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On the existence of solutions of a set-valued functional integral equation of Volterra–Stieltjes type and some applications

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Abstract

This paper is concerned with the existence of continuous solutions of a set-valued functional integral equation of Volterra–Stieltjes type. The continuous dependence of the solution on the set of selections of the set-valued function will be proven. As an application, we study the existence of solutions to an initial-value problem of arbitrary fractional-order differential inclusion.

MSC: 74H10; 45G10

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1 Introduction

Consider the set-valued functional integral equation of Volterra–Stieltjes type

$$x(t) \in p(t) + \int_0^t F_1\left(s, \int_0^s f_2\bigl(\theta, x\bigl(\varphi(\theta)\bigr)\bigr)\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s), \quad t,s \in [0,T], \tag{1.1}$$

and the initial-value problem

$$\frac{dx(t)}{dt} \in I^\alpha F_1\bigl(t, D^\gamma x(t)\bigr), \quad t \in (0,T], \gamma \in (0,1], \tag{1.2}$$

$$x(0) = x_o. \tag{1.3}$$

Here we study the existence of continuous solutions of the set-valued functional integral equation of Volterra–Stieltjes type (1.1). The continuous dependence of the solution on the set of selections of the set-valued function F_1 will be proven. As an application, we study the existence of solutions of the initial-value problem of arbitrary (fractional) order differential inclusion (1.2)–(1.3).

2 Preliminaries

This section is devoted to providing the notation, definitions, and preliminary facts from the set-valued analysis, which will be needed in our further study.

First, we establish some notation.

We will denote by $I = [0, T]$ a fixed interval, where $T > 0$ is arbitrarily fixed and by $C(I) = C[0, T]$ the Banach space consisting of all continuous functions acting from the interval I into R with the standard norm

$$\|x\|_C = \sup_{t\in I}\bigl|x(t)\bigr|.$$

Define the Banach space $X = C(I) \times C(I)$ with the norm

$$\bigl\|(x,y)\bigr\|_X = \|x\|_C + \|y\|_C.$$

Definition 2.1 Let F be a set-valued map defined on a Banach space E, f is called a selection of F if $f(x) \in F(x)$, for every $x \in E$ and we denote by

$$S_F = \bigl\{f : f(x) \in F(x), x \in E\bigr\}$$

the set of all selections of F (for the properties of the selection of F see [1–3]).

Definition 2.2 ([4]) A set-valued map F from $I \times E$ to family of all nonempty closed subsets of E is called Lipschitzian if there exists $k > 0$ such that, for all $t \in I$ and all $x_1, x_2 \in E$, we have

$$h\bigl(F(t,x_1), F(s,x_2)\bigr) \le k\bigl(|t-s| + |x_1 - x_2|\bigr), \tag{2.1}$$

where $h(A,B)$ is the Hausdorff distance between the two subsets $A, B \in I \times E$.

(For properties of the Hausdorff distance see [5].)

The following theorem [5, Sect. 9, Chap. 1, Th. 1] assumes the existence of a Lipschitzian selection.

Theorem 2.3 ([6]) *Let M be a metric space and F be Lipschitzian set-valued function from M into the nonempty compact convex subsets of R^n. Assume, moreover, that, for some $\lambda > 0$, $F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball on R^n. Then there exist a constant c and a single-valued function $f : M \to R^n$, $f(x) \in F(x)$ for $x \in M$; this function is Lipschitzian with constant k.*

In what follows, we discuss a few auxiliary facts concerning functions of bounded variation (cf. [7]). To this end assumes that x is a real function defined on a fixed interval $[a,b]$. By the symbol $\bigvee_a^b x$ we will denote the variation of the function x on the interval $[a,b]$. In the case when $\bigvee_a^b x$ is finite we say that x is of bounded variation on $[a,b]$. In the case of a function $u(t,s) =: [a,b] \times [c,d] \to R$ we can consider the variation $\bigvee_{t=p}^q u(t,s)$ of the function $t \to u(t,s)$ (i.e., the variation of the function $u(t,s)$ with respect to the variable t) on the interval $[p,q] \subset [a,b]$. Similarly, we define the quantity $\bigvee_{s=p}^q u(t,s)$. We will not discuss the properties of the variation of functions of bounded variation, we refer to [7] for the mentioned properties. Furthermore, assume that x and ϕ are two real functions defined on the interval $[a,b]$. Then, under some extra conditions (cf. [7]), we can define the Stieltjes integral (more precisely, the Riemann–Stieltjes integral) of the function x with respect to the function ϕ on the interval $[a,b]$ which is denoted by the symbol

$$\int_a^b x(t)\,d_\phi(t).$$

In such a case, we say that x is Stieltjes integrable on the interval $[a,b]$ with respect to ϕ.

In the relevant literature, we may encounter a lot of conditions guaranteeing the Stieltjes integrability [7–9]. One of the most frequently exploited condition requires that x is continuous and ϕ is of bounded variation on $[a,b]$.

Next, we recall a few properties of the Stieltjes integral which will be used in our considerations (cf. [7]).

Lemma 2.4 *Assume that x is Stieltjes integrable on the interval $[a,b]$ with respect to a function ϕ of bounded variation. Then*

$$\left|\int_a^b x(t)\,d_\phi(t)\right| \le \int_a^b \bigl|x(t)\bigr|\,d\left(\bigvee_a^t \phi\right).$$

Lemma 2.5 *Let x_1 and x_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function ϕ such that $x_1(t) \le x_2(t)$ for $t \in [a,b]$. Then the following inequality is satisfied*:

$$\int_a^b x_1(t)\,d_\phi(t) \le \int_a^b x_2(t)\,d_\phi(t).$$

In the sequel, we will also consider the Stieltjes integrals of the form

$$\int_a^b x(s)\,d_s g(t,s),$$

where $g : [a,b] \times [a,b] \to R$ and the symbol d_s indicates the integration with respect to the variable s. The details concerning the integral of such a type will be given later.

3 Existence of at least one continuous solution

Consider now the set-valued integral equation (1.1) under the following assumptions.

(i) $p : I \to I$ is continuous function, where $p^* = \sup_{t\in I} |p(t)|$.

(ii) $F_1 : I \times R \to P(R)$ is a Lipschitzian set-valued map with a nonempty compact convex subset of 2^{R^+}.

(iii) $\varphi : I \to I$ is continuous function.

(iv) $f_2 : I \times R \to R$ is continuous and there exist two constants a and b such that

$$\bigl|f_2(t,x)\bigr| \le a + b|x|, \quad \forall t \in [0,T] \text{ and } x \in R.$$

(v) The function g_i is continuous on the triangle $\triangle_i$, for $i = 1, 2$, where

$$\triangle_1 = \bigl\{(t,s) : 0 \le s \le t \le T\bigr\},$$
$$\triangle_2 = \bigl\{(s,\theta) : 0 \le \theta \le s \le T\bigr\}.$$

(vi) The function $s \to g_i(t,s)$ is of bounded variation on $[0,t]$ for each $t \in I$ $(i = 1,2)$.

(vii) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t_1; t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, the following inequality holds:

$$\bigvee_{0}^{t_1} \bigl[g_i(t_2,s) - g_i(t_1,s)\bigr] \leq \epsilon$$

for $i = 1,2$.

(viii) $g_i(t,0) = 0$ for any $t \in I$ $(i = 1,2)$.

It is clear that, from Theorem 2.3 and assumption (ii), the set of Lipschitz selection of F_1 is non-empty. So, the solution of the single-valued integral equation

$$x(t) = p(t) + \int_0^t f_1\left(s, \int_0^s f_2\bigl(\theta, x\bigl(\varphi(\theta)\bigr)\bigr)\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s), \quad t,s \in [0,T], \tag{3.1}$$

where $f_1 \in S_{F_1}$, is a solution of inclusion (1.1).

It must be noted that f_1 satisfies the Lipschitz selection

$$\bigl|f_1(t,x) - f_1(s,y)\bigr| \leq k\bigl(|t-s| + |x-y|\bigr).$$

Obviously, we will assume that g_i satisfies assumptions (v)–(viii). For our purposes, we only need the following lemmas.

Lemma 3.1 ([10]) *The function* $z \to \bigvee_{s=0}^{z} g_i(t,s)$ *is continuous on* $[0,t]$ *for any* $t \in I$ $(i = 1,2)$.

Lemma 3.2 ([10]) *Let the assumptions* (v)–(vii) *be satisfied. Then, for arbitrary fixed number* $0 < t_2 \in I$ *and for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $t_1 \in I$; $t_1 < t_2$ *and* $t_2 - t_1 \leq \delta$ *then* $\bigvee_{s=t_1}^{t_2} g_i(t_2,s) \leq \epsilon$ $(i = 1,2)$.

Lemma 3.3 ([10]) *Under the assumptions* (v)–(vii), *the function* $t \to \bigvee_{s=0}^{t} g_i(t,s)$ *is continuous on* I $(i = 1,2)$.

Further, let us observe that based on Lemma 3.3 we infer that there exists a finite positive constant K_i, such that

$$K_i = \sup\left\{\bigvee_{s=0}^{t} g_i(t,s) : t \in [0,T]\right\},$$

where $T > 0$ is arbitrarily fixed and $i = 1,2$.

We now introduce some functions that will be useful in our further studies:

$$N_i(\epsilon) = \sup\left\{\bigvee_{s=0}^{t_1} \bigl(g_i(t_2,s) - g_i(t_1,s)\bigr) : t_1, t_2 \in [0,T], t_1 < t_2; t_2 - t_1 \leq \epsilon, i = 1,2\right\}.$$

In our considerations, we will examine the double Stieltjes integral of the form

$$\int_c^d \left(\int_c^d f(t,x)\, d_y g_2(x,y)\right) d_s g_1(t,s) = \int_c^d \int_c^d f(t,x)\, d_y g_2(x,y)\, d_s g_1(t,s),$$

where $g_i : [a, b] \times [c, d] \to R (i = 1, 2)$ and the symbol d_y indicates the integration with respect to the variable y (similarly, we define the symbol d_s).

Now, let

$$y(t) = \int_0^t f_2\big(s, x\big(\varphi(s)\big)\big)\, d_s g_2(t, s), \quad t \in [0, T], \tag{3.2}$$

then the nonlinear functional integral equation (3.1) can be written in the form

$$x(t) = p(t) + \int_0^t f_1\big(s, y(s)\big)\, d_s g_1(t, s), \quad t \in [0, T]. \tag{3.3}$$

Hence, the functional integral equation (3.1) is equivalent to the coupled system (3.2) and (3.3).

Now, we study the existence of a continuous solution of the functional integral equation (3.1), which is a solution of the functional integral inclusion (1.1), by getting the continuous solution of the coupled system (3.2) and (3.3).

Definition 3.4 By a solution of the coupled system (3.2), (3.3) we mean the functions $x, y \in C[0, T]$ satisfying (3.2), (3.3).

Remark 3.5 From the Lipschitz condition of f_1, we have

$$\big|f_1(t, x)\big| - \big|f_1(t, 0)\big| \le \big|f_1(t, x) - f_1(t, 0)\big| \le k|x|,$$

i.e.,

$$\big|f_1(t, x)\big| \le k|x| + \sup_{t\in[0,T]} \big|f_1(t, 0)\big| \le k|x| + f_1^*,$$

where

$$f_1^* = \sup_{t\in[0,T]} \big|f_1(t, 0)\big|.$$

Now for the existence of at least one solution $u = (x, y)$, $x, y \in C[0; T]$ of the coupled system (3.3), (3.2) we have the following theorem.

Theorem 3.6 *Under assumptions* (i)–(viii), *there exists at least one solution* $u = (x, y)$, $x, y \in C[0, T]$ *of the coupled system* (3.3), (3.2).

Proof Define the set Q_r by

$$Q_r = \big\{u = (x, y) \in R^2, \|x\| \le r_1, \|y\| \le r_2, \big\|(x, y)\big\| \le r_1 + r_2 = r\big\},$$

where $r = \frac{p^* + f_1^* K_1}{1 - kK_1} + \frac{aK_2}{1 - bK_2}$ with $kK_1 < 1$, $bK_2 < 1$.

It is clear that the set Q_r is nonempty, bounded, closed and convex. Let A be any operator defined by

$$Au(t) = A(x, y)(t) = \big(A_1 y(t), A_2 x(t)\big),$$

$$A_1y(t) = p(t) + \int_0^t f_1\big(s, y(s)\big)\, d_sg_1(t,s), \quad t \in [0,T],$$

and

$$A_2x(t) = \int_0^t f_2(s, x\big(\varphi(s)\big)\, d_sg_2(t,s), \quad t \in [0,T],$$

where for $u = (x, y) \in Q_r$, and from Remark 3.5 we have

$$\begin{aligned}
\big|A_1y(t)\big| &= \left|p(t) + \int_0^t f_1\big(s; y(s)\big)\, d_sg_1(t,s)\right| \\
&\le \big|p(t)\big| + \int_0^t \big|f_1\big(s; y(s)\big)\big|\big|\, d_sg_1(t,s)\big| \\
&\le p^* + \int_0^t \big(k|y| + f_1^*\big)\, d_s\left(\bigvee_{p=0}^{s} g_1(t,p)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\|A_1y\| &\le p^* + \big(kr_1 + f_1^*\big)\left(\bigvee_{s=0}^{t} g_1(t,s)\right) \\
&\le p^* + \big(kr_1 + f_1^*\big) \sup_{t\in I}\left(\bigvee_{s=0}^{t} g_1(t,s)\right) \\
&\le p^* + \big(kr_1 + f_1^*\big)K_1 = r_1, \quad r_1 = \frac{p^* + f_1^* K_1}{1 - kK_1}.
\end{aligned}$$

Also

$$\begin{aligned}
\big|A_2x(t)\big| &= \left|\int_0^t f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t,s)\right| \\
&\le \int_0^t \big|f_2\big(s, x\big(\varphi(s)\big)\big)\big|\big|\, d_sg_2(t,s)\big| \\
&\le \int_0^t \big[a + b\big|x\big(\varphi(s)\big)\big|\big]\, d_s\left(\bigvee_{p=0}^{s} g_2(t,p)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\|A_2x\| &\le (a + br_2)\left(\bigvee_{s=0}^{t} g_2(t,s)\right) \\
&\le (a + br_2) \sup_{t\in I}\left(\bigvee_{s=0}^{t} g_2(t,s)\right) \\
&\le (a + br_2)K_2 = r_2, \quad r_2 = \frac{aK_2}{1 - bK_2}.
\end{aligned}$$

From the above estimate we derive the following inequality:

$$\begin{aligned}\|Au\|_X &= \|A_1 y\|_C + \|A_2 x\|_C \\ &\le r_1 + r_2 \\ &= \frac{p^* + f_1^* K_1}{1 - kK_1} + \frac{aK_2}{1 - bK_2} = r.\end{aligned}$$

Hence, $AQ_r \subset Q_r$ and the class $\{Au\}$, $u \in Q_r$ is uniformly bounded.

Now, for $u = (x, y) \in Q_r$, for all $\epsilon > 0$, $\delta > 0$ and for each $t_1, t_2 \in [0, T]$, $t_1 < t_2$, such that $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}\left|A_1 y(t_2) - A_1 y(t_1)\right| &= \left|p(t_2) + \int_0^{t_2} f_1\big(s; y(s)\big)\, d_s g_1(t_2, s)\right. \\ &\quad \left. - p(t_1) - \int_0^{t_1} f_1\big(s; y(s)\big)\, d_s g_1(t_1, s)\right| \\ &\le \left|p(t_2) - p(t_1)\right| + \left|\int_0^{t_2} f_1\big(s, y(s)\big)\, d_s g_1(t_2, s)\right. \\ &\quad \left. - \int_0^{t_1} f_1\big(s, y(s)\big)\, d_s g_1(t_2, s)\right| \\ &\quad + \left|\int_0^{t_1} f_1\big(s, y(s)\big)\, d_s g_1(t_2, s) - \int_0^{t_1} f_1\big(s, y(s)\big)\, d_s g_1(t_1, s)\right| \\ &\le \left|p(t_2) - p(t_1)\right| + \int_{t_1}^{t_2} \left|f_1\big(s, y(s)\big)\right| \left| d_s g_1(t_2, s)\right| \\ &\quad + \int_0^{t_1} \left|f_1\big(s, y(s)\big)\right| \left|\big[d_s g_1(t_2, s) - d_s g_1(t_1, s)\big]\right| \\ &\le \left|p(t_2) - p(t_1)\right| + \int_{t_1}^{t_2} \big[k\left|y(s)\right| + f_1(s, 0)\big]\, d_s\left(\bigvee_{p=0}^{s} g_1(t_2, p)\right) \\ &\quad + \int_0^{t_1} \big[k\left|y(s)\right| + f_1(s, 0)\big]\, d_s\left(\bigvee_{p=0}^{s} \big[g_1(t_2, p) - g_1(t_1, p)\big]\right) \\ &\le \left|p(t_2) - p(t_1)\right| + \big[kr_1 + f_1^*\big] \int_{t_1}^{t_2} d_s\left(\bigvee_{p=0}^{s} g_1(t_2, p)\right) \\ &\quad + \int_0^{t_1} d_s\left(\bigvee_{p=0}^{s} \big[g_1(t_2, p) - g_1(t_1, p)\big]\right) \\ &\le \left|p(t_2) - p(t_1)\right| + \big[kr_1 + f_1^*\big] \bigvee_{s=t_1}^{t_2} g_1(t_2, s) \\ &\quad + \bigvee_{s=0}^{t_1} \big[g_1(t_2, s) - g_1(t_1, s)\big] \\ &\le \left|p(t_2) - p(t_1)\right| + \big[kr_1 + f_1^*\big] \left[\bigvee_{s=t_1}^{t_2} g_1(t_2, s) + N_1(\epsilon)\right]\end{aligned}$$

and

$$\begin{aligned}
&\left|A_2x(t_2) - A_2x(t_1)\right| \\
&\quad\leq \left|\int_0^{t_2} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_2, s) - \int_0^{t_1} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_1, s)\right| \\
&\quad\leq \left|\int_0^{t_2} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_2, s) - \int_0^{t_1} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_2, s)\right| \\
&\qquad+ \left|\int_0^{t_1} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_2, s) - \int_0^{t_1} f_2\big(s, x\big(\varphi(s)\big)\big)\, d_sg_2(t_1, s)\right| \\
&\quad\leq \int_{t_1}^{t_2} \big|f_2\big(s, x\big(\varphi(s)\big)\big)\big|\big|\, d_sg_2(t_2, s)\big| \\
&\qquad+ \int_0^{t_1} \big|f_2\big(s, x\big(\varphi(s)\big)\big)\big|\big|\big[d_sg_2(t_2, s) - d_sg_2(t_1, s)\big]\big| \\
&\quad\leq \int_{t_1}^{t_2} \big[a + b\big|x\big(\varphi(s)\big)\big|\big]\, d_s\left(\bigvee_{p=0}^{s} g_2(t_2, p)\right) \\
&\qquad+ \int_0^{t_1} \big[a + b\big|x\big(\varphi(s)\big)\big|\big]\, d_s\left(\bigvee_{p=0}^{s}\big[g_2(t_2, p) - g_2(t_1, p)\big]\right) \\
&\quad\leq (a + br_2)\left[\int_{t_1}^{t_2} d_s\left(\bigvee_{p=0}^{s} g_2(t_2, p)\right) + \int_0^{t_1} d_s\left(\bigvee_{p=0}^{s}\big[g_2(t_2, p) - g_2(t_1, p)\big]\right)\right] \\
&\quad\leq (a + br_2)\left[\bigvee_{s=0}^{t_2} g_2(t_2, s) - \bigvee_{s=0}^{t_1} g(t_2, s)\right] + \bigvee_{s=0}^{t_1}\big[g_2(t_2, s) - g_2(t_1, s)\big] \\
&\quad\leq (a + br_2)\left[\bigvee_{s=t_1}^{t_2} g_2(t_2, s) + N_2(\epsilon)\right].
\end{aligned}$$

Further, for the operator A and $u \in Q_r$ we have

$$\begin{aligned}
Au(t_2) - Au(t_1) &= A(x, y)(t_2) - A(x, y)(t_1) \\
&= \big(A_1y(t_2), A_2x(t_2)\big) - \big(A_1y(t_1), A_2x(t_1)\big) \\
&= \big(A_1y(t_2) - A_1y(t_1), A_2x(t_2) - A_2x(t_1)\big).
\end{aligned}$$

Then

$$\begin{aligned}
\big\|Au(t_2) - Au(t_1)\big\|_X &= \big\|\big(A_1y(t_2) - A_1y(t_2), A_2x(t_2) - A_2x(t_1)\big)\big\|_X \\
&= \big\|A_1y(t_2) - A_1y(t_2)\big\|_C + \big\|A_2x(t_2) - A_2x(t_1)\big\|_C \\
&= \big\|p(t_2) - p(t_1)\big\| + \big[kr_1 + f_1^*\big]\left[\bigvee_{s=t_1}^{t_2} g(t_2, s) + N_1(\epsilon)\right] \\
&\quad+ (a + br_2)\left[\bigvee_{s=t_1}^{t_2} g(t_2, s) + N_2(\epsilon)\right]. \qquad (*)
\end{aligned}$$

This means that the class of functions Au is equi-continuous on Q_r. Then by the Arzela–Ascoli theorem [11] the operator A is compact.

It remains to prove the continuity of $A : Q_r \to Q_r$. Let $u_n = (x_n, y_n)$ is a sequence in Q_r with $x_n \to x$, and $y_n \to x$ and since $f_1(t, y(t))$ and $f_2(t, x(t))$ is continuous in $C[0, T] \times R$ then $f_1(t, y_n(t))$ and $f_2(t, x_n(t))$ converge to $f_1(t, y(t))$ and $f_2(t, x(t))$, thus $f_2(t, x_n(\varphi(t)))$ converges to $f_2(t, x(\varphi(t)))$ (see assumption (ii)). Using assumption (iii) and applying Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^t f_2\bigl(s, x_n\bigl(\varphi(s)\bigr)\bigr)\, d_s g_2(t,s) = \int_0^t f_2\bigl(s, x\bigl(\varphi(s)\bigr)\bigr)\, d_s g_2(t,s)$$

and

$$\lim_{n\to\infty} \int_0^t f_1\bigl(s, y_n(s)\bigr)\, d_s g_1(t,s) = \int_0^t f_1\bigl(s, y(s)\bigr)\, d_s g_1(t,s);$$

then

$$\begin{aligned}
\lim_{n\to\infty} A_1 y_n(t) &= p(t) + \lim_{n\to\infty} \int_0^t f_1\bigl(s, y_n(s)\bigr)\, d_s g_1(t,s) \\
&= p(t) + \int_0^t f_1\bigl(s, y(s)\bigr)\, d_s g_1(t,s) = A_1 y(t), \quad t \in [0, T], \\
\lim_{n\to\infty} A_2 x_n(t) &= \int_0^t \lim_{n\to\infty} f_2\bigl(s, x_n\bigl(\varphi(s)\bigr)\bigr)\, d_s g_2(t,s) \\
&= \int_0^t f_2\bigl(s, x\bigl(\varphi(s)\bigr)\bigr)\, d_s g_2(t,s) = A_2 x(t), \quad t \in [0, T], \\
\lim_{n\to\infty} A u_n(t) &= \lim_{n\to\infty} \bigl(A_1 y_n(t), A_2 x_n(t)\bigr) \\
&= \Bigl(\lim_{n\to\infty} A_1 y_n(t), \lim_{n\to\infty} A_2 x_n(t)\Bigr) = \bigl(A_1 y(t), A_2 x(t)\bigr) = Au(t).
\end{aligned}$$

Since all conditions of the Schauder fixed-point theorem [12] hold, A has a fixed point $u \in Q_r$, and then the system (3.3), (3.2) has at least one continuous solution $u = (x, y) \in Q_r$, $x; y \in C[0, T]$.

Consequently, the functional integral equation (3.1) has at least one solution $x \in C[0, T]$. □

4 Existence of a unique solution

In this section, we study the uniqueness of the solutions $x \in C[0, T]$ of the functional integral inclusion (1.1).

Theorem 4.1 *Consider the assumptions of Theorem 3.6 satisfied with replacing condition* (iv) *by assuming that the f_2 satisfies the Lipschitz condition with respect to the second variable*; *that is, there exists a constant c such that*

$$\bigl|f_2(t,x) - f_2(t,y)\bigr| \le b|x - y|.$$

If $kbK_1K_2 < 1$, then the functional integral inclusion (1.1) *has a unique solution $x \in C[0, T]$, where k is Lipschitz constant of functions f_1 and K_i ($i = 1, 2$) as defined in Lemma 3.3.*

Proof Let x_1 and x_2 be two solutions of Eq. (3.1), then

$$\begin{aligned}\left|x_1(t)-x_2(t)\right| \le \int_0^t \biggl| f_1\left(s, \int_0^s f_2\bigl(\theta, x_1(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta)\right) \\ - f_1\left(s, \int_0^s f_2\bigl(\theta, x_2(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta)\right)\biggr| \, d_s g_1(t,s).\end{aligned}$$

Using the Lipschitz condition for f_1, we obtain

$$\begin{aligned}&\left|x_1(t)-x_2(t)\right| \\ &\le k\left|\int_0^t \left[\int_0^s f_2\bigl(\theta, x_1(\varphi(\theta)\bigr)\, d_\theta g_2(s,\theta)\bigr) \right.\right. \\ &\quad \left.\left. - \int_0^s f_2\bigl(\theta, x_2(\varphi(\theta)\bigr)\, d_\theta g_2(s,\theta)\bigr)\right]\right| d_s g_1(t,s) \\ &\le k\int_0^t \int_0^s \left|f_2\bigl(\theta, x_1(\varphi(\theta))\bigr) - f_2\bigl(\theta, x_2(\varphi(\theta))\bigr)\right| \left|d_\theta g_2(s,\theta)\right| \left|d_s g_1(t,s)\right|.\end{aligned}$$

Using Lipschitz condition for f_2, we obtain

$$\begin{aligned}\left|x_1(t)-x_2(t)\right| &\le kb\int_0^t \int_0^s \left|x_1\bigl(\varphi(\theta)\bigr) - x_2\bigl(\varphi(\theta)\bigr)\right| d_\theta\left(\bigvee_{p=0}^{\theta} g_2(s,p)\right) d_s\left(\bigvee_{q=0}^{s} g_1(t,q)\right) \\ &\le kb\|x_1-x_2\| \int_0^t \int_0^s d_\theta\left(\bigvee_{p=0}^{\theta} g_2(s,p)\right) d_s\left(\bigvee_{q=0}^{s} g_1(t,q)\right) \\ &\le kb\|x_1-x_2\| \bigvee_{s=0}^{t} g_1(t,s) \bigvee_{\theta=0}^{s} g_2(s,\theta) \\ &\le kb\|x_1-x_2\| \sup_{t\in[0,T]} \bigvee_{s=0}^{t} g(t,s) \sup_{s\in[0,T]} \bigvee_{\theta=0}^{s} g(s,\theta) \\ &\le kb\|x_1-x_2\| K_1K_2, \\ \left\|x_1(t)-x_2(t)\right\| &\le kbK_1K_2\|x_1-x_2\|.\end{aligned}$$

Then

$$(1-kbK_1K_2)\|x_1-x_2\| < 0.$$

This proves the uniqueness of the solution of the functional integral equation (3.1). □

4.1 Continuous dependence

Theorem 4.2 *The solution of the inclusion* (1.1) *depends continuously on the* S_{F_1} *of all Lipschitzian selections of* F_1.

Proof Let $f_1(t,x(t))$ and $f_1^*(t,x(t))$ be two different Lipschitzian selections of $F_1(t,x(t))$ such that

$$\left|f_1\bigl(t,x(t)\bigr) - f_1^*\bigl(t,x(t)\bigr)\right| < \delta, \quad \delta > 0, t \in [0,T],$$

then for the two corresponding solutions $x_{f_1}(t)$ and $x_{f_1^*}(t)$ of (1.1) we have

$$\begin{aligned}
&x_{f_1}(t) - x_{f_1^*}(t) \\
&\quad = \int_0^t \biggl[f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \\
&\qquad - f_1^*\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \biggr]\, d_s g_1(t,s), \\
&\bigl| x_{f_1}(t) - x_{f_1^*}(t) \bigr| \\
&\quad \leq \biggl| \int_0^t \biggl[f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \\
&\qquad - f_1^*\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \biggr]\, d_s g_1(t,s) \biggr| \\
&\quad \leq \int_0^t \biggl| f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \\
&\qquad - f_1^*\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_s g_2(s,\theta) \biggr) \biggr| \bigl| d_s g_1(t,s) \bigr| \\
&\quad \leq \int_0^t \biggl| f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \\
&\qquad - f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \biggr| \bigl| d_s g_1(t,s) \bigr| \\
&\qquad + \biggl| \int_0^t \biggl| f_1\biggl(s, \int_0^s f_2\bigl(s, x_{f_1^*}(\varphi(s))\bigr)\, d_s g(t,s) \biggr) \\
&\qquad - f_1^*\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \biggr| \bigl| d_s g_1(t,s) \bigr| \\
&\quad \leq \int_0^t \biggl| f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \\
&\qquad - f_1\biggl(s, \int_0^s f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr)\, d_\theta g_2(s,\theta) \biggr) \biggr| \bigl| d_s g_1(t,s) \bigr| + \delta \int_0^t \bigl| d_s g_1(t,s) \bigr| \\
&\quad \leq k \int_0^t \int_0^s \bigl| f_2\bigl(\theta, x_{f_1}(\varphi(\theta))\bigr) - f_2\bigl(\theta, x_{f_1^*}(\varphi(\theta))\bigr) \bigr| \bigl| d_\theta g_2(s,\theta)) \bigr| \bigl| d_s g_1(t,s) \bigr| \\
&\qquad + \delta \int_0^t \bigl| d_s g_1(t,s) \bigr| \\
&\quad \leq kb \int_0^t \int_0^s \bigl| x_{f_1}(\varphi(\theta)) - x_{f_1^*}(\varphi(\theta)) \bigr| \bigl| d_\theta g_2(s,\theta)) \bigr| \bigl| d_s g_1(t,s) \bigr| + \delta \int_0^t \bigl| d_s g_1(t,s) \bigr| \\
&\quad \leq kb \| x_{f_1} - x_{f_1^*} \| \int_0^t \int_0^s d_\theta \Biggl(\bigvee_{q=0}^{\theta} g_2(s,q) \Biggr) d_s \Biggl(\bigvee_{p=0}^{s} g_1(t,p) \Biggr) + \delta \int_0^t d_s \Biggl(\bigvee_{p=0}^{s} g_1(t,p) \Biggr) \\
&\quad \leq kb \| x_{f_1} - x_{f_1^*} \| \bigvee_{\theta=0}^{s} g_2(s,\theta) \bigvee_{s=0}^{t} g_1(t,s) + \delta \bigvee_{s=0}^{t} g_1(t,s) \\
&\quad \leq kb \| x_{f_1} - x_{f_1^*} \| \sup_{s\in[0,T]} \bigvee_{\theta=0}^{s} g_2(s,\theta) \sup_{t\in[0,T]} \bigvee_{s=0}^{t} g_1(t,s) + \delta \sup_{t\in[0,T]} \bigvee_{s=0}^{t} g_1(t,s) \\
&\quad \leq kb \| x_{f_1} - x_{f_1^*} \| K_2 K_1 + \delta K_1,
\end{aligned}$$

$$\|x_{f_1} - x_{f_1^*}\| \le \delta K_1 (1 - kbK_1K_2)^{-1} = \epsilon.$$

Thus from last inequality, we get

$$\|x_{f_1} - x_{f_1^*}\| \le \epsilon.$$

This proves the continuous dependence of the solution on the set S_{F_1} of all Lipschitzian selections of F_1. This completes the proof. □

5 Volterra integral inclusion of fractional order

In this section, we will consider the fractional integral inclusion, which has the form

$$x(t) \in p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1\left(s, \int_0^s \frac{(s-\theta)^{\beta-1}}{\Gamma(\beta)} f_2\big(\theta, x(\varphi(\theta))\big)\, d\theta\right) ds, \quad t,s \in [0,T], \quad (5.1)$$

where $t \in I = [0,T]$ and $\alpha \in (0,1)$. Moreover, $\Gamma(\alpha)$ denotes the gamma function. Let us mention that (5.1) represents the so-called nonlinear Volterra integral inclusion of fractional orders. Recently, the inclusion of such a type was intensively investigated in some papers [13–18].

Now, we show that the functional integral inclusion of fractional orders (5.1) can be treated as a particular case of the set-valued functional integral equation of Volterra–Stieltjes (1.1) studied in Sect. 3.

Indeed, we can consider the functions $g_i(w,z) = g_i : \triangle_i \to R$ $(i = 1,2)$ defined by the formulas

$$g_1(t,s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha+1)}, \qquad g_2(s,\theta) = \frac{s^\beta - (s-\theta)^\beta}{\Gamma(\beta+1)}.$$

Note that the functions g_1 and g_2 satisfy assumptions (v)–(viii) in Theorem 3.6; see [10, 19].

Now, we can formulate the following existence results concerning with the Volterra integral inclusion of fractional order (5.1).

Theorem 5.1 *Under the assumptions* (i)–(iv) *of Theorem* 3.6, *the fractional integral inclusion* (5.1) *has at least one continuous solution* $x \in C[0,T]$.

Theorem 5.2 *Under the assumptions of Theorem* 4.1, *the fractional integral inclusion* (5.1) *has exactly one unique solution* $x \in C[0,T]$.

6 Existence of the maximal and minimal solutions

In this section, we establish the existence of the maximal and minimal solutions of the nonlinear Volterra integral inclusion of fractional order (5.1). It is clear that, from Theorem 2.3 and assumption (ii) of Theorem 3.6, the set of Lipschitz selections of F_1 is non-empty. So, the solution of the nonlinear functional integral equation of fractional order

$$x(t) = p(t) + I^\alpha f_1\big(t, I^\beta f_2\big(t, x(\varphi(t))\big)\big), \quad t,s \in [0,T], \quad (6.1)$$

where $f_1 \in S_{F_1}$, is a solution of inclusion (5.1).

Definition 6.1 ([12]) Let $m(t)$ be a solution of the non-linear functional integral equation (6.1), then $m(t)$ is said to be a maximal solution of (6.1), if for every solution x of inclusion (6.1) existing on $[0, T]$ the inequality $x(t) < m(t)$, $t \in [0, T]$ holds. A minimal solution $s(t)$ may be defined similarly by reversing the last inequality i.e. $x(t) > s(t)$, $\forall t \in [0, T]$.

Consider the following lemma.

Lemma 6.2 *Let $p(t)$, $f_i(t; x)$ $(i = 1, 2)$ and $\varphi(t)$ satisfy the assumptions in Theorem 5.1 and let $x(t)$, $y(t)$ be two continuous functions on $[0, T]$ satisfying*

$$x(t) \leq p(t) + I^{\alpha} f_1\bigl(t, I^{\beta} f_2\bigl(t, x\bigl(\varphi(t)\bigr)\bigr)\bigr), \quad t \in [0, T],$$
$$y(t) \geq p(t) + I^{\alpha} f_1\bigl(t, I^{\beta} f_2\bigl(t, y\bigl(\varphi(t)\bigr)\bigr)\bigr), \quad t \in [0, T],$$

where one of them is strict.

Suppose f_1 and f_2 are monotonic nondecreasing functions in x, then

$$x(t) < y(t), \quad t > 0. \tag{6.2}$$

Proof Let the conclusion (6.2) be false, then there exists t_1 such that

$$x(t_1) = y(t_1), \quad t_1 > 0,$$

and

$$x(t) < y(t), \quad 0 < t < t_1, t \in [0, T].$$

From the monotonicity of the functions f_1 and f_2 in x, we get

$$\begin{aligned} x(t_1) &\leq p(t_1) + I^{\alpha} f_1\bigl(t_1, I^{\beta} f_2\bigl(t_1, x\bigl(\varphi(t_1)\bigr)\bigr)\bigr) \\ &= p(t_1) + \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1\bigl(s, I^{\beta} f_2\bigl(s, x\bigl(\varphi(s)\bigr)\bigr)\bigr)\, ds \\ &< p(t_1) + \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1\bigl(s, I^{\beta} f_2\bigl(s, y\bigl(\varphi(s)\bigr)\bigr)\bigr)\, ds, \\ x(t_1) &< y(t_1). \end{aligned}$$

This contradicts the fact that $x(t_1) = y(t_1)$.

Then

$$x(t) < y(t).$$ □

Now, for the existence of the continuous maximal and minimal solutions of the nonlinear functional integral inclusion (6.1) we have the following theorem.

Theorem 6.3 *Consider the assumptions* (i)–(iv) *of Theorem 5.1 satisfied, furthermore, if f_1 and f_2 are monotonic nondecreasing functions in x for each $t \in [0, T]$, then the nonlinear functional integral inclusion (6.1) has maximal and minimal solutions.*

Proof Firstly, we shall prove the existence of the maximal solution of (6.1).

Let $\epsilon > 0$ be given such that $0 < \epsilon < \frac{T}{2}$, and consider the nonlinear functional integral equation of fractional order

$$x_\epsilon(t) = p(t) + I^\alpha f_{1_\epsilon}\big(t, I^\beta f_{2_\epsilon}\big(t, x_\epsilon\big(\varphi(t)\big)\big)\big),$$

where

$$\begin{aligned} f_{1_\epsilon}\big(t, I^\beta f_{2_\epsilon}\big(t, x_\epsilon\big(\varphi(t)\big)\big)\big) &= f_1\big(t, I^\beta f_{2_\epsilon}\big(t, x_\epsilon\big(\varphi(t)\big)\big)\big) + \epsilon, \\ f_{2_\epsilon}\big(t, x_\epsilon\big(\varphi(t)\big)\big) &= f_2\big(t, x_\epsilon\big(\varphi(t)\big)\big) + \epsilon. \end{aligned}$$

Clearly the functions $f_{1_\epsilon}(t, I^\beta f_{2_\epsilon}(t, x_\epsilon(\varphi(t))))$ and $f_{2_\epsilon}(t, x_\epsilon(\varphi(t)))$ satisfy the assumptions of Theorem 5.1 and therefore, Eq. (6.1) has a continuous solution x_ϵ according to Theorem 5.1. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$\begin{aligned} x_{\epsilon_2}(t) &= p(t) + I^\alpha f_{1_{\epsilon_2}}\big(t, I^\beta f_{2_{\epsilon_2}}\big(t, x_{\epsilon_2}\big(\varphi(t)\big)\big)\big), \\ x_{\epsilon_2}(t) &= p(t) + I^\alpha f_1\big(t, I^\beta f_2\big(t, x_{\epsilon_2}\big(\varphi(t)\big)\big) + I^\beta \epsilon_2\big) + I^\alpha \epsilon_2, \end{aligned} \tag{6.3}$$

also

$$\begin{aligned} x_{\epsilon_1}(t) &= p(t) + I^\alpha f_{1_{\epsilon_1}}\big(t, I^\beta f_{2_{\epsilon_1}}\big(t, x_{\epsilon_1}\big(\varphi(t)\big)\big)\big), \\ x_{\epsilon_1}(t) &= p(t) + I^\alpha f_1\big(t, I^\beta f_2\big(t, x_{\epsilon_1}\big(\varphi(t)\big)\big) + I^\beta \epsilon_1\big) + I^\alpha \epsilon_1, \\ x_{\epsilon_1}(t) &> p(t) + I^\alpha f_1\big(t, I^\beta f_2\big(t, x_{\epsilon_1}\big(\varphi(t)\big)\big) + I^\beta \epsilon_2\big) + I^\alpha \epsilon_2. \end{aligned} \tag{6.4}$$

Applying Lemma 6.2, and (6.3) and (6.4), we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \quad \text{for } t \in [0, T].$$

As shown before in the proof of Theorem 3.6, the family of functions $x_\epsilon(t)$ is uniformly bounded and equi-continuous. Hence by the Arzela–Ascoli theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and we denote this limit by $m(t)$. From the continuity of the functions $f_{i\epsilon_n}$ for $i = 1, 2$ and in the second argument, we get

$$\begin{aligned} & f_{2_{\epsilon_n}}\big(t, x_{\epsilon_n}\big(\varphi(t)\big)\big) \to f_2\big(t, x\big(\varphi(t)\big)\big), \quad \text{as } n \to \infty, \\ & f_{1_{\epsilon_n}}\big(t, I^\beta f_{2_{\epsilon_n}}\big(t, x_{\epsilon_n}\big(\varphi(t)\big)\big)\big) \to f_1\big(t, I^\beta f_2\big(t, x\big(\varphi(t)\big)\big)\big), \quad \text{as } n \to \infty, \end{aligned}$$

and

$$m(t) = \lim_{n\to\infty} x_{\epsilon_n} = p(t) + I^\alpha f_1\big(t, I^\beta f_2\big(t, x\big(\varphi(t)\big)\big)\big),$$

which implies that $m(t)$ is a solution of the quadratic integral equation (6.1). Finally, we shall show that $m(t)$ is the maximal solution of (6.1).

To do this let $x(t)$ be any solution of (6.1), then

$$x(t) = p(t) + I^{\alpha} f_1\bigl(t, I^{\beta} f_2\bigl(t, x\bigl(\varphi(t)\bigr)\bigr)\bigr), \tag{6.5}$$

and also

$$\begin{aligned} x_{\epsilon}(t) &= p(t) + I^{\alpha} f_{1_{\epsilon}}\bigl(t, I^{\beta} f_{2_{\epsilon}}\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr)\bigr), \\ x_{\epsilon}(t) &= p(t) + I^{\alpha} f_1\bigl(t, I^{\beta} f\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr) + I^{\beta}\epsilon\bigr) + I^{\alpha}\epsilon, \\ x_{\epsilon}(t) &> p(t) + I^{\alpha} f_1\bigl(t, I^{\beta} f_2\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr)\bigr). \end{aligned} \tag{6.6}$$

Applying Lemma 6.2 and (6.5) and (6.6), we get

$$x(t) < x_{\epsilon}(t), \quad \forall t \in [0, T].$$

From the uniqueness of the maximal solution (see [12]), it is clear that $x_{\epsilon}(t)$ tends to $m(t)$ uniformly in $[0, T]$ as $\epsilon \to \infty$.

Similarly, we can prove the existence of the minimal solution. We set

$$\begin{aligned} f_{1_{\epsilon}}\bigl(t, I^{\beta} f_{2_{\epsilon}}\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr)\bigr) &= f_1\bigl(t, I^{\beta} f_{2_{\epsilon}}\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr) - I^{\beta}\epsilon\bigr) - \epsilon, \\ f_{2_{\epsilon}}\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr) &= f_2\bigl(t, x_{\epsilon}\bigl(\varphi(t)\bigr)\bigr) - \epsilon, \end{aligned}$$

and thus we prove the existence of a minimal solution. □

7 Differential inclusion

Consider now the initial-value problem of the differential inclusion (1.2) with the initial data (1.3).

Theorem 7.1 *Consider the assumptions of Theorem 5.1 satisfied, then the initial-value problem (1.2)–(1.3) has at least one positive solution $x \in C([0, 1])$.*

Proof Let $y(t) = \frac{dx(t)}{dt}$, then the inclusion (1.2) will be

$$y(t) \in I^{\alpha} F_1\bigl(t, I^{1-\tau} y(t)\bigr). \tag{7.1}$$

Letting $f_2(t, x) = x$, $\varphi(t) = t$, and $\beta = 1 - \tau$, applying Theorem 5.2 to the functional inclusion (7.1) we deduce that there exists a continuous solution $y \in C[0, T]$ of the functional inclusion (7.1) and this solution depends on the set S_{F_1}.

This implies the existence of a solution $x \in C[0, T]$,

$$x(t) = x_{\circ} + \int_0^t y(s)\, ds$$

of the initial-value problem (1.2)–(1.3). □

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References

1. Cellina, A., Solimini, S.: Continuous extensions of selections Bull. Pol. Acad. Sci., Math. **35**, 573–581 (1987)
2. Fryszkowski, A.: Continuous selections for a class of non-convex multivalued maps. Stud. Math. **76**(2), 163–174 (1983)
3. Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**(6), 397–403 (1965)
4. El-Sayed, A.M.A., Ibrahim, A.G.: Multivalued fractional differential equations. Appl. Math. Comput. **68**(1), 15–25 (1995)
5. Aubin, J.-P., Cellina, A.: Differential Inclusions: Set-Valued Maps and Viability Theory. Grundlehren der Mathematischen Wissenschaften, vol. 264. Springer, Berlin (2012)
6. El-Sayed, A.M.A., Ibrahim, A.G.: Set-valued integral equations of fractional-orders. Appl. Math. Comput. **118**, 113–121 (2001)
7. Appell, J., Banaś, J., Díaz, N.J.M.: Bounded Variation and Around. De Gruyter Series in Nonlinear Analysis and Applications, vol. 17 (2013)
8. Lojasiewicz, S., Ojasiewicz, S.: An Introduction to the Theory of Real Functions (1988)
9. Natanson, I.P.: Theory of Functions of a Real Variable. Ungar, New York (1961)
10. Banaś, J., Zajaç, T.: A new approach to the theory of functional integral equations of fractional order. J. Math. Anal. Appl. **375**(2), 375–387 (2011)
11. Curtain, R.F., Pritchard, A.J.: Functional Analysis in Modern Applied Mathematics. Mathematics in Science and Engineering, vol. 21, pp. 409–410 (1979)
12. Lakshmikantham, V., Leela, S.: Differential and Integral Inequalities: Theory and Applications: Volume I: Ordinary Differential Equations (1969)
13. Al-Issa, S.M., El-Sayed, A.M.A.: Positive integrable solutions for nonlinear integral and differential inclusions of fractional-orders. Comment. Math. **49**(2), 171–177 (2009)
14. El-Sayed, A.M.A., Al-Issa, S.M.: Global integrable solution for a nonlinear functional integral inclusion. SRX Math. **2010**, Article ID 891982 (2010)
15. El-Sayed, A.M.A., Al-Issa, S.M.: Monotonic continuous solution for a mixed type integral inclusion of fractional order. J. Math. **33**, 27–34 (2010)
16. El-Sayed, A.M.A., Al-Issa, S.M.: Monotonic integrable solution for a mixed type integral and differential inclusion of fractional orders. Int. J. Differ. Equ. Appl. **18**(1), 1–9 (2019)
17. El-Sayed, A.M.A., Al-Issa, S.M.: Monotonic solutions for a quadratic integral equation of fractional order. AIMS Math. **4**(3), 821–830 (2019)
18. El-Sayed, A.M.A., Al-Issa, S.M.: Existence of continuous solutions for nonlinear functional differential and integral inclusions. Malaya J. Mat. **7**(3), 541–544 (2019)
19. El-Sayed, A.M.A., El-Sayed, W.G., Abd El-Mowla, A.A.H.: Weak solutions of fractional order differential equations via Volterra–Stieltjes integral operator. J. Math. Appl. **40**, 85–96 (2017)