-\psi_p(s)\bigr\|^2.
\end{aligned}$$

Therefore, we have

$$\left\|x(t)-y(t)\right\|^2 \le e^{-2\lambda t}\left\{1+\frac{\sum_{q=1}^n {c_{qp}^+}^2 (L_p^g)^2 (e^{2\lambda\tau_{qp}^+}-1)}{2\lambda(1-\dot{\tau}_{qp}^+)}\right\}\|\varphi-\psi\|_\tau^2.$$

Take $\Theta = \{1+\frac{\sum_{q=1}^n {c_{qp}^+}^2 (L_p^g)^2 (e^{2\lambda\tau_{qp}^+}-1)}{2\lambda(1-\dot{\tau}_{qp}^+)}\}$, we have

$$\left\|x(t)-y(t)\right\| \le \sqrt{\Theta} e^{-\lambda t}\|\varphi-\psi\|_\tau.$$

Hence, the system (1) is globally exponentially stable. The uniqueness follows from the global stability. The proof is complete. □

Similarly, we have the following.

Theorem 4.2 *In system* (1), *let* $\eta_p(t)\equiv 0, a_p \in C(\mathbb{R},\mathbb{R}^+)$ *with* $a_p^- = \inf_{t\in[0,2T]} a_p(t) > 0, p = 1,2,\ldots,n$. *Assume that* (H_1), (H_2), (H_4), (H_6) *and* (H_7) *hold. Then system* (1) *has a unique* T*-anti-periodic solution that is globally exponentially stable.*

5 Numerical examples

Example 5.1 Consider the following quaternion-valued cellular neural network with time-varying transmission delays and leakage delays:

$$\begin{aligned}\dot{x}_p(t) &= -a_p(t)x_p\big(t-\eta_p(t)\big)+\sum_{q=1}^n b_{pq}(t)f_q\big(x_q(t)\big)\\ &\quad+\sum_{q=1}^n c_{pq}(t)g_q\big(x_q\big(t-\tau_{pq}(t)\big)\big)+Q_p(t), \quad p=1,2, \end{aligned}\tag{12}$$

where

$$\begin{aligned}
&\dot{x}_p(t) = \dot{x}_p^R(t)+i\dot{x}_p^I(t)+j\dot{x}_p^J(t)+k\dot{x}_p^K(t)\in\mathbb{H},\\
&f_q(x_q) = \frac{1}{20}\sin 8x_q^R+\frac{1}{30}i\sin 8x_q^I+\frac{1}{20}j\sin 8x_q^J+\frac{1}{24}k\sin 8x_q^K,\\
&g_q(x_q) = \frac{1}{15}\sin 8x_q^R+\frac{1}{25}i\sin 8x_q^I+\frac{1}{25}j\sin 8x_q^J+\frac{1}{25}k\sin 8x_q^K,\\
&\begin{pmatrix}a_1(t)\\ a_2(t)\end{pmatrix} = \begin{pmatrix}1.6+0.1j\sin 8t+0.1k\cos 8t\\ 1.5+0.1i\sin 8t+0.2j\cos 8t\end{pmatrix},\\
&\begin{pmatrix}b_{11}(t) & b_{12}(t)\\ b_{21}(t) & b_{22}(t)\end{pmatrix} = \begin{pmatrix}0.05\sin 8t+0.01i\sin^2 4t & 0.01+0.02j\cos 8t+0.01k\sin 8t\\ 0.05-0.02i\cos 8t+0.01k\cos 8t & 0.01+0.02i\sin 8t+0.01j\sin 8t\end{pmatrix},\\
&\begin{pmatrix}c_{11}(t) & c_{12}(t)\\ c_{21}(t) & c_{22}(t)\end{pmatrix} = \begin{pmatrix}0.02\sin 4t+0.01i\sin^2 4t\\ 0.03-0.01i\cos^2 4t+0.01j\cos 8t\end{pmatrix}
\end{aligned}$$

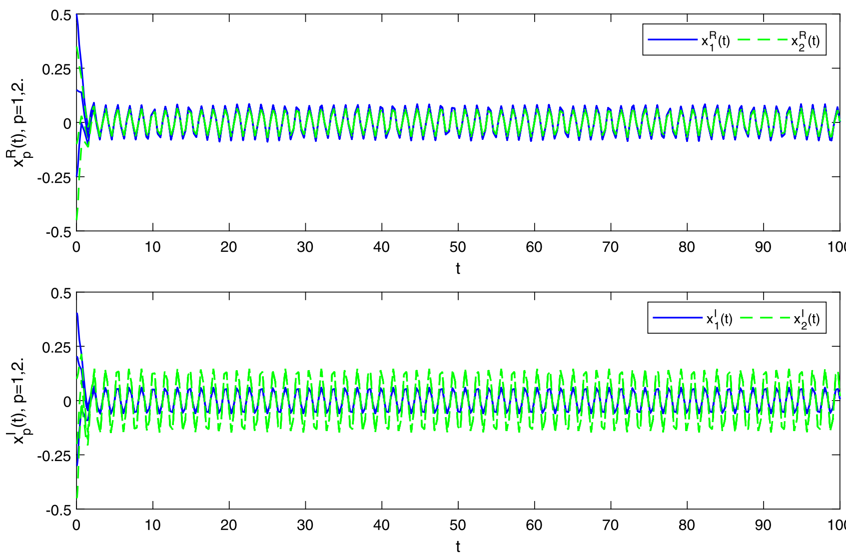

Figure 1 Curves of $x_p^R(t) = (x_1^R(t), x_2^R(t))^T$ and $x_p^I(t) = (x_1^I(t), x_2^I(t))^T$ of system (12) with the initial values $(x_1^R(0), x_2^R(0))^T = (0.5, -0.1)^T, (-0.25, 0.35)^T, (0.15, -0.45)^T$ and $(x_1^I(0), x_2^I(0))^T = (0.4, -0.2)^T, (0.2, -0.45)^T, (-0.3, 0.1)^T$

$$\begin{aligned}
&\left.\begin{matrix} 0.01 + 0.03j\cos 8t + 0.01k \\ 0.01 + 0.02j\sin^2 4t + 0.01k\sin 8t \end{matrix}\right), \\
&\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} 0.11 + 0.01\sin 8t \\ 0.1 + 0.01\sin 8t \end{pmatrix}, \\
&\begin{pmatrix} \tau_{11}(t) & \tau_{12}(t) \\ \tau_{21}(t) & \tau_{22}(t)3 \end{pmatrix} = \begin{pmatrix} 0.03\sin 8t + 0.001 & 0 \\ 0 & 0.01\sin 8t + 0.001 \end{pmatrix}, \\
&\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\sin^2 2t + \frac{1}{5}i\sin^2 2t + \frac{1}{12}j\cos 4t + \frac{1}{15}k\sin 4t \\ \frac{1}{5}\sin 4t + \frac{1}{2}i\sin 4t + \frac{1}{10}j\cos 4t + \frac{1}{20}k\sin^2 2t \end{pmatrix}.
\end{aligned}$$

By computing, $T = \frac{\pi}{4}$, $\|f_1(x)\| = \|f_2(x)\| \leq 0.089$, $\|g_1(x)\| = \|g_2(x)\| \leq 0.097$, $a_1^+ = 1.7, a_2^+ = 1.8$, $\eta_1^+ = 0.13, \eta_2^+ = 0.11$, $\dot{\eta}_1^+ = 0.16, \dot{\eta}_2^+ = 0.08$, $b_{11}^+ \leq 0.051, b_{12}^+ \leq 0.0245, b_{21}^+ \leq 0.055, b_{22}^+ \leq 0.0245$, $c_{11}^+ \leq 0.023, c_{12}^+ \leq 0.034, c_{21}^+ \leq 0.034, c_{22}^+ \leq 0.025$, $Q_1 \leq 0.5, Q_2 \leq 0.55$. So $(H_1), (H_2)$ and (H_3) are satisfied. Besides, it is easy to obtain

$$\begin{aligned}
&\frac{a_1^+ T}{\pi\sqrt{1 - \dot{\eta}_1^+}} \approx 0.464 < 1, \\
&\frac{a_2^+ T}{\pi\sqrt{1 - \dot{\eta}_2^+}} \approx 0.469 < 1.
\end{aligned}$$

Therefore, all of the conditions of Theorem 3.1 are satisfied. Hence, system (12) has at least one $\frac{\pi}{4}$-anti-periodic solution. Setting the three different initial values, the transient states of four parts of system (12) are shown in Figs. 1 and 2.

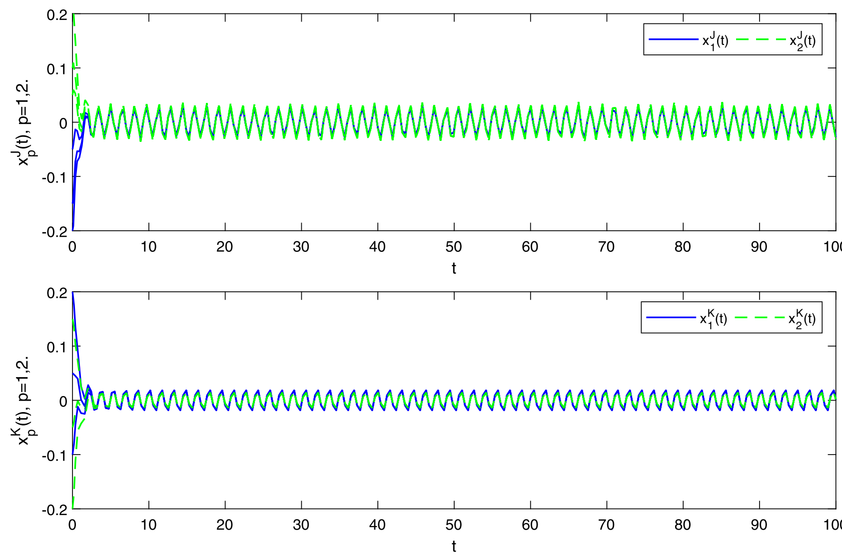

Figure 2 Curves of $x_p^J(t) = (x_1^J(t), x_2^J(t))^T$ and $x_p^K(t) = (x_1^K(t), x_2^K(t))^T$ of system (12) with the initial values $(x_1^J(0), x_2^J(0))^T = (-0.05, 0.05)^T, (-0.15, 0.2)^T, (-0.2, 0.1)^T$ and $(x_1^K(0), x_2^K(0))^T = (-0.1, -0.05)^T, (0.05, 0.15)^T, (0.2, -0.2)^T$

Example 5.2 Consider the following quaternion-valued cellular neural network with time-varying transmission delays:

$$\begin{aligned} \dot{x}_p(t) &= -a_p(t)x_p(t) + \sum_{q=1}^{2} b_{pq}(t)f_q\bigl(x_q(t)\bigr) + \sum_{q=1}^{2} c_{pq}(t)g_q\bigl(x_q\bigl(t-\tau_{pq}(t)\bigr)\bigr) \\ &\quad + Q_p(t), \quad p = 1, 2, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\dot{x}_p(t) = \dot{x}_p^R(t) + i\dot{x}_p^I(t) + j\dot{x}_p^J(t) + k\dot{x}_p^K(t) \in \mathbb{H}, \\ &f_q(x_q) = \frac{1}{30}\sin 16x_q^R + \frac{1}{40}i\sin 16x_q^I + \frac{1}{20}j\sin 16x_q^J + \frac{1}{34}k\sin 16x_q^K(t), \\ &g_q(x_q) = \frac{1}{35}\sin 16x_q^R + \frac{1}{45}i\sin 16x_q^I + \frac{1}{25}j\sin 16x_q^J + \frac{1}{65}k\sin 16x_q^K(t), \\ &\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} 1.4 + 0.01\cos 8t \\ 1.6 + 0.02\cos 8t \end{pmatrix}, \\ &\begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} = \left(\begin{matrix} 0.01\sin 16t + 0.02i\sin 16t \\ 0.05 - 0.01i\cos 16t + 0.02k\cos 16t \end{matrix} \right. \\ &\qquad\qquad\qquad\qquad \left. \begin{matrix} 0.03 + 0.01j\cos 16t + 0.01k\sin 16t \\ 0.01 + 0.01i\sin 8t + 0.02j\sin 16t \end{matrix} \right), \\ &\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \left(\begin{matrix} 0.02\sin 4t + 0.01i\sin 16t \\ 0.05 - 0.02i\cos^2 8t + 0.01j\cos 16t \end{matrix} \right. \\ &\qquad\qquad\qquad\qquad \left. \begin{matrix} 0.01 + 0.01j\cos 16t + 0.03k\sin 16t \\ 0.02 + 0.01j\sin^2 8t + 0.01k\sin 16t \end{matrix} \right), \end{aligned}$$

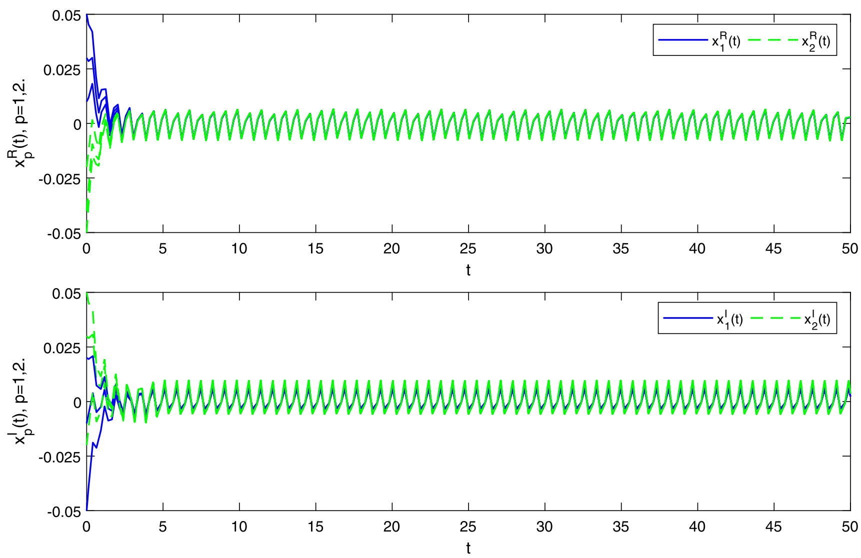

Figure 3 Curves of $x_p^R(t) = (x_1^R(t), x_2^R(t))^T$ and $x_p^I(t) = (x_1^I(t), x_2^I(t))^T$ of system (13) with the initial values $(x_1^R(0), x_2^R(0))^T = (0.01, -0.02)^T, (0.03, -0.04)^T, (0.05, -0.05)^T$ and $(x_1^I(0), x_2^I(0))^T = (-0.01, 0.03)^T, (0.02, 0.05)^T, (-0.05, -0.02)^T$

$$\begin{pmatrix} \tau_{11}(t) & \tau_{12}(t) \\ \tau_{21}(t) & \tau_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.01\sin 8t + 0.081 & 0 \\ 0 & 0.005\sin 8t + 0.01 \end{pmatrix},$$

$$\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{16}\sin 8t + \frac{1}{20}i\sin 8t + \frac{1}{12}j\cos 8t + \frac{1}{15}k\sin 8t \\ \frac{1}{15}\sin 8t + \frac{1}{12}i\sin 8t + \frac{1}{10}j\cos 8t + \frac{1}{20}k\sin^2 4t \end{pmatrix}.$$

By computing, $T = \frac{\pi}{8}$, $\|f_q(x)\| \leq 0.0715$, $\|g_q(x)\| \leq 0.0561$, $\|f_q(x)\| \leq \frac{1}{20}\|x-y\|$, $\|g_q(x)\| \leq \frac{1}{25}\|x-y\|$, $a_1^- = 1.4, a_2^- = 1.6, a_1^+ = 1.41, a_2^+ = 1.62$, $b_{11}^+ \leq 0.0224, b_{12}^+ \leq 0.0332, b_{21}^+ \leq 0.0548$, $b_{22}^+ \leq 0.0245$, $c_{11}^+ \leq 0.0224, c_{12}^+ \leq 0.0332, c_{21}^+ \leq 0.0548, c_{22}^+ \leq 0.0245$, $Q_1^+ \leq 0.1334, Q_2^+ \leq 0.1546$. So (H_1)–(H_4) are satisfied. Besides, $\tau_{11}^+ = 0.091, \tau_{12}^+ = 0, \tau_{21}^+ = 0, \tau_{22}^+ = 0.015, \dot{\tau}_{11}^+ = 0.08, \dot{\tau}_{12}^+ = 0, \dot{\tau}_{21}^+ = 0, \dot{\tau}_{22}^+ = 0.04$, and it is easy to obtain

$$\frac{a_1^+ T}{\pi} \approx 0.17625 < 1,$$
$$\frac{a_2^+ T}{\pi} \approx 0.2025 < 1.$$

Therefore, all of the conditions of Theorem 3.1 are satisfied. Hence, system (13) has at least one $\frac{\pi}{8}$-anti-periodic solution. Furthermore, take $\lambda = 0.1$, we have

$$\Gamma = \max_{1\leq p\leq 2}\left\{2\lambda + 2 - 2a_p^- + \sum_{q=1}^{2} {b_{qp}^+}^2 \left(L_p^f\right)^2 + \sum_{q=1}^{2} {c_{qp}^+}^2 \left(L_p^g\right)^2 \frac{e^{2\lambda\tau_{qp}^+}}{1-\dot{\tau}_{qp}^+}\right\} \approx -0.6993 < 0.$$

Therefore, all of the conditions of Theorem 4.1 are satisfied. Thus, system (13) has at least one $\frac{\pi}{8}$-anti-periodic solution that is globally exponentially stable. Figures 3 and 4 show the time responses of four parts of state variables of system (13) with three different initial values. Figure 5 depicts the curves of neurons $x_p^R(t)$, $x_p^I(t)$, $x_p^J(t)$ and $x_p^K(t)$ with two random initial conditions in three-dimensional space for stable case.

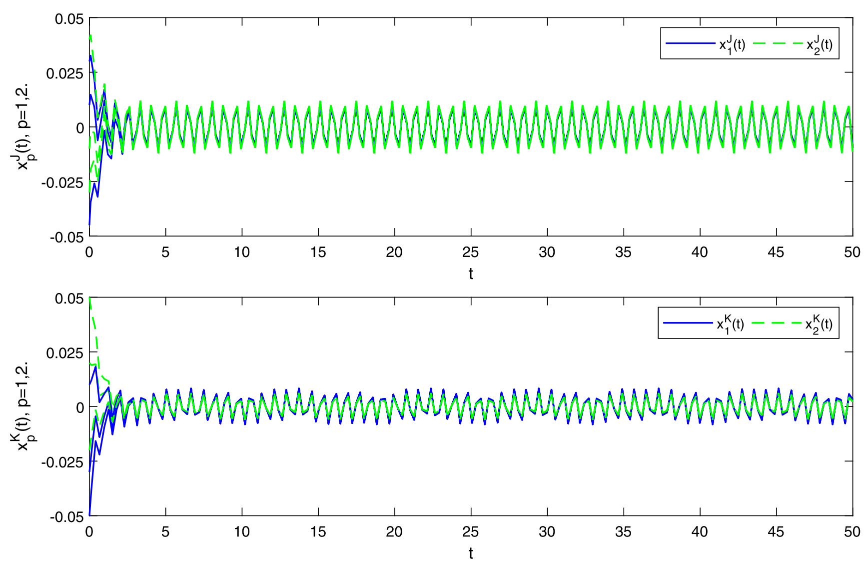

Figure 4 Curves of $x_p^J(t) = (x_1^J(t), x_2^J(t))^T$ and $x_p^K(t) = (x_1^K(t), x_2^K(t))^T$ of system (13) with the initial values $(x_1^J(0), x_2^J(0))^T = (0.01, 0.04)^T, (0.03, -0.01)^T, (-0.045, -0.03)^T$ and $(x_1^K(0), x_2^K(0))^T = (-0.03, 0.02)^T, (-0.05, -0.02)^T, (0.01, 0.05)^T$

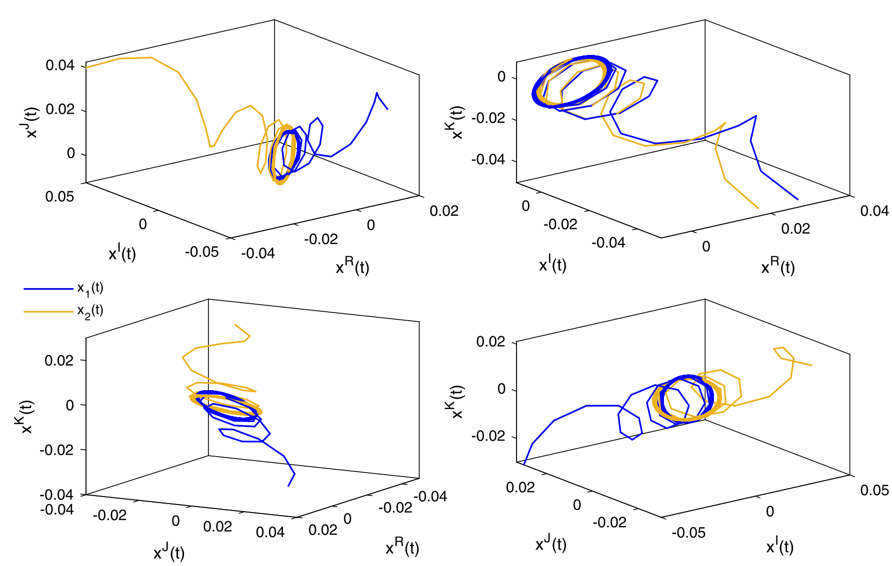

Figure 5 Curves of $x_2^R(t)$, $x_2^I(t)$, $x_2^J(t)$ and $x_2^K(t)$ in three-dimensional space for stable case

6 Conclusion

In this paper, we investigated the existence and global exponential stability of QVNNs with time-varying delays by applying a continuation theorem of coincidence degree theory and by constructing an appropriate Lyapunov functional via direct methods. Our results are new and our proposed methods can be used to study the anti-periodic problem for other types of QVNNs.

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