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Using of PQWs for solving NFID in the complex plane

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Abstract

We approximate the solution of the nonlinear Fredholm integro-differential equation (NFID) in the complex plane by periodic quasi-wavelets (PQWs). This kind of wavelets possesses orthonormality properties, the numbers of terms in the decomposition and reconstruction formulas are strictly limited, and the localization is not emphasized. To the best of our knowledge, there are no numerical methods to obtain the solution of the NFID by PQWs. Here, we attempt to obtain the numerical solution of the NFID based on B-spline functions. Finally, the simulation results are shown for three examples.

Keywords: Nonlinear integro Fredholm integral equation; PQWs; B-spline functions; complex plane

1 Introduction

In this paper, we use a kind of wavelet playing a key role in solving integral equations, which is named the periodic quasi-wavelets (PQWs) based on B-spline functions that approximate smooth functions very well. Some researchers focused on investigating PQWs and approximating of Fredholm integral equation [1, 2] and mixed Volterra–Fredholm integral [3]. The aim of this study is to present a numerical method for approximating the nonlinear Fredholm integro-differential equation (NFIDE) defined as the following form:

$$\omega'(x) = \mu(x) + \lambda \int_0^T R\bigl(x, t, \omega(t)\bigr)\, dt, \quad \lambda \in \mathbb{R}, 0 \le x \le T, \tag{1}$$

where $\omega(x)$ is an unknown complex function to be found, $\mu(x) : [0, T] \to \mathbb{C}$ and $R(x, t, \omega(t)) : [0, T]^2 \times \mathbb{C} \to \mathbb{C}$ are continuous and Lipschitzian periodic functions such that

$$\bigl|R\bigl(x, t, \omega_1(t)\bigr) - R\bigl(x, t, \omega_2(t)\bigr)\bigr| \le M\bigl|\omega_1(t) - \omega_2(t)\bigr|,$$

where M is a Lipschitz constant.

We approximate the solution of NFIDE by using the B-spline basis functions and applying the iterative method [4] in each iteration on the complex plane.

Every integro-differential equation (IDE) is an ordinary differential equation in which one of the variables is integral. There are many equations in mathematical modeling, such as Maxwell's equations, biological, radiative energy, engineering problems, potential theory, and transfers problems of oscillations that can be formulated by this equation and fractional integro-differential equations; see [5–7].

Some numerical algorithms that discuss the approximation of the solution of IDE can be listed such as the nonsmooth initial data arising method [8], Haar and RH methods [9–11], cubic B-spline finite element method [12], Runge–Kutta–Nystrom methods [13, 14], and high-rank constant terms [15]. Furthermore, in [16, 17], by using a system of Cauchy type and numerical method with graded meshes, singular integral equations were solved.

This article is organized as follows. Section 2 contains the notation and some properties of B-spline and PQWs. Then, in Sect. 3, we formulate a problem and approximate the solution for the NFIDE in the complex plane. In Sect. 4, we analyze the error of the suggested approach. In fact, we investigate the convergence analysis in that section. Finally, in Sect. 5, we illustrate the proposed methodology in numerical examples. We conclude our work in Sect. 6.

2 Preliminaries

We can define the B-spline $B_i^n(x)$ as follows:

$$B_i^n(x) = \frac{1}{h^n n!} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (x - y_{i+j})_+^n, \tag{2}$$

where $y_j = y_0 + jh$, $y_0 = -\frac{(n+1)h}{2}$, $j = 1, 2, \dots$, and

$$(x-y)_+^n = \begin{cases} x - y, & x \geq y, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

and $n \in \mathbb{N}$ denotes the degree of splines. If we use y_j^m instead of y_j and h_m instead of h, then the family of $\{y_j^m\}_{j\in\mathbb{Z}}$ will be denoted by $S_n(h_m)$, where the length of step is h_m and

$$k_m = 2^m p, \qquad h_m = \frac{T}{k_m}, \qquad T = hp, \quad p \geq k+1, m \in \mathbb{N}.$$

Definition 2.1 ([18]) The periodic B-spline is defined by

$$B_0^{\circ n,m}(x) = B_0^n(x, h_m) = k_m^n \sum_{l\in\mathbb{Z}} \left(\frac{\sin(\frac{l\pi}{k_m})}{l\pi} \right)^{n+1} \exp\left(\frac{i2\pi lx}{T} \right). \tag{4}$$

Definition 2.2 ([18]) The functions $\{A_r^{n,m}(x)\}_{r=0}^{k_m-1}$ are defined by

$$A_r^{n,m}(x) = C_r^{n,m} \sum_{l=0}^{k_m-1} \exp\left(\frac{i2\pi lr}{k_m} \right) B_0^{\circ n,m}(x - lh_m), \quad x \in \mathbb{R}, \tag{5}$$

where

$$C_r^{n,m} = \left\{ t_0 + 2 \sum_{\lambda=1}^{n} t_\lambda \cos\left(\frac{2\pi\lambda r h_m}{T} \right) \right\}^{-1/2} \tag{6}$$

and

$$t_\lambda = B_0^{\circ 2n+1}(\lambda, 1). \tag{7}$$

The functions $\{A_r^{n,m}(x)\}_{r=0}^{k_m-1}$ are an orthonormal basis for $S_n^{\sim}(h_m)$.

Also, $S_n^{\sim}(h_m)$ is a class of periodic spline functions in $S_n(h_m)$, which is a set of polynomials of degree n such as $f \in C^{n-1}[0,T]$, on each interval $[y_j^m, y_j^m + h_m]$ that $j = 0, 1, \ldots, k_m - 1$ and

$$S^i(0) = S^i(T), \quad i = 0, 1, \ldots, n-1.$$

Also, we can rewrite $A_r^{n,m}(x)$ by using the Fourier expansion, so we have

$$A_r^{n,m}(x) = C_r^{n,m} {k_m}^{n+1} \sum_{\lambda \in \mathbb{Z}} \left(\frac{\sin(r\pi/k_m)}{(r + \lambda k_m)\pi} \right)^{n+1} \exp\left(\frac{i2\pi(r + \lambda k_m)x}{T} \right). \tag{8}$$

Definition 2.3 ([19]) Let $m \in \mathbb{N}$. Then we define V_m and W_m as two spaces of functions as follows:

$$V_m = S_n^{\sim}(h_m), \qquad W_m = V_{m+1} - V_m. \tag{9}$$

Definition 2.4 ([19]) The function $D_r^{n,m}(x) \in W_m$ is defined by

$$D_r^{n,m}(x) := -b_r^{n,m+1} A_r^{n,m+1}(x) + a_r^{n,m+1} A_{r+k_m}^{n,m+1}(x), \quad x \in \mathbb{R}, \tag{10}$$

where

$$a_r^{n,m+1} = \frac{C_r^{n,m}}{C_r^{n,m+1}} \left(\cos \frac{r\pi}{k_{m+1}} \right)^{n+1}, \tag{11}$$

$$b_r^{n,m+1} = \frac{C_r^{n,m}}{C_{r+k_m}^{n,m+1}} \left(\sin \frac{r\pi}{k_{m+1}} \right)^{n+1}, \tag{12}$$

and

$$\begin{aligned} &\langle D_{r_1}^{n,m}, D_{r_2}^{n,m} \rangle = \delta_{r_1,r_2} \quad \text{for } r_1, r_2 = 0, \ldots, k_m - 1, \\ &\langle D_{r_1}^{n,m}, A_{r_2}^{n,m} \rangle = 0 \quad \text{for } 0 \le r_1, r_2 \le k_m - 1, \end{aligned} \tag{13}$$

which is called the periodic quasi-wavelet.

3 Approximation of the solutions of NFIDE

Integrating Eq. (1) from 0 to x yields

$$\omega(x) = \omega(0) + \int_0^x \mu(s)\,ds + \alpha \int_0^x \int_0^T R\bigl(s, t, \omega(t)\bigr)\,dt\,ds. \tag{14}$$

Moreover, in Banach spaces, we present a continuous integral operator P such that the Banach fixed point theorem guarantees that P has a unique fixed point; see [20]. That means that the NFIDE has exactly one solution. Let P be a contraction map and let P be defined for Eq. (14) as

$$P\omega(x) = \omega(0) + \int_0^x \mu(s)\,ds + \alpha \int_0^x \int_0^T R\bigl(s,t,\omega(t)\bigr)\,dt\,ds. \tag{15}$$

According to Eqs. (14) and (15), for every $x, s \in [0,T]$ in the $(k+1)$th iteration, we have

$$\omega_{k+1}(x) = \omega(0) + \int_0^x \mu(s)\,ds + \alpha \int_0^x \int_0^T R\bigl(s,t,\omega_k(t)\bigr)\,dt\,ds. \tag{16}$$

We define the function $\psi_m(s,t)$ as

$$\psi_m(s,t) = R\bigl(s,t,\omega_m(t)\bigr). \tag{17}$$

We assume that $Q_m \in V_m$ is an orthogonal projection. Using Definition 2.4, Eqs. (8) and (10), and the interpolation property, we have

$$Q_m(\psi)(s,t) = R\left(s,t,\sum_{r=0}^{K_m-1} \alpha_r^m A_r^{n,m}(t)\right)$$

or

$$Q_m(\psi)(s,t) = R\left(s,t,\sum_{r=0}^{K_m-1} \beta_r^m D_r^{n,m}(t)\right), \tag{18}$$

where

$$\begin{pmatrix} \alpha^{m+1} \\ \beta^{m+1} \end{pmatrix} = \begin{pmatrix} L_{m+1}^T & H_{m+1}^T \end{pmatrix} \begin{pmatrix} \alpha^m \\ \beta^m \end{pmatrix},$$

and L_m and H_m are, respectively, given as follows:

$$L_m = \begin{pmatrix} a_0^{n,m} & 0 & 0 & \cdots & 0 \\ 0 & a_1^{n,m} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k_m-1}^{n,m} \end{pmatrix},$$

$$H_m = \begin{pmatrix} b_0^{n,m} & 0 & 0 & \cdots & 0 \\ 0 & b_1^{n,m} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{k_m-1}^{n,m} \end{pmatrix},$$

and

$$\alpha^m = \bigl(\alpha_0^m, \ldots, \alpha_{k_m-1}^m\bigr)^T, \qquad \beta^m = \bigl(\beta_0^m, \ldots, \beta_{k_m-1}^m\bigr)^T.$$

Thus

$$\Omega_{k+1}(x)=\Omega(0)+\int_0^x \mu(s)\,ds+\alpha\int_0^x\int_0^T Q_m\bigl(\psi_k(s,t)\bigr)\,dt\,ds,\quad k=1,2,\ldots. \tag{19}$$

We can approximate the integral of any function of ω on $[-1,1]$ by $L_{M+1}(\xi)$ (the Legendre polynomial of order $M+1$) as

$$\int_{-1}^1 \omega(\xi)\,d\xi \simeq \sum_{j=0}^M \gamma_j\omega(\xi_j), \tag{20}$$

where $\{\xi_j\}_{j=0}^M$ are the zeros of Legendre polynomial of order $M+1$ on $[-1,1]$ and

$$\gamma_j=\frac{2}{(1-\xi_j^2)[L'_{M+1}(\xi_j)]^2},\quad j=0,1,\ldots,M. \tag{21}$$

By changing the variable $t=\frac{T}{2}(\xi+1)$, it can be written as

$$\begin{aligned}\Omega_{m+1}(x)&=\Omega(0)+\int_0^x \mu(s)\,ds\\&\quad+\frac{T\alpha}{2}\int_0^x\left(\int_{-1}^1 R\left(s,\frac{T}{2}(\xi+1),\sum_{r=0}^{K_m-1}\alpha_r^m A_r^{n,m}\left(\frac{T}{2}(\xi+1)\right)\right)d\xi\right)ds.\end{aligned}$$

Applying (20) implies that

$$\begin{aligned}\Omega_{m+1}(x)&\simeq\Omega(0)+\int_0^x \mu(s)\,ds\\&\quad+\frac{T\alpha}{2}\int_0^x\sum_{j=0}^M\gamma_j R\left(s,\frac{T}{2}(\xi_j+1),\sum_{r=0}^{K_m-1}\alpha_r^m A_r^{n,m}\left(\frac{T}{2}(\xi_j+1)\right)\right)ds\end{aligned}$$

or that

$$\begin{aligned}\Omega_{m+1}(x)&\simeq\Omega(0)+\int_0^x \mu(s)\,ds\\&\quad+\frac{T\alpha}{2}\sum_{j=0}^M\gamma_j\int_0^x R\left(s,\frac{T}{2}(\xi_j+1),\sum_{r=0}^{K_m-1}\alpha_r^m A_r^{n,m}\left(\frac{T}{2}(\xi_j+1)\right)\right)ds.\end{aligned} \tag{22}$$

By changing the variable $s=\frac{x}{2}(\tau+1)$ in (22), we have

$$\begin{aligned}&\Omega_{m+1}(x)\\&\quad\simeq\Omega(0)+\int_0^x \mu(s)\,ds\\&\quad\quad+\frac{T\alpha x}{4}\sum_{j=0}^M\gamma_j\int_{-1}^1(\tau+1)R\left(\frac{x(\tau+1)}{2},\frac{T}{2}(\xi_j+1),\sum_{r=0}^{K_m-1}\alpha_r^m A_r^{n,m}\left(\frac{T}{2}(\xi_j+1)\right)\right)d\tau,\end{aligned}$$

and by applying (20) again, we have

$$\begin{aligned}
&\Omega_{m+1}(x)\\
&\quad\simeq \Omega(0)+\int_0^x \mu(s)\,ds\\
&\qquad+\frac{T\alpha x}{4}\sum_{j=0}^{M}\gamma_j\sum_{i=0}^{M}\lambda_i(\tau_i+1)\\
&\qquad\times R\left(\frac{x(\tau_i+1)}{2},\frac{T}{2}(\xi_j+1),\sum_{r=0}^{K_m-1}\alpha_r^m A_r^{n,m}\left(\frac{T}{2}(\xi_j+1)\right)\right),
\end{aligned}\tag{23}$$

where

$$\gamma_j=\frac{2}{(1-\xi_j^2)[L'_{M+1}(\xi_j)]^2},\qquad \lambda_i=\frac{2}{(1-\tau_i^2)[L'_{M+1}(\tau_i)]^2}\quad i,j=0,1,\dots,M.\tag{24}$$

4 Convergence analysis and error estimates

In this section, we discuss the convergence and compute the order of convergence of (1) by using the following lemma and theorem.

Lemma 4.1 *Let $R(s,t,\omega(s)):[0,T]\times[0,T]\to\mathbb{C}$ be a continuous and Lipschitzian function such that*

$$\bigl|R\bigl(s,t,\omega_1(t)\bigr)-R\bigl(s,t,\omega_2(t)\bigr)\bigr|\le M\bigl|\omega_1(t)-\omega_2(t)\bigr|,$$

where M is a Lipschitz constant. Then P defined in (15) has a unique fixed point and

$$\bigl\|\omega-P^n(\omega_0)\bigr\|_\infty\le\bigl\|P(\omega_0)-\Omega_0\bigr\|_\infty\sum_{j=n}^{\infty}\beta^j,\tag{25}$$

for all $\omega_0\in C([0,T])$, where $\beta=|\alpha|M<1$.

Proof Applying (15) gives

$$\begin{aligned}
\bigl|P\omega_1(x)-P\omega_2(x)\bigr|&=\left|\alpha\int_0^x\int_0^T\bigl(R\bigl(s,t,\omega_1(t)\bigr)-R\bigl(s,t,\omega_2(t)\bigr)\bigr)\,dt\,ds\right|\\
&\le|\alpha|\int_0^x\int_0^T\bigl|R\bigl(s,t,\omega_1(t)\bigr)-R\bigl(s,t,\omega_2(t)\bigr)\bigr|\,dt\,ds\\
&\le|\alpha|\int_0^x\int_0^T M\bigl|\omega_1(t)-\omega_2(t)\bigr|\,ds\,dx\\
&\le|\alpha|M\bigl\|\omega_1(t)-\omega_2(t)\bigr\|_\infty.
\end{aligned}$$

Thus

$$\bigl|P\omega_1(x)-P\omega_2(x)\bigr|\le|\alpha|M\bigl\|\omega_1(x)-\omega_2(x)\bigr\|_\infty.\tag{26}$$

Induction on $n \in \mathbb{N}$ implies that

$$\left\| P^n \omega_1 - P^n \omega_2 \right\|_\infty \leq \left(|\alpha| M \right)^n \|\omega_1 - \omega_2\|_\infty.$$

If we set $\beta = |\alpha| M < 1$, then

$$\sum_{n=1}^{\infty} \left\| P^n \omega_1 - P^n \omega_2 \right\|_\infty < \infty.$$

Thus, P has a unique fixed point, which means that Eq. (12) has a unique solution. □

Theorem 4.2 *Assume that* $\psi_{i-1} \in \mathbb{C}([0,T]^2)$, *that* $\{\omega_i\}_{i \geq 1}$ *is a subset of* $C([0,T])$, *and that* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i > 0$ *for* $i \geq 1$. *Then*

$$\|\omega - \Omega_i\|_\infty \leq \left\| P(\omega_0) - \omega_0 \right\|_\infty \sum_{j=i}^{\infty} \beta^j + \sum_{j=1}^{i} \beta^{i-j} \varepsilon_j.$$

Proof Let

$$\begin{aligned} \left\| P(\omega_{i-1}) - \Omega_i \right\|_\infty &\leq |\alpha| \left\| \int_0^x \int_0^T \psi_{i-1}(s,t) - Q_m(\psi_{i-1})(s,t)\, dt\, ds \right\|_\infty \\ &\leq |\alpha| \left\| \psi_{i-1} - Q_m(\psi_{i-1}) \right\|_\infty. \end{aligned} \tag{27}$$

Suppose that

$$L_{i-1} = \max\left\{ \left\| \frac{\partial \psi_{i-1}}{\partial t} \right\|_\infty, \left\| \frac{\partial \psi_{i-1}}{\partial s} \right\|_\infty \right\} \tag{28}$$

for $i = 1, 2, \ldots$. Since L_{i-1} is uniformly bounded, we have $|L_{i-1}| \leq \xi$ for any ξ. We set $g(x,s) := \psi_{i-1} - Q_m(\psi_{i-1})$,

$$\begin{aligned} x_l &= \frac{1}{2^{n_1+1}} + \frac{\nu_1}{2^{n_1}}, \quad l = 2^{n_1} + \nu_1, n_1, n_2 \geq 1, \\ s_j &= \frac{1}{2^{n_2+1}} + \frac{\nu_2}{2^{n_2}}, \quad j = 2^{n_2} + \nu_2, s_0 = x_0 = 0. \end{aligned}$$

Applying the interpolating property and the mean-value theorem implies

$$\begin{aligned} &\left\| \psi_{i-1} - Q_m(\psi_{i-1}) \right\|_\infty \\ &\quad = \left\| g(x_l, s_j) + \frac{\partial g}{\partial x}(\xi, \gamma)(\xi - x_l) + \frac{\partial g}{\partial s}(\xi, \gamma)(\gamma - s_j) \right\|_\infty \\ &\quad = \left\| (I - Q_m) \frac{\partial \psi_{i-1}}{\partial x}(\xi, \gamma) + (I - Q_m) \frac{\partial \psi_{i-1}}{\partial s}(\xi, \gamma) \right\|_\infty \\ &\qquad \times \max\left\{ \|\xi - x_l\|_\infty, \|\gamma - s_j\|_\infty \right\} \\ &\quad \leq \frac{2}{2^i} \left\| (I - Q_m) \right\|_\infty \left\| \frac{\partial \psi_{i-1}}{\partial x}(\xi, \gamma) + \frac{\partial \psi_{i-1}}{\partial s}(\xi, \gamma) \right\|_\infty. \end{aligned}$$

So, we have

$$\left\| \psi_{i-1} - Q_m(\psi_{i-1}) \right\|_\infty \le |\alpha| \frac{4L_{i-1}}{2^i}.$$

Therefore, inequality (27) can be expressed as follows:

$$\left\| P(\omega_{i-1}) - \Omega_i \right\|_\infty \le |\alpha| \frac{4L_{i-1}}{2^i}.$$

If

$$|\alpha| \frac{4L_{k-1}}{2^k} < \varepsilon_k, \quad k = 1, 2, \ldots, i,$$

and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i > 0$ for $i \ge 1$, then

$$\left\| P(\omega_{i-1}) - \Omega_i \right\|_\infty < \varepsilon_i. \tag{29}$$

Applying the triangle inequality, we achieve

$$\|\omega - \Omega_i\|_\infty \le \left\| \omega - P^i(\omega_0) \right\|_\infty + \sum_{j=1}^{i} \beta^j \left\| P(\omega_{j-1}) - \omega_j \right\|_\infty.$$

By using (21) and (25) and Lemma 4.1, we have

$$\|\omega - \Omega_i\|_\infty \le \left\| P(\omega_0) - \Omega_0 \right\|_\infty \sum_{j=i}^{\infty} \beta^j + \sum_{j=1}^{i} \beta^{i-j} \varepsilon_j. \tag{30}$$

If we set $\beta = \frac{1}{2} - \frac{1}{2^{l+1}} < \frac{1}{2}$ at the geometric series

$$\sum_{j=n}^{\infty} \beta^j = \frac{\beta^n}{1-\beta}$$

and

$$\|\omega - \Omega_i\|_\infty \le \left\| P(\omega_0) - \Omega_0 \right\|_\infty \frac{\beta^n}{1-\beta} + \sum_{j=1}^{i} \left(\frac{1}{2} - \frac{1}{2^{l+1}} \right)^{i-j} \frac{4|\alpha| L_{j-1}}{2^j}, \tag{31}$$

then from (31) and (28), we have

$$\begin{aligned} \|\omega - \Omega_i\|_\infty &\le \left\| P(\omega_0) - \Omega_0 \right\|_\infty \frac{\beta^n}{1-\beta} + \sum_{j=1}^{i} \left(\frac{1}{2} - \frac{1}{2^{l+1}} \right)^{i-j} \frac{4|\alpha| L_{j-1}}{2^j} \\ &= \left\| P(\omega_0) - \Omega_0 \right\|_\infty \frac{\beta^n}{1-\beta} + 4\xi |\alpha| \beta^n \sum_{j=1}^{n} \left(\frac{1}{2} - \frac{1}{2^{l+1}} \right)^{-j} \frac{1}{2^j} \\ &= \left\| P(\omega_0) - \Omega_0 \right\|_\infty \frac{\beta^n}{1-\beta} + 4\xi |\alpha| \beta^n \sum_{j=1}^{n} \left(1 + \frac{1}{2^l - 1} \right)^j. \end{aligned} \tag{32}$$

Since $(1 + \frac{1}{2^l-1}) \le 2$ for any $l \in \mathbb{N}$, inequality (32) implies

$$\begin{aligned}\|\omega - \Omega_i\|_\infty &\le \|P(\omega_0) - \Omega_0\|_\infty \frac{\beta^n}{1-\beta} + 4\xi\beta^n|\alpha|\sum_{j=1}^{n} 2^j \\ &\le \|P(\omega_0) - \Omega_0\|_\infty \frac{\beta^n}{1-\beta} + 4\xi\beta^n|\alpha|n2^n. \end{aligned} \tag{33}$$

Since $\beta < \frac{1}{4}$, we have

$$\|\omega - \Omega_i\|_\infty \le 4\xi|\alpha|n(2\beta)^n. \tag{34}$$

Therefore the order of convergence is $O(n(2\beta)^n)$. □

5 Numerical results

In this section, we consider three examples to demonstrate the efficiency of the PQWs based on B-spline functions. In fact, using Eqs. (8) and (15), we define the absolute error for nodes

$$x_i = \frac{2i\pi}{k_m} \quad \text{for } i = 0, 1, \dots, k_m - 1.$$

The corresponding computations are performed by Maple 18 software on a Intel core i7 Duo processor 2.4 GHz and 8 GB memory.

So far, to the best of our knowledge, no researcher has yet been attempted to solve this integral equation by PQWs. Thus we use the rational Haar (RH) wavelet method for comparing results of the solution of integral equations. First we apply the change of the variable $t = \frac{x}{2\pi}$ and the interval of integral changes to $[0, 1]$. Then, by using of $m = 4$ or a 2^5 Haar wavelet basis, we approximate the solution of integral equations.

Example 5.1 Consider the NFID of the second kind as

$$\omega'(x) = \mu(x) + \int_0^{2\pi} \sin(x+t)\omega^2(t)\,dt. \tag{35}$$

The exact solution of (35) is

$$\omega(x) = 5\cos(x) + 2\sin(4x) + i\bigl(5\sin(x) + 2\cos(4x)\bigr).$$

The absolute error for $m = 2, 4$ with different values of node $x_i = \frac{2i\pi}{k_m}$ for $i = 0, 1, \dots, k_m - 1$, is shown in Table 1. Moreover, their running time is 1.250 and 11.890 seconds, respectively. Also, in Fig. 1, we compare the numerical solution and the exact solution, and in Fig. 2, the absolute errors of Example 5.1 are depicted.

Example 5.2 Consider the NFID of the second kind as

$$\omega'(x) = \mu(x) + \int_0^{2\pi} \sin\bigl(\omega(t) + ix\bigr)\cos(x+t)\,dt. \tag{36}$$

Then the exact solution of (36) is $\omega(x) = -\cos(2x) + i\sin(2x)$.

Table 1 Numerical results of Example 5.1

x_i	$m = 2$	$m = 4$	2^5 Haar wavelet basis
0.523	9.88E–8	1.55E–11	3.62E–6
1.047	1.92E–7	3.00E–11	8.90E–5
2.094	3.34E–7	5.20E–11	4.88E–4
3.141	3.79E–7	6.02E–11	9.83E–3
4.188	3.22E–7	5.23E–11	7.45E–3
5.235	1.86E–7	3.03E–11	5.40E–3
5.759	9.75E–8	1.57E–11	4.97E–3
CPU-Time (s)	1.250	11.890	15

Figure 1 Comparison between the exact and numerical solution for $m = 4$ of Example 5.1

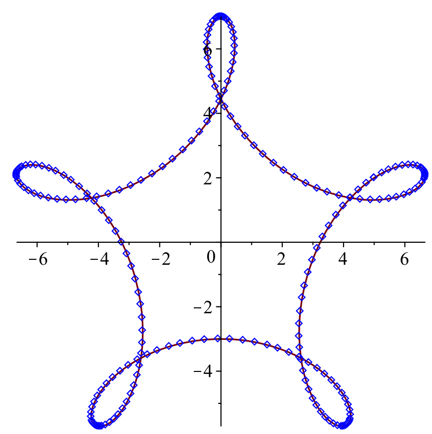
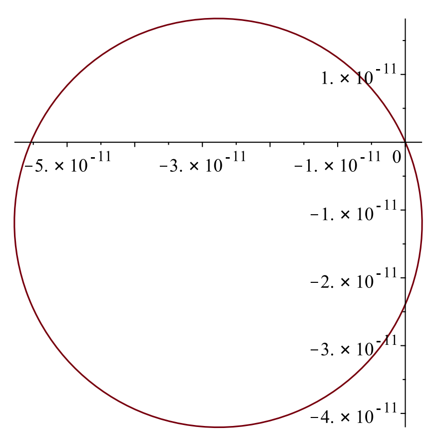


Figure 2 Plot of the absolute errors for $m = 4$ of Example 5.1



For different values of x_i, $i = 1, 2, \ldots, k_m - 1$, in Table 2, the absolute errors for $m = 2, 4$ are given. Comparison between the numerical and exact solution and absolute errors of Example 5.2 are shown in Figs. 3 and 4, respectively.

Example 5.3 Consider the NFID of the second kind

$$\omega'(x) = \mu(x) + \int_0^{2\pi} \sin(t)\bigl(11 + \sin(x)\bigr)\omega^2(t)\,dt. \tag{37}$$

Table 2 Numerical results of Example 5.2

x_i	$m = 2$	$m = 4$	2^5 Haar wavelet basis
0.523	1.56E–8	4.44E–15	6.60E–6
1.047	2.70E–8	6.16E–15	8.93E–6
2.094	1.74E–8	9.04E–15	3.94E–5
3.141	1.01E–7	7.61E–14	8.21E–5
4.188	3.36E–7	1.72E–13	3.41E–4
5.235	2.31E–7	2.89E–13	5.30E–4
5.759	9.31E–7	7.97E–13	3.92E–4
CPU-Time (s)	3.610	17.922	17.45

Figure 3 The plot of comparison between the exact and numerical solution for $m = 4$ of Example 5.2

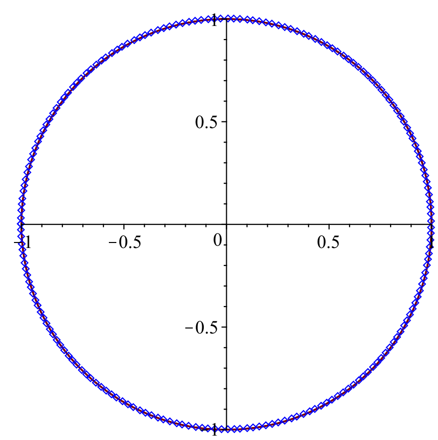
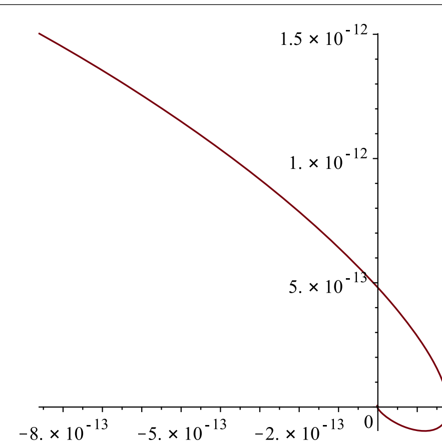


Figure 4 Plot of the absolute errors for $m = 4$ of Example 5.2



In this example, we choose the exact solution as

$$\omega(x) = 3\cos(x)\cos(4x) + 3i\sin(x)\sin(4x).$$

Similar to the previous examples, the absolute errors are shown in Table 3. Furthermore, the comparison between the numerical and exact solutions and absolute errors of Example 5.3 are drawn in Figs. 5 and 6, respectively.

Table 3 Numerical results of Example 5.3

x_i	$m = 2$	$m = 4$	2^5 Haar wavelet basis
0.523	8.61E–7	2.23E–19	1.66E–5
1.047	1.75E–6	6.00E–20	7.08E–5
2.094	3.58E–6	6.05E–19	3.70E–4
3.141	5.34E–6	3.14E–19	5.00E–4
4.188	6.95E–6	7.14E–19	4.19E–3
5.235	8.49E–6	1.22E–18	8.38E–3
5.759	9.28E–6	1.22E–18	1.09E–2
CPU-Time (s)	3.869	21.490	36

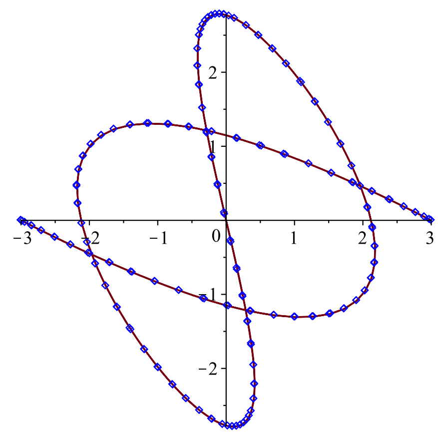


Figure 5 The plot of comparison between the exact and numerical solution for $m = 4$ of Example 5.3

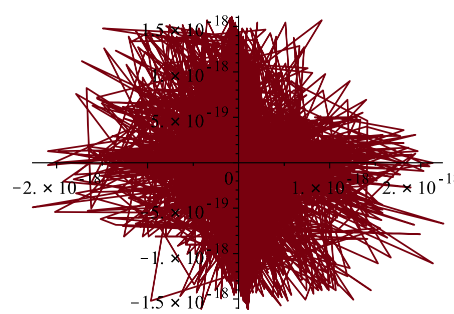


Figure 6 Plot of the absolute errors for $m = 4$ of Example 5.3

6 Conclusion

In this research article, we have proposed a new idea by introducing PQWs for solving a class of NFID. In each iteration of this method, by using these basis functions and the iterative method, we approximated the solution. We discussed the convergence and computed the order of convergence of Eq. (1) by using some lemmas and theorems. Finally, we demonstrated the efficiency and accuracy of the proposed method with several numerical examples.

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Authors' contributions
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