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# Stability of a discrete-time general delayed viral model with antibody and cell-mediated immune responses

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## Abstract

We propose a discrete-time viral model with antibody and cell-mediated immune responses. Two types of infected cells are incorporated into the model, namely latently infected and actively infected. The incidence rate of infection as well as the production and removal rates of all compartments are modeled by general nonlinear functions. The model contains three types of intracellular time delays. We utilize nonstandard finite difference (NSFD) method to discretize the continuous-time model. We prove that NSFD preserves the positivity and boundedness of the solutions of the model. Based on four threshold parameters, the existence of the five equilibria of the model is established. We perform global stability of all equilibria of the model by using Lyapunov approach. Numerical simulations are carried out to illustrate our theoretical results. The impact of time delay on the viral dynamics is established.

**Keywords:** Viral dynamics; Latency; Time delay; Global stability; Antibody; Cell-mediated response; Discrete-time model; Lyapunov function

## 1 Introduction

Mathematical modeling and analysis of within host human viral infections have provided useful insights into the understanding of interplay between three main compartments: viruses, target cells, and infected cells. Nowak and Bangham [1] have proposed the basic virus infection model in the form

$$\begin{cases} \dot{F}(t) = \delta - \beta F(t) - \kappa F(t)H(t), \\ \dot{S}(t) = \kappa F(t)H(t) - aS(t), \\ \dot{H}(t) = \theta S(t) - cH(t), \end{cases} \tag{1}$$

where $F$, $S$, and $H$ are the healthy (uninfected) target cells, actively infected cells, and free virus particles. $\delta$ and $\beta F$ are the production and natural death rates of the healthy cells, respectively. The incidence rate is modeled by $\kappa FH$. The death rate of actively infected cells is given by $aS$. The free viruses are generated at rate $\theta S$ and cleared at rate $cH$. The model has been developed to describe within host dynamics of different viruses such as human immunodeficiency virus (HIV) [2, 3], hepatitis C virus (HCV) [4, 5], hepatitis B virus (HBV) [6, 7], chikungunya virus (CHIKV) [8], and dengue virus [9].

The immune system works to defend the body against attacks by foreign invaders. B cells and CTL cells play a central role in the specific immune response. In general, B cells produce antibodies that neutralize the viruses, namely antibody immune response and CTL cells attack and kill virus-infected cells, namely cell-mediated immune response. The impact of cell-mediated immune response on the virus dynamics has been modeled [1] as follows:

$$\begin{cases} \dot{F}(t) = \delta - \beta F(t) - \kappa F(t)H(t), \\ \dot{S}(t) = \kappa F(t)H(t) - aS(t) - \lambda S(t)Z(t), \\ \dot{H}(t) = \theta S(t) - cH(t), \\ \dot{Z}(t) = gS(t)Z(t) - \xi Z(t), \end{cases} \tag{2}$$

where $Z$ is the concentration of the CTL cells. The CTL cells are proliferated at rate $gSZ$, die at rate $\xi Z$, and kill infected cells at rate $\lambda SZ$. Model (2) has been extended in several works (see, e.g., [10–13]). In other words, the basic model (1) has been modified to take into account the effect of antibody immune response in [14] as follows:

$$\begin{cases} \dot{F}(t) = \delta - \beta F(t) - \kappa F(t)H(t), \\ \dot{S}(t) = \kappa F(t)H(t) - aS(t), \\ \dot{H}(t) = \theta S(t) - cH(t) - dH(t)Y(t), \\ \dot{Y}(t) = qH(t)Y(t) - \eta Y(t), \end{cases} \tag{3}$$

where $Y$ is the concentration of antibodies. The free viruses are removed by the antibodies at rate $dHY$. The antibodies are proliferated at rate $qHY$ and die at rate $\eta Y$. In the literature, the effect of antibody immune response has been incorporated into mathematical models of various virus infections (see, e.g., [15–17]).

In models (1)–(3) it has been assumed that once a healthy cell is contacted by a virus it becomes productive instantaneously. In fact, a number of intracellular processes is needed to produce new viruses. In case of HIV infection the intracellular processes take approximately 0.9 days [18]. Therefore, the intracellular time delay has a significant effect on the virus dynamics. Delayed viral infection models have been constructed and analyzed in several works (see, e.g., [19–32]). To incorporate both cell-mediated and antibody immune responses as well as the time delay into the virus dynamics, Wodarz [33] has proposed the following model:

$$\begin{cases} \dot{F}(t) = \delta - \beta F(t) - \kappa F(t)H(t), \\ \dot{S}(t) = \kappa e^{-\mu\tau} F(t-\tau)H(t-\tau) - aS(t) - \lambda S(t)Z(t), \\ \dot{H}(t) = \theta S(t) - cH(t) - dH(t)Y(t), \\ \dot{Y}(t) = qH(t)Y(t) - \eta Y(t), \\ \dot{Z}(t) = gS(t)Z(t) - \xi Z(t), \end{cases} \tag{4}$$

where, $e^{-\mu\tau}$ represents the survival rate of infected cells after the interval $\tau$, and $\mu$ is a positive constant. Model (4) has been extended in [34–36]. It has been reported in [37, 38] that in case of HIV infection, latent HIV reservoirs serve as a major obstruction in treating HIV infection. Latently infected cells have been incorporated into the virus dynamics model with both cell-mediated and antibody in [25] and [39]. The bilinear incidence rate associated with the mass action principle can be insufficient to describe infection process in detail. The most obvious reason for a nonlinear incidence rate is that the number of free pathogen particles can vary in a very wide range, from a few particles up to hundreds of millions of them in the case of virus. The bilinear interaction term, which may be considered as an approximation, is hardly able to adequately describe the process that runs over such a range of variables. For instance, if the number of free pathogen particles is very high, so that exposure of a susceptible host is virtually certain, then the incidence rate may respond more slowly than linear to the further increase in the number of the pathogen particles [40].

In all the above mentioned works the viral infection is modeled by a system of ordinary or delay differential equations. These models are nonlinear, and calculating the exact analytical solution is difficult or impossible. Therefore, only approximate discrete-time models can be obtained by using suitable numerical approximation methods. One of the discretization methods which has been widely used to discretize viral infection models is called nonstandard finite difference (NSFD) [41]. It has been established that NSFD has the advantage of preserving the essential qualitative features of these models such as equilibria, positivity, boundedness, and global behaviors of solutions independently of the chosen step-size [42–46]. The impact of cell-mediated immune response has been incorporated into the discrete-time virus dynamics models in [47, 48]. In very recent works, Elaiw and Alshaikh [49–51] have proposed and investigated a class of discrete-time virus infection models with antibody immune response. However, the impact of both antibody and cell-mediated immune responses on the discrete-time virus infection model has not been investigated before.

The aim of the present paper is to formulate and analyze a discrete-time viral infection model with both antibody and cell-mediated immune responses. The model considers both latently infected cells and actively infected cells. The incidence rate of infection as well as the production and removal rates of all compartments are modeled by general nonlinear functions. We discretize the continuous-time model by using NSFD method. We first show that the solutions of the discrete-time model are positive and bounded, then we prove the global stability of the equilibria by constructing Lyapunov functions. Moreover, we perform numerical simulations to support the global stability results.

The achievements in this present paper look ahead to research perspectives focused on pattern formation induced by the action of the external environments, for instance, by Keller Segel dynamics [52]. The present literature is focused simply on the original SIR model [52], while it appears interesting extending the qualitative and computational analysis to more advanced models such as the one treated in our paper.

## 2 The model

We introduce the following general viral infection model with three types of time delays and both antibody and cell-mediated immune responses:

$$\dot{F}(t) = \Theta\bigl(F(t)\bigr) - \Lambda\bigl(F(t), H(t)\bigr), \tag{5}$$

$$\dot{K}(t) = (1-\varepsilon)e^{-\mu_1\tau_1}\Lambda\bigl(F(t-\tau_1), H(t-\tau_1)\bigr) - (\alpha+m)\digamma_1\bigl(K(t)\bigr), \tag{6}$$

$$\begin{aligned}\dot{S}(t) &= \varepsilon e^{-\mu_2\tau_2}\Lambda\big(F(t-\tau_2),H(t-\tau_2)\big) + m\digamma_1\big(K(t)\big) - a\digamma_2\big(S(t)\big)\\ &\quad - \lambda\digamma_2\big(S(t)\big)\digamma_5\big(Z(t)\big), \end{aligned} \tag{7}$$

$$\dot{H}(t) = \theta e^{-\mu_3\tau_3}\digamma_2\big(S(t-\tau_3)\big) - c\digamma_3\big(H(t)\big) - d\digamma_3\big(H(t)\big)\digamma_4\big(Y(t)\big), \tag{8}$$

$$\dot{Y}(t) = q\digamma_3\big(H(t)\big)\digamma_4\big(Y(t)\big) - \eta\digamma_4\big(Y(t)\big), \tag{9}$$

$$\dot{Z}(t) = g\digamma_2\big(S(t)\big)\digamma_5\big(Z(t)\big) - \xi\digamma_5\big(Z(t)\big), \tag{10}$$

where $K$ is the concentration of latently infected cells. The parameter $\varepsilon$, with $0 < \varepsilon < 1$, is a fraction of the healthy cells that become latently infected. The terms $\alpha\digamma_1(K)$ and $m\digamma_1(K)$ represent the death and activation rates of the latently infected cells. We suppose that the viruses contact the healthy cells at times $t-\tau_1$ and $t-\tau_2$, respectively, the cells become latently infected and actively infected at time $t$, where $\tau_1$ and $\tau_2$ are positive constants. The immature viruses at time $t-\tau_3$ are assumed to be mature at time $t$, where $\tau_3$ is a positive constant. $e^{-\mu_j\tau_j}$, $j = 1,2,3$, is the probability of the cells and viruses survival during the delay periods, where $\mu_1$, $\mu_2$, and, $\mu_3$ are positive constants. Here $\Theta$, $\Lambda$, and $\digamma_i$, $i = 1,\dots,5$, are general nonlinear functions satisfy the following conditions:

C1 (i) There exists $F^0 > 0$ such that $\Theta(F^0) = 0$, $\Theta(F) > 0$ for $F \in [0,F^0)$;
(ii) $\Theta'(F) < 0$ for all $F > 0$;
(iii) $\exists b,\bar{b} > 0$ such that $\Theta(F) \le b - \bar{b}F$ for all $F \ge 0$.

Here, $F^0$ is the equilibrium susceptible cell concentration in the absence of viral infection. Condition C1 implies that $F(t) \to F^0$ as $t \to \infty$ in the absence of the infection.

C2 (i) $\Lambda(F,H) > 0$, and $\Lambda(0,H) = \Lambda(F,0) = 0$ for all $F > 0$, $H > 0$;
(ii) $\frac{\partial\Lambda(F,H)}{\partial F} > 0$, $\frac{\partial\Lambda(F,H)}{\partial H} > 0$, $\frac{\partial\Lambda(F,0)}{\partial H} > 0$ for all $F > 0$, $H > 0$;
(iii) $\frac{d}{dF}(\frac{\partial\Lambda(F,0)}{\partial H}) > 0$ for all $F > 0$.

Furthermore, C2(i) means there are no incidences if there are no susceptible cells or free virus particles. For C2(ii), the number of new cases monotonically grows with growth in the numbers of susceptible cells and free pathogen particles. Moreover, condition C2(iii) accounts that the infection rate starts growing even if the number of pathogens is very small.

C3 (i) $\digamma_j(\rho) > 0$ for $\rho > 0$, $\digamma_j(0) = 0$, $j = 1,\dots,5$;
(ii) $\digamma_j'(\rho) > 0$ for $\rho > 0$, $j = 1,2,4,5$ and $\digamma_3'(\rho) > 0$ for $\rho \ge 0$;
(iii) There are $\upsilon_j > 0$, $j = 1,\dots,5$, such that $\digamma_j(\rho) \ge \upsilon_j\rho$ for $\rho \ge 0$.

Condition C3 indicates that the natural mortality rates of the infected cells, pathogens, CTL, and antibodies monotonically grow with growth in their populations.

C4 $\frac{\partial}{\partial H}(\frac{\Lambda(F,H)}{\digamma_3(H)}) \le 0$ for all $H > 0$.

The quantity $\frac{\Lambda(F,H)}{\digamma_3(H)}$ may be interpreted as the efficiency of the pathogen, that is, the ratio of its infectivity to its removal. Condition C4 mentions that the efficiency of the pathogen is nonincreasing with respect to the population of the pathogens [39].

*Remark* 1 There are several forms of the general functions which can satisfy C1–C4 such as:

(i) Intrinsic growth rate function $\Theta(F)$: linear form $\Theta(F) = \delta - \beta F$ [1] and logistic growth form $\Theta(F) = \delta - \beta F + rF(1 - \frac{F}{F_{\max}})$, where $r < \beta$ [53, 54].
(ii) Incidence rate function $\Lambda(F,H)$: bilinear incidence $\kappa FH$ [55], saturated incidence $\frac{\kappa FH}{1+uH}$ [56], (iii) Holling-type II incidence $\frac{\kappa FH}{1+wF}$ [57], Beddington–DeAngelis

incidence $\frac{\kappa FH}{1+uH+wF}$ [58], Crowley–Martin incidence $\frac{\kappa FH}{(1+uH)(1+wF)}$ [59], Hill-type incidence $\frac{\kappa F^{\ell}H}{\zeta^{\ell}+F^{\ell}}$ [60], where $\kappa$, $u$, $w$, $\zeta$, and $\ell$ are positive constants.

(iii) Function $\digamma_i(\rho)$: linear $\digamma_i(\rho) = \upsilon_i\rho$ [1] and quadratic $\digamma_i(\rho) = \upsilon_i\rho + \overline{\upsilon}_i\rho^2$ [15], where $\upsilon_i$ and $\overline{\upsilon}_i$ are positive constants.

We use the NSFD method [41] to discretize model (5)–(10). Let $t_n = nh$, where $h > 0$ is the time step size and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Let $(F_n, K_n, S_n, H_n, Y_n, Z_n)$ be the approximations of the solution $(F(t_n), K(t_n), S(t_n), H(t_n), Y(t_n), Z(t_n))$ of system (5)–(10) at the discrete time points $t_n$. Assume that there exist integers $m_i \in \mathbb{N}$, $i = 1, 2, 3$, with $\tau_i = hm_i$.

$$\frac{F_{n+1} - F_n}{\phi(h)} = \Theta(F_{n+1}) - \Lambda(F_{n+1}, H_n), \tag{11}$$

$$\frac{K_{n+1} - K_n}{\phi(h)} = (1 - \varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{n-m_1+1}, H_{n-m_1}) - (\alpha + m)\digamma_1(K_{n+1}), \tag{12}$$

$$\begin{aligned}\frac{S_{n+1} - S_n}{\phi(h)} &= \varepsilon e^{-\mu_2\tau_2}\Lambda(F_{n-m_2+1}, H_{n-m_2}) + m\digamma_1(K_{n+1}) - a\digamma_2(S_{n+1}) \\ &\quad - \lambda\digamma_2(S_{n+1})\digamma_5(Z_{n+1}),\end{aligned} \tag{13}$$

$$\frac{H_{n+1} - H_n}{\phi(h)} = \theta e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1}) - c\digamma_3(H_{n+1}) - d\digamma_3(H_{n+1})\digamma_4(Y_{n+1}), \tag{14}$$

$$\frac{Y_{n+1} - Y_n}{\phi(h)} = q\digamma_3(H_{n+1})\digamma_4(Y_{n+1}) - \eta\digamma_4(Y_{n+1}), \tag{15}$$

$$\frac{Z_{n+1} - Z_n}{\phi(h)} = g\digamma_2(S_{n+1})\digamma_5(Z_{n+1}) - \xi\digamma_5(Z_{n+1}). \tag{16}$$

The function $\phi(h)$ is a denominator function [61, 62] where $\phi(h) = h + O(h^2)$.

The initial conditions of system (11)–(16) are

$$\begin{aligned} &F_\omega = \psi^1_\omega \geq 0, \qquad K_\omega = \psi^2_\omega \geq 0, \qquad S_\omega = \psi^3_\omega \geq 0, \\ &H_\omega = \psi^4_\omega \geq 0, \qquad Y_\omega = \psi^5_\omega \geq 0, \qquad Z_\omega = \psi^6_\omega \geq 0 \end{aligned}$$

for all $\omega = -\overline{m}, -\overline{m} + 1, \ldots, 0,$

$$\psi^i_0 > 0, \quad i = 1, \ldots, 6, \tag{17}$$

where $\overline{m} = \max\{m_1, m_2, m_3\}$.

### 2.1 Preliminaries

We define a compact set

$$\Gamma = \bigl\{(F, K, S, H, Y, Z) : 0 < F, K, S < \vartheta_1, 0 < H < \vartheta_3, 0 < Y < \vartheta_4, 0 < Z < \vartheta_2\bigr\},$$

where $\vartheta_1 = \frac{b}{\sigma_1}$, $\vartheta_2 = \frac{g\vartheta_1}{\lambda}$, $\vartheta_3 = \frac{\theta\vartheta_1}{\sigma_2}$, $\vartheta_4 = \frac{q\theta\vartheta_1}{d\sigma_2}$, $\sigma_1 = \min\{\overline{b}, \alpha\upsilon_1, a, \xi\}$, and $\sigma_2 = \min\{c, \eta\}$.

**Lemma 1** *Suppose that Conditions* C1–C3 *are satisfied and* $\digamma_j(\rho) = \rho$, $j = 2, 3, 4, 5$, *then any solution* $(F_n, K_n, S_n, H_n, Y_n, Z_n)$ *of model* (11)–(16) *with initial conditions* (17) *is positive and ultimately bounded.*

**Lemma 2** *For system* (11)–(16), *let Conditions* C1–C4 *hold true, then there exist four threshold parameters* $\mathcal{R}_0 > 0$, $\mathcal{R}_1^Y > 0$, $\mathcal{R}_1^Z > 0$, *and* $\mathcal{R}_2^Z > 0$ *such that*

(i) *if* $\mathcal{R}_0 \leq 1$, *then the system has only one equilibrium* $Q^0$.

(ii) *if* $\mathcal{R}_1^Y \leq 1 < \mathcal{R}_0$ *and* $\mathcal{R}_1^Z \leq 1 < \mathcal{R}_0$, *then the system has two equilibria* $Q^0$ *and* $Q^*$.

(iii) *if* $\mathcal{R}_1^Y > 1$ *and* $\mathcal{R}_2^Z \leq 1$, *then the system has three equilibria* $Q^0$, $Q^*$, *and* $\overline{Q}$.

(iv) *if* $\mathcal{R}_1^Z > 1$ *and* $\mathcal{R}_1^Y/\mathcal{R}_2^Z \leq 1$, *then the system has three equilibria* $Q^0$, $Q^*$, *and* $\widehat{Q}$.

(v) *if* $\mathcal{R}_1^Y > \mathcal{R}_2^Z > 1$, *then the system has five equilibria* $Q^0$, $Q^*$, $\overline{Q}$, $\widehat{Q}$, *and* $\widetilde{Q}$.

The proofs of Lemmas 1–2 are given in Appendix.

## 2.2 Global stability

We define the function

$$G(\rho) = \rho - \ln \rho - 1.$$

Clearly, $G(\rho) \geq 0$ and $G(1) = 0$ for all $\rho > 0$, therefore

$$\ln \rho \leq \rho - 1. \tag{18}$$

**Theorem 1** *Suppose that Conditions* C1–C4 *are satisfied and* $\mathcal{R}_0 \leq 1$, *then* $Q^0$ *of model* (11)–(16) *is G.A.S.*

*Remark* 2 Conditions C2–C4 imply that

$$\big(\Lambda(F,H) - \Lambda\big(F,H^*\big)\big)\left(\frac{\Lambda(F,H)}{\digamma_3(H)} - \frac{\Lambda(F,H^*)}{\digamma_3(H^*)}\right) \leq 0,$$

which yields

$$\left(1 - \frac{\Lambda(F,H^*)}{\Lambda(F,H)}\right)\left(\frac{\Lambda(F,H)}{\Lambda(F,H^*)} - \frac{\digamma_3(H)}{\digamma_3(H^*)}\right) \leq 0. \tag{19}$$

**Lemma 3** *Suppose that Conditions* C1–C4 *are satisfied and* $\mathcal{R}_0 > 1$, *then*

(i) $\operatorname{sgn}\big(\overline{F} - F^*\big) = \operatorname{sgn}\big(H^* - \overline{H}\big) = \operatorname{sgn}\big(\mathcal{R}_1^Y - 1\big)$,

(ii) $\operatorname{sgn}\big(\widehat{F} - F^*\big) = \operatorname{sgn}\big(H^* - \widehat{H}\big) = \operatorname{sgn}\big(S^* - \widehat{S}\big) = \operatorname{sgn}\big(\mathcal{R}_1^Z - 1\big)$.

**Theorem 2** *Suppose that Conditions* C1–C4 *are satisfied,* $\mathcal{R}_1^Y \leq 1 < \mathcal{R}_0$ *and* $\mathcal{R}_1^Z \leq 1 < \mathcal{R}_0$, *then* $Q^*$ *of system* (11)–(16) *is G.A.S.*

**Theorem 3** *If Conditions* C1–C4 *hold,* $\mathcal{R}_1^Y > 1$ *and* $\mathcal{R}_2^Z \leq 1$, *then* $\overline{Q}$ *of system* (11)–(16) *is G.A.S.*

**Theorem 4** *If Conditions* C1–C4 *are satisfied,* $\mathcal{R}_1^Z > 1$ *and* $\mathcal{R}_1^Y/\mathcal{R}_2^Z \leq 1$, *then* $\widehat{Q}$ *of system* (11)–(16) *is G.A.S.*

**Theorem 5** *If Conditions* C1–C4 *are satisfied and* $\mathcal{R}_1^Y > \mathcal{R}_2^Z > 1$, *then* $\widetilde{Q}$ *of system* (11)–(16) *is G.A.S.*

The proofs of Theorems 1–5 and Lemma 3 are given in Appendix.

## 3 Numerical simulations

We perform our simulation by choosing

$$\Theta(F) = \delta - \beta F + rF\left(1 - \frac{F}{F_{\max}}\right), \qquad \Lambda(F,H) = \frac{\kappa FH}{1 + c_1 H}, \qquad (20)$$
$$F_i(\rho) = \rho, \quad i = 1,\dots,5,$$

where $r > 0$ is the maximum proliferation rate of the healthy cells and $F_{\max} > 0$ is the maximum level of healthy cells concentration in the body. If $F$ reaches $F_{\max}$, it should decrease. Therefore, system (11)–(16) becomes

$$\frac{F_{n+1} - F_n}{\phi(h)} = \delta - \beta F_{n+1} + rF_{n+1}\left(1 - \frac{F_{n+1}}{F_{\max}}\right) - \frac{\kappa F_{n+1} H_n}{1 + c_1 H_n}, \qquad (21)$$

$$\frac{K_{n+1} - K_n}{\phi(h)} = \frac{(1-\varepsilon)e^{-\mu_1\tau_1}\kappa F_{n-m_1+1} H_{n-m_1}}{1 + c_1 H_{n-m_1}} - (\alpha + m)K_{n+1}, \qquad (22)$$

$$\frac{S_{n+1} - S_n}{\phi(h)} = \frac{\varepsilon e^{-\mu_2\tau_2}\kappa F_{n-m_2+1} H_{n-m_2}}{1 + c_1 H_{n-m_2}} + mK_{n+1} - aS_{n+1} - \lambda S_{n+1} Z_{n+1}, \qquad (23)$$

$$\frac{H_{n+1} - H_n}{\phi(h)} = \theta e^{-\mu_3\tau_3} S_{n-m_3+1} - cH_{n+1} - dH_{n+1} Y_{n+1}, \qquad (24)$$

$$\frac{Y_{n+1} - Y_n}{\phi(h)} = qH_{n+1} Y_{n+1} - \eta Y_{n+1}, \qquad (25)$$

$$\frac{Z_{n+1} - Z_n}{\phi(h)} = gS_{n+1} Z_{n+1} - \xi Z_{n+1}. \qquad (26)$$

The denominator function $\phi(h)$ can take the form [61, 62]

$$\phi(h) = \frac{1 - e^{-\beta h}}{\beta}.$$

We assume that $r < \beta$ [54]. Now we check the validity of Conditions C1–C4 for the functions given by (20). We have $\Theta(0) = \delta > 0$, $\Theta(F^0) = 0$, where

$$F^0 = \frac{F_{\max}}{2r}\left(r - \beta + \sqrt{(r-\beta)^2 + \frac{4r\delta}{F_{\max}}}\right).$$

Since $r < \beta$, then

$$\Theta'(F) = -\beta + r - \frac{2rF}{F_{\max}} < 0.$$

It follows that $\Theta(F) > 0$ for all $F \in [0, F^0)$ and

$$\Theta(F) = \delta - (\beta - r)F - r\frac{F^2}{F_{\max}} < \delta - (\beta - r)F.$$

Let $b = \delta > 0$ and $\overline{b} = \beta - r > 0$. Thus, C1 is satisfied. We have also

$$\begin{aligned}
&\Lambda(F,H) = \frac{\kappa FH}{1+c_1H} > 0 \quad \text{and} \quad \Lambda(0,H) = \Lambda(F,0) = 0 \quad \text{for all } F>0, H>0,\\
&\frac{\partial\Lambda(F,H)}{\partial F} = \frac{\kappa H}{1+c_1H} > 0 \quad \text{for all } F>0 \text{ and } H>0,\\
&\frac{\partial\Lambda(F,H)}{\partial H} = \frac{\kappa F}{(1+c_1H)^2} > 0 \quad \text{for all } F>0 \text{ and } H>0,\\
&\frac{\partial\Lambda(F,0)}{\partial H} = \kappa F > 0 \quad \text{for all } F>0,\\
&\frac{d}{dF}\left(\frac{\partial\Lambda(F,0)}{\partial H}\right) = \kappa > 0 \quad \text{for all } F>0.
\end{aligned}$$

Therefore, Condition C2 is satisfied. We have $\digamma_j(\rho) = \rho > 0$ for all $\rho > 0$ and $\digamma_j(0) = 0$, $j = 1,\dots,5$, and $\digamma_j'(\rho) = 1 > 0$, $j = 1,\dots,5$, for all $\rho \geq 0$. Then Condition C3 is satisfied, where $\upsilon_j = 1$, $j = 1,\dots,5$. Finally, we have

$$\frac{\partial}{\partial H}\left(\frac{\Lambda(F,H)}{\digamma_3(H)}\right) = \frac{-\kappa Fc_1}{(1+c_1H)^2} < 0 \quad \text{for all } F>0 \text{ and } H>0.$$

Therefore, Condition C4 holds true and hence Theorems 1–5 are applicable.

For this system, the threshold parameters are given by

$$\begin{aligned}
&\mathcal{R}_0 = \frac{\theta\gamma\kappa F^0}{ac}, \qquad \mathcal{R}_1^Y = \frac{\theta\gamma\kappa\overline{F}}{ac(1+c_1\overline{H})},\\
&\mathcal{R}_1^Z = \frac{\theta\gamma\kappa\widehat{F}}{ac(1+c_1\widehat{H})}, \qquad \mathcal{R}_2^Z = \frac{e^{\mu_3\tau_3}\gamma\kappa\widetilde{F}\widetilde{H}}{a(1+c_1\widetilde{H})\widetilde{S}},\\
&\mathcal{R}_1^Y\backslash\mathcal{R}_2^Z = \frac{\theta e^{-\mu_3\tau_3}\xi g}{cg\eta},
\end{aligned}$$

where

$$\begin{aligned}
&\overline{F} = \frac{-\overline{B}+\sqrt{\overline{B}^2-4\overline{A}\overline{C}}}{2\overline{A}} = \widetilde{F},\\
&\overline{A} = r + c_1r\overline{H},\\
&\overline{B} = -F_{\max}\bigl(r-\beta-\kappa\overline{H}+c_1\overline{H}(r-\beta)\bigr),\\
&\overline{C} = -F_{\max}\delta(1+c_1\overline{H}),\\
&\overline{H} = \frac{\eta}{q} = \widetilde{H},
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{F} = \frac{-\widehat{B}+\sqrt{\widehat{B}^2-4\widehat{A}\widehat{C}}}{2\widehat{A}},\\
&\widehat{A} = r + c_1r\widehat{H},\\
&\widehat{B} = -F_{\max}\bigl(r-\beta-\kappa\widehat{H}+c_1\widehat{H}(r-\beta)\bigr),
\end{aligned}$$

$$\widehat{C} = -F_{\max}\delta(1 + c_1\widehat{H}),$$
$$\widehat{H} = \frac{\theta e^{-\mu_3\tau_3}\xi}{cg}, \qquad \widetilde{S} = \frac{\xi}{g}.$$

Numerical simulations for system (21)–(26) are performed using the following values: $\delta = 10$, $\beta = 0.01$, $r = 0.009$, $F_{\max} = 1200$, $\varepsilon = 0.3$, $\alpha = 0.4$, $m = 0.1$, $a = 0.6$, $\lambda = 0.1$, $\theta = 1.2$, $c = 4$, $\eta = 0.1$, $\xi = 0.03$, $d = 0.8$, $h = 0.1$, and $\mu_i = 0.1$ $(i = 1, 2, 3)$. The other parameters will be chosen in what follows.

Let us consider the initial values:

IV1: $\psi^1_\omega = 800$, $\psi^2_\omega = 8$, $\psi^3_\omega = 1$, $\psi^4_\omega = 0.5$, $\psi^5_\omega = 1$, $\psi^6_\omega = 0.5$,

IV2: $\psi^1_\omega = 600$, $\psi^2_\omega = 10$, $\psi^3_\omega = 3$, $\psi^4_\omega = 1$, $\psi^5_\omega = 3$, $\psi^6_\omega = 1$, and

IV3: $\psi^1_\omega = 400$, $\psi^2_\omega = 12$, $\psi^3_\omega = 5$, $\psi^4_\omega = 2$, $\psi^5_\omega = 5$, $\psi^6_\omega = 2$, $\omega = -\overline{m}, -\overline{m} + 1, \dots, 0$.

- *Stability of equilibria*

We choose $\tau_1 = 0.1$, $\tau_2 = 0.5$, $\tau_3 = 0.9$ and choose $\kappa$, $\eta$, and $\xi$ are varied as follows.

*Case (1)* $\kappa = 0.003$, $q = 0.05$, and $g = 0.002$. This yields $\mathcal{R}_0 = 0.634 < 1$. Figure 1 shows that the concentration of healthy cells increases and tends to the value $F^0 = 1089.96$. Moreover, the concentrations of infected cells, free viruses, antibodies, and CTL cells decay and reach zero for IV1–IV3. Consequently, there exists only one equilibrium that is $Q^0$ and it is G.A.S. This result supports the result of Theorem 1.

*Case (2)* $\kappa = 0.01$, $q = 0.05$, and $g = 0.002$. With these values we obtain $\mathcal{R}^Y_1 = 0.799 < 1 < \mathcal{R}_0 = 2.112$ and $\mathcal{R}^Z_1 = 0.440 < 1 < \mathcal{R}_0 = 2.112$. Figure 2 displays that for all the three initial values IV1–IV3, the solutions of the system reach the equilibrium $Q^* = (523.57, 10.285, 5.244, 1.438, 0, 0)$. Consequently, $Q^*$ exists and it is G.A.S. This agrees with the result of Theorem 2.

*Case (3)* $\kappa = 0.01$, $q = 0.2$, and $g = 0.002$ and then $\mathcal{R}_0 = 2.112 > 1$, $\mathcal{R}^Y_1 = 1.587 > 1$, and $\mathcal{R}^Z_1 = 0.440 < 1$. Figure 3 displays that the solutions of the system reach the equilibrium $\overline{Q} = (823.137, 5.676, 2.894, 0.5, 2.934, 0)$ for all the initial values IV1–IV3. Thus $\overline{Q}$ exists and it is G.A.S. This result is consistent with the result of Theorem 3.

*Case (4)* $\kappa = 0.01$, $q = 0.005$, and $g = 0.01$ and then $\mathcal{R}_0 = 2.112 > 1$, $\mathcal{R}^Z_1 = 1.337 > 1$, and $\mathcal{R}^Y_1/\mathcal{R}^Z_2 = 0.041 < 1$. From Fig. 4 we can see that, for all the initial values IV1–IV3, the solutions of the system tend to the equilibrium $\widehat{Q} = (695.628, 7.866, 3, 0.823, 0, 2.020)$. This result shows that $\widehat{Q}$ exists and it is G.A.S, and this agrees with the result of Theorem 4.

*Case (5)* $\kappa = 0.01$, $q = 0.3$, and $g = 0.02$ and then $\mathcal{R}_0 = 2.112 > 1$, $\mathcal{R}^Y_1 = 1.742 > 1$, and $\mathcal{R}^Z_2 = 1.412 > 1$. From Fig. 5 we observe that the solutions of the system reach the equilibrium $\widetilde{Q} = (901.96, 4.153, 1.5, 0.333, 1.169, 2.469)$. This yields that $\widetilde{Q}$ exists and it is G.A.S. This *illustrates* the result of Theorem 5.

- *Impact of time delay on the viral dynamics*

Without loss of generality we let $\tau = \tau_1 = \tau_2 = \tau_3$. We fix the values $\kappa = 0.01$, $q = 0.3$, and $g = 0.02$ and select different values of $\tau$. We solve the system with initial IV2. Figure 6 shows the influence of the time delay parameter $\tau$ on the stability of the equilibria. One can see that as $\tau$ is increased, the concentration of healthy cells is increased, while the concentrations of infected cells, free viruses, CTL cells, and antibodies are decreased. Let us write $\mathcal{R}_0$ as follows:

$$\mathcal{R}_0(\tau) = F^0\left(\frac{m(1-\varepsilon)e^{-(\mu_1+\mu_3)\tau}}{\alpha + m} + \varepsilon e^{-(\mu_2+\mu_3)\tau}\right)\left(\frac{\kappa\theta}{ac}\right).$$

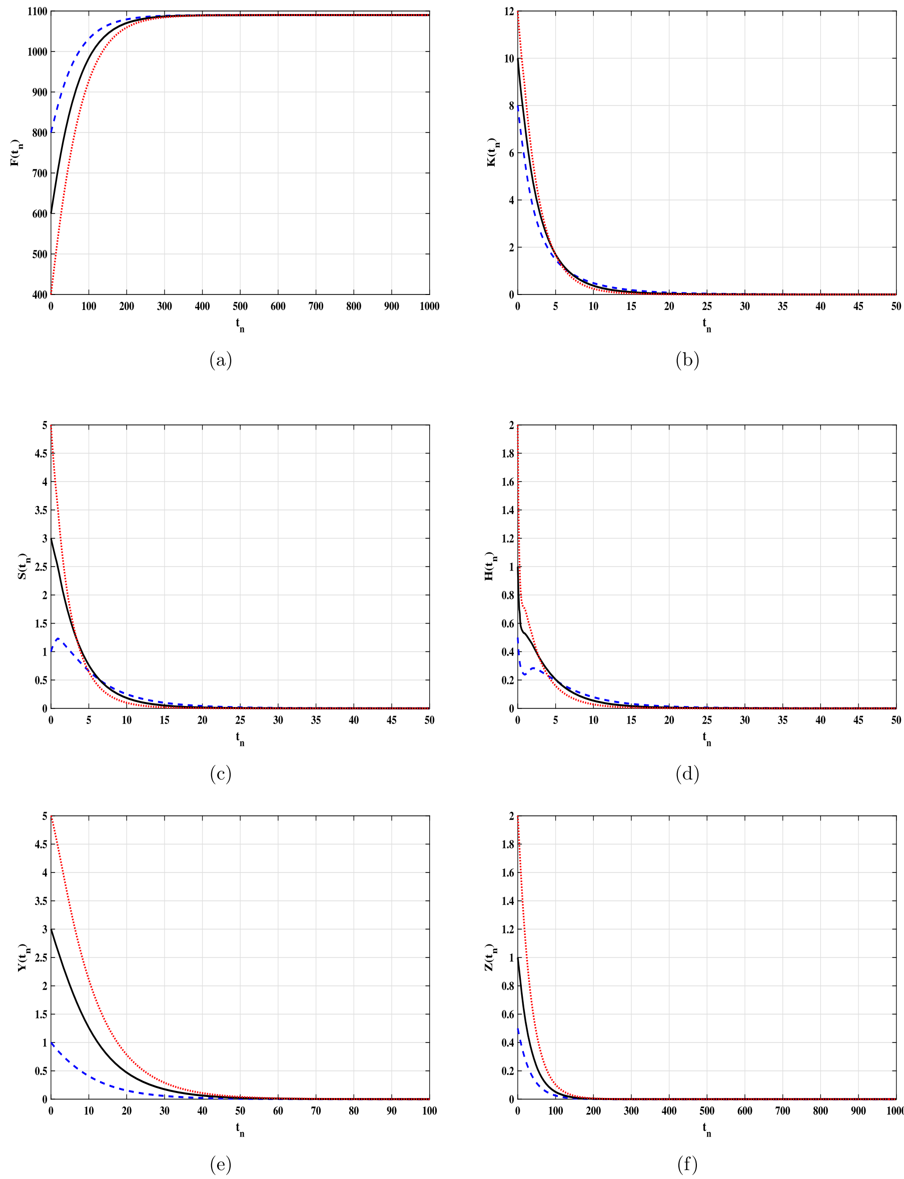


**Figure 1** The simulation of trajectories of system (21)–(26) for Case $\mathcal{R}_0 \leq 1$

Since $\mathcal{R}_0$ is a decreasing function of $\tau$, then the time delay can change the stability properties of equilibria. Using the values of the parameters and from Fig. 6 we can see that if $\tau \geq 4.373$, then $Q^0$ is G.A.S. Biologically, the time delays play a similar role of antiviral treatment in eliminating the viruses from the body.

## 4 Conclusion

In this paper, we formulated and analyzed a discrete-time viral infection model with both antibody and cell-mediated immune responses. We incorporated two categories of infected cells, namely latently infected cells (such cells contain the virus but are not producing) and actively infected cells (such cells produce new viruses). The production and removal rates of the cells and viruses as well as the incidence rate were modeled by general

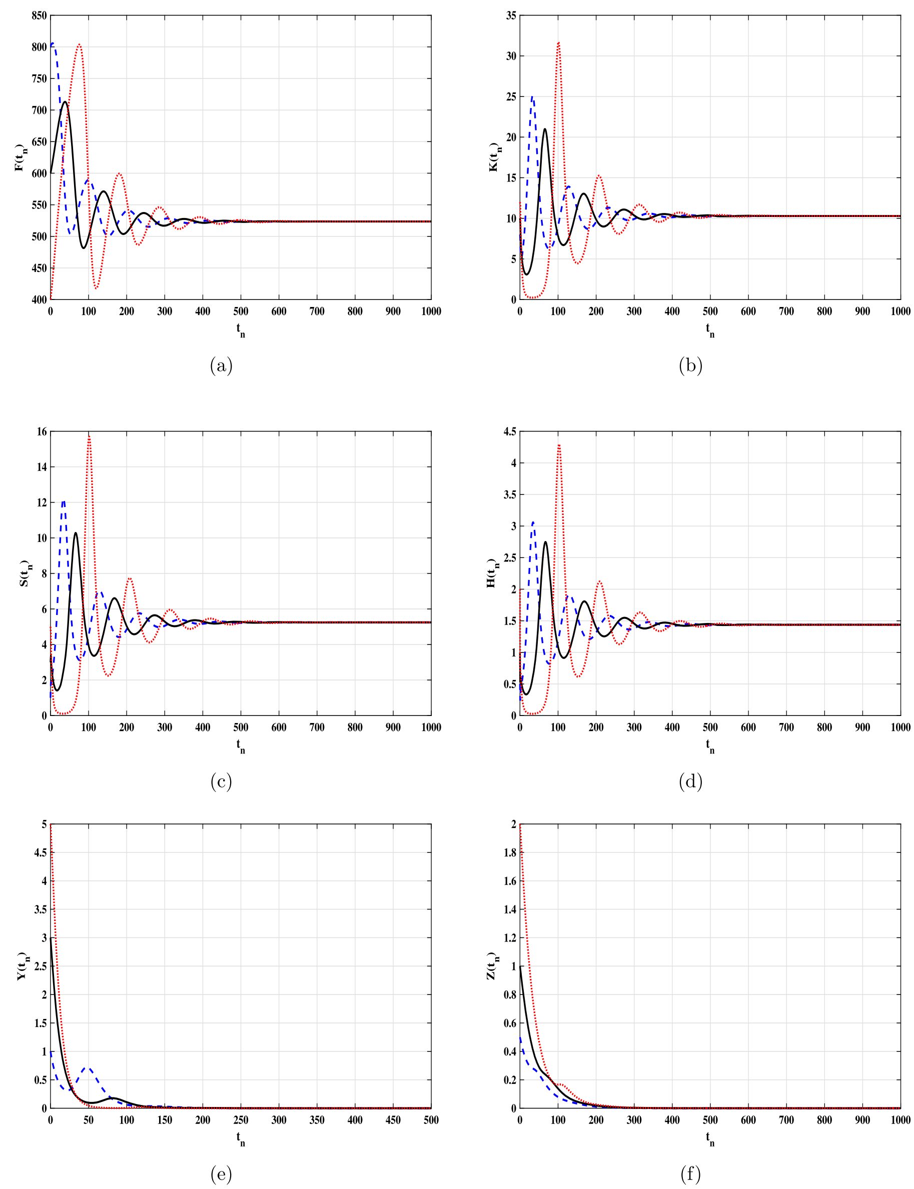


**Figure 2** The simulation of trajectories of system (21)–(26) for Case $\mathcal{R}_1^Y \le 1 < \mathcal{R}_0$ and $\mathcal{R}_1^Z \le 1 < \mathcal{R}_0$

nonlinear functions which satisfy a set of conditions. These general functions encompass several specific forms commonly used in the virus dynamics literature. We incorporated three types of time delays, in which the first and second delays describe the times between a virus contacts a susceptible cell and the cell becomes latently infected and actively infected cell, respectively. The third delay is the time from death of an infected cell until the virus is active. We used nonstandard finite difference scheme to discretize the continuous-time model. We showed that the solutions of the discrete-time model with given initial states are positive and bounded. We derived four threshold parameters which fully determine the existence and stability of the five equilibria of the model. Then, we proved the global stability of the equilibria by constructing Lyapunov functions. Moreover, we performed numerical simulations to support the global stability results. We studied the

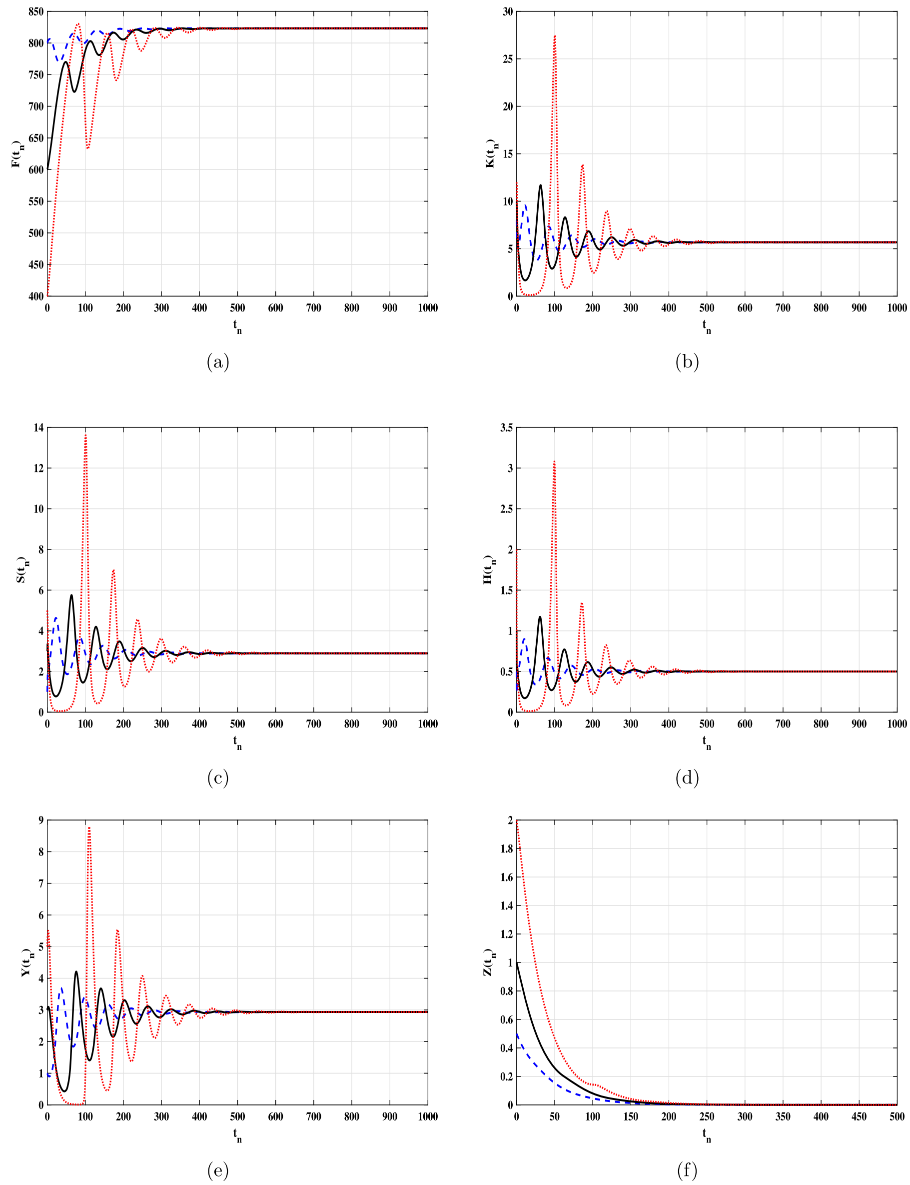


**Figure 3** The simulation of trajectories of system (21)–(26) for Case $\mathcal{R}_1^Y > 1$ and $\mathcal{R}_1^Z \le 1$

effect of time delay on the virus dynamics. Since $Q^0$ is the desired equilibrium to be stabilized, we determined the critical time delay parameter $\tau^{\text{critical}}$ by solving the equation $\mathcal{R}_0(\tau^{\text{critical}}) = 1$ and showed that $Q^0$ is globally asymptotically stable when $\tau \ge \tau^{\text{critical}}$. This shows that the time delay can have a similar effect as the antiviral drugs. This gives some impression to develop a new class of treatment to increase the delay period and then suppress the viral replication. It is worth emphasizing that the role of the delay term does not only take into account the delay in the dynamical response of the interacting entities, but also their heterogeneity. This can be accounted for by modeling interactions as shown in [63]. Recently, many authors have argued that the virus moves freely in body and follows the Fickian diffusion (see, e.g., [64–66]). Therefore, it is more reasonable to study reaction-diffusion versions of our model. We leave these points as possible future works.

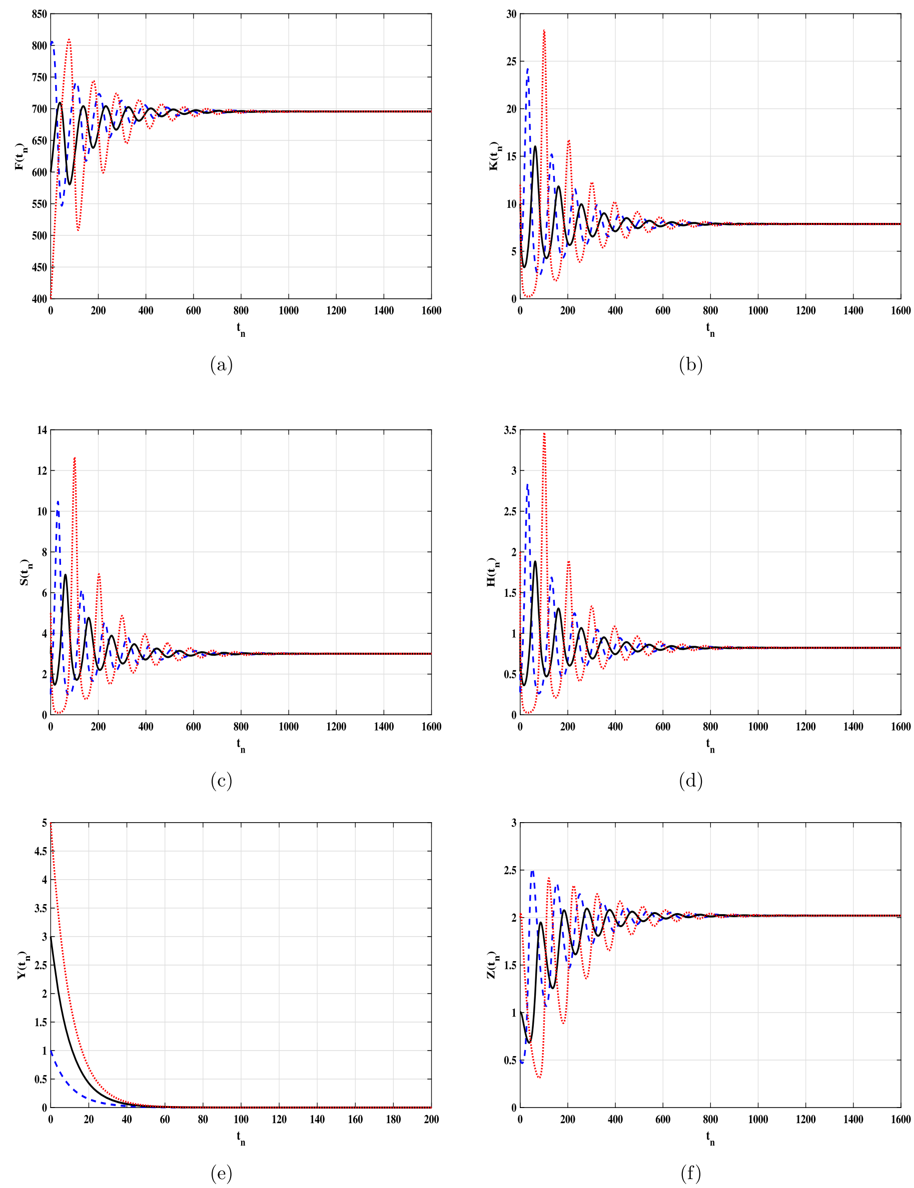


**Figure 4** The simulation of trajectories of system (21)–(26) for Case $\mathcal{R}_1^Z > 1$ and $\mathcal{R}_1^Y/\mathcal{R}_2^Z \le 1$

## Appendix

*Proof of Lemma* 1 First we put $n = 0$ and show that there exists positive and unique $(F_1, K_1, S_1, H_1, Y_1, Z_1)$. From Eq. (11) we have

$$F_1 - F_0 - \phi(h)\Theta(F_1) + \phi(h)\Lambda(F_1, H_0) = 0.$$

We define a function $\chi_1(F)$ as follows:

$$\chi_1(F) = F - F_0 - \phi(h)\Theta(F) + \phi(h)\Lambda(F, H_0) = 0.$$

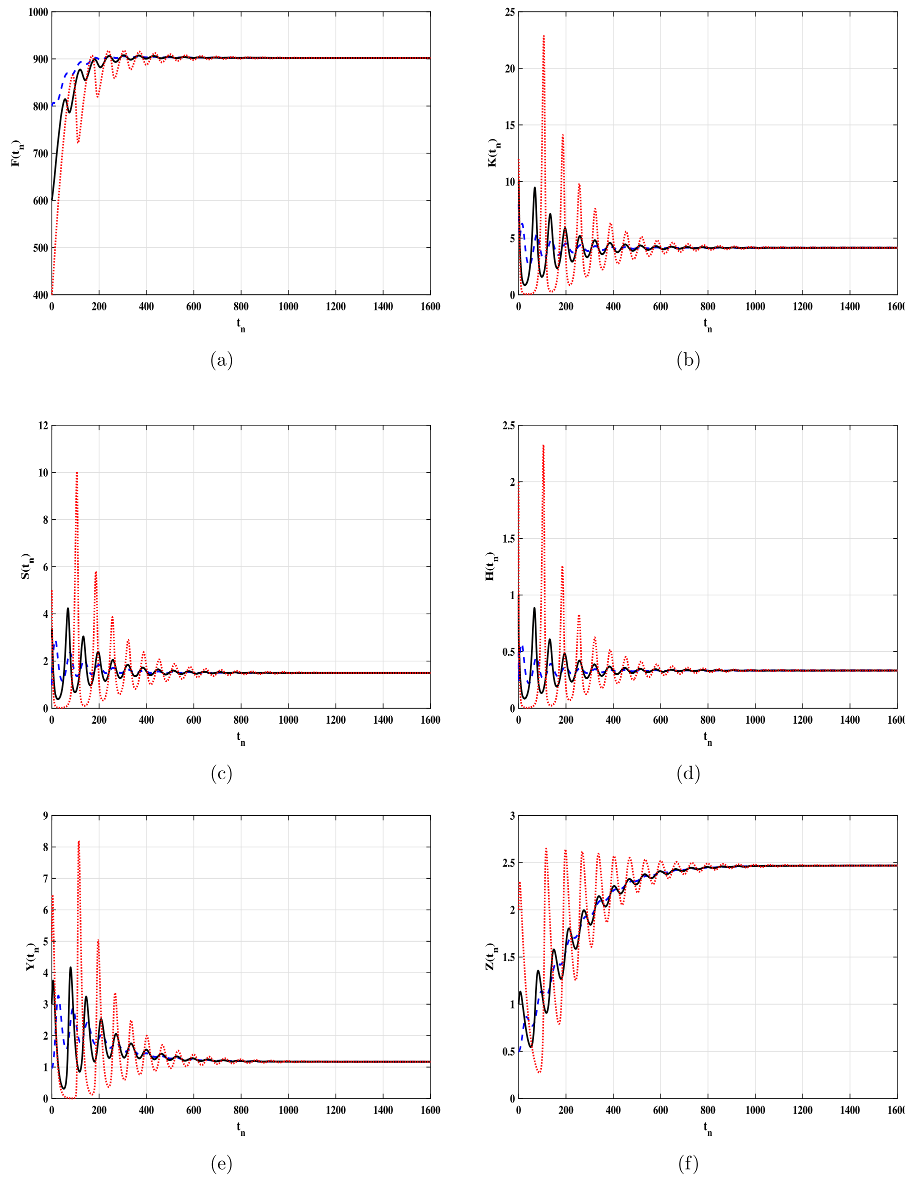


**Figure 5** The simulation of trajectories of system (21)–(26) for Case $\mathcal{R}_1^Y > \mathcal{R}_2^Z > 1$

According to Conditions C1–C2 we have $\chi_1$ is a strictly increasing function in $F$ and

$$\chi_1(0) = -F_0 - \phi(h)\Theta(0) < 0,$$

$$\lim_{F\to\infty} \chi_1(F) = \infty.$$

Hence, there exists unique $F_1 > 0$ such that $\chi_1(F_1) = 0$.

From Eq. (12) we have

$$K_1 - K_0 - \phi(h)(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{-m_1+1}, H_{-m_1}) + \phi(h)(\alpha + m)\Gamma_1(K_1) = 0.$$

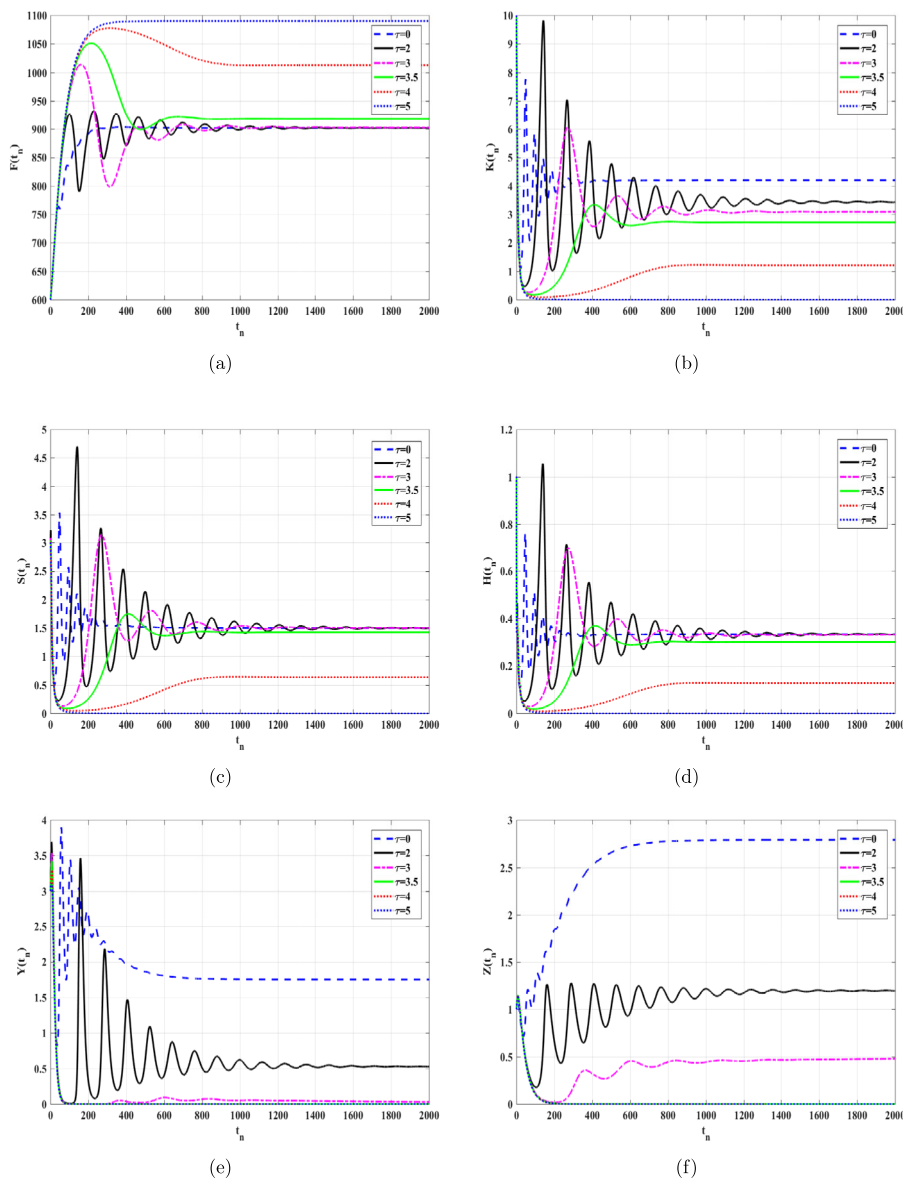


**Figure 6** The simulation of trajectories of system (21)–(26) with different values of $\tau$

Define a function $\chi_2(K)$ as follows:

$$\chi_2(K) = K - K_0 - \phi(h)(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{-m_1+1}, H_{-m_1}) + \phi(h)(\alpha + m)F_1^{-1}(K) = 0.$$

Conditions C2–C3 imply that $\chi_2$ is a strictly increasing function in $K$. In addition,

$$\chi_2(0) = -K_0 - \phi(h)(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{-m_1+1}, H_{-m_1}) < 0,$$

$$\lim_{K\to\infty} \chi_2(K) = \infty.$$

It follows that there exists unique $K_1 \in (0,\infty)$ such that $\chi_2(K_1) = 0$.

Since $\digamma_2(S_1) = S_1$, $\digamma_3(H_1) = H_1$, $\digamma_4(Y_1) = Y_1$, and $\digamma_5(Z_1) = Z_1$, then from Eqs. (13) and (16) we get

$$\begin{aligned} Z_1 &= Z_0 + \frac{\phi(h)g[S_0 + \phi(h)\varepsilon e^{-\mu_2\tau_2}\Lambda(F_{-m_2+1}, H_{-m_2}) + \phi(h)m\digamma_1(K_1)]}{1+\phi(h)(a+\lambda Z_1)} Z_1 \\ &\quad - \phi(h)\xi Z_1. \end{aligned}$$

Then

$$A_1 Z_1^2 + B_1 Z_1 + C_1 = 0, \tag{27}$$

where

$$\begin{aligned} A_1 &= \bigl(1+\phi(h)\xi\bigr)\phi(h)\lambda, \\ B_1 &= \bigl(1+\phi(h)\xi\bigr)\bigl(1+\phi(h)a\bigr) - \phi(h)\lambda Z_0 \\ &\quad - \phi(h)g\bigl[S_0 + \phi(h)e^{-\mu_2\tau_2}\Lambda(F_{-m_2+1}, H_{-m_2}) + \phi(h)m\digamma_1(K_1)\bigr], \\ C_1 &= -\bigl(1+\phi(h)a\bigr)Z_0. \end{aligned}$$

Since $A_1 > 0$, $C_1 < 0$, then $B_1^2 - 4A_1C_1 > 0$, and hence there exists a unique positive root of Eq. (27) $Z_1 > 0$. It follows from Eq. (13)

$$S_1 = \frac{S_0 + \phi(h)\varepsilon e^{-\mu_2\tau_2}\Lambda(F_{-m_2+1}, H_{-m_2}) + \phi(h)m\digamma_1(K_1)}{1+\phi(h)(a+\lambda Z_1)} > 0.$$

Then we have $S_1 > 0$.

Now we show that $Y_1 > 0$. From Eqs. (14)–(15) we get

$$Y_1 = Y_0 + \frac{\phi(h)q(H_0 + \phi(h)\theta e^{-\mu_3\tau_3}S_{-m_3+1})}{1+\phi(h)(c+dY_1)} Y_1 - \phi(h)\eta Y_1.$$

Then we get

$$A_2 Y_1^2 + B_2 Y_1 + C_2 = 0, \tag{28}$$

where

$$\begin{aligned} A_2 &= \bigl(1+\phi(h)\eta\bigr)\phi(h)d, \\ B_2 &= \bigl(1+\phi(h)\eta\bigr)\bigl(1+\phi(h)c\bigr) - \phi(h)\,dY_0 - \phi(h)q\bigl[H_0 + \phi(h)\theta e^{-\mu_3\tau_3}\digamma_2(S_{-m_3+1})\bigr], \\ C_2 &= -\bigl(1+\phi(h)c\bigr)Y_0. \end{aligned}$$

Since $A_2 > 0$, $C_2 < 0$, then $B_2^2 - 4A_2C_2 > 0$, and hence there exists a unique positive root of Eq. (28) $Y_1 > 0$.

From Eq. (14) we get

$$H_1 = \frac{H_0 + \phi(h)\theta e^{-\mu_3\tau_3}S_{-m_3+1}}{1+\phi(h)(c+dY_1)} > 0.$$

Therefore, the solution $(F_1, K_1, S_1, H_1, Y_1, Z_1)$ exists uniquely and is positive.

Repeating the above process for $n = 1$, we can prove that $(F_2, K_2, S_2, H_2, Y_2, Z_2)$ exists uniquely and is positive. Therefore, mathematical induction yields that, for all $n \in \mathbb{N}$, $(F_n, K_n, S_n, H_n, Y_n, Z_n)$ exists uniquely and is positive.

By induction, we obtain $F_n > 0$, $K_n > 0$, $S_n > 0$, $H_n > 0$, $Y_n > 0$, and $Z_n > 0$ $\forall n \geq 0$. To investigate the boundedness of solution, from Eq. (11) we have

$$\frac{F_{n+1} - F_n}{\phi(h)} \leq \Theta(F_{n+1}) \leq b - \overline{b}F_{n+1}.$$

Hence

$$F_{n+1} \leq \frac{F_n}{1 + \phi(h)\overline{b}} + \frac{\phi(h)b}{1 + \phi(h)\overline{b}}.$$

By Lemma 2.2 in [67] we have

$$F_n \leq \left(\frac{1}{1 + \phi(h)\overline{b}}\right)^n F_0 + \frac{b}{\overline{b}}\left[1 - \left(\frac{1}{1 + \phi(h)\overline{b}}\right)^n\right],$$

which implies that $\lim_{n\to\infty} \sup F_n \leq b/\overline{b} \leq \vartheta_1$. Define

$$\Omega_n = (1 - \varepsilon)e^{-\mu_1\tau_1}F_{n-m_1} + \varepsilon e^{-\mu_2\tau_2}F_{n-m_2} + K_n + S_n + \frac{\lambda}{g}Z_n.$$

Then

$$\begin{aligned}
\Omega_{n+1} - \Omega_n &= (1 - \varepsilon)e^{-\mu_1\tau_1}(F_{n-m_1+1} - F_{n-m_1}) + \varepsilon e^{-\mu_2\tau_2}(F_{n-m_2+1} - F_{n-m_2}) \\
&\quad + K_{n+1} - K_n + S_{n+1} - S_n + \frac{\lambda}{g}(Z_{n+1} - Z_n) \\
&= (1 - \varepsilon)e^{-\mu_1\tau_1}\phi(h)\big[\Theta(F_{n-m_1+1}) - \Lambda(F_{n-m_1+1}, H_{n-m_1})\big] \\
&\quad + \varepsilon e^{-\mu_2\tau_2}\phi(h)\big[\Theta(F_{n-m_2+1}) - \Lambda(F_{n-m_2+1}, H_{n-m_2})\big] \\
&\quad + \phi(h)\big[(1 - \varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{n-m_1+1}, H_{n-m_1}) - (\alpha + m)\digamma_1(K_{n+1})\big] \\
&\quad + \phi(h)\big[\varepsilon e^{-\mu_2\tau_2}\Lambda(F_{n-m_2+1}, H_{n-m_2}) + m\digamma_1(K_{n+1}) - a\digamma_2(S_{n+1}) \\
&\quad - \lambda\digamma_2(S_{n+1})\digamma_5(Z_{n+1})\big] \\
&\quad + \phi(h)\frac{\lambda}{g}\big[g\digamma_2(S_{n+1})\digamma_5(Z_{n+1}) - \xi\digamma_5(Z_{n+1})\big] \\
&= \phi(h)\bigg[(1 - \varepsilon)e^{-\mu_1\tau_1}\Theta(F_{n-m_1+1}) + \varepsilon e^{-\mu_2\tau_2}\Theta(F_{n-m_2+1}) - \alpha\digamma_1(K_{n+1}) \\
&\quad - aS_{n+1} - \frac{\lambda\xi}{g}Z_{n+1}\bigg].
\end{aligned}$$

According to Conditions C1 and C3, we have

$$\begin{aligned}
\Omega_{n+1} - \Omega_n \leq \phi(h)\bigg[&(1 - \varepsilon)e^{-\mu_1\tau_1}(b - \overline{b}F_{n-m_1+1}) + \varepsilon e^{-\mu_2\tau_2}(b - \overline{b}F_{n-m_2+1}) - \alpha\upsilon_1K_{n+1} \\
&- aS_{n+1} - \frac{\lambda\xi}{g}Z_{n+1}\bigg].
\end{aligned}$$

We have

$$(1-\varepsilon)e^{-\mu_1\tau_1}b+\varepsilon e^{-\mu_2\tau_2}b \le b(1-\varepsilon)+b\varepsilon=b.$$

Then

$$\begin{aligned}\Omega_{n+1}-\Omega_n &\le \phi(h)b-\phi(h)\sigma_1\left[(1-\varepsilon)e^{-\mu_1\tau_1}F_{n-m_1+1}+\varepsilon e^{-\mu_2\tau_2}F_{n-m_2+1}+K_{n+1}+S_{n+1}+\frac{\lambda}{g}Z_{n+1}\right]\\ &= \phi(h)b-\phi(h)\sigma_1\Omega_{n+1}.\end{aligned}$$

Hence

$$\Omega_{n+1}\le \frac{\Omega_n}{1+\phi(h)\sigma_1}+\frac{\phi(h)b}{1+\phi(h)\sigma_1}.$$

Consequently, we get $\lim_{n\to\infty}\sup\Omega_n\le\vartheta_1$, $\lim_{n\to\infty}\sup K_n\le\vartheta_1$, $\lim_{n\to\infty}\sup S_n\le\vartheta_1$, and $\lim_{n\to\infty}\sup Z_n\le\vartheta_2$. Thus, for any $\varpi>0$, there exists an integer $\varrho_\varpi>0$ such that $S_n\le\vartheta_1+\varpi$ for $n\ge\varrho_\varpi$. We define

$$\Psi_n=H_n+\frac{d}{q}Y_n.$$

Then

$$\begin{aligned}\Psi_{n+1}-\Psi_n &= H_{n+1}-H_n+\frac{d}{q}(Y_{n+1}-Y_n)\\ &= \phi(h)\left(\theta e^{-\mu_3\tau_3}S_{n-m_3+1}-cH_{n+1}-\frac{d\eta}{q}Y_{n+1}\right)\\ &\le \phi(h)\left(\theta e^{-\mu_3\tau_3}(\vartheta_1+\varpi)-cH_{n+1}-\frac{d\eta}{q}Y_{n+1}\right)\\ &\le \phi(h)\big(\theta e^{-\mu_3\tau_3}(\vartheta_1+\varpi)-\sigma_2\Psi_{n+1}\big)\quad \text{for } n\ge\varrho_\varpi+m_3.\end{aligned}$$

Then $\lim_{n\to\infty}\sup\Psi_n\le\frac{\theta e^{-\mu_3\tau_3}(\vartheta_1+\varpi)}{\sigma_2}\le\frac{\theta(\vartheta_1+\varpi)}{\sigma_2}$. The arbitrariness of $\varpi$ yields that $\lim_{n\to\infty}\sup\Psi_n\le\frac{\theta\vartheta_1}{\sigma_2}=\vartheta_3$. Hence, $\lim_{n\to\infty}\sup H_n\le\vartheta_3$ and $\lim_{n\to\infty}\sup Y_n\le\vartheta_4$. Therefore, the solution $(F_n,K_n,S_n,H_n,Y_n,Z_n)$ converges to $\Gamma$ as $n\to\infty$. □

*Proof of Lemma* 2 The equilibria of system (11)–(16) satisfy:

$$\Theta(F)-\Lambda(F,H)=0, \tag{29}$$

$$(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F,H)-(\alpha+m)\digamma_1(K)=0, \tag{30}$$

$$\varepsilon e^{-\mu_2\tau_2}\Lambda(F,H)+m\digamma_1(K)-a\digamma_2(S)-\lambda\digamma_2(S)\digamma_5(Z)=0, \tag{31}$$

$$\theta e^{-\mu_3\tau_3}\digamma_2(S)-c\digamma_3(H)-d\digamma_3(H)\digamma_4(Y)=0, \tag{32}$$

$$q\digamma_3(H)\digamma_4(Y)-\eta\digamma_4(Y)=0, \tag{33}$$

$$g\lambda\digamma_2(S)\digamma_5(Z)-\xi\digamma_5(Z)=0. \tag{34}$$

From Eq. (33), either $\digamma_4(Y)=0$ or $\digamma_4(Y)\neq 0$ ($\digamma_3(H)=\frac{\eta}{q}$). By solving Eq. (34), we get $\digamma_5(Z)=0$ or $\digamma_5(Z)\neq 0$ ($\digamma_2(S)=\frac{\xi}{g}$). If Condition C3 $\digamma_4(Y)=0$ and $\digamma_5(Z)=0$ imply that $Y=0$ and $Z=0$, thus we have the following possibilities:

*1.* $Y=Z=0$, then Condition C3 implies that $\digamma_i^{-1}$, $i=1,\dots,5$, exist and they are strictly increasing functions. From Eqs. (29)–(32) we get

$$K=\digamma_1^{-1}\left(\frac{(1-\varepsilon)e^{-\mu_1\tau_1}\Theta(F)}{\alpha+m}\right)=\pi_1(F), \tag{35}$$

$$S=\digamma_2^{-1}\left(\frac{\gamma e^{\mu_3\tau_3}\Theta(F)}{a}\right)=\pi_2(F), \tag{36}$$

$$H=\digamma_3^{-1}\left(\frac{\theta\gamma\Theta(F)}{ac}\right)=\pi_3(F), \tag{37}$$

where

$$\gamma=\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}. \tag{38}$$

Obviously, $\pi_i(0)>0$ and $\pi_i(F^0)=0$, $i=1,2,3$. Substituting Eq. (37) into Eq. (32), we obtain

$$\frac{\theta\gamma}{a}\Lambda\bigl(F,\pi_3(F)\bigr)-c\digamma_3\bigl(\pi_3(F)\bigr)=0. \tag{39}$$

Equation (39) admits a solution $F=F^0$, which gives $K=S=H=0$ and leads to the pathogen-free equilibrium $Q^0=(F^0,0,0,0,0,0)$. Let

$$\psi_1(F)=\frac{\theta\gamma}{a}\Lambda\bigl(F,\pi_3(F)\bigr)-c\digamma_3\bigl(\pi_3(F)\bigr)=0.$$

Then from Conditions C1–C3 we have

$$\psi_1(0)=-c\digamma_3\bigl(\pi_3(0)\bigr)<0,$$
$$\psi_1\bigl(F^0\bigr)=0.$$

Moreover,

$$\psi_1'(F)=\frac{\theta\gamma}{a}\left[\frac{\partial\Lambda}{\partial F}+\pi_3'(F)\frac{\partial\Lambda}{\partial H}\right]-c\pi_3'(F)\digamma_3'\bigl(\pi_3(F)\bigr),$$
$$\psi_1'\bigl(F^0\bigr)=\frac{\theta\gamma}{a}\left[\frac{\partial\Lambda(F^0,0)}{\partial F}+\pi_3'\bigl(F^0\bigr)\frac{\partial\Lambda(F^0,0)}{\partial H}\right]-c\pi_3'\bigl(F^0\bigr)\digamma_3'(0).$$

Condition C2 implies that $\frac{\partial\Lambda(F^0,0)}{\partial F}=0$. Also, from Condition C3, we have $\digamma_3'(0)>0$, then

$$\begin{aligned}\psi_1'\bigl(F^0\bigr)&=c\pi_3'\bigl(F^0\bigr)\digamma_3'(0)\left(\frac{\theta\gamma}{ac\digamma_3'(0)}\frac{\partial\Lambda(F^0,0)}{\partial H}-1\right)\\&=\frac{\theta\gamma\Theta'(F^0)}{a}\left(\frac{\theta\gamma}{ac\digamma_3'(0)}\frac{\partial\Lambda(F^0,0)}{\partial H}-1\right).\end{aligned}$$

From Condition C1, we have $\Theta'(F^0) < 0$. Therefore, if

$$\frac{\theta\gamma}{ac\digamma_3'(0)}\frac{\partial\Lambda(F^0,0)}{\partial H} > 1,$$

hence $\psi_1'(F^0) < 0$ and there exists $F^* \in (0,F^0)$ such that $\psi_1(F^*) = 0$. From Eqs. (35)–(37) we obtain $K^* = \pi_1(F^*) > 0$, $S^* = \pi_2(F^*) > 0$, and $H^* = \pi_3(F^*) > 0$. Therefore, a persistent infection equilibrium without immune response $Q^*(F^*,K^*,H^*,S^*,0,0)$ exists when $\frac{\theta\gamma}{ac\digamma_3'(0)}\frac{\partial\Lambda(F^0,0)}{\partial H} > 1$. Let us define

$$\mathcal{R}_0 = \frac{\theta\gamma}{ac\digamma_3'(0)}\frac{\partial\Lambda(F^0,0)}{\partial H}.$$

*2.* $Y \neq 0$ and $Z = 0$, we have $\overline{H} = \digamma_3^{-1}(\frac{\eta}{q}) > 0$. Let $H = \overline{H}$ in Eq. (29) and define $\psi_2$ as follows:

$$\psi_2(F) = \Theta(F) - \Lambda(F,\overline{H}) = 0.$$

According to Conditions C1 and C2, we have

$$\psi_2(0) = \Theta(0) > 0 \quad \text{and} \quad \psi_2\bigl(F^0\bigr) = -\Lambda\bigl(F^0,\overline{H}\bigr) < 0.$$

Since $\psi_2(F)$ is a strictly decreasing function of $F$, then there exists unique $\overline{F} \in (0,F^0)$ such that $\psi_2(\overline{F}) = 0$. Now from Eqs. (32), (35), and (36) we obtain

$$\overline{K} = \pi_1(\overline{F}), \qquad \overline{S} = \pi_2(\overline{F}), \qquad \overline{Y} = \digamma_4^{-1}\left(\frac{c}{d}\left(\frac{\theta\gamma\Lambda(\overline{F},\overline{H})}{ac\digamma_3(\overline{H})} - 1\right)\right).$$

Clearly, $\overline{K} > 0$ and $\overline{S} > 0$; moreover, $\overline{Y} > 0$ when $\frac{\theta\gamma\Lambda(\overline{F},\overline{H})}{ac\digamma_3(\overline{H})} > 1$. Now we define

$$\mathcal{R}_1^Y = \frac{\theta\gamma\Lambda(\overline{F},\overline{H})}{ac\digamma_3(\overline{H})}.$$

Hence, $\overline{Y}$ can be rewritten as $\overline{Y} = \digamma_4^{-1}(\frac{c}{d}(\mathcal{R}_1^Y - 1))$. It follows that there exists a persistent infection equilibrium with only humoral immune response $\overline{Q}(\overline{F},\overline{K},\overline{S},\overline{H},\overline{Y},0)$ if $\mathcal{R}_1^Y > 1$.

*3.* $Z \neq 0$ and $Y = 0$, we have $\widehat{S} = \digamma_2^{-1}(\frac{\xi}{g}) > 0$. Let $S = \widehat{S}$ in Eq. (32), then we have

$$\widehat{H} = \digamma_3^{-1}\left(\frac{\theta e^{-\mu_3\tau_3}\xi}{cg}\right) > 0.$$

Let $H = \widehat{H}$ in Eq. (29) and define $\psi_3$ as follows:

$$\psi_3(F) = \Theta(F) - \Lambda(F,\widehat{H}) = 0.$$

Clearly,

$$\psi_3(0) = \Theta(0) > 0 \quad \text{and} \quad \psi_3\bigl(F^0\bigr) = -\Lambda\bigl(F^0,\widehat{H}\bigr) < 0.$$

According to Conditions C1 and C2, there exists unique $\widehat{F} \in (0, F^0)$ such that $\psi_3(\widehat{F}) = 0$. From Eq. (35) we conclude that $\widehat{K} = \pi_1(\widehat{F}) > 0$. Now from Eqs. (30)–(32) we have $\widehat{Z} = \digamma_5^{-1}(\frac{a}{\lambda}(\mathcal{R}_1^Z - 1))$, where

$$\mathcal{R}_1^Z = \frac{\theta\gamma\Lambda(\widehat{F},\widehat{H})}{ac\digamma_3(\widehat{H})}.$$

Consequently, there exists a persistent infection equilibrium with only CTL immune response $\widehat{Q}(\widehat{F},\widehat{K},\widehat{S},\widehat{H},0,\widehat{Z})$ if $\mathcal{R}_1^Z > 1$.

*4.* $Z \neq 0$ and $Y \neq 0$, we have $\widetilde{H} = \overline{H} = \digamma_3^{-1}(\frac{\eta}{q}) > 0$ and $\widetilde{S} = \widehat{S} = \digamma_2^{-1}(\frac{\xi}{g}) > 0$. Let $H = \widetilde{H}$ in Eq. (29) and define $\psi_4$ as follows:

$$\psi_4(F) = \Theta(F) - \Lambda(F,\widetilde{H}) = 0.$$

Clearly,

$$\psi_4(0) = \Theta(0) > 0 \quad \text{and} \quad \psi_4(F^0) = -\Lambda(F^0,\widetilde{H}) < 0.$$

According to C1 and C2, there exists unique $\widetilde{F} \in (0, F^0)$ such that $\psi_4(\widetilde{F}) = 0$. Thus, we conclude from Eq. (35) that $\widetilde{K} = \pi_1(\widetilde{F}) > 0$. Now from Eqs. (30)–(32) we have

$$\widetilde{Z} = \digamma_5^{-1}\left(\frac{a}{\lambda}(\mathcal{R}_2^Z - 1)\right) \quad \text{and} \quad \widetilde{Y} = \digamma_4^{-1}\left(\frac{c}{d}\big((\mathcal{R}_1^Y/\mathcal{R}_2^Z) - 1\big)\right) > 0,$$

where

$$\mathcal{R}_2^Z = \frac{e^{\mu_3\tau_3}\gamma\Lambda(\widetilde{F},\widetilde{H})}{a\digamma_2(\widetilde{S})} \quad \text{and} \quad \mathcal{R}_1^Y/\mathcal{R}_2^Z = \frac{\theta\digamma_2(\widetilde{S})\Lambda(\overline{F},\overline{H})}{c\digamma_3(\widetilde{H})\Lambda(\widetilde{F},\widetilde{H})} = \frac{\theta e^{-\mu_3\tau_3}\xi q}{cg\eta}.$$

It follows that there exists a persistent infection equilibrium with both humoral and CTL immune responses $\widetilde{Q}(\widetilde{F},\widetilde{K},\widetilde{S},\widetilde{H},\widetilde{Y},\widetilde{Z})$ if $\mathcal{R}_1^Y > \mathcal{R}_2^Z > 1$. □

*Proof of Theorem 1* Define

$$\begin{aligned}
\mathcal{L}_n &= \frac{1}{\phi(h)}\Bigg[\gamma\left(F_n - F^0 - \int_{F^0}^{F_n}\lim_{H\to 0^+}\frac{\Lambda(F^0,H)}{\Lambda(\varsigma,H)}\,d\varsigma\right) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}K_n + e^{-\mu_3\tau_3}S_n + \frac{a}{\theta}H_n \\
&\quad + \frac{ad}{\theta q}Y_n + \frac{\lambda e^{-\mu_3\tau_3}}{g}Z_n\Bigg] + \frac{ac}{\theta}\digamma_3(H_n) + \frac{ad\eta}{\theta q}\digamma_4(Y_n) + \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n) \\
&\quad + \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\sum_{j=n-m_1}^{n-1}\Lambda(F_{j+1},H_j) + \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\sum_{j=n-m_2}^{n-1}\Lambda(F_{j+1},H_j) \\
&\quad + ae^{-\mu_3\tau_3}\sum_{j=n-m_3}^{n-1}\digamma_2(S_{j+1}),
\end{aligned}$$

where $\gamma$ is defined by Eq. (38). Hence, $\mathcal{L}_n > 0$ for all $F_n, K_n, S_n, H_n, Y_n, Z_n > 0$, and $\mathcal{L}_n = 0$ if and only if $F_n = F^0$, $K_n = 0$, $S_n = 0$, $H_n = 0$, $Y_n = 0$, and $Z_n = 0$. We compute the difference

$\Delta\mathcal{L}_n = \mathcal{L}_{n+1} - \mathcal{L}_n$ as follows:

$$\begin{aligned}
\Delta\mathcal{L}_n ={}& \frac{1}{\phi(h)}\Bigg[\gamma\Bigg(F_{n+1} - F^0 - \int_{F^0}^{F_{n+1}} \lim_{H\to 0^+} \frac{\Lambda(F^0,H)}{\Lambda(\varsigma,H)}\,d\varsigma\Bigg) \\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}K_{n+1} + e^{-\mu_3\tau_3}S_{n+1} + \frac{a}{\theta}H_{n+1} \\
&+ \frac{ad}{\theta q}Y_{n+1} + \frac{\lambda e^{-\mu_3\tau_3}}{g}Z_{n+1}\Bigg] + \frac{ac}{\theta}\digamma_3(H_{n+1}) + \frac{ad\eta}{\theta q}\digamma_4(Y_{n+1}) \\
&+ \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_{n+1}) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\sum_{j=n-m_1+1}^{n}\Lambda(F_{j+1},H_j) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\sum_{j=n-m_2+1}^{n}\Lambda(F_{j+1},H_j) + ae^{-\mu_3\tau_3}\sum_{j=n-m_3+1}^{n}\digamma_2(S_{j+1}) \\
&- \frac{1}{\phi(h)}\Bigg[\gamma\Bigg(F_n - F^0 - \int_{F^0}^{F_n} \lim_{H\to 0^+} \frac{\Lambda(F^0,H)}{\Lambda(\varsigma,H)}\,d\varsigma\Bigg) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}K_n \\
&+ e^{-\mu_3\tau_3}S_n + \frac{a}{\theta}H_n + \frac{ad}{\theta q}Y_n \\
&+ \frac{\lambda e^{-\mu_3\tau_3}}{g}Z_n\Bigg] - \frac{ac}{\theta}\digamma_3(H_n) - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\sum_{j=n-m_1}^{n-1}\Lambda(F_{j+1},H_j) \\
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\sum_{j=n-m_2}^{n-1}\Lambda(F_{j+1},H_j) - ae^{-\mu_3\tau_3}\sum_{j=n-m_3}^{n-1}\digamma_2(S_{j+1}), \\
\Delta\mathcal{L}_n ={}& \frac{1}{\phi(h)}\Bigg[\gamma\Bigg(F_{n+1} - F_n - \int_{F_n}^{F_{n+1}} \lim_{H\to 0^+} \frac{\Lambda(F^0,H)}{\Lambda(\varsigma,H)}\,d\varsigma\Bigg) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}(K_{n+1}-K_n) \\
&+ e^{-\mu_3\tau_3}(S_{n+1}-S_n) + \frac{a}{\theta}(H_{n+1}-H_n) + \frac{ad}{\theta q}(Y_{n+1}-Y_n) \\
&+ \frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1}-Z_n)\Bigg] \\
&+ \frac{ac}{\theta}\big[\digamma_3(H_{n+1}) - \digamma_3(H_n)\big] + \frac{ad\eta}{\theta q}\big[\digamma_4(Y_{n+1}) - \digamma_4(Y_n)\big] \\
&+ \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\big[\digamma_5(Z_{n+1}) - \digamma_5(Z_n)\big] \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Bigg(\sum_{j=n-m_1+1}^{n}\Lambda(F_{j+1},H_j) - \sum_{j=n-m_1}^{n-1}\Lambda(F_{j+1},H_j)\Bigg) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Bigg(\sum_{j=n-m_2+1}^{n}\Lambda(F_{j+1},H_j) - \sum_{j=n-m_2}^{n-1}\Lambda(F_{j+1},H_j)\Bigg) \\
&+ ae^{-\mu_3\tau_3}\Bigg(\sum_{j=n-m_3+1}^{n}\digamma_2(S_{j+1}) - \sum_{j=n-m_3}^{n-1}\digamma_2(S_{j+1})\Bigg).
\end{aligned}$$

Using Lemma 3.1 [68], we get

$$\begin{aligned} \lim_{H\to 0^+}\frac{\varLambda(F^0,H)}{\varLambda(F_{n+1},H)}(F_{n+1}-F_n) &\le \int_{F_n}^{F_{n+1}}\lim_{H\to 0^+}\frac{\varLambda(F^0,H)}{\varLambda(\varsigma,H)}\,d\varsigma \\ &\le \lim_{H\to 0^+}\frac{\varLambda(F^0,H)}{\varLambda(F_n,H)}(F_{n+1}-F_n). \end{aligned}$$

Hence

$$\begin{aligned} \Delta\mathcal{L}_n \le{}& \frac{1}{\phi(h)}\Bigg[\gamma\left(1-\lim_{H\to 0^+}\frac{\varLambda(F^0,H)}{\varLambda(F_{n+1},H)}\right)(F_{n+1}-F_n) \\ &+\frac{me^{-\mu_3\tau_3}}{\alpha+m}(K_{n+1}-K_n)+e^{-\mu_3\tau_3}(S_{n+1}-S_n) \\ &+\frac{a}{\theta}(H_{n+1}-H_n)+\frac{ad}{\theta q}(Y_{n+1}-Y_n)+\frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1}-Z_n)\Bigg] \\ &+\frac{ac}{\theta}\big[\digamma_3(H_{n+1})-\digamma_3(H_n)\big] \\ &+\frac{ad\eta}{\theta q}\big[\digamma_4(Y_{n+1})-\digamma_4(Y_n)\big]+\frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\big[\digamma_5(Z_{n+1})-\digamma_5(Z_n)\big] \\ &+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\big[\varLambda(F_{n+1},H_n)-\varLambda(F_{n-m_1+1},H_{n-m_1})\big] \\ &+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\big[\varLambda(F_{n+1},H_n)-\varLambda(F_{n-m_2+1},H_{n-m_2})\big] \\ &+ae^{-\mu_3\tau_3}\big[\digamma_2(S_{n+1})-\digamma_2(S_{n-m_3+1})\big]. \end{aligned}$$

From Eqs. (11)–(16) we have

$$\begin{aligned} \Delta\mathcal{L}_n \le{}& \gamma\left(1-\lim_{H\to 0^+}\frac{\varLambda(F^0,H)}{\varLambda(F_{n+1},H)}\right)\big(\varTheta(F_{n+1})-\varLambda(F_{n+1},H_n)\big) \\ &+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\big[(1-\varepsilon)e^{-\mu_1\tau_1}\varLambda(F_{n-m_1+1},H_{n-m_1})-(\alpha+m)\digamma_1(K_{n+1})\big] \\ &+e^{-\mu_3\tau_3}\big[\varepsilon e^{-\mu_2\tau_2}\varLambda(F_{n-m_2+1},H_{n-m_2})+m\digamma_1(K_{n+1})-a\digamma_2(S_{n+1}) \\ &-\lambda\digamma_2(S_{n+1})\digamma_5(Z_{n+1})\big] \\ &+\frac{a}{\theta}\big[\theta e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})-c\digamma_3(H_{n+1})-d\digamma_3(H_{n+1})\digamma_4(Y_{n+1})\big] \\ &+\frac{ad}{\theta q}\big[q\digamma_3(H_{n+1})\digamma_4(Y_{n+1})-\eta\digamma_4(Y_{n+1})\big] \\ &+\frac{\lambda e^{-\mu_3\tau_3}}{g}\big[g\digamma_2(S_{n+1})\digamma_5(Z_{n+1})-\xi\digamma_5(Z_{n+1})\big] \\ &+\frac{ac}{\theta}\big[\digamma_3(H_{n+1})-\digamma_3(H_n)\big]+\frac{ad\eta}{\theta q}\big[\digamma_4(Y_{n+1})-\digamma_4(Y_n)\big] \\ &+\frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\big[\digamma_5(Z_{n+1})-\digamma_5(Z_n)\big] \\ &+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\big[\varLambda(F_{n+1},H_n)-\varLambda(F_{n-m_1+1},H_{n-m_1})\big] \\ &+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\big[\varLambda(F_{n+1},H_n)-\varLambda(F_{n-m_2+1},H_{n-m_2})\big] \end{aligned}$$

$$
\begin{aligned}
&+ ae^{-\mu_3\tau_3}\big[\digamma_2(S_{n+1}) - \digamma_2(S_{n-m_3+1})\big] \\
&= \gamma\left(1 - \lim_{H\to 0^+}\frac{\Lambda(F^0,H)}{\Lambda(F_{n+1},H)}\right)\Theta(F_{n+1}) + \gamma\lim_{H\to 0^+}\frac{\Lambda(F^0,H)}{\Lambda(F_{n+1},H)}\Lambda(F_{n+1},H_n) \\
&- \frac{ac}{\theta}\digamma_3(H_n) - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n).
\end{aligned}
$$

Using $\Theta(F^0) = 0$, we obtain

$$
\begin{aligned}
\Delta\mathcal{L}_n &\le \gamma\left(1 - \frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^0\big)\big) + \gamma\frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\Lambda(F_{n+1},H_n) \\
&\quad - \frac{ac}{\theta}\digamma_3(H_n) - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n) \\
&= \gamma\left(1 - \frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^0\big)\big) \\
&\quad + \frac{ac}{\theta}\left(\frac{\gamma\theta}{ac}\frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\frac{\Lambda(F_{n+1},H_n)}{\digamma_3(H_n)} - 1\right)\digamma_3(H_n) \\
&\quad - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n).
\end{aligned}
$$

From Condition C4 we have

$$
\frac{\Lambda(F_{n+1},H_n)}{\digamma_3(H_n)} \le \lim_{H\to 0^+}\frac{\Lambda(F_{n+1},H)}{\digamma_3(H)} = \frac{\partial\Lambda(F_{n+1},0)/\partial H}{\digamma_3'(0)}.
$$

Then we get

$$
\begin{aligned}
\Delta\mathcal{L}_n &\le \gamma\left(1 - \frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^0\big)\big) \\
&\quad + \frac{ac}{\theta}\left(\frac{\gamma\theta}{ac}\frac{\partial\Lambda(F^0,0)/\partial H}{\digamma_3'(0)} - 1\right)\digamma_3(H_n) \\
&\quad - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n) \\
&= \gamma\left(1 - \frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^0\big)\big) + \frac{ac}{\theta}(\mathcal{R}_0 - 1)\digamma_3(H_n) \\
&\quad - \frac{ad\eta}{\theta q}\digamma_4(Y_n) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_n).
\end{aligned}
$$

Conditions C1 and C2 imply that

$$
\left(1 - \frac{\partial\Lambda(F^0,0)/\partial H}{\partial\Lambda(F_{n+1},0)/\partial H}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^0\big)\big) \le 0.
$$

Hence, if $\mathcal{R}_0 \le 1$, then we have $\Delta\mathcal{L}_n \le 0$ for all $n \ge 0$. Obviously, $\lim_{n\to\infty}\Delta\mathcal{L}_n = 0$ if $\lim_{n\to\infty} F_n = F^0$, $\lim_{n\to\infty}(\mathcal{R}_0 - 1)\digamma_3(H_n) = 0$, $\lim_{n\to\infty}\digamma_4(Y_n) = 0$, and $\lim_{n\to\infty}\digamma_5(Z_n) = 0$. We have two cases:

- If $\mathcal{R}_0 < 1$, then $\lim_{n\to\infty}\digamma_3(H_n) = 0$, $\lim_{n\to\infty}\digamma_4(Y_n) = 0$, $\lim_{n\to\infty}\digamma_5(Z_n) = 0$, and from Condition C3 we get $\lim_{n\to\infty} H_n = 0$, $\lim_{n\to\infty} Y_n = 0$, and $\lim_{n\to\infty} Z_n = 0$, then we get from Eqs. (13)–(14) $\lim_{n\to\infty} K_n = 0$ and $\lim_{n\to\infty} S_n = 0$.

- If $\mathcal{R}_0 = 1$, then $\lim_{n\to\infty} \Delta K_n = 0$ when $\lim_{n\to\infty} Y_n = 0$, $\lim_{n\to\infty} Z_n = 0$, and $\lim_{n\to\infty} F_n = F^0$, and from Eq. (11) we obtain $\lim_{n\to\infty} \Lambda(F^0, H_n) = 0$, then $\lim_{n\to\infty} H_n = 0$. Moreover, from Eqs. (13)–(14) we get $\lim_{n\to\infty} K_n = 0$, $\lim_{n\to\infty} S_n = 0$. Hence $Q^0$ is G.A.S. □

*Proof of Lemma* 3 From Conditions C1 and C2, for $F^*, \overline{F}, \widehat{F}, S^*, \widehat{S}, H^*, \overline{H}, \widehat{H} > 0$, we have

$$(F^* - \overline{F})(\Theta(\overline{F}) - \Theta(F^*)) > 0, \tag{40}$$

$$(\overline{F} - F^*)(\Lambda(\overline{F},\overline{H}) - \Lambda(F^*,\overline{H})) > 0, \tag{41}$$

$$(\overline{H} - H^*)(\Lambda(F^*,\overline{H}) - \Lambda(F^*,H^*)) > 0, \tag{42}$$

$$(\overline{H} - H^*)(\Lambda(\overline{F},\overline{H}) - \Lambda(\overline{F},H^*)) > 0. \tag{43}$$

Using Condition C4, we get

$$(H^* - \overline{H})\left(\frac{\Lambda(F^*,\overline{H})}{\digamma_3(\overline{H})} - \frac{\Lambda(F^*,H^*)}{\digamma_3(H^*)}\right) > 0. \tag{44}$$

Suppose that $\operatorname{sgn}(\overline{F} - F^*) = \operatorname{sgn}(\overline{H} - H^*)$. For the equilibria $Q^*$ and $\overline{Q}$, we have

$$\begin{aligned}\Theta(\overline{F}) - \Theta(F^*) &= \Lambda(\overline{F},\overline{H}) - \Lambda(F^*,H^*)\\ &= (\Lambda(\overline{F},\overline{H}) - \Lambda(F^*,\overline{H})) + (\Lambda(F^*,\overline{H}) - \Lambda(F^*,H^*)).\end{aligned}$$

Therefore, from inequalities (40)–(43) we get

$$\operatorname{sgn}(F^* - \overline{F}) = \operatorname{sgn}(\overline{F} - F^*),$$

which leads to a contradiction. Thus, $\operatorname{sgn}(\overline{F} - F^*) = \operatorname{sgn}(H^* - \overline{H})$. Using the equilibrium conditions for $Q^*$, we have $\frac{\theta\gamma\Lambda(F^*,H^*)}{ac\digamma_3(H^*)} = 1$, then

$$\begin{aligned}\mathcal{R}_1^Y - 1 &= \frac{\theta\gamma\Lambda(\overline{F},\overline{H})}{ac\digamma_3(\overline{H})} - \frac{\theta\gamma\Lambda(F^*,H^*)}{ac\digamma_3(H^*)}\\ &= \frac{\theta\gamma}{ac}\left[\frac{\Lambda(\overline{F},\overline{H})}{\digamma_3(\overline{H})} - \frac{\Lambda(F^*,H^*)}{\digamma_3(H^*)}\right]\\ &= \frac{\theta\gamma}{ac}\left[\frac{1}{\digamma_3(\overline{H})}(\Lambda(\overline{F},\overline{H}) - \Lambda(F^*,\overline{H})) + \frac{\Lambda(F^*,\overline{H})}{\digamma_3(\overline{H})} - \frac{\Lambda(F^*,H^*)}{\digamma_3(H^*)}\right].\end{aligned}$$

Thus, from inequalities (41)–(44) we get $\operatorname{sgn}(\mathcal{R}_1^Y - 1) = \operatorname{sgn}(H^* - \overline{H})$. Similarly, one can show that $\operatorname{sgn}(\widehat{F} - F^*) = \operatorname{sgn}(H^* - \widehat{H}) = \operatorname{sgn}(\mathcal{R}_1^Z - 1)$. Moreover, we have

$$\digamma_2(S^*) - \digamma_2(\widehat{S}) = \frac{ce^{\mu_3\tau_3}}{\theta}(\digamma_3(H^*) - \digamma_3(\widehat{H})),$$

which gives us $\operatorname{sgn}(H^* - \widehat{H}) = \operatorname{sgn}(S^* - \widehat{S})$. □

*Proof of Theorem* 2 Consider a function $\mathcal{U}_n(F_n, K_n, S_n, H_n, Y_n, Z_n)$ as follows:

$$\begin{aligned}
\mathcal{U}_n ={}& \frac{1}{\phi(h)}\left[\gamma\left(F_n - F^* - \int_{F^*}^{F_n}\frac{\Lambda(F^*,H^*)}{\Lambda(\varsigma,H^*)}\,d\varsigma\right) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - K^* - \int_{K^*}^{K_n}\frac{\digamma_1(K^*)}{\digamma_1(\varsigma)}\,d\varsigma\right)\right.\\
&+ e^{-\mu_3\tau_3}\left(S_n - S^* - \int_{S^*}^{S_n}\frac{\digamma_2(S^*)}{\digamma_2(\varsigma)}\,d\varsigma\right)\\
&\left.+ \frac{a}{\theta}\left(H_n - H^* - \int_{H^*}^{H_n}\frac{\digamma_3(H^*)}{\digamma_3(\varsigma)}\,d\varsigma\right) + \frac{ad}{\theta q}Y_n + \frac{\lambda e^{-\mu_3\tau_3}}{g}Z_n\right]\\
&+ \frac{ac}{\theta}\digamma_3(H^*)G\left(\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\right)\\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda\left(F^*,H^*\right)\sum_{j=n-m_1}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right)\\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda\left(F^*,H^*\right)\sum_{j=n-m_2}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right)\\
&+ ae^{-\mu_3\tau_3}\digamma_2\left(S^*\right)\sum_{j=n-m_3}^{n-1}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(S^*)}\right).
\end{aligned}$$

We have $\mathcal{U}_n(F_n, K_n, S_n, H_n, Y_n, Z_n) > 0$ for all $F_n, K_n, S_n, H_n, Y_n, Z_n > 0$; moreover, $\mathcal{U}_n(F^*, K^*, S^*, H^*, 0, 0) = 0$. We compute $\Delta\mathcal{U}_n = \mathcal{U}_{n+1} - \mathcal{U}_n$ as follows:

$$\begin{aligned}
\Delta\mathcal{U}_n ={}& \frac{1}{\phi(h)}\left[\gamma\left(F_{n+1} - F^* - \int_{F^*}^{F_{n+1}}\frac{\Lambda(F^*,H^*)}{\Lambda(\varsigma,H^*)}\,d\varsigma\right)\right.\\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1} - K^* - \int_{K^*}^{K_{n+1}}\frac{\digamma_1(K^*)}{\digamma_1(\varsigma)}\,d\varsigma\right)\\
&+ e^{-\mu_3\tau_3}\left(S_{n+1} - S^* - \int_{S^*}^{S_{n+1}}\frac{\digamma_2(S^*)}{\digamma_2(\varsigma)}\,d\varsigma\right) + \frac{a}{\theta}\left(H_{n+1} - H^* - \int_{H^*}^{H_{n+1}}\frac{\digamma_3(H^*)}{\digamma_3(\varsigma)}\,d\varsigma\right)\\
&\left.+ \frac{ad}{\theta q}Y_{n+1} + \frac{\lambda e^{-\mu_3\tau_3}}{g}Z_{n+1}\right] + \frac{ac}{\theta}\digamma_3(H^*)G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(H^*)}\right)\\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda\left(F^*,H^*\right)\sum_{j=n-m_1+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right)\\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda\left(F^*,H^*\right)\sum_{j=n-m_2+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right)\\
&+ ae^{-\mu_3\tau_3}\digamma_2\left(S^*\right)\sum_{j=n-m_3+1}^{n}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(S^*)}\right)\\
&- \frac{1}{\phi(h)}\left[\gamma\left(F_n - F^* - \int_{F^*}^{F_n}\frac{\Lambda(F^*,H^*)}{\Lambda(\varsigma,H^*)}\,d\varsigma\right)\right.\\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - K^* - \int_{K^*}^{K_n}\frac{\digamma_1(K^*)}{\digamma_1(\varsigma)}\,d\varsigma\right)\\
&+ e^{-\mu_3\tau_3}\left(S_n - S^* - \int_{S^*}^{S_n}\frac{\digamma_2(S^*)}{\digamma_2(\varsigma)}\,d\varsigma\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{a}{\theta}\left(H_n - H^* - \int_{H^*}^{H_n} \frac{\digamma_3(H^*)}{\digamma_3(\varsigma)}\, d\varsigma\right) + \frac{ad}{\theta q} Y_n + \frac{\lambda e^{-\mu_3\tau_3}}{g} Z_n\Bigg] \\
&- \frac{ac}{\theta}\digamma_3(H^*) G\left(\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\right) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(F^*,H^*) \sum_{j=n-m_1}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right) \\
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*) \sum_{j=n-m_2}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(F^*,H^*)}\right) \\
&- a e^{-\mu_3\tau_3}\digamma_2(S^*) \sum_{j=n-m_3}^{n-1} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(S^*)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\triangle\mathcal{U}_n = {}& \frac{1}{\phi(h)}\Bigg[\gamma\left(F_{n+1} - F_n - \int_{F_n}^{F_{n+1}} \frac{\Lambda(F^*,H^*)}{\Lambda(\varsigma,H^*)}\, d\varsigma\right) \\
&+ \frac{m e^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1} - K_n - \int_{K_n}^{K_{n+1}} \frac{\digamma_1(K^*)}{\digamma_1(\varsigma)}\, d\varsigma\right) \\
&+ e^{-\mu_3\tau_3}\left(S_{n+1} - S_n - \int_{S_n}^{S_{n+1}} \frac{\digamma_2(S^*)}{\digamma_2(\varsigma)}\, d\varsigma\right) + \frac{a}{\theta}\left(H_{n+1} - H_n - \int_{H_n}^{H_{n+1}} \frac{\digamma_3(H^*)}{\digamma_3(\varsigma)}\, d\varsigma\right) \\
&+ \frac{ad}{\theta q}(Y_{n+1} - Y_n) + \frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1} - Z_n)\Bigg] \\
&+ \frac{ac}{\theta}\digamma_3(H^*)\left[G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(H^*)}\right) - G\left(\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\right)\right] \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(F^*,H^*)\Bigg[G\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F^*,H^*)}\right) \\
&- G\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F^*,H^*)}\right)\Bigg] \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*)\left[G\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F^*,H^*)}\right) - G\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F^*,H^*)}\right)\right] \\
&+ a e^{-\mu_3\tau_3}\digamma_2(S^*)\left[G\left(\frac{\digamma_2(S_{n+1})}{\digamma_2(S^*)}\right) - G\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S^*)}\right)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
\left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_n,H^*)}\right)(F_{n+1} - F_n) &\le F_{n+1} - F_n - \int_{F_n}^{F_{n+1}} \frac{\Lambda(F^*,H^*)}{\Lambda(\varsigma,H^*)}\, d\varsigma \\
&\le \left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)(F_{n+1} - F_n), \qquad (45) \\
\left(1 - \frac{\digamma_i(\rho^*)}{\digamma_i(\rho_n)}\right)(\rho_{n+1} - \rho_n) &\le \rho_{n+1} - \rho_n - \int_{\rho_n}^{\rho_{n+1}} \frac{\digamma_i(\rho^*)}{\digamma_i(\varsigma)}\, d\varsigma \\
&\le \left(1 - \frac{\digamma_i(\rho^*)}{\digamma_i(\rho_{n+1})}\right)(\rho_{n+1} - \rho_n), \qquad (46)
\end{aligned}
$$

$i = 1, \dots, 5$, $\rho^* \in \{K^*, S^*, H^*\}$.

Then

$$\begin{aligned}
\Delta\mathcal{U}_n \le{}& \frac{1}{\phi(h)}\biggl[\gamma\biggl(1-\frac{\varLambda(F^*,H^*)}{\varLambda(F_{n+1},H^*)}\biggr)(F_{n+1}-F_n)+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\biggl(1-\frac{\digamma_1(K^*)}{\digamma_1(K_{n+1})}\biggr)(K_{n+1}-K_n)\\
&+e^{-\mu_3\tau_3}\biggl(1-\frac{\digamma_2(S^*)}{\digamma_2(S_{n+1})}\biggr)(S_{n+1}-S_n)+\frac{a}{\theta}\biggl(1-\frac{\digamma_3(H^*)}{\digamma_3(H_{n+1})}\biggr)(H_{n+1}-H_n)\\
&+\frac{ad}{\theta q}(Y_{n+1}-Y_n)\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1}-Z_n)\biggr]+\frac{ac}{\theta}\digamma_3\bigl(H^*\bigr)\biggl[\frac{\digamma_3(H_{n+1})}{\digamma_3(H^*)}-\frac{\digamma_3(H_n)}{\digamma_3(H^*)}+\ln\biggl(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\biggr)\biggr]\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda\bigl(F^*,H^*\bigr)\biggl[\frac{\varLambda(F_{n+1},H_n)}{\varLambda(F^*,H^*)}-\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F^*,H^*)}\\
&+\ln\biggl(\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F_{n+1},H_n)}\biggr)\biggr]\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\varLambda\bigl(F^*,H^*\bigr)\biggl[\frac{\varLambda(F_{n+1},H_n)}{\varLambda(F^*,H^*)}-\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(F^*,H^*)}\\
&+\ln\biggl(\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(F_{n+1},H_n)}\biggr)\biggr]\\
&+ae^{-\mu_3\tau_3}\digamma_2\bigl(S^*\bigr)\biggl[\frac{\digamma_2(S_{n+1})}{\digamma_2(S^*)}-\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S^*)}+\ln\biggl(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\biggr)\biggr].
\end{aligned}$$

From Eqs. (11)–(16) we have

$$\begin{aligned}
\Delta\mathcal{U}_n \le{}& \gamma\biggl(1-\frac{\varLambda(F^*,H^*)}{\varLambda(F_{n+1},H^*)}\biggr)\bigl(\varTheta(F_{n+1})-\varLambda(F_{n+1},H_n)\bigr)\\
&+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\biggl(1-\frac{\digamma_1(K^*)}{\digamma_1(K_{n+1})}\biggr)\bigl[(1-\varepsilon)e^{-\mu_1\tau_1}\varLambda(F_{n-m_1+1},H_{n-m_1})\\
&-(\alpha+m)\digamma_1(K_{n+1})\bigr]\\
&+e^{-\mu_3\tau_3}\biggl(1-\frac{\digamma_2(S^*)}{\digamma_2(S_{n+1})}\biggr)\bigl[\varepsilon e^{-\mu_2\tau_2}\varLambda(F_{n-m_2+1},H_{n-m_2})+m\digamma_1(K_{n+1})-a\digamma_2(S_{n+1})\\
&-\lambda\digamma_2(S_{n+1})\digamma_5(Z_{n+1})\bigr]\\
&+\frac{a}{\theta}\biggl(1-\frac{\digamma_3(H^*)}{\digamma_3(H_{n+1})}\biggr)\bigl[\theta e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})-c\digamma_3(H_{n+1})-d\digamma_3(H_{n+1})\digamma_4(Y_{n+1})\bigr]\\
&+\frac{ad}{\theta q}\bigl[q\digamma_3(H_{n+1})\digamma_4(Y_{n+1})-\eta\digamma_4(Y_{n+1})\bigr]\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\bigl[g\digamma_2(S_{n+1})\digamma_5(Z_{n+1})-\xi\digamma_5(Z_{n+1})\bigr]\\
&+\frac{ac}{\theta}\digamma_3\bigl(H^*\bigr)\biggl[\frac{\digamma_3(H_{n+1})}{\digamma_3(H^*)}-\frac{\digamma_3(H_n)}{\digamma_3(H^*)}+\ln\biggl(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\biggr)\biggr]\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda\bigl(F^*,H^*\bigr)\biggl[\frac{\varLambda(F_{n+1},H_n)}{\varLambda(F^*,H^*)}-\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F^*,H^*)}\\
&+\ln\biggl(\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F_{n+1},H_n)}\biggr)\biggr]
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3} \Lambda\big(F^*,H^*\big)\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F^*,H^*)} - \frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F^*,H^*)}\right.\\
&\left.+ \ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&+ ae^{-\mu_3\tau_3}\digamma_2\big(S^*\big)\left[\frac{\digamma_2(S_{n+1})}{\digamma_2(S^*)} - \frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S^*)} + \ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right)\right].
\end{aligned} \tag{47}
$$

Collecting terms of Eq. (47), we get

$$
\begin{aligned}
\Delta\mathcal{U}_n \le{}& \gamma\left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^*\big)\big) + \gamma\Theta\big(F^*\big)\left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)\\
&+ \gamma\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\Lambda(F_{n+1},H_n) - \frac{m(1-\varepsilon)}{\alpha+m}e^{-\mu_1\tau_1-\mu_3\tau_3}\Lambda(F_{n-m_1+1},H_{n-m_1})\frac{\digamma_1(K^*)}{\digamma_1(K_{n+1})}\\
&+ me^{-\mu_3\tau_3}\digamma_1\big(K^*\big) - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F_{n-m_2+1},H_{n-m_2})\frac{\digamma_2(S^*)}{\digamma_2(S_{n+1})}\\
&- me^{-\mu_3\tau_3}\digamma_1(K_{n+1})\frac{\digamma_2(S^*)}{\digamma_2(S_{n+1})}\\
&+ ae^{-\mu_3\tau_3}\digamma_2\big(S^*\big) + \lambda e^{-\mu_3\tau_3}\digamma_2\big(S^*\big)\digamma_5(Z_{n+1}) - ae^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})\frac{\digamma_3(H^*)}{\digamma_3(H_{n+1})}\\
&+ \frac{ac}{\theta}\digamma_3\big(H^*\big)\\
&+ \frac{ad}{\theta}\digamma_3\big(H^*\big)\digamma_4(Y_{n+1}) - \frac{ad\eta}{\theta q}\digamma_4(Y_{n+1}) - \frac{\lambda\xi e^{-\mu_3\tau_3}}{g}\digamma_5(Z_{n+1}) - \frac{ac}{\theta}\digamma_3(H_n)\\
&+ \frac{ac}{\theta}\digamma_3\big(H^*\big)\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda\big(F^*,H^*\big)\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda\big(F^*,H^*\big)\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\\
&+ ae^{-\mu_3\tau_3}\digamma_2\big(S^*\big)\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right).
\end{aligned}
$$

Using the conditions of $Q^*$

$$
\begin{cases}
\Theta(F^*) = \Lambda(F^*,H^*),\\
(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F^*,H^*) = (\alpha+m)\digamma_1(K^*),\\
\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*) + me^{-\mu_3\tau_3}\digamma_1(K^*) = \gamma\Lambda(F^*,H^*) = ae^{-\mu_3\tau_3}\digamma_2(S^*),\\
\theta e^{-\mu_3\tau_3}\digamma_2(S^*) = c\digamma_3(H^*),
\end{cases} \tag{48}
$$

we get

$$
\begin{aligned}
\Delta\mathcal{U}_n \le{}& \gamma\left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)\big(\Theta(F_{n+1}) - \Theta\big(F^*\big)\big)\\
&+ \gamma\Lambda\big(F^*,H^*\big)\left(1 - \frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\Lambda(F^*,H^*)\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},H^*)}\\
&-\frac{m(1-\varepsilon)}{\alpha+m}e^{-\mu_1\tau_1-\mu_3\tau_3}\Lambda(F^*,H^*)\frac{\digamma_1(K^*)\Lambda(F_{n-m_1+1},H_{n-m_1})}{\digamma_1(K_{n+1})\Lambda(F^*,H^*)}\\
&+\frac{m(1-\varepsilon)}{\alpha+m}e^{-\mu_1\tau_1-\mu_3\tau_3}\Lambda(F^*,H^*)\\
&-\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*)\frac{\digamma_2(S^*)\Lambda(F_{n-m_2+1},H_{n-m_2})}{\digamma_2(S_{n+1})\Lambda(F^*,H^*)}\\
&-\frac{m(1-\varepsilon)}{\alpha+m}e^{-\mu_1\tau_1-\mu_3\tau_3}\Lambda(F^*,H^*)\frac{\digamma_2(S^*)\digamma_1(K_{n+1})}{\digamma_2(S_{n+1})\digamma_1(K^*)}+\gamma\Lambda(F^*,H^*)\\
&-\gamma\Lambda(F^*,H^*)\frac{\digamma_3(H^*)\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(S^*)}+\gamma\Lambda(F^*,H^*)\\
&+\frac{ad}{\theta}\left(\digamma_3(H^*)-\frac{\eta}{q}\right)\digamma_4(Y_{n+1})\\
&+\lambda e^{-\mu_3\tau_3}\left(\digamma_2(S^*)-\frac{\xi}{g}\right)\digamma_5(Z_{n+1})-\gamma\Lambda(F^*,H^*)\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\\
&+\gamma\Lambda(F^*,H^*)\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(F^*,H^*)\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*)\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\\
&+\gamma\Lambda(F^*,H^*)\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Delta\mathcal{U}_n\le{}&\gamma\left(1-\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\right)\left(\Theta(F_{n+1})-\Theta(F^*)\right)\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(F^*,H^*)\\
&\times\left[5-\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}-\frac{\digamma_1(K^*)\Lambda(F_{n-m_1+1},H_{n-m_1})}{\digamma_1(K_{n+1})\Lambda(F^*,H^*)}\right.\\
&-\frac{\digamma_2(S^*)\digamma_1(K_{n+1})}{\digamma_2(S_{n+1})\digamma_1(K^*)}-\frac{\digamma_3(H^*)\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(S^*)}-\frac{\Lambda(F_{n+1},H^*)\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(H^*)}\\
&\left.+\ln\left(\frac{\digamma_3(H_n)\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})}\right)\right]\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F^*,H^*)\left[4-\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}-\frac{\digamma_2(S^*)\Lambda(F_{n-m_2+1},H_{n-m_2})}{\digamma_2(S_{n+1})\Lambda(F^*,H^*)}\right.\\
&-\frac{\digamma_3(H^*)\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(S^*)}-\frac{\Lambda(F_{n+1},H^*)\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(H^*)}\\
&\left.+\ln\left(\frac{\digamma_3(H_n)\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})}\right)\right]\\
&+\frac{ad}{\theta}\left(\digamma_3(H^*)-\digamma_3(\overline{H})\right)\digamma_4(Y_{n+1})+\lambda e^{-\mu_3\tau_3}\left(\digamma_2(S^*)-\digamma_2(\widehat{S})\right)\digamma_5(S_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\Lambda\big(F^*,H^*\big)\bigg[-1+\frac{\Lambda(F_{n+1},H^*)\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(H^*)}+\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},H^*)}-\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\bigg]\\
&=\gamma\bigg(1-\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\bigg)\big(\Theta(F_{n+1})-\Theta\big(F^*\big)\big)\\
&\quad-\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda\big(F^*,H^*\big)\bigg[G\bigg(\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\bigg)\\
&\quad+G\bigg(\frac{\digamma_1(K^*)\Lambda(F_{n-m_1+1},H_{n-m_1})}{\digamma_1(K_{n+1})\Lambda(F^*,H^*)}\bigg)\\
&\quad+G\bigg(\frac{\digamma_2(S^*)\digamma_1(K_{n+1})}{\digamma_2(S_{n+1})\digamma_1(K^*)}\bigg)+G\bigg(\frac{\digamma_3(H^*)\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(S^*)}\bigg)\\
&\quad+G\bigg(\frac{\Lambda(F_{n+1},H^*)\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(H^*)}\bigg)\bigg]\\
&\quad-\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda\big(F^*,H^*\big)\bigg[G\bigg(\frac{\Lambda(F^*,H^*)}{\Lambda(F_{n+1},H^*)}\bigg)\\
&\quad+G\bigg(\frac{\digamma_2(S^*)\Lambda(F_{n-m_2+1},H_{n-m_2})}{\digamma_2(S_{n+1})\Lambda(F^*,H^*)}\bigg)\\
&\quad+G\bigg(\frac{\digamma_3(H^*)\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(S^*)}\bigg)+G\bigg(\frac{\Lambda(F_{n+1},H^*)\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(H^*)}\bigg)\bigg]\\
&\quad+\frac{ad}{\theta}\big(\digamma_3\big(H^*\big)-\digamma_3(\overline{H})\big)\digamma_4(Y_{n+1})+\lambda e^{-\mu_3\tau_3}\big(\digamma_2\big(S^*\big)-\digamma_2(\widehat{S})\big)\digamma_5(Z_{n+1})\\
&\quad+\gamma\Lambda\big(F^*,H^*\big)\bigg(1-\frac{\Lambda(F_{n+1},H^*)}{\Lambda(F_{n+1},H_n)}\bigg)\bigg(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},H^*)}-\frac{\digamma_3(H_n)}{\digamma_3(H^*)}\bigg).
\end{aligned}
\tag{49}
$$

Conditions C1–C4 imply that the first and last terms of Eq. (49) are less than or equal to zero. If $\mathcal{R}_1^Y \le 1$, then from Lemma 3 we have $H^* \le \overline{H}$ and from Condition C3 we get $\digamma_3(H^*) \le \digamma_3(\overline{H})$. Moreover, if $\mathcal{R}_1^Z \le 1$, then $\digamma_2(S^*) \le \digamma_2(\widehat{S})$. Therefore, $\Delta\mathcal{U}_n \le 0$, and thus $\mathcal{U}_n$ is a monotone decreasing sequence. Since $\mathcal{U}_n \ge 0$, then there is a limit $\lim_{n\to\infty}\mathcal{U}_n \ge 0$. Therefore, $\lim_{n\to\infty}\Delta\mathcal{U}_n = 0$, which implies that $\lim_{n\to\infty}F_n = F^*$, $\lim_{n\to\infty}K_n = K^*$, $\lim_{n\to\infty}S_n = S^*$, $\lim_{n\to\infty}H_n = H^*$, $\lim_{n\to\infty}Y_n = 0$, and $\lim_{n\to\infty}Z_n = 0$. We have four cases as follows:

- $\mathcal{R}_1^Y = 1$, $\mathcal{R}_1^Z = 1$, then from Eq. (13)

$$
\begin{aligned}
0 &= \varepsilon e^{-\mu_2\tau_2}\Lambda\big(F^*,H^*\big)+m\digamma_1\big(K^*\big)-a\digamma_2\big(S^*\big)\\
&\quad-\lambda\digamma_2\big(S^*\big)\lim_{n\to\infty}\digamma_5(Z_{n+1}).
\end{aligned}
\tag{50}
$$

  Using equilibrium condition (48), we get $\lim_{n\to\infty}Z_n = 0$. Moreover, from Eq. (14) we have

$$
0 = \theta e^{-\mu_3\tau_3}\digamma_2\big(S^*\big)-c\digamma_3\big(H^*\big)-d\digamma_3\big(H^*\big)\lim_{n\to\infty}\digamma_4(Y_{n+1}).
\tag{51}
$$

  From Eq. (48) we get $\lim_{n\to\infty}Y_n = 0$.
- $\mathcal{R}_1^Y = 1$, $\mathcal{R}_1^Z < 1$, and $\lim_{n\to\infty}Z_n = 0$. From Eq. (51) we get $\lim_{n\to\infty}Y_n = 0$.
- $\mathcal{R}_1^Y < 1$, $\mathcal{R}_1^Z = 1$, $\lim_{n\to\infty}Y_n = 0$. From Eq. (50) we get $\lim_{n\to\infty}Z_n = 0$.

- $\mathcal{R}_1^Y < 1$, $\mathcal{R}_1^Z < 1$, $\lim_{n\to\infty} Y_n = 0$, and $\lim_{n\to\infty} Z_n = 0$. It follows that if $\mathcal{R}_1^Z \leq 1$ and $\mathcal{R}_1^Y \leq 1$, then $\lim_{n\to\infty} F_n = F^*$, $\lim_{n\to\infty} K_n = K^*$, $\lim_{n\to\infty} S_n = S^*$, $\lim_{n\to\infty} H_n = H^*$, $\lim_{n\to\infty} Y_n = 0$, and $\lim_{n\to\infty} Z_n = 0$. Then $Q^*$ is G.A.S. □

*Proof of Theorem 3* Define $\mathcal{W}_n(F_n, K_n, S_n, H_n, Y_n, Z_n)$

$$\begin{aligned}
\mathcal{W}_n ={}& \frac{1}{\phi(h)}\Bigg[\gamma\left(F_n - \overline{F} - \int_{\overline{F}}^{F_n} \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(\varsigma,\overline{H})}\,d\varsigma\right) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - \overline{K} - \int_{\overline{K}}^{K_n} \frac{\digamma_1(\overline{K})}{\digamma_1(\varsigma)}\,d\varsigma\right) \\
&+ e^{-\mu_3\tau_3}\left(S_n - \overline{S} - \int_{\overline{S}}^{S_n} \frac{\digamma_2(\overline{S})}{\digamma_2(\varsigma)}\,d\varsigma\right) + \frac{a}{\theta}\left(H_n - \overline{H} - \int_{\overline{H}}^{H_n} \frac{\digamma_3(\overline{H})}{\digamma_3(\varsigma)}\,d\varsigma\right) \\
&+ \frac{ad}{\theta q}\left(Y_n - \overline{Y} - \int_{\overline{Y}}^{Y_n} \frac{\digamma_4(\overline{Y})}{\digamma_4(\varsigma)}\,d\varsigma\right) + \frac{\lambda e^{-\mu_3\tau_3}}{g} Z_n\Bigg] \\
&+ \frac{a}{\theta}\big(c + d\digamma_4(\overline{Y})\big)\digamma_3(\overline{H})G\left(\frac{\digamma_3(H_n)}{\digamma_3(\overline{H})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\sum_{j=n-m_1}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\sum_{j=n-m_2}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\right) \\
&+ ae^{-\mu_3\tau_3}\digamma_2(\overline{S})\sum_{j=n-m_3}^{n-1} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\overline{S})}\right).
\end{aligned}$$

Clearly, $\mathcal{W}_n(F_n, K_n, S_n, H_n, Y_n, Z_n) > 0$ for all $F_n, K_n, S_n, H_n, Y_n, Z_n > 0$ and $\mathcal{W}_n(\overline{F}, \overline{K}, \overline{S}, \overline{H}, \overline{Y}, 0) = 0$. We compute $\Delta\mathcal{W}_n = \mathcal{W}_{n+1} - \mathcal{W}_n$ as follows:

$$\begin{aligned}
\Delta\mathcal{W}_n ={}& \frac{1}{\phi(h)}\Bigg[\gamma\left(F_{n+1} - \overline{F} - \int_{\overline{F}}^{F_{n+1}} \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(\varsigma,\overline{H})}\,d\varsigma\right) \\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1} - \overline{K} - \int_{\overline{K}}^{K_{n+1}} \frac{\digamma_1(\overline{K})}{\digamma_1(\varsigma)}\,d\varsigma\right) \\
&+ e^{-\mu_3\tau_3}\left(S_{n+1} - \overline{S} - \int_{\overline{S}}^{S_{n+1}} \frac{\digamma_2(\overline{S})}{\digamma_2(\varsigma)}\,d\varsigma\right) + \frac{a}{\theta}\left(H_{n+1} - \overline{H} - \int_{\overline{H}}^{H_{n+1}} \frac{\digamma_3(\overline{H})}{\digamma_3(\varsigma)}\,d\varsigma\right) \\
&+ \frac{ad}{\theta q}\left(Y_{n+1} - \overline{Y} - \int_{\overline{Y}}^{Y_{n+1}} \frac{\digamma_4(\overline{Y})}{\digamma_4(\varsigma)}\,d\varsigma\right) + \frac{\lambda e^{-\mu_3\tau_3}}{g} Z_{n+1}\Bigg] \\
&+ \frac{a}{\theta}\big(c + d\digamma_4(\overline{Y})\big)\digamma_3(\overline{H})G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(\overline{H})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\sum_{j=n-m_1+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\sum_{j=n-m_2+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\right) \\
&+ ae^{-\mu_3\tau_3}\digamma_2(\overline{S})\sum_{j=n-m_3+1}^{n} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\overline{S})}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\phi(h)}\biggl[\gamma\biggl(F_n - \overline{F} - \int_{\overline{F}}^{F_n} \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(\varsigma,\overline{H})}\,d\varsigma\biggr) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\biggl(K_n - \overline{K} - \int_{\overline{K}}^{K_n} \frac{F_1(\overline{K})}{F_1(\varsigma)}\,d\varsigma\biggr) \\
&+ e^{-\mu_3\tau_3}\biggl(S_n - \overline{S} - \int_{\overline{S}}^{S_n} \frac{F_2(\overline{S})}{F_2(\varsigma)}\,d\varsigma\biggr) + \frac{a}{\theta}\biggl(H_n - \overline{H} - \int_{\overline{H}}^{H_n} \frac{F_3(\overline{H})}{F_3(\varsigma)}\,d\varsigma\biggr) \\
&+ \frac{ad}{\theta q}\biggl(Y_n - \overline{Y} - \int_{\overline{Y}}^{Y_n} \frac{F_4(\overline{Y})}{F_4(\varsigma)}\,d\varsigma\biggr) + \frac{\lambda e^{-\mu_3\tau_3}}{g} Z_n\biggr] \\
&- \frac{a}{\theta}\bigl(c + dF_4(\overline{Y})\bigr)F_3(\overline{H})G\biggl(\frac{F_3(H_n)}{F_3(\overline{H})}\biggr) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H}) \sum_{j=n-m_1}^{n-1} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr) \\
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H}) \sum_{j=n-m_2}^{n-1} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr) \\
&- ae^{-\mu_3\tau_3}F_2(\overline{S}) \sum_{j=n-m_3}^{n-1} G\biggl(\frac{F_2(S_{j+1})}{F_2(\overline{S})}\biggr),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathcal{W}_n = {}& \frac{1}{\phi(h)}\biggl[\gamma\biggl(F_{n+1} - F_n - \int_{F_n}^{F_{n+1}} \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(\varsigma,\overline{H})}\,d\varsigma\biggr) \\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}\biggl(K_{n+1} - K_n - \int_{K_n}^{K_{n+1}} \frac{F_1(\overline{K})}{F_1(\varsigma)}\,d\varsigma\biggr) \\
&+ e^{-\mu_3\tau_3}\biggl(S_{n+1} - S_n - \int_{S_n}^{S_{n+1}} \frac{F_2(\overline{S})}{F_2(\varsigma)}\,d\varsigma\biggr) + \frac{a}{\theta}\biggl(H_{n+1} - H_n - \int_{H_n}^{H_n+1} \frac{F_3(\overline{H})}{F_3(\varsigma)}\,d\varsigma\biggr) \\
&+ \frac{ad}{\theta q}\biggl(Y_{n+1} - Y_n - \int_{Y_n}^{Y_{n+1}} \frac{F_4(\overline{Y})}{F_4(\varsigma)}\,d\varsigma\biggr) + \frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1} - Z_n)\biggr] \\
&+ \frac{a}{\theta}\bigl(c + dF_4(\overline{Y})\bigr)F_3(\overline{H})\biggl[G\biggl(\frac{F_3(H_{n+1})}{F_3(\overline{H})}\biggr) - G\biggl(\frac{F_3(H_n)}{F_3(\overline{H})}\biggr)\biggr] \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\biggl(\sum_{j=n-m_1+1}^{n} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr) \\
&- \sum_{j=n-m_1}^{n-1} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr)\biggr) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\biggl(\sum_{j=n-m_2+1}^{n} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr) - \sum_{j=n-m_2}^{n-1} G\biggl(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\overline{F},\overline{H})}\biggr)\biggr) \\
&+ ae^{-\mu_3\tau_3}F_2(\overline{S})\biggl(\sum_{j=n-m_3+1}^{n} G\biggl(\frac{F_2(S_{j+1})}{F_2(\overline{S})}\biggr) - \sum_{j=n-m_3}^{n-1} G\biggl(\frac{F_2(S_{j+1})}{F_2(\overline{S})}\biggr)\biggr).
\end{aligned}
$$

Using inequalities (45) and (46) by replacing $F^*, H^*, \rho^*$ with $\overline{F}, \overline{H}, \overline{\rho}$, we obtain

$$
\begin{aligned}
\Delta\mathcal{W}_n \le {}& \frac{1}{\phi(h)}\biggl[\gamma\biggl(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\biggr)(F_{n+1} - F_n) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\biggl(1 - \frac{F_1(\overline{K})}{F_1(K_{n+1})}\biggr)(K_{n+1} - K_n) \\
&+ e^{-\mu_3\tau_3}\biggl(1 - \frac{F_2(\overline{S})}{F_2(S_{n+1})}\biggr)(S_{n+1} - S_n) + \frac{a}{\theta}\biggl(1 - \frac{F_3(\overline{H})}{F_3(H_{n+1})}\biggr)(H_{n+1} - H_n)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{ad}{\theta q}\left(1-\frac{F_4(\overline{Y})}{F_4(Y_{n+1})}\right)(Y_{n+1}-Y_n)+\frac{\lambda e^{-\mu_3\tau_3}}{g}(Z_{n+1}-Z_n)\Bigg] \\
&+\frac{a}{\theta}\big(c+dF_4(\overline{Y})\big)F_3(\overline{H})\left[\frac{F_3(H_{n+1})}{F_3(\overline{H})}-\frac{F_3(H_n)}{F_3(\overline{H})}+\ln\left(\frac{F_3(H_n)}{F_3(H_{n+1})}\right)\right] \\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\overline{F},\overline{H})}-\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(\overline{F},\overline{H})}\right. \\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\right) \\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\overline{F},\overline{H})}-\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(\overline{F},\overline{H})}\right. \\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\right) \\
&+ae^{-\mu_3\tau_3}F_2(\overline{S})\left(\frac{F_2(S_{n+1})}{F_2(\overline{S})}-\frac{F_2(S_{n-m_3+1})}{F_2(\overline{S})}+\ln\left(\frac{F_2(S_{n-m_3+1})}{F_2(S_{n+1})}\right)\right).
\end{aligned}
$$

From Eqs. (11)–(16) we have

$$
\begin{aligned}
\Delta\mathcal{W}_n \le{}& \gamma\left(1-\frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)\big[\Theta(F_{n+1})-\Lambda(F_{n+1},H_n)\big] \\
&+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(1-\frac{F_1(\overline{K})}{F_1(K_{n+1})}\right)\big[(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{n-m_1+1},H_{n-m_1}) \\
&-(\alpha+m)F_1(K_{n+1})\big] \\
&+e^{-\mu_3\tau_3}\left(1-\frac{F_2(\overline{S})}{F_2(S_{n+1})}\right)\big[\varepsilon e^{-\mu_2\tau_2}\Lambda(F_{n-m_2+1},H_{n-m_2})+mF_1(K_{n+1})-aF_2(S_{n+1}) \\
&-\lambda F_2(S_{n+1})F_5(Z_{n+1})\big] \\
&+\frac{a}{\theta}\left(1-\frac{F_3(\overline{H})}{F_3(H_{n+1})}\right)\big[\theta e^{-\mu_3\tau_3}F_2(S_{n-m_3+1})-cF_3(H_{n+1})-dF_3(H_{n+1})F_4(Y_{n+1})\big] \\
&+\frac{ad}{\theta q}\left(1-\frac{F_4(\overline{Y})}{F_4(Y_{n+1})}\right)\big[qF_3(H_{n+1})F_4(Y_{n+1})-\eta F_4(Y_{n+1})\big] \\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\big[gF_2(S_{n+1})F_5(Z_{n+1})-\xi F_5(Z_{n+1})\big] \\
&+\frac{a}{\theta}\big(c+dF_4(\overline{Y})\big)F_3(\overline{H})\left[\frac{F_3(H_{n+1})}{F_3(\overline{H})}-\frac{F_3(H_n)}{F_3(\overline{H})}+\ln\left(\frac{F_3(H_n)}{F_3(H_{n+1})}\right)\right] \\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\overline{F},\overline{H})}-\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(\overline{F},\overline{H})}\right. \\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\right) \\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\overline{F},\overline{H})}-\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(\overline{F},\overline{H})}\right. \\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\right) \\
&+ae^{-\mu_3\tau_3}F_2(\overline{S})\left(\frac{F_2(S_{n+1})}{F_2(\overline{S})}-\frac{F_2(S_{n-m_3+1})}{F_2(\overline{S})}+\ln\left(\frac{F_2(S_{n-m_3+1})}{F_2(S_{n+1})}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma \left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)\left(\Theta(F_{n+1}) - \Theta(\overline{F})\right) + \gamma \Theta(\overline{F})\left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right) \\
&\quad + \gamma \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\Lambda(F_{n+1},H_n) - \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha + m}\Lambda(F_{n-m_1+1},H_{n-m_1})\frac{\digamma_1(\overline{K})}{\digamma_1(K_{n+1})} \\
&\quad + me^{-\mu_3\tau_3}\digamma_1(\overline{K}) - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F_{n-m_2+1},H_{n-m_2})\frac{\digamma_2(\overline{S})}{\digamma_2(S_{n+1})} \\
&\quad - me^{-\mu_3\tau_3}\digamma_1(K_{n+1})\frac{\digamma_2(\overline{S})}{\digamma_2(S_{n+1})} \\
&\quad + ae^{-\mu_3\tau_3}\digamma_2(\overline{S}) + \lambda e^{-\mu_3\tau_3}\digamma_2(\overline{S})\digamma_5(Z_{n+1}) - ae^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})\frac{\digamma_3(\overline{H})}{\digamma_3(H_{n+1})} \\
&\quad + \frac{ac}{\theta}\digamma_3(\overline{H}) \\
&\quad + \frac{ad}{\theta}\digamma_3(\overline{H})\digamma_4(Y_{n+1}) - \frac{ad\eta}{\theta q}\digamma_4(Y_{n+1}) + \frac{ad\eta}{\theta q}\digamma_4(\overline{Y}) - \frac{\lambda e^{-\mu_3\tau_3}\xi}{g}\digamma_5(Z_{n+1}) \\
&\quad - \frac{a}{\theta}\left(c + d\digamma_4(\overline{Y})\right)\digamma_3(H_n) + \frac{a}{\theta}\left(c + d\digamma_4(\overline{Y})\right)\digamma_3(\overline{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right) \\
&\quad + \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha + m}\Lambda(\overline{F},\overline{H})\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right) \\
&\quad + \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right) \\
&\quad + ae^{-\mu_3\tau_3}\digamma_2(\overline{S})\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right).
\end{aligned}
$$

Using the conditions of $\overline{Q}$

$$
\begin{aligned}
&\Theta(\overline{F}) = \Lambda(\overline{F},\overline{H}), \\
&(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(\overline{F},\overline{H}) = (\alpha + m)\digamma_1(\overline{K}), \\
&\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H}) + me^{-\mu_3\tau_3}\digamma_1(\overline{K}) = \gamma\Lambda(\overline{F},\overline{H}) = ae^{-\mu_3\tau_3}\digamma_2(\overline{S}), \\
&\theta e^{-\mu_3\tau_3}\digamma_2(\overline{S}) = \left(c + d\digamma_4(\overline{Y})\right)\digamma_3(\overline{H}), \\
&\eta = q\digamma_3(\overline{H}),
\end{aligned}
$$

we get

$$
\begin{aligned}
\Delta\mathcal{W}_n &\le \gamma \left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)\left(\Theta(F_{n+1}) - \Theta(\overline{F})\right) + \gamma\Lambda(\overline{F},\overline{H})\left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right) \\
&\quad + \gamma\Lambda(\overline{F},\overline{H})\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\overline{H})} \\
&\quad - \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha + m}\Lambda(\overline{F},\overline{H})\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\overline{K})}{\Lambda(\overline{F},\overline{H})\digamma_1(K_{n+1})} \\
&\quad + \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha + m}\Lambda(\overline{F},\overline{H}) \\
&\quad - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\overline{S})}{\Lambda(\overline{F},\overline{H})\digamma_2(S_{n+1})}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\frac{\digamma_1(K_{n+1})\digamma_2(\overline{S})}{\digamma_1(\overline{K})\digamma_2(S_{n+1})} + \gamma\Lambda(\overline{F},\overline{H}) \\
&- \gamma\Lambda(\overline{F},\overline{H})\frac{\digamma_3(\overline{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\overline{S})} \\
&+ \gamma\Lambda(\overline{F},\overline{H}) - \gamma\Lambda(\overline{F},\overline{H})\frac{\digamma_3(H_n)}{\digamma_3(\overline{H})} + \lambda e^{-\mu_3\tau_3}\left(\digamma_2(\overline{S}) - \frac{\xi}{g}\right)\digamma_5(Z_{n+1}) \\
&+ \gamma\Lambda(\overline{F},\overline{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \gamma\Lambda(\overline{F},\overline{H})\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right) \\
= {}& \gamma\left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)\big(\Theta(F_{n+1}) - \Theta(\overline{F})\big) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\Bigg[5 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})} \\
&- \frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\overline{K})}{\Lambda(\overline{F},\overline{H})\digamma_1(K_{n+1})} \\
&- \frac{\digamma_1(K_{n+1})\digamma_2(\overline{S})}{\digamma_1(\overline{K})\digamma_2(S_{n+1})} - \frac{\digamma_3(\overline{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\overline{S})} - \frac{\Lambda(F_{n+1},\overline{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\overline{H})} \\
&+ \ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\Bigg] \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\Bigg[4 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})} \\
&- \frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\overline{S})}{\Lambda(\overline{F},\overline{H})\digamma_2(S_{n+1})} - \frac{\digamma_3(\overline{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\overline{S})} \\
&- \frac{\Lambda(F_{n+1},\overline{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\overline{H})} + \ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\Bigg] \\
&+ \lambda e^{-\mu_3\tau_3}\big(\digamma_2(\overline{S}) - \digamma_2(\widetilde{S})\big)\digamma_5(Z_{n+1}) \\
&+ \gamma\Lambda(\overline{F},\overline{H})\left[-1 + \frac{\Lambda(F_{n+1},\overline{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\overline{H})} + \frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\overline{H})} - \frac{\digamma_3(H_n)}{\digamma_3(\overline{H})}\right],
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\Delta\mathcal{W}_n \le {}& \gamma\left(1 - \frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)\big(\Theta(F_{n+1}) - \Theta(\overline{F})\big) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\overline{F},\overline{H})\Bigg[G\left(\frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right) \\
&+ G\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\overline{K})}{\Lambda(\overline{F},\overline{H})\digamma_1(K_{n+1})}\right) \\
&+ G\left(\frac{\digamma_1(K_{n+1})\digamma_2(\overline{S})}{\digamma_1(\overline{K})\digamma_2(S_{n+1})}\right) + G\left(\frac{\digamma_3(\overline{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\overline{S})}\right)
\end{aligned}
$$

$$+ G\left(\frac{\Lambda(F_{n+1},\overline{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\overline{H})}\right)\Bigg] - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\overline{F},\overline{H})\Bigg[G\left(\frac{\Lambda(\overline{F},\overline{H})}{\Lambda(F_{n+1},\overline{H})}\right)$$
$$+ G\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\overline{S})}{\Lambda(\overline{F},\overline{H})\digamma_2(S_{n+1})}\right) + G\left(\frac{\digamma_3(\overline{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\overline{S})}\right)$$
$$+ G\left(\frac{\Lambda(F_{n+1},\overline{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\overline{H})}\right)\Bigg]$$
$$+ \lambda e^{-\mu_3\tau_3}\digamma_2(\widetilde{S})(\mathcal{R}_2^Z - 1)\digamma_5(Z_{n+1})$$
$$+ \gamma\Lambda(\overline{F},\overline{H})\left(1 - \frac{\Lambda(F_{n+1},\overline{H})}{\Lambda(F_{n+1},H_n)}\right)\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\overline{H})} - \frac{\digamma_3(H_n)}{\digamma_3(\overline{H})}\right).$$

Using Conditions C1–C4, we get that the first and last terms of Eq. (52) are less than or equal to zero. Moreover, if $\mathcal{R}_2^Z \le 1$, we get $\Delta\mathcal{W}_n \le 0$, and thus $\mathcal{W}_n$ is a monotone decreasing sequence. Since $\mathcal{W}_n \ge 0$, then there is a limit $\lim_{n\to\infty}\mathcal{W}_n \ge 0$. Therefore, $\lim_{n\to\infty}\Delta\mathcal{W}_n = 0$, which implies that $\lim_{n\to\infty}F_n = \overline{F}$, $\lim_{n\to\infty}K_n = \overline{K}$, $\lim_{n\to\infty}S_n = \overline{S}$, $\lim_{n\to\infty}H_n = \overline{H}$, and $\lim_{n\to\infty}(\mathcal{R}_2^Z - 1)\digamma_5(Z_{n+1}) = 0$ We have two cases:

- $\mathcal{R}_2^Z = 1$, then from Eq. (13)

$$0 = \varepsilon e^{-\mu_2\tau_2}\Lambda(\overline{F,H}) + m\digamma_1(\overline{K}) - a\digamma_2(\overline{S}) - \lambda\digamma_2(\overline{S})\lim_{n\to\infty}\digamma_5(Z_{n+1}), \tag{53}$$

  we get $\lim_{n\to\infty}Z_n = 0$. From Eq. (14) we get

$$0 = \theta e^{-\mu_3\tau_3}\digamma_2(\overline{S}) - c\digamma_3(\overline{H}) - d\digamma_3(\overline{H})\lim_{n\to\infty}\digamma_4(Y_{n+1}). \tag{54}$$

  This gives $\lim_{n\to\infty}Y_n = \overline{Y}$.
- $\mathcal{R}_2^Z < 1$, $\lim_{n\to\infty}\digamma_5(Z_n) = 0$. From Eq. (54) we get $\lim_{n\to\infty}Y_n = \overline{Y}$. Hence, $\overline{Q}$ is G.A.S. □

*Proof of Theorem 4* Define $\mathcal{M}_n(F_n,K_n,S_n,H_n,Y_n,Z_n)$:

$$\mathcal{M}_n = \frac{1}{\phi(h)}\Bigg[\gamma\left(F_n - \widehat{F} - \int_{\widehat{F}}^{F_n}\frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(\varsigma,\widehat{H})}d\varsigma\right) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - \widehat{K} - \int_{\widehat{K}}^{K_n}\frac{\digamma_1(\widehat{K})}{\digamma_1(\varsigma)}d\varsigma\right)$$
$$+ e^{-\mu_3\tau_3}\left(S_n - \widehat{S} - \int_{\widehat{S}}^{S_n}\frac{\digamma_2(\widehat{S})}{\digamma_2(\varsigma)}d\varsigma\right) + \frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(H_n - \widehat{H} - \int_{\widehat{H}}^{H_n}\frac{\digamma_3(\widehat{H})}{\digamma_3(\varsigma)}d\varsigma\right)$$
$$+ \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}Y_n + \frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_n - \widehat{Z} - \int_{\widehat{Z}}^{Z_n}\frac{\digamma_5(\widehat{Z})}{\digamma_5(\varsigma)}d\varsigma\right)\Bigg]$$
$$+ \frac{c}{\theta}(a+\lambda\digamma_5(\widehat{Z}))\digamma_3(\widehat{H})G\left(\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}\right)$$
$$+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_1}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right)$$
$$+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_2}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right) + \gamma\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_3}^{n-1}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widehat{S})}\right).$$

Clearly, $\mathcal{M}_n(F_n, K_n, S_n, H_n, Y_n, Z_n) > 0$ for all $F_n, K_n, S_n, H_n, Y_n, Z_n > 0$ and $\mathcal{M}_n(\widehat{F}, \widehat{K}, \widehat{S}, \widehat{H}, 0, \widehat{Z}) = 0$. We compute $\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_n$ as follows:

$$\begin{aligned}
\Delta\mathcal{M}_n &= \frac{1}{\phi(h)}\Bigg[\gamma\left(F_{n+1} - \widehat{F} - \int_{\widehat{F}}^{F_{n+1}} \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(\varsigma,\widehat{H})}\,d\varsigma\right) \\
&\quad + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1} - \widehat{K} - \int_{\widehat{K}}^{K_{n+1}} \frac{\digamma_1(\widehat{K})}{\digamma_1(\varsigma)}\,d\varsigma\right) \\
&\quad + e^{-\mu_3\tau_3}\left(S_{n+1} - \widehat{S} - \int_{\widehat{S}}^{S_{n+1}} \frac{\digamma_2(\widehat{S})}{\digamma_2(\varsigma)}\,d\varsigma\right) \\
&\quad + \frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(H_{n+1} - \widehat{H} - \int_{\widehat{H}}^{H_{n+1}} \frac{\digamma_3(\widehat{H})}{\digamma_3(\varsigma)}\,d\varsigma\right) \\
&\quad + \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}Y_{n+1} + \frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_{n+1} - \widehat{Z} - \int_{\widehat{Z}}^{Z_{n+1}} \frac{\digamma_5(\widehat{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\Bigg] \\
&\quad + \frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(\widehat{H})}\right) \\
&\quad + \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_1+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right) \\
&\quad + \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_2+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right) \\
&\quad + \gamma\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_3+1}^{n} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widehat{S})}\right) \\
&\quad - \frac{1}{\phi(h)}\Bigg[\gamma\left(F_n - \widehat{F} - \int_{\widehat{F}}^{F_n} \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(\varsigma,\widehat{H})}\,d\varsigma\right) \\
&\quad + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - \widehat{K} - \int_{\widehat{K}}^{K_n} \frac{\digamma_1(\widehat{K})}{\digamma_1(\varsigma)}\,d\varsigma\right) \\
&\quad + e^{-\mu_3\tau_3}\left(S_n - \widehat{S} - \int_{\widehat{S}}^{S_n} \frac{\digamma_2(\widehat{S})}{\digamma_2(\varsigma)}\,d\varsigma\right) \\
&\quad + \frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(H_n - \widehat{H} - \int_{\widehat{H}}^{H_n} \frac{\digamma_3(\widehat{H})}{\digamma_3(\varsigma)}\,d\varsigma\right) \\
&\quad + \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}Y_n + \frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_n - \widehat{Z} - \int_{\widehat{Z}}^{Z_n} \frac{\digamma_5(\widehat{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\Bigg] \\
&\quad - \frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})G\left(\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}\right) \\
&\quad - \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_1}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right) \\
&\quad - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_2}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right) \\
&\quad - \gamma\Lambda(\widehat{F},\widehat{H})\sum_{j=n-m_3}^{n-1} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widehat{S})}\right),
\end{aligned}$$

$$\begin{aligned}
\triangle\mathcal{M}_n &= \frac{1}{\phi(h)}\Bigg[\gamma\left(F_{n+1}-F_n-\int_{F_n}^{F_{n+1}}\frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(\varsigma,\widehat{H})}\,d\varsigma\right)\\
&\quad+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1}-K_n-\int_{K_n}^{K_{n+1}}\frac{\digamma_1(\widehat{K})}{\digamma_1(\varsigma)}\,d\varsigma\right)\\
&\quad+e^{-\mu_3\tau_3}\left(S_{n+1}-S_n-\int_{S_n}^{S_{n+1}}\frac{\digamma_2(\widehat{S})}{\digamma_2(\varsigma)}\,d\varsigma\right)\\
&\quad+\frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(H_{n+1}-H_n-\int_{H_n}^{H_{n+1}}\frac{\digamma_3(\widehat{H})}{\digamma_3(\varsigma)}\,d\varsigma\right)\\
&\quad+\frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}(Y_{n+1}-Y_n)+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_{n+1}-Z_n-\int_{Z_n}^{Z_{n+1}}\frac{\digamma_5(\widehat{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\Bigg]\\
&\quad+\frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})\left[G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(\widehat{H})}\right)-G\left(\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}\right)\right]\\
&\quad+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\left(\sum_{j=n-m_1+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right)\right.\\
&\quad\left.-\sum_{j=n-m_1}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right)\right)\\
&\quad+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\left(\sum_{j=n-m_2+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right)-\sum_{j=n-m_2}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widehat{F},\widehat{H})}\right)\right)\\
&\quad+\gamma\Lambda(\widehat{F},\widehat{H})\left(\sum_{j=n-m_3+1}^{n}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widehat{S})}\right)-\sum_{j=n-m_3}^{n-1}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widehat{S})}\right)\right).
\end{aligned}$$

Using inequalities (45) and (46) by replacing $F^*$, $H^*$, $\rho^*$ with $\widehat{F}$, $\widehat{H}$, $\widehat{\rho}$, we get

$$\begin{aligned}
\triangle\mathcal{M}_n &\le \frac{1}{\phi(h)}\Bigg[\gamma\left(1-\frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right)(F_{n+1}-F_n)\\
&\quad+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(1-\frac{\digamma_1(\widehat{K})}{\digamma_1(K_{n+1})}\right)(K_{n+1}-K_n)\\
&\quad+e^{-\mu_3\tau_3}\left(1-\frac{\digamma_2(\widehat{S})}{\digamma_2(S_{n+1})}\right)(S_{n+1}-S_n)\\
&\quad+\frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(1-\frac{\digamma_3(\widehat{H})}{\digamma_3(H_{n+1})}\right)(H_{n+1}-H_n)\\
&\quad+\frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}(Y_{n+1}-Y_n)+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(1-\frac{\digamma_5(\widehat{Z})}{\digamma_5(Z_{n+1})}\right)(Z_{n+1}-Z_n)\Bigg]\\
&\quad+\frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})\left[\frac{\digamma_3(H_{n+1})}{\digamma_3(\widehat{H})}-\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}+\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\right]\\
&\quad+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widehat{F},\widehat{H})}-\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(\widehat{F},\widehat{H})}\right.\\
&\quad\left.+\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&\quad+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widehat{F},\widehat{H})}-\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(\widehat{F},\widehat{H})}\right.
\end{aligned}$$

$$+\ln\left(\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(F_{n+1},H_n)}\right)\bigg]$$
$$+\gamma\varLambda(\widehat{F},\widehat{H})\left[\frac{\digamma_2(S_{n+1})}{\digamma_2(\widehat{S})}-\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(\widehat{S})}+\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right)\right].$$

From Eqs. (11)–(16) we have

$$\begin{aligned}
\Delta\mathcal{M}_n \le{}& \gamma\left(1-\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\right)\big(\Theta(F_{n+1})-\varLambda(F_{n+1},H_n)\big)\\
&+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(1-\frac{\digamma_1(\widehat{K})}{\digamma_1(K_{n+1})}\right)\big[(1-\varepsilon)e^{-\mu_1\tau_1}\varLambda(F_{n-m_1+1},H_{n-m_1})\\
&-(\alpha+m)\digamma_1(K_{n+1})\big]\\
&+e^{-\mu_3\tau_3}\left(1-\frac{\digamma_2(\widehat{S})}{\digamma_2(S_{n+1})}\right)\big[\varepsilon e^{-\mu_2\tau_2}\varLambda(F_{n-m_2+1},H_{n-m_2})+m\digamma_1(K_{n+1})\\
&-a\digamma_2(S_{n+1})-\lambda\digamma_2(S_{n+1})\digamma_5(Z_{n+1})\big]\\
&+\frac{(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(1-\frac{\digamma_3(\widehat{H})}{\digamma_3(H_{n+1})}\right)\big[\theta e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})-c\digamma_3(H_{n+1})\\
&-d\digamma_3(H_{n+1})\digamma_4(Y_{n+1})\big]\\
&+\frac{d(a+\lambda\digamma_5(\widehat{Z}))}{q\theta}\big[q\digamma_3(H_{n+1})\digamma_4(Y_{n+1})-\eta\digamma_4(Y_{n+1})\big]\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(1-\frac{\digamma_5(\widehat{Z})}{\digamma_5(Z_{n+1})}\right)\big[g\digamma_2(S_{n+1})\digamma_5(Z_{n+1})-\xi\digamma_5(Z_{n+1})\big]\\
&+\frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})\left[\frac{\digamma_3(H_{n+1})}{\digamma_3(\widehat{H})}-\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}+\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\right]\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda(\widehat{F},\widehat{H})\left[\frac{\varLambda(F_{n+1},H_n)}{\varLambda(\widehat{F},\widehat{H})}-\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(\widehat{F},\widehat{H})}\right.\\
&\left.+\ln\left(\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F_{n+1},H_n)}\right)\right]\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\varLambda(\widehat{F},\widehat{H})\left[\frac{\varLambda(F_{n+1},H_n)}{\varLambda(\widehat{F},\widehat{H})}-\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(\widehat{F},\widehat{H})}\right.\\
&\left.+\ln\left(\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(F_{n+1},H_n)}\right)\right]\\
&+\gamma\varLambda(\widehat{F},\widehat{H})\left[\frac{\digamma_2(S_{n+1})}{\digamma_2(\widehat{S})}-\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(\widehat{S})}+\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right)\right].
\end{aligned}\tag{55}$$

Collecting terms of Eq. (55), we get

$$\begin{aligned}
\Delta\mathcal{M}_n \le{}& \gamma\left(1-\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widehat{F})\big)+\gamma\Theta(\widehat{F})\left(1-\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\right)\\
&+\gamma\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\varLambda(F_{n+1},H_n)\\
&-\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda(F_{n-m_1+1},H_{n-m_1})\frac{\digamma_1(\widehat{K})}{\digamma_1(K_{n+1})}
\end{aligned}$$

$$
\begin{aligned}
&+ me^{-\mu_3\tau_3}\digamma_1(\widehat{K}) - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\varLambda(F_{n-m_2+1},H_{n-m_2})\frac{\digamma_2(\widehat{S})}{\digamma_2(S_{n+1})} \\
&- me^{-\mu_3\tau_3}\digamma_1(K_{n+1})\frac{\digamma_2(\widehat{S})}{\digamma_2(S_{n+1})} \\
&- ae^{-\mu_3\tau_3}\digamma_2(S_{n+1}) + \lambda e^{-\mu_3\tau_3}\digamma_2(\widehat{S})\digamma_5(Z_{n+1}) + ae^{-\mu_3\tau_3}\digamma_2(\widehat{S}) \\
&+ \big(a+\lambda\digamma_5(\widehat{Z})\big)e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1}) \\
&- \big(a+\lambda\digamma_5(\widehat{Z})\big)e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})\frac{\digamma_3(\widehat{H})}{\digamma_3(H_{n+1})} \\
&+ \frac{c(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\digamma_3(\widehat{H}) + \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\digamma_3(\widehat{H})\digamma_4(Y_{n+1}) \\
&- \frac{d(a+\lambda\digamma_5(\widehat{Z}))\eta}{q\theta}\digamma_4(Y_{n+1}) \\
&- \lambda e^{-\mu_3\tau_3}\digamma_5(\widehat{Z})\digamma_2(S_{n+1}) - \frac{\lambda e^{-\mu_3\tau_3}\xi}{g}\digamma_5(Z_{n+1}) + \frac{\lambda e^{-\mu_3\tau_3}\xi}{g}\digamma_5(\widehat{Z}) \\
&- \frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(H_n) + \frac{c}{\theta}\big(a+\lambda\digamma_5(\widehat{Z})\big)\digamma_3(\widehat{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda(\widehat{F},\widehat{H})\ln\left(\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})}{\varLambda(F_{n+1},H_n)}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\varLambda(\widehat{F},\widehat{H})\ln\left(\frac{\varLambda(F_{n-m_2+1},H_{n-m_2})}{\varLambda(F_{n+1},H_n)}\right) \\
&+ \gamma\varLambda(\widehat{F},\widehat{H})\left(\frac{\digamma_2(S_{n+1})}{\digamma_2(\widehat{S})} - \frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(\widehat{S})} + \ln\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right).
\end{aligned}
$$

Using the conditions of $\widehat{Q}$

$$
\begin{aligned}
&\Theta(\widehat{F}) = \varLambda(\widehat{F},\widehat{H}), \\
&(1-\varepsilon)e^{-\mu_1\tau_1}\varLambda(\widehat{F},\widehat{H}) = (\alpha+m)\digamma_1(\widehat{K}), \\
&\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\varLambda(\widehat{F},\widehat{H}) + me^{-\mu_3\tau_3}\digamma_1(\widehat{K}) = \gamma\varLambda(\widehat{F},\widehat{H}) = \big(a+\lambda\digamma_5(\widehat{Z})\big)e^{-\mu_3\tau_3}\digamma_2(\widehat{S}), \\
&\theta e^{-\mu_3\tau_3}\digamma_2(\widehat{S}) = c\digamma_3(\widehat{H}), \\
&\digamma_2(\widehat{S}) = \frac{\xi}{g},
\end{aligned}
$$

we get

$$
\begin{aligned}
\Delta\mathcal{M}_n \le{}& \gamma\left(1-\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widehat{F})\big) + \gamma\varLambda(\widehat{F},\widehat{H})\left(1-\frac{\varLambda(\widehat{F},\widehat{H})}{\varLambda(F_{n+1},\widehat{H})}\right) \\
&+ \gamma\varLambda(\widehat{F},\widehat{H})\frac{\varLambda(F_{n+1},H_n)}{\varLambda(F_{n+1},\widehat{H})} \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda(\widehat{F},\widehat{H})\frac{\varLambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widehat{K})}{\varLambda(\widehat{F},\widehat{H})\digamma_1(K_{n+1})} \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\varLambda(\widehat{F},\widehat{H})
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\widehat{S})}{\Lambda(\widehat{F},\widehat{H})\digamma_2(S_{n+1})} \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\frac{\digamma_1(K_{n+1})\digamma_2(\widehat{S})}{\digamma_1(\widehat{K})\digamma_2(S_{n+1})} + \gamma\Lambda(\widehat{F},\widehat{H}) \\
&- \gamma\Lambda(\widehat{F},\widehat{H})\frac{\digamma_3(\widehat{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widehat{S})} \\
&+ \gamma\Lambda(\widehat{F},\widehat{H}) - \gamma\Lambda(\widehat{F},\widehat{H})\frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})} + \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(\digamma_3(\widehat{H}) - \frac{\eta}{q}\right)\digamma_4(Y_{n+1}) \\
&+ \gamma\Lambda(\widehat{F},\widehat{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \gamma\Lambda(\widehat{F},\widehat{H})\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right) \\
&= \gamma\left(1 - \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right)\left(\Theta(F_{n+1}) - \Theta(\widehat{F})\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\left[5 - \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right. \\
&- \frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widehat{K})}{\Lambda(\widehat{F},\widehat{H})\digamma_1(K_{n+1})} - \frac{\digamma_1(K_{n+1})\digamma_2(\widehat{S})}{\digamma_1(\widehat{K})\digamma_2(S_{n+1})} \\
&- \frac{\digamma_3(\widehat{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widehat{S})} - \frac{\Lambda(F_{n+1},\widehat{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widehat{H})} \qquad (56) \\
&\left.+ \ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\right] \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\left[4 - \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})} - \frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\widehat{S})}{\Lambda(\widehat{F},\widehat{H})\digamma_2(S_{n+1})}\right. \\
&- \frac{\digamma_3(\widehat{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widehat{S})} \\
&\left.- \frac{\Lambda(F_{n+1},\widehat{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widehat{H})} + \ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\right] \\
&+ \frac{d(a+\lambda\digamma_5(\widehat{Z}))}{\theta}\left(\digamma_3(\widehat{H}) - \digamma_3(\widetilde{H})\right)\digamma_4(Y_{n+1}) \\
&+ \gamma\Lambda(\widehat{F},\widehat{H})\left[-1 + \frac{\Lambda(F_{n+1},\widehat{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widehat{H})} + \frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\widehat{H})} - \frac{\digamma_3(H_n)}{\digamma_3(\widehat{H})}\right], \\
\Delta\mathcal{M}_n &\le \gamma\left(1 - \frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right)\left(\Theta(F_{n+1}) - \Theta(\widehat{F})\right) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widehat{F},\widehat{H})\left[G\left(\frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right)\right. \\
&+ G\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widehat{K})}{\Lambda(\widehat{F},\widehat{H})\digamma_1(K_{n+1})}\right) + G\left(\frac{\digamma_1(K_{n+1})\digamma_2(\widehat{S})}{\digamma_1(\widehat{K})\digamma_2(S_{n+1})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ G\left(\frac{F_3(\widehat{H})F_2(S_{n-m_3+1})}{F_3(H_{n+1})F_2(\widehat{S})}\right) \\
&+ G\left(\frac{\Lambda(F_{n+1},\widehat{H})F_3(H_n)}{\Lambda(F_{n+1},H_n)F_3(\widehat{H})}\right)\Bigg] - \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widehat{F},\widehat{H})\Bigg[G\left(\frac{\Lambda(\widehat{F},\widehat{H})}{\Lambda(F_{n+1},\widehat{H})}\right) \\
&+ G\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})F_2(\widehat{S})}{\Lambda(\widehat{F},\widehat{H})F_2(S_{n+1})}\right) + G\left(\frac{F_3(\widehat{H})F_2(S_{n-m_3+1})}{F_3(H_{n+1})F_2(\widehat{S})}\right) \\
&+ G\left(\frac{\Lambda(F_{n+1},\widehat{H})F_3(H_n)}{\Lambda(F_{n+1},H_n)F_3(\widehat{H})}\right)\Bigg] \\
&+ \frac{d(a+\lambda F_5(\widehat{Z}))}{\theta}F_3(\widetilde{H})\left(\mathcal{R}_1^Y/\mathcal{R}_2^Z - 1\right)F_4(Y_{n+1}) \\
&+ \gamma\Lambda(\widehat{F},\widehat{H})\left(1 - \frac{\Lambda(F_{n+1},\widehat{H})}{\Lambda(F_{n+1},H_n)}\right)\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\widehat{H})} - \frac{F_3(H_n)}{F_3(\widehat{H})}\right).
\end{aligned}
$$

Using Conditions C1–C4, we get that the first and last terms of Eq. (56) are less than or equal to zero. Moreover, if $\mathcal{R}_1^Y/\mathcal{R}_2^Z \le 1$, we get $\Delta\mathcal{M}_n \le 0$, and thus $\mathcal{M}_n$ is a monotone decreasing sequence. Since $\mathcal{M}_n \ge 0$, then there is a limit $\lim_{n\to\infty}\mathcal{M}_n \ge 0$. Therefore, $\lim_{n\to\infty}\Delta\mathcal{M}_n = 0$, which implies that $\lim_{n\to\infty} F_n = \widehat{F}$, $\lim_{n\to\infty} K_n = \widehat{K}$, $\lim_{n\to\infty} S_n = \widehat{S}$, $\lim_{n\to\infty} H_n = \widehat{H}$, and $\lim_{n\to\infty}(\mathcal{R}_1^Y/\mathcal{R}_2^Z - 1)Y_{n+1} = 0$. We have two cases as follows:

- $\mathcal{R}_1^Y/\mathcal{R}_2^Z = 1$, from Eq. (13)

$$0 = \varepsilon e^{-\mu_2\tau_2}\Lambda(\widehat{F},\widehat{H}) + mF_1(\widehat{K}) - aF_2(\widehat{S}) - \lambda F_2(\widehat{S})\lim_{n\to\infty} F_5(Z_{n+1}), \tag{57}$$

  and this gives $\lim_{n\to\infty} Z_n = \widehat{Z}$. Moreover, from Eq. (14) we have

$$0 = \theta e^{-\mu_3\tau_3}F_2(\widehat{S}) - cF_3(\widehat{H}) - dF_3(\widehat{H})\lim_{n\to\infty} F_4(Y_{n+1}), \tag{58}$$

  then we get $\lim_{n\to\infty} Y_n = 0$.
- $\mathcal{R}_1^Y/\mathcal{R}_2^Z < 1$, $\lim_{n\to\infty} Y_n = 0$. From Eq. (57) we get $\lim_{n\to\infty} Z_n = \widehat{Z}$. Then we get that $\widehat{Q}$ is G.A.S. □

*Proof of Theorem 5* Define $\mathcal{V}_n(F_n,K_n,S_n,H_n,Y_n,Z_n)$:

$$
\begin{aligned}
\mathcal{V}_n = {}& \frac{1}{\phi(h)}\Bigg[\gamma\left(F_n - \widetilde{F} - \int_{\widetilde{F}}^{F_n}\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(\varsigma,\widetilde{H})}\,d\varsigma\right) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n - \widetilde{K} - \int_{\widetilde{K}}^{K_n}\frac{F_1(\widetilde{K})}{F_1(\varsigma)}\,d\varsigma\right) \\
&+ e^{-\mu_3\tau_3}\left(S_n - \widetilde{S} - \int_{\widetilde{S}}^{S_n}\frac{F_2(\widetilde{S})}{F_2(\varsigma)}\,d\varsigma\right) \\
&+ \frac{(a+\lambda F_5(\widetilde{Z}))}{\theta}\left(H_n - \widetilde{H} - \int_{\widetilde{H}}^{H_n}\frac{F_3(\widetilde{H})}{F_3(\varsigma)}\,d\varsigma\right) \\
&+ \frac{d(a+\lambda F_5(\widetilde{Z}))}{q\theta}\left(Y_n - \widetilde{Y} - \int_{\widetilde{Y}}^{Y_n}\frac{F_4(\widetilde{Y})}{F_4(\varsigma)}\,d\varsigma\right) \\
&+ \frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_n - \widetilde{Z} - \int_{\widetilde{Z}}^{Z_n}\frac{F_5(\widetilde{Z})}{F_5(\varsigma)}\,d\varsigma\right)\Bigg] \\
&+ \frac{(c+dF_4(\widetilde{Y}))}{\theta}(a+\lambda F_5(\widetilde{Z}))F_3(\widetilde{H})G\left(\frac{F_3(H_n)}{F_3(\widetilde{H})}\right)
\end{aligned}
$$

$$+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_1}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)$$
$$+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_2}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)+\gamma\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_3}^{n-1}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widetilde{S})}\right).$$

Clearly, $\mathcal{V}_n(F_n,K_n,S_n,H_n,Y_n,Z_n)>0$ for all $F_n,K_n,S_n,H_n,Y_n,Z_n>0$ and $\mathcal{V}_n(\widetilde{F},\widetilde{K},\widetilde{S},\widetilde{H},\widetilde{Y},\widetilde{Z})=0$. We compute $\triangle\mathcal{V}_n=\mathcal{V}_{n+1}-\mathcal{V}_n$ as follows:

$$\begin{aligned}
\triangle\mathcal{V}_n={}&\frac{1}{\phi(h)}\Bigg[\gamma\left(F_{n+1}-\widetilde{F}-\int_{\widetilde{F}}^{F_{n+1}}\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(\varsigma,\widetilde{H})}\,d\varsigma\right)\\
&+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1}-\widetilde{K}-\int_{\widetilde{K}}^{K_{n+1}}\frac{\digamma_1(\widetilde{K})}{\digamma_1(\varsigma)}\,d\varsigma\right)\\
&+e^{-\mu_3\tau_3}\left(S_{n+1}-\widetilde{S}-\int_{\widetilde{S}}^{S_{n+1}}\frac{\digamma_2(\widetilde{S})}{\digamma_2(\varsigma)}\,d\varsigma\right)\\
&+\frac{(a+\lambda\digamma_5(\widetilde{Z}))}{\theta}\left(H_{n+1}-\widetilde{H}-\int_{\widetilde{H}}^{H_{n+1}}\frac{\digamma_3(\widetilde{H})}{\digamma_3(\varsigma)}\,d\varsigma\right)\\
&+\frac{d(a+\lambda\digamma_5(\widetilde{Z}))}{q\theta}\left(Y_{n+1}-\widetilde{Y}-\int_{\widetilde{Y}}^{Y_{n+1}}\frac{\digamma_4(\widetilde{Y})}{\digamma_4(\varsigma)}\,d\varsigma\right)\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_{n+1}-\widetilde{Z}-\int_{\widetilde{Z}}^{Z_{n+1}}\frac{\digamma_5(\widetilde{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\Bigg]\\
&+\frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(\widetilde{H})G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(\widetilde{H})}\right)\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_1+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_2+1}^{n}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_3+1}^{n}G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widetilde{S})}\right)\\
&-\frac{1}{\phi(h)}\Bigg[\gamma\left(F_n-\widetilde{F}-\int_{\widetilde{F}}^{F_n}\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(\varsigma,\widetilde{H})}\,d\varsigma\right)+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_n-\widetilde{K}-\int_{\widetilde{K}}^{K_n}\frac{\digamma_1(\widetilde{K})}{\digamma_1(\varsigma)}\,d\varsigma\right)\\
&+e^{-\mu_3\tau_3}\left(S_n-\widetilde{S}-\int_{\widetilde{S}}^{S_n}\frac{\digamma_2(\widetilde{S})}{\digamma_2(\varsigma)}\,d\varsigma\right)+\frac{(a+\lambda\digamma_5(\widetilde{Z}))}{\theta}\left(H_n-\widetilde{H}-\int_{\widetilde{H}}^{H_n}\frac{\digamma_3(\widetilde{H})}{\digamma_3(\varsigma)}\,d\varsigma\right)\\
&+\frac{d(a+\lambda\digamma_5(\widetilde{Z}))}{q\theta}\left(Y_n-\widetilde{Y}-\int_{\widetilde{Y}}^{Y_n}\frac{\digamma_4(\widetilde{Y})}{\digamma_4(\varsigma)}\,d\varsigma\right)\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_n-\widetilde{Z}-\int_{\widetilde{Z}}^{Z_n}\frac{\digamma_5(\widetilde{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\Bigg]\\
&-\frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(\widetilde{H})G\left(\frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})}\right)\\
&-\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\sum_{j=n-m_1}^{n-1}G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H}) \sum_{j=n-m_2}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right) - \gamma\Lambda(\widetilde{F},\widetilde{H}) \sum_{j=n-m_3}^{n-1} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widetilde{S})}\right), \\
\Delta\mathcal{V}_n = {}& \frac{1}{\phi(h)}\left[\gamma\left(F_{n+1} - F_n - \int_{F_n}^{F_{n+1}} \frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(\varsigma,\widetilde{H})}\,d\varsigma\right)\right. \\
&+ \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(K_{n+1} - K_n - \int_{K_n}^{K_{n+1}} \frac{\digamma_1(\widetilde{K})}{\digamma_1(\varsigma)}\,d\varsigma\right) \\
&+ e^{-\mu_3\tau_3}\left(S_{n+1} - S_n - \int_{S_n}^{S_{n+1}} \frac{\digamma_2(\widetilde{S})}{\digamma_2(\varsigma)}\,d\varsigma\right) \\
&+ \frac{(a+\lambda\digamma_5(\widetilde{Z}))}{\theta}\left(H_{n+1} - H_n - \int_{H_n}^{H_n+1} \frac{\digamma_3(\widetilde{H})}{\digamma_3(\varsigma)}\,d\varsigma\right) \\
&+ \frac{d(a+\lambda\digamma_5(\widetilde{Z}))}{q\theta}\left(Y_{n+1} - Y_n - \int_{Y_n}^{Y_{n+1}} \frac{\digamma_4(\widetilde{Y})}{\digamma_4(\varsigma)}\,d\varsigma\right) \\
&+ \left.\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(Z_{n+1} - Z_n - \int_{Z_n}^{Z_{n+1}} \frac{\digamma_5(\widetilde{Z})}{\digamma_5(\varsigma)}\,d\varsigma\right)\right] \\
&+ \frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(\widetilde{H})\left[G\left(\frac{\digamma_3(H_{n+1})}{\digamma_3(\widetilde{H})}\right) - G\left(\frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})}\right)\right] \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\left(\sum_{j=n-m_1+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)\right. \\
&- \left.\sum_{j=n-m_1}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\left(\sum_{j=n-m_2+1}^{n} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right) - \sum_{j=n-m_2}^{n-1} G\left(\frac{\Lambda(F_{j+1},H_j)}{\Lambda(\widetilde{F},\widetilde{H})}\right)\right) \\
&+ \gamma\Lambda(\widetilde{F},\widetilde{H})\left(\sum_{j=n-m_3+1}^{n} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widetilde{S})}\right) - \sum_{j=n-m_3}^{n-1} G\left(\frac{\digamma_2(S_{j+1})}{\digamma_2(\widetilde{S})}\right)\right).
\end{aligned}
$$

Using inequalities (45) and (46) by replacing $F^*$, $H^*$, $\rho^*$ with $\widetilde{F}$, $\widetilde{H}$, $\widetilde{\rho}$, we obtain

$$
\begin{aligned}
\Delta\mathcal{V}_n \le {}& \frac{1}{\phi(h)}\left[\gamma\left(1 - \frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)(F_{n+1} - F_n) + \frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(1 - \frac{\digamma_1(\widetilde{K})}{\digamma_1(K_{n+1})}\right)(K_{n+1} - K_n)\right. \\
&+ e^{-\mu_3\tau_3}\left(1 - \frac{\digamma_2(\widetilde{S})}{\digamma_2(S_{n+1})}\right)(S_{n+1} - S_n) \\
&+ \frac{(a+\lambda\digamma_5(\widetilde{Z}))}{\theta}\left(1 - \frac{\digamma_3(\widetilde{H})}{\digamma_3(H_{n+1})}\right)(H_{n+1} - H_n) \\
&+ \frac{d(a+\lambda\digamma_5(\widetilde{Z}))}{q\theta}\left(1 - \frac{\digamma_4(\widetilde{Y})}{\digamma_4(Y_{n+1})}\right)(Y_{n+1} - Y_n) \\
&+ \left.\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(1 - \frac{\digamma_5(\widetilde{Z})}{\digamma_5(Z_{n+1})}\right)(Z_{n+1} - Z_n)\right] \\
&+ \frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(\widetilde{H})\left[\frac{\digamma_3(H_{n+1})}{\digamma_3(\widetilde{H})} - \frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})} + \ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widetilde{F},\widetilde{H})}-\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(\widetilde{F},\widetilde{H})}\right.\\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widetilde{F},\widetilde{H})}-\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(\widetilde{F},\widetilde{H})}\right.\\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\left[\frac{F_2(S_{n+1})}{F_2(\widetilde{S})}-\frac{F_2(S_{n-m_3+1})}{F_2(\widetilde{S})}+\ln\left(\frac{F_2(S_{n-m_3+1})}{F_2(S_{n+1})}\right)\right].
\end{aligned}
$$

From Eqs. (11)–(16) we have

$$
\begin{aligned}
\Delta\mathcal{V}_n\le{}&\gamma\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\big(\Theta(F_{n+1})-\Lambda(F_{n+1},H_n)\big)\\
&+\frac{me^{-\mu_3\tau_3}}{\alpha+m}\left(1-\frac{F_1(\widetilde{K})}{F_1(K_{n+1})}\right)\big[(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(F_{n-m_1+1},H_{n-m_1})\\
&-(\alpha+m)F_1(K_{n+1})\big]\\
&+e^{-\mu_3\tau_3}\left(1-\frac{F_2(\widetilde{S})}{F_2(S_{n+1})}\right)\big[\varepsilon e^{-\mu_2\tau_2}\Lambda(F_{n-m_2+1},H_{n-m_2})+mF_1(K_{n+1})-aF_2(S_{n+1})\\
&-\lambda F_2(S_{n+1})F_5(Z_{n+1})\big]\\
&+\frac{(a+\lambda F_5(\widetilde{Z}))}{\theta}\left(1-\frac{F_3(\widetilde{H})}{F_3(H_{n+1})}\right)\big[\theta e^{-\mu_3\tau_3}F_2(S_{n-m_3+1})-cF_3(H_{n+1})\\
&-dF_3(H_{n+1})F_4(Y_{n+1})\big]\\
&+\frac{d(a+\lambda F_5(\widetilde{Z}))}{q\theta}\left(1-\frac{F_4(\widetilde{Y})}{F_4(Y_{n+1})}\right)\big[qF_3(H_{n+1})F_4(Y_{n+1})-\eta F_4(Y_{n+1})\big]\\
&+\frac{\lambda e^{-\mu_3\tau_3}}{g}\left(1-\frac{F_5(\widetilde{Z})}{F_5(Z_{n+1})}\right)\big[gF_2(S_{n+1})F_5(Z_{n+1})-\xi F_5(Z_{n+1})\big]\\
&+\frac{(c+dF_4(\widetilde{Y}))}{\theta}(a+\lambda F_5(\widetilde{Z}))F_3(\widetilde{H})\left[\frac{F_3(H_{n+1})}{F_3(\widetilde{H})}-\frac{F_3(H_n)}{F_3(\widetilde{H})}+\ln\left(\frac{F_3(H_n)}{F_3(H_{n+1})}\right)\right]\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widetilde{F},\widetilde{H})}-\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(\widetilde{F},\widetilde{H})}\right.\\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\left[\frac{\Lambda(F_{n+1},H_n)}{\Lambda(\widetilde{F},\widetilde{H})}-\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(\widetilde{F},\widetilde{H})}\right.\\
&\left.+\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\right]\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\left[\frac{F_2(S_{n+1})}{F_2(\widetilde{S})}-\frac{F_2(S_{n-m_3+1})}{F_2(\widetilde{S})}+\ln\left(\frac{F_2(S_{n-m_3+1})}{F_2(S_{n+1})}\right)\right],\\
\Delta\mathcal{V}_n\le{}&\gamma\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widetilde{F})\big)+\gamma\Theta(\widetilde{F})\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\Lambda(F_{n+1},H_n)-\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(F_{n-m_1+1},H_{n-m_1})\frac{\digamma_1(\widetilde{K})}{\digamma_1(K_{n+1})}\\
&+me^{-\mu_3\tau_3}\digamma_1(\widetilde{K})-\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(F_{n-m_2+1},H_{n-m_2})\frac{\digamma_2(\widetilde{S})}{\digamma_2(S_{n+1})}\\
&-me^{-\mu_3\tau_3}\digamma_1(K_{n+1})\frac{\digamma_2(\widetilde{S})}{\digamma_2(S_{n+1})}\\
&-ae^{-\mu_3\tau_3}\digamma_2(S_{n+1})+\lambda e^{-\mu_3\tau_3}\digamma_2(\widetilde{S})\digamma_5(Z_{n+1})+ae^{-\mu_3\tau_3}\digamma_2(\widetilde{S})\\
&+\big(a+\lambda\digamma_5(\widetilde{Z})\big)e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})\\
&-\big(a+\lambda\digamma_5(\widetilde{Z})\big)e^{-\mu_3\tau_3}\digamma_2(S_{n-m_3+1})\frac{\digamma_3(\widetilde{H})}{\digamma_3(H_{n+1})}+\frac{(a+\lambda\digamma_5(\widetilde{Z}))c}{\theta}\digamma_3(\widetilde{H})\\
&+\frac{d(a+\lambda\digamma_5(\widetilde{Z}))}{q\theta}\digamma_3(\widetilde{H})\digamma_4(Y_{n+1})-\frac{d(a+\lambda\digamma_5(\widetilde{Z}))\eta}{q\theta}\digamma_4(Y_{n+1})\\
&+\frac{d(a+\lambda\digamma_5(\widetilde{Z}))\eta}{q\theta}\digamma_4(\widetilde{Y})-\lambda e^{-\mu_3\tau_3}\digamma_5(\widetilde{Z})\digamma_2(S_{n+1})-\frac{\lambda e^{-\mu_3\tau_3}\xi}{g}\digamma_5(Z_{n+1})\\
&+\frac{\lambda e^{-\mu_3\tau_3}\xi}{g}\digamma_5(\widetilde{Z})\\
&-\frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(H_n)\\
&+\frac{(c+d\digamma_4(\widetilde{Y}))}{\theta}\big(a+\lambda\digamma_5(\widetilde{Z})\big)\digamma_3(\widetilde{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right)\\
&+\frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right)\\
&+\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right)\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\left(\frac{\digamma_2(S_{n+1})}{\digamma_2(\widetilde{S})}-\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(\widetilde{S})}+\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right)\right).
\end{aligned}
$$

Using the conditions of $\widetilde{Q}$

$$
\begin{aligned}
&\Theta(\widetilde{F})=\Lambda(\widetilde{F},\widetilde{H}),\\
&(1-\varepsilon)e^{-\mu_1\tau_1}\Lambda(\widetilde{F},\widetilde{H})=(\alpha+m)\digamma_1(\widetilde{K}),\\
&\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})+me^{-\mu_3\tau_3}\digamma_1(\widetilde{K})=\gamma\Lambda(\widetilde{F},\widetilde{H})=\big(a+\lambda\digamma_5(\widetilde{Z})\big)e^{-\mu_3\tau_3}\digamma_2(\widetilde{S}),\\
&\theta e^{-\mu_3\tau_3}\digamma_2(\widetilde{S})=\big(c+d\digamma_4(\widetilde{Y})\big)\digamma_3(\widetilde{H}),\\
&\digamma_2(\widetilde{S})=\frac{\xi}{g},\qquad \digamma_3(\widetilde{H})=\frac{\eta}{q},
\end{aligned}
$$

we get

$$
\begin{aligned}
\Delta\mathcal{V}_n\le{}&\gamma\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widetilde{F})\big)+\gamma\Lambda(\widetilde{F},\widetilde{H})\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\widetilde{H})}
\end{aligned}
$$

$$\begin{aligned}
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widetilde{K})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_1(K_{n+1})} \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H}) \\
&- \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\widetilde{S})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_2(S_{n+1})} \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\frac{\digamma_1(K_{n+1})\digamma_2(\widetilde{S})}{\digamma_1(\widetilde{K})\digamma_2(S_{n+1})} + \gamma\Lambda(\widetilde{F},\widetilde{H}) \\
&- \gamma\Lambda(\widetilde{F},\widetilde{H})\frac{\digamma_3(\widetilde{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widetilde{S})} \\
&+ \gamma\Lambda(\widetilde{F},\widetilde{H}) - \gamma\Lambda(\widetilde{F},\widetilde{H})\frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})} \\
&+ \gamma\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\digamma_3(H_n)}{\digamma_3(H_{n+1})}\right) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})}{\Lambda(F_{n+1},H_n)}\right) \\
&+ \gamma\Lambda(\widetilde{F},\widetilde{H})\ln\left(\frac{\digamma_2(S_{n-m_3+1})}{\digamma_2(S_{n+1})}\right) \\
= {}& \gamma\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widetilde{F})\big) \\
&+ \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\Bigg[5-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})} \\
&- \frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widetilde{K})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_1(K_{n+1})} - \frac{\digamma_1(K_{n+1})\digamma_2(\widetilde{S})}{\digamma_1(\widetilde{K})\digamma_2(S_{n+1})} \\
&- \frac{\digamma_3(\widetilde{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widetilde{S})} - \frac{\Lambda(F_{n+1},\widetilde{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widetilde{H})} \\
&+ \ln\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\Bigg] \\
&+ \varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\Bigg[4-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})} - \frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\widetilde{S})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_2(S_{n+1})} \\
&- \frac{\digamma_3(\widetilde{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widetilde{S})} \\
&- \frac{\Lambda(F_{n+1},\widetilde{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widetilde{H})} + \ln\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(S_{n-m_3+1})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_2(S_{n+1})\digamma_3(H_{n+1})}\right)\Bigg] \\
&+ \gamma\Lambda(\widetilde{F},\widetilde{H})\left[-1+\frac{\Lambda(F_{n+1},\widetilde{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widetilde{H})} + \frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\widetilde{H})} - \frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})}\right],
\end{aligned} \tag{59}$$

$$\begin{aligned}
\Delta\mathcal{V}_n \le {}& \gamma\left(1-\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\big(\Theta(F_{n+1})-\Theta(\widetilde{F})\big) \\
&- \frac{m(1-\varepsilon)e^{-\mu_1\tau_1-\mu_3\tau_3}}{\alpha+m}\Lambda(\widetilde{F},\widetilde{H})\Bigg[G\left(\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)
\end{aligned}$$

$$
\begin{aligned}
&+G\left(\frac{\Lambda(F_{n-m_1+1},H_{n-m_1})\digamma_1(\widetilde{K})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_1(K_{n+1})}\right)+G\left(\frac{\digamma_1(K_{n+1})\digamma_2(\widetilde{S})}{\digamma_1(\widetilde{K})\digamma_2(S_{n+1})}\right)\\
&+G\left(\frac{\digamma_3(\widetilde{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widetilde{S})}\right)\\
&+G\left(\frac{\Lambda(F_{n+1},\widetilde{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widetilde{H})}\right)\Bigg]-\varepsilon e^{-\mu_2\tau_2-\mu_3\tau_3}\Lambda(\widetilde{F},\widetilde{H})\Bigg[G\left(\frac{\Lambda(\widetilde{F},\widetilde{H})}{\Lambda(F_{n+1},\widetilde{H})}\right)\\
&+G\left(\frac{\Lambda(F_{n-m_2+1},H_{n-m_2})\digamma_2(\widetilde{S})}{\Lambda(\widetilde{F},\widetilde{H})\digamma_2(S_{n+1})}\right)+G\left(\frac{\digamma_3(\widetilde{H})\digamma_2(S_{n-m_3+1})}{\digamma_3(H_{n+1})\digamma_2(\widetilde{S})}\right)\\
&+G\left(\frac{\Lambda(F_{n+1},\widetilde{H})\digamma_3(H_n)}{\Lambda(F_{n+1},H_n)\digamma_3(\widetilde{H})}\right)\Bigg]\\
&+\gamma\Lambda(\widetilde{F},\widetilde{H})\left(1-\frac{\Lambda(F_{n+1},\widetilde{H})}{\Lambda(F_{n+1},H_n)}\right)\left(\frac{\Lambda(F_{n+1},H_n)}{\Lambda(F_{n+1},\widetilde{H})}-\frac{\digamma_3(H_n)}{\digamma_3(\widetilde{H})}\right).
\end{aligned}
$$

Using Conditions C1–C4, we get that the first and last terms of Eq. (59) are less than or equal to zero. Thus, $\mathcal{V}_n$ is a monotone decreasing sequence. Since $\mathcal{V}_n \geq 0$, then there is a limit $\lim_{n\to\infty}\mathcal{V}_n \geq 0$. Therefore, $\lim_{n\to\infty}\Delta\mathcal{V}_n = 0$, which implies that $\lim_{n\to\infty}F_n = \widetilde{F}$, $\lim_{n\to\infty}K_n = \widetilde{K}$, $\lim_{n\to\infty}S_n = \widetilde{S}$, $\lim_{n\to\infty}H_n = \widetilde{H}$. From Eqs. (13) and (14) we have

$$
\begin{aligned}
0 &= \varepsilon e^{-\mu_2\tau_2}\Lambda(\widetilde{F},\widetilde{H}) + m\digamma_1(\widetilde{K}) - a\digamma_2(\widetilde{S}) - \lambda\digamma_2(\widetilde{S})\lim_{n\to\infty}\digamma_5(Z_{n+1}),\\
0 &= \theta e^{-\mu_3\tau_3}\digamma_2(\widetilde{S}) - c\digamma_3(\widetilde{H}) - d\digamma_2(\widetilde{H})\lim_{n\to\infty}\digamma_4(Y_{n+1}),
\end{aligned}
$$

then $\lim_{n\to\infty}Y_n = \widetilde{Y}$ and $\lim_{n\to\infty}Z_n = \widetilde{Z}$. Then we get $\widetilde{Q}$ is G.A.S. □

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**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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