Zada et al. *Advances in Difference Equations* (2020) 2020:64
https://doi.org/10.1186/s13662-020-2534-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Ulam–Hyers stability of impulsive integrodifferential equations with Riemann–Liouville boundary conditions

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Abstract

This paper is concerned with a class of impulsive implicit fractional integrodifferential equations having the boundary value problem with mixed Riemann–Liouville fractional integral boundary conditions. We establish some existence and uniqueness results for the given problem by applying the tools of fixed point theory. Furthermore, we investigate different kinds of stability such as Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. Finally, we give two examples to demonstrate the validity of main results.

MSC: 34A08; 34B37

Keywords: Caputo derivative; Riemann–Liouville integral; Impulse; Ulam–Hyers stability; Fixed point theory

1 Introduction

During the last few decades, boundary value problems of fractional differential equations have been utilized in different problems of applied nature; for example, we can find it in analytical formulations of systems and processes. Due to a more accurate behavior of fractional differential equations, it got the interest of research community in various applied fields of sciences such as chemistry, engineering, mechanics, physics, and so on. For the readers' convenience, we refer to the monographs [9, 11, 15, 23] and their references. Also, an experimental study was presented in [21].

For boundary value problems of fractional differential equations, the existence of solutions is an important and basic requirement. Furthermore, the uniqueness of solutions is the next important feature for more specific behavior of solutions. In the literature, many results are available about these two necessary properties of solutions; see, for example [2, 7, 8, 20, 22, 27]. Integral boundary conditions are very important in the solutions of many practical systems [1, 51].

The impulsive phenomena and their models are investigated and analyzed in different practical problems. The theory of impulsive mathematical models based on fractional differential equations has very significant applications in many applied problems in natural sciences and engineering. Many evolutionary processes that possess abrupt changes at certain moments can be described with the help of aforesaid models. The abrupt changes in evolutionary processes can be of two types. The first one, characterized by short-term perturbations with negligible duration in comparison with the duration of the whole processes, is called instantaneous impulses. The second one is characterized by abrupt changes that remain active for a finite interval of time is called noninstantaneous impulses. Many evolutionary processes can be modeled using noninstantaneous impulses such as the flow of drugs in blood streams (hemodynamic equilibrium of a person), decompensation, and many others. In this context, impulsive fractional differential equations are studied in different aspects; see, for example [13, 14, 17, 24, 26, 30, 32, 34, 41, 49].

Stability analysis, which has been solely studied for differential equations of arbitrary order and abundantly discussed by the researchers, is the theory related to the stability of differential equations. In stability theory, the Ulam stability was first established by Ulam [35] in 1940 and then was extended by Hyers and Rassias [12, 25]. More recent results on the so-called Hyers–Ulam stability have relaxed the stability conditions. Many mathematicians extended the Hyers results in different directions [4, 18, 19, 28–31, 33, 36, 37, 39, 41–45, 47–49]. The monographs [5, 6, 16, 38] treated fractional differential equations with instantaneous impulses of the following form:

$$\begin{cases} {}^{c}\mathcal{D}^{r}\nu(\tau) = u(\tau,\nu(\tau)), & \tau \in [0,\mathrm{T}], \mathrm{T} > 0, \tau \neq \tau_k, k = 1,2,\ldots,m, \\ \Delta\nu(\tau) = \Upsilon_k(\nu(\tau_k^-)), & k = 1,2,\ldots,m, \end{cases}$$

where ${}^{c}\mathcal{D}^{r}$ is the Caputo fractional derivative of order $r \in (n-1,n)$, n is any natural number with lower bound 0, $u : [0,\mathrm{T}] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\Upsilon_k : \mathbb{R} \to \mathbb{R}$ is instantaneous impulse, and τ_k satisfies $0 = \tau_0 < \tau_1 < \cdots < \tau_m = \mathrm{T}$, $\nu(\tau_k^+) = \lim_{\epsilon\to 0}\nu(\tau_k + \epsilon)$ and $\nu(\tau_k^-) = \lim_{\epsilon\to 0}\nu(\tau_k + \epsilon)$ denotes the right and left limits of $\nu(\tau)$ at $\tau = \tau_k$, respectively.

Ahmad et al. [3] studied an implicit type of nonlinear impulsive fractional differential equations given by

$$\begin{cases} {}^{c}\mathcal{D}^{r}y(\tau) = f(\tau,y(\tau),{}^{c}\mathcal{D}^{r}y(\tau)), & \tau \in [0,1], \tau \neq \tau_k, k = 1,2,\ldots,m, \\ \Delta y(\tau) = \Upsilon_k(y(\tau_k)), \qquad \Delta y'(\tau) = \hat{\Upsilon}_k(y(\tau_k)), & k = 1,2,\ldots,m, \\ y(0) = g(y), \qquad y(1) = h(y), \end{cases}$$

where ${}^{c}\mathcal{D}^{r}$ is Caputo fractional derivative of order $1 < r \leq 2, f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\Upsilon_k, \hat{\Upsilon}_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, and

$$\begin{aligned} \Delta y(\tau_k) &= y(\tau_k^+) - y(\tau_k^-), \\ \Delta y'(\tau_k) &= y'(\tau_k^+) - y'(\tau_k^-), \end{aligned}$$

where $(\tau_k^+), y'(\tau_k^+), y(\tau_k^-), y'(\tau_k^-)$ are the respective left and right limits of $y(\tau_k)$ at $\tau = \tau_k$.

Recently, Wang et al. [39] studied the existence, uniqueness, and different kinds of stability in the sense of Ulam for the following nonlinear implicit fractional integrodifferential equation of the form

$$\begin{cases} {}^c\mathcal{D}^p u(\tau) = \alpha(\tau, u(\tau), {}^c\mathcal{D}^p u(\tau)) + \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau - s)^{\sigma-1} g(s, u(s), {}^c\mathcal{D}^p u(s))\,ds, \\ \quad \tau \in \mathcal{J}, \\ u(\tau)|_{\tau=0} = -u(\tau)|_{\tau=T}, \qquad {}^c\mathcal{D}^r u(\tau)|_{\tau=0} = -{}^c\mathcal{D}^r u(\tau)|_{\tau=\mathrm{T}}, \end{cases} \tag{1.1}$$

where ${}^c\mathcal{D}^p$ and ${}^c\mathcal{D}^r$ is the Caputo fractional derivatives of orders $1 < p \leq 2$ and $0 \leq r \leq 2$, $\mathcal{J} = [0, \mathrm{T}]$ with $\mathrm{T}, \sigma, \delta > 0$, and the functions $\alpha, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. Also, they performed the same analysis for the proposed implicit coupled system:

$$\begin{cases} {}^c\mathcal{D}^p u(\tau) - \alpha(\tau, y(\tau), {}^c\mathcal{D}^p u(\tau)) - \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau - s)^{\sigma-1} g(s, y(s), {}^c\mathcal{D}^p u(s))\,ds = 0, \\ \quad \tau \in \mathcal{J}, \\ {}^c\mathcal{D}^q y(\tau) - \chi(\tau, u(\tau), {}^c\mathcal{D}^q y(\tau)) - \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau - s)^{\sigma-1} f(s, u(s), {}^c\mathcal{D}^p y(s))\,ds = 0, \\ \quad \tau \in \mathcal{J}, \\ u(\tau)|_{\tau=0} = -u(\tau)|_{\tau=\mathrm{T}}, \qquad {}^c\mathcal{D}^r u(\tau)|_{\tau=0} = -{}^c\mathcal{D}^r u(\tau)|_{\tau=\mathrm{T}}, \\ y(\tau)|_{\tau=0} = -y(\tau)|_{\tau=\mathrm{T}}, \qquad {}^c\mathcal{D}^\omega y(\tau)|_{\tau=0} = -{}^c\mathcal{D}^\omega y(\tau)|_{\tau=\mathrm{T}}, \end{cases} \tag{1.2}$$

where ${}^c\mathcal{D}^p$, ${}^c\mathcal{D}^r$, ${}^c\mathcal{D}^q$, and ${}^c\mathcal{D}^\omega$ are the Caputo fractional derivatives of orders $1 < p, q \leq 2$ and $0 \leq r, \omega \leq 2$, $\sigma, \delta > 0$, $\mathcal{J} = [0, \mathrm{T}]$, $\mathrm{T} > 0$, and the functions $\alpha, \chi, g, f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

In the present study, we extend models (1.1) and (1.2) to impulsive systems with Riemann–Liouville boundary conditions instead of antiperiodic boundary condition. More precisely, we study the model

$$\begin{cases} {}^c\mathcal{D}^r \omega(\tau) = \mathcal{A}(\tau, \omega(\tau), {}^c\mathcal{D}^r \omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, \omega(s), {}^c\mathcal{D}^r \omega(s))\,ds, \\ \quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_i, i = 1, 2, \ldots, m, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\ \eta_1\omega(0) + \xi_1 I^r \omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r \omega(\mathrm{T}) = \nu_2, \end{cases} \tag{1.3}$$

where ${}^c\mathcal{D}^r$ is the Caputo fractional derivative with $1 < r \leq 2, \mathcal{J} = [0, \mathrm{T}]$ with $\mathrm{T} > 0$, and $\sigma, \delta > 0$, the functions $\mathcal{A}, \mathcal{B} : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and $\eta_1, \eta_2, \xi_1, \xi_2$ are positive constants.

The first results of this paper establish the existence and uniqueness of solution for this problem. Also, we investigate the following implicit coupled system:

$$\begin{cases} {}^c\mathcal{D}^r \omega(\tau) = \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r \omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, y(s), {}^c\mathcal{D}^r \omega(s))\,ds, \\ \quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_i, i = 1, 2, \ldots, m, \\ {}^c\mathcal{D}^p y(\tau) = \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\,ds, \\ \quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_j, j = 1, 2, \ldots, n, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\ \Delta y(\tau_j) = \Upsilon_j(y(\tau_j)), \qquad \Delta y'(\tau_j) = \hat{\Upsilon}_j(y(\tau_j)), \quad j = 1, 2, \ldots, n, \\ \eta_1\omega(0) + \xi_1 I^r \omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r \omega(\mathrm{T}) = \nu_2, \\ \eta_3 y(0) + \xi_3 I^p y(0) = \nu_3, \qquad \eta_4 y(\mathrm{T}) + \xi_4 I^p y(\mathrm{T}) = \nu_4, \end{cases} \tag{1.4}$$

where ${}^{c}\mathcal{D}^{r}$ and ${}^{c}\mathcal{D}^{p}$ are the Caputo fractional derivatives with $1 < r, p \leq 2, \mathcal{J} = [0, \mathrm{T}]$ with $\mathrm{T} > 0$, $\sigma, \delta > 0$, the functions $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}' : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and $\eta_1, \eta_2, \eta_3, \eta_4, \xi_1, \xi_2, \xi_3, \xi_4$ are positive constants. Coupled systems of fractional integrodifferential equations have also been extensively studied due to their applications. Some recent works dealing with coupled systems of Caputo fractional differential equations involving different kinds of integral boundary conditions can be found in [50].

The second main results are devoted to the study of stability results for both systems. There are two main classes of stability results considered here, Ulam–Hyers and Ulam–Hyers–Rassias stability, and their generalized equivalents. To be more specific, our aim is to build connections between stability results in both systems.

It is important to note that problem (1.3) and the coupled one (1.4) considered in this paper extend the study of fractional integrodifferential systems, and from this point of view, we believe that the obtained results will contribute to the existing literature on the topic.

The rest of the paper is organized as follows: In Sect. 2, we first establish an equivalent integral equation for the fractional integrodifferential equations with impulse, and we obtain existence results by using the Banach contraction principle, Schauder's fixed point theorem, and Krasnoselskii's fixed point theorem to the proposed problems (1.3) and (1.4), respectively. In Sect. 3, we consider four types of Ulam–Hyers stability concepts. Finally, in Sect. 4, we construct two examples to illustrate the obtained results. Fundamental definitions, essential lemmas, and the proofs of the main theorems are given in Appendices 1, 2, and 3.

Notation: We denote by $\mathcal{M}$ the space of all piecewise continuous functions $\mathrm{PC}(\mathcal{J}, \mathbb{R})$; $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_i$, where $\mathcal{J}_0 = [0, \tau_1], \mathcal{J}_1 = (\tau_1, \tau_2], \mathcal{J}_2 = (\tau_2, \tau_3], \ldots, \mathcal{J}_i = (\tau_i, \tau_{i+1}], i = 1, 2, \ldots, m$, and $\mathcal{J}' = \mathcal{J} - \{\tau_1, \tau_2, \tau_3, \ldots, \tau_i\}$.

We define $\mathcal{M} = \{\omega : \mathcal{J} \to \mathbb{R} : \omega \in C(\mathcal{J}_i, \mathbb{R})$ and $\omega(\tau_i^+), \omega(\tau_i^-)$ exist such that $\Delta\omega(\tau_i) = \omega(\tau_i^+) - \omega(\tau_i^-)$ for $i = 1, 2, \ldots, m\}$.

2 Existence and uniqueness

The aim of this section is giving conditions under which the fractional integrodifferential equation (1.3) and coupled system (1.4) provide existence and uniqueness results.

2.1 Existence and uniqueness solution for system (1.3)

Our first result is stated as follows.

Theorem 2.1 *Let $1 < r \leq 2$, and let $\alpha \in \mathcal{M}$ be a continuous function. Then a function $\omega \in \mathcal{M}$ is solution to the problem*

$$\begin{cases} {}^{c}\mathcal{D}^{r}\omega(\tau) = \alpha(\tau), \quad \tau \in \mathcal{J}, \tau \neq \tau_i, i = 1, 2, \ldots, m, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\ \eta_1\omega(0) + \xi_1 I^r\omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r\omega(\mathrm{T}) = \nu_2, \end{cases} \tag{2.1}$$

where

$$\alpha(\tau) = \mathcal{A}\big(\tau, \omega(\tau), {}^{c}\mathcal{D}^{r}\omega(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}\big(s, \omega(s), {}^{c}\mathcal{D}^{r}\omega(s)\big)\, ds,$$

if and only if ω satisfies

$$\omega(\tau)=\begin{cases}\frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds-\frac{\tau}{\eta_2\mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds\\ \quad+\frac{\nu_1}{\eta_1}-\frac{\tau\nu_1}{\mathrm{T}\eta_1}+\frac{\tau\nu_2}{\mathrm{T}\eta_2}\\ \quad-\frac{\tau}{\mathrm{T}}[\frac{1}{\Gamma(r)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds+\frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds\\ \quad+\frac{(\mathrm{T}-\tau_1)}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds+(\mathrm{T}-\tau_1)\hat{\Upsilon}_1(\omega(\tau_1))+\Upsilon_1(\omega(\tau_1))],\\ \quad\tau\in\mathcal{J}_0,\\ \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds-\frac{\tau}{\eta_2\mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds\\ \quad+\frac{\nu_1}{\eta_1}-\frac{\tau\nu_1}{\mathrm{T}\eta_1}+\frac{\tau\nu_2}{\mathrm{T}\eta_2}\\ \quad-\frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds+\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds\\ \quad+\frac{(\mathrm{T}-\tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds+(\mathrm{T}-\tau_i)\hat{\Upsilon}_i(\omega(\tau_i))+\Upsilon_i(\omega(\tau_i))],\\ \quad\tau\in\mathcal{J}_i, i=1,2,\ldots,m.\end{cases}\tag{2.2}$$

Proof Applying Lemma A.3 (see Appendix 1) to (2.1) with $a_0, a_1 \in \mathbb{R}$, we have

$$\omega(\tau)=I^r\alpha(\tau)-a_0-a_1\tau=\frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds-a_0-a_1\tau,\quad \tau\in[0,\tau_1].\tag{2.3}$$

Furthermore, we obtain

$$\omega'(\tau)=I^{r-1}\alpha(\tau)-a_1=\frac{1}{\Gamma(r-1)}\int_0^{\tau}(\tau-s)^{r-2}\alpha(s)\,ds-a_1,\quad \tau\in[0,\tau_1].$$

For $\tau \in (\tau_1, \tau_2]$, there are $b_0, b_1 \in \mathbb{R}$ such that

$$\begin{cases}\omega(\tau)=\frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds-b_0-b_1(\tau-\tau_1),\\ \omega'(\tau)=\frac{1}{\Gamma(r-1)}\int_{\tau_1}^{\tau}(\tau-s)^{r-2}\alpha(s)\,ds-b_1.\end{cases}$$

Hence it follows that

$$\begin{cases}\omega(\tau_1^-)=\frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds-a_0-a_1\tau_1,\\ \omega(\tau_1^+)=-b_0,\\ \omega'(\tau_1^-)=\frac{1}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds-a_1,\\ \omega'(\tau_1^+)=-b_1.\end{cases}$$

Using

$$\begin{cases}\Delta\omega(\tau_1)=\omega(\tau_1^+)-\omega(\tau_1^-)=\Upsilon_1(\omega(\tau_1)),\\ \Delta\omega'(\tau_1)=\omega'(\tau_1^+)-\omega'(\tau_1^-)=\hat{\Upsilon}_1(\omega(\tau_1)),\end{cases}$$

we obtain

$$\begin{cases}-b_0=\frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds-a_0-a_1\tau_1+\Upsilon_1(\omega(\tau_1)),\\ -b_1=\frac{1}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds-a_1+\hat{\Upsilon}_1(\omega(\tau_1)).\end{cases}$$

Thus

$$\begin{aligned}\omega(\tau) &= \frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds \\ &\quad + \frac{\tau-\tau_1}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds \\ &\quad + (\tau-\tau_1)\hat{\Upsilon}_1\big(\omega(\tau_1)\big) + \Upsilon_1\big(\omega(\tau_1)\big) - a_0 - a_1\tau, \quad \tau\in(\tau_1,\tau_2].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\omega(\tau) &= \sum_{i=1}^{m}\bigg[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds \\ &\quad + \frac{\tau-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds + (\tau-\tau_i)\hat{\Upsilon}_i\big(\omega(\tau_i)\big) + \Upsilon_i\big(\omega(\tau_i)\big)\bigg] \\ &\quad - a_0 - a_1\tau, \\ &\quad \tau\in(\tau_i,\tau_{i+1}], i=1,2,\dots,m. \end{aligned} \tag{2.4}$$

Finally, after applying $\eta_1\omega(0)+\xi_1 I^r\omega(0)=\nu_1$ and $\eta_2\omega(\mathrm{T})+\xi_2 I^r\omega(\mathrm{T})=\nu_2$ to (2.4) and calculating the values of a_0 and a_1, we obtain equation (2.2).

Conversely, if $\omega(\tau)$ is a solution of (2.2), then it is obvious that ${}^c\mathcal{D}^r\omega(\tau)=\alpha(\tau)$ and $\eta_1\omega(0)+\xi_1 I^r\omega(0)=\nu_1, \eta_2\omega(\mathrm{T})+\xi_2 I^r\omega(\mathrm{T})=\nu_2, \Delta\omega(\tau_i)=\Upsilon_i(\omega(\tau_i)), \Delta\omega'(\tau_i)=\hat{\Upsilon}_i(\omega(\tau_i))$, $i=1,2,\dots,m$. □

Corollary 2.2 *In light of Theorem 2.1, problem (1.3) has the solution*

$$\omega(\tau)=\begin{cases}\frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds - \frac{\tau}{\eta_2\mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau\nu_1}{\mathrm{T}\eta_1} + \frac{\tau\nu_2}{\mathrm{T}\eta_2} \\ \quad - \frac{\tau}{\mathrm{T}}[\frac{1}{\Gamma(r)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds \\ \quad + \frac{(\mathrm{T}-\tau_1)}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(\omega(\tau_1)) + \Upsilon_1(\omega(\tau_1))], \quad \tau\in\mathcal{J}_0, \\ \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds - \frac{\tau}{\eta_2\mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau\nu_1}{\mathrm{T}\eta_1} + \frac{\tau\nu_2}{\mathrm{T}\eta_2} \\ \quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds \\ \quad + \frac{(\mathrm{T}-\tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i(\omega(\tau_i)) + \Upsilon_i(\omega(\tau_i))], \\ \quad \tau\in\mathcal{J}_i, i=1,2,\dots,m,\end{cases}$$

where

$$\alpha(\tau)=\mathcal{A}\big(\tau,\omega(\tau),{}^c\mathcal{D}^r\omega(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\omega(s),{}^c\mathcal{D}^r\omega(s)\big)\,ds.$$

Let

$$\begin{aligned}\nu(\tau) &= \mathcal{A}\big(\tau,\omega(\tau),{}^c\mathcal{D}^r\omega(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\omega(s),{}^c\mathcal{D}^r\omega(s)\big)\,ds \\ &= \mathcal{A}\big(\tau,\omega(\tau),\nu(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\omega(s),\nu(s)\big)\,ds.\end{aligned}$$

Also, we consider $\mathcal{M} = \mathrm{PC}(\mathcal{J}, \mathbb{R})$ endowed with the norm

$$\|\omega\|_{\mathcal{M}} = \max\bigl\{\bigl|\omega(\tau)\bigr| : \tau \in \mathcal{J}\bigr\}.$$

We can easily see that $\mathcal{M}$ is a Banach space. Further, if ω is a solution of problem (1.3), then

$$\begin{aligned}\omega(\tau) ={}& \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\alpha(s)\,ds - \frac{\tau}{\eta_2 \mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau\nu_1}{\mathrm{T}\eta_1} + \frac{\tau\nu_2}{\mathrm{T}\eta_2} \\ & - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds \\ & + \frac{(\mathrm{T}-\tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr) + \Upsilon_i\bigl(\omega(\tau_i)\bigr)\biggr], \\ & \tau \in \mathcal{J}_i, i = 1,2,\ldots,m.\end{aligned}$$

Now, to study (1.3) by fixed point theory, let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be the operator defined as

$$\bigl(\mathcal{T}\omega(\tau)\bigr) = \begin{cases} \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\nu(s)\,ds - \frac{\tau}{\eta_2 \mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds \\ \quad + \frac{\nu_1}{\eta_1} - \frac{\tau\nu_1}{\mathrm{T}\eta_1} + \frac{\tau\nu_2}{\mathrm{T}\eta_2} \\ \quad - \frac{\tau}{\mathrm{T}}[\frac{1}{\Gamma(r)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\nu(s)\,ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\nu(s)\,ds \\ \quad + \frac{(\mathrm{T}-\tau_1)}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\nu(s)\,ds + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(\omega(\tau_1)) + \Upsilon_1(\omega(\tau_1))], \\ \quad \tau \in \mathcal{J}_0, \\ \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\nu(s)\,ds - \frac{\tau}{\eta_2 \mathrm{T}}\frac{\xi_2}{\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds \\ \quad + \frac{\nu_1}{\eta_1} - \frac{\tau\nu_1}{\mathrm{T}\eta_1} + \frac{\tau\nu_2}{\mathrm{T}\eta_2} \\ \quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\nu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\nu(s)\,ds \\ \quad + \frac{(\mathrm{T}-\tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\nu(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i(\omega(\tau_i)) + \Upsilon_i(\omega(\tau_i))], \\ \quad \tau \in \mathcal{J}_i, i = 1,2,\ldots,m, \end{cases} \tag{2.5}$$

where

$$\nu(\tau) = \mathcal{A}\bigl(\tau, \omega(\tau), \nu(\tau)\bigr) + \int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, \omega(s), \nu(s)\bigr)\,ds.$$

Let us assume the following hypotheses:

- $[A_1]$ There exist constants $\mathrm{M}_1 > 0$ and $\mathrm{N}_1 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{M}$, and $w, \overline{w} \in \mathbb{R}$,

$$\bigl|\mathcal{A}(\tau, u, w) - \mathcal{A}(\tau, \overline{u}, \overline{w})\bigr| \le \mathrm{M}_1|u - \overline{u}| + \mathrm{N}_1|w - \overline{w}|.$$

Similarly, there exist constants $\mathrm{M}_2 > 0$ and $\mathrm{N}_2 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{M}$, and $w, \overline{w} \in \mathbb{R}$,

$$\bigl|\mathcal{B}(\tau, u, w) - \mathcal{B}(\tau, \overline{u}, \overline{w})\bigr| \le \mathrm{M}_2|u - \overline{u}| + \mathrm{N}_2|w - \overline{w}|;$$

- $[A_2]$ For any $u, \overline{u} \in \mathcal{M}$, there exist constants $\mathbb{A}, \mathbb{B} > 0$ such that

$$\begin{aligned} &\left|\Upsilon_i\big(u(\tau_i)\big) - \Upsilon_i\big(\overline{u}(\tau_i)\big)\right| \le \mathbb{A}\left|u(\tau_i) - \overline{u}(\tau_i)\right|, \\ &\left|\hat{\Upsilon}_i\big(u(\tau_i)\big) - \hat{\Upsilon}_i\big(\overline{u}(\tau_i)\big)\right| \le \mathbb{B}\left|u(\tau_i) - \overline{u}(\tau_i)\right|, \quad i = 1, 2, \ldots, m; \end{aligned}$$

- $[A_3]$ There exist bounded functions $l_1, m_1, n_1 \in \mathcal{M}$ such that

$$\left|\mathcal{A}\big(\tau, u(\tau), w(\tau)\big)\right| \le l_1(\tau) + m_1(\tau)\left|u(\tau)\right| + n_1(\tau)\left|w(\tau)\right|$$

  with $l_1^* = \sup_{\tau\in\mathcal{J}} l_1(\tau), m_1^* = \sup_{\tau\in\mathcal{J}} m_1(\tau)$ and $n_1^* = \sup_{\tau\in\mathcal{J}} n_1(\tau) < 1$.

  Similarly, there exist bounded functions $l_2, m_2, n_2 \in \mathcal{M}$ such that

$$\left|\mathcal{B}\big(\tau, u(\tau), w(\tau)\big)\right| \le l_2(\tau) + m_2(\tau)\left|u(\tau)\right| + n_2(\tau)\left|w(\tau)\right|$$

  with $l_2^* = \sup_{\tau\in\mathcal{J}} l_2(\tau), m_2^* = \sup_{\tau\in\mathcal{J}} m_2(\tau)$, and
  $n_2^* = \sup_{\tau\in\mathcal{J}} n_2(\tau) < 1$ with $1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)} > 0$;
- $[A_4]$ The functions $\Upsilon_i : \mathbb{R} \to \mathbb{R}, i = 1, 2, \ldots, m$, are continuous for each $u \in \mathbb{R}$. There exist constants $\mathcal{K}_{\Upsilon_i}, \mathcal{L}_{\Upsilon_i} > 0$ such that $|\Upsilon_i(u(\tau_i))| \le \mathcal{K}_{\Upsilon_i}|u(\tau)| + \mathcal{L}_{\Upsilon_i}$.

  Similarly, for each $u \in \mathbb{R}$, the functions $\hat{\Upsilon}_i : \mathbb{R} \to \mathbb{R}; i = 1, 2, \ldots, m$, are continuous, and for constants $\mathcal{K}'_{\hat{\Upsilon}_i}, \mathcal{L}'_{\hat{\Upsilon}_i} > 0$, we have the inequality $|\hat{\Upsilon}_i(u(\tau_i))| \le \mathcal{K}'_{\hat{\Upsilon}_i}|u(\tau)| + \mathcal{L}'_{\hat{\Upsilon}_i}$.

The main results of this section are presented in the following theorems.

Theorem 2.3 *If hypotheses* $[A_1]$–$[A_4]$ *are satisfied, then problem* (1.3) *has at least one solution.*

Proof See Appendix 2. □

Theorem 2.4 *If hypotheses* $[A_1]$–$[A_2]$ *and the inequality*

$$\begin{aligned} &\left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)\right. \\ &\quad \left. + \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A} + \mathbb{B})\right] < 1 \quad \text{with } 1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)} > 0 \end{aligned} \tag{2.6}$$

are satisfied, then problem (1.3) *has a unique solution.*

Proof See Appendix 2. □

Our approach to prove the existence of the solution for problem (1.3) from Theorem 2.3 is based on Theorem A.5 (see Appendix 1). Also, the proof of the uniqueness for problem (1.3) treated in Theorem 2.4 is based on the arguments from Theorem A.6 (see Appendix 1).

In Sect. 4, we will provide an example demonstrating how (2.6) can be computed in a specific case.

2.2 Existence and uniqueness solution for system (1.4)

In this section, we consider the coupled system of nonlinear implicit fractional differential equation with impulsive conditions from (1.4). First, we have the following:

Theorem 2.5 *The system*

$$\begin{cases} {}^cD^r\omega(\tau) = \alpha(\tau), \quad \tau \in \mathcal{J}, \\ {}^cD^p y(\tau) = \beta(\tau), \quad \tau \in \mathcal{J}, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1,2,\ldots,m, \\ \Delta y(\tau_j) = \Upsilon_j(y(\tau_j)), \qquad \Delta y'(\tau_j) = \hat{\Upsilon}_j(y(\tau_j)), \quad j = 1,2,\ldots,n, \\ \eta_1\omega(0) + \xi_1 I^r\omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r\omega(\mathrm{T}) = \nu_2, \\ \eta_3 y(0) + \xi_3 I^p y(0) = \nu_3, \qquad \eta_4 y(\mathrm{T}) + \xi_4 I^p y(\mathrm{T}) = \nu_4 \end{cases}$$

has a solution (ω, y) *if and only if*

$$\omega(\tau) = \begin{cases} \frac{1}{\Gamma(r)}\int_0^\tau(\tau-s)^{r-1}\alpha(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds \\ \quad + \frac{1}{\Gamma(r)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds \\ \quad + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\alpha(s)\,ds + \frac{\mathrm{T}-\tau_1}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\alpha(s)\,ds \\ \quad + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(\omega(\tau_1)) + \Upsilon_1(\omega(\tau_1)) + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}], \quad \tau \in \mathcal{J}_0, \\ \frac{1}{\Gamma(r)}\int_0^\tau(\tau-s)^{r-1}\alpha(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}] \\ \quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^m[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds \\ \quad + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds \\ \quad + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i(\omega(\tau_i)) + \Upsilon_i(\omega(\tau_i))], \quad \tau \in \mathcal{J}_i, \end{cases}$$

and

$$y(\tau) = \begin{cases} \frac{1}{\Gamma(p)}\int_0^\tau(\tau-s)^{p-1}\beta(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}[\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y(s)\,ds \\ \quad + \frac{1}{\Gamma(p)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\beta(s)\,ds \\ \quad + \frac{1}{\Gamma(p)}\int_0^{\tau_1}(\tau_1-s)^{p-1}\beta(s)\,ds + \frac{\mathrm{T}-\tau_1}{\Gamma(p-1)}\int_0^{\tau_1}(\tau_1-s)^{p-2}\beta(s)\,ds \\ \quad + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(y(\tau_1)) + \Upsilon_1(y(\tau_1)) + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4}], \quad \tau \in \mathcal{J}_0, \\ \frac{1}{\Gamma(p)}\int_0^\tau(\tau-s)^{p-1}\beta(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}[\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4}] \\ \quad - \frac{\tau}{\mathrm{T}}\sum_{j=1}^n[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\beta(s)\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}\beta(s)\,ds \\ \quad + \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\beta(s)\,ds \\ \quad + (\mathrm{T}-\tau_j)\hat{\Upsilon}_j(y(\tau_j)) + \Upsilon_j(y(\tau_j))], \quad \tau \in \mathcal{J}_j, \end{cases}$$

where

$$\alpha(\tau) = \mathcal{A}\big(\tau, y(\tau), {}^cD^r\omega(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, y(s), {}^cD^r\omega(s)\big)\,ds$$

and

$$\beta(\tau) = \mathcal{A}'\big(\tau, \omega(\tau), {}^cD^p y(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'\big(s, \omega(s), {}^cD^p y(s)\big)\,ds.$$

Proof The proof is similar to that given in Theorem 2.1 and hence is not included here. □

For $\tau_i \in \mathcal{J}$ such that $\tau_1 < \tau_2 < \cdots < \tau_m$ and $\mathcal{J}' = \mathcal{J} - \{\tau_1, \tau_2, \ldots, \tau_m\}$, we define the space $\mathcal{X} = \{\omega : \mathcal{J} \to \mathbb{R} | \omega \in \mathcal{C}(\mathcal{J}')$, right limit $\omega(\tau_i^+)$ and left limit $\omega(\tau_i^-)$ exist, and $\Delta\omega(\tau_i) = \omega(\tau_i^-) - \omega(\tau_i^+), 1 < i \leq m\}$. Clearly, $(\mathcal{X}, \|\cdot\|)$ is a Banach space endowed with the norm $\|\omega\| = \max_{\tau\in\mathcal{J}} |\omega|$.

Similarly, for $\tau_j \in \mathcal{J}$ such that $\tau_1 < \tau_2 < \cdots < \tau_n$ and $\mathcal{J}' = \mathcal{J} - \{\tau_1, \tau_2, \ldots, \tau_n\}$, we define the space $\mathcal{Y} = \{y : \mathcal{J} \to \mathbb{R} | y \in \mathcal{C}(\mathcal{J}')$, right limit $y(\tau_j^+)$ and left limit $y(\tau_j^-)$ exist, and $\Delta y(\tau_i) = y(\tau_j^-) - y(\tau_j^+), 1 < j \leq n\}$, which is a Banach space endowed with the norm $\|y\| = \max_{\tau\in\mathcal{J}} |y|$.

Consequently, the product space $\mathcal{X} \times \mathcal{Y}$ is a Banach space with the norm $\|(\omega, y)\| = \|\omega\| + \|y\|$ or $\|(\omega, y)\| = \max\{\|\omega\|, \|y\|\}$.

Theorem 2.6 *Let $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ be continuous functions. Then $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ is a solution of problem (1.4) if and only if (ω, y) is a solution of*

$$\begin{aligned}
\omega(\tau) ={}& \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \alpha(s)\, ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}} \biggl[\frac{\xi_2}{\eta_2 \Gamma(r)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \omega(s)\, ds \\
&+ \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} \biggr] - \frac{\tau}{\mathrm{T}} \sum_{i=1}^{m} \biggl[\frac{1}{\Gamma(r)} \int_{\tau_i}^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \alpha(s)\, ds + \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \alpha(s)\, ds \\
&+ \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2} \alpha(s)\, ds + (\mathrm{T} - \tau_i) \hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr) + \Upsilon_i\bigl(\omega(\tau_i)\bigr) \biggr], \\
&\tau \in \mathcal{J}_i,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
y(\tau) ={}& \frac{1}{\Gamma(p)} \int_0^\tau (\tau - s)^{p-1} \beta(s)\, ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}} \biggl[\frac{\xi_4}{\eta_4 \Gamma(p)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{p-1} y(s)\, ds \\
&+ \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4} \biggr] - \frac{\tau}{\mathrm{T}} \sum_{j=1}^{n} \biggl[\frac{1}{\Gamma(p)} \int_{\tau_j}^{\mathrm{T}} (\mathrm{T} - s)^{p-1} \beta(s)\, ds + \frac{1}{\Gamma(p)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{p-1} \beta(s)\, ds \\
&+ \frac{\mathrm{T} - \tau_j}{\Gamma(p-1)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{p-2} \beta(s)\, ds + (\mathrm{T} - \tau_j) \hat{\Upsilon}_j\bigl(y(\tau_j)\bigr) + \Upsilon_j\bigl(y(\tau_j)\bigr) \biggr], \quad \tau \in \mathcal{J}_j.
\end{aligned}$$

Proof If (ω, y) is a solution of system (1.4), then it is a solution of (2.7). Conversely, if (ω, y) is a solution of (2.7), then

$$\begin{cases}
{}^c\mathcal{D}^r \omega(\tau) = \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r \omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, y(s), {}^c\mathcal{D}^r \omega(s))\, ds \\
\quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_i \text{ for } i = 1, 2, \ldots, m, \\
{}^c\mathcal{D}^p y(\tau) = \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\, ds \\
\quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_j \text{ for } j = 1, 2, \ldots, n, \\
\Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\
\Delta y(\tau_j) = \Upsilon_j(y(\tau_j)), \qquad \Delta y'(\tau_j) = \hat{\Upsilon}_j(y(\tau_j)), \quad j = 1, 2, \ldots, n, \\
\eta_1 \omega(0) + \xi_1 I^r \omega(0) = \nu_1, \qquad \eta_2 \omega(\mathrm{T}) + \xi_2 I^r \omega(\mathrm{T}) = \nu_2, \\
\eta_3 y(0) + \xi_3 I^p y(0) = \nu_3, \qquad \eta_4 y(\mathrm{T}) + \xi_4 I^p y(\mathrm{T}) = \nu_4.
\end{cases}$$

Thus (ω, y) is a solution of (1.4). □

For convenience, we use the following notations:

$$\begin{aligned}
\nu(\tau) &= \mathcal{A}\big(\tau, y(\tau), {}^cD^r\omega(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, y(s), {}^cD^r\omega(s)\big)\,ds \\
&= \mathcal{A}\big(\tau, y(\tau), \nu(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, y(s), \nu(s)\big)\,ds, \\
z(\tau) &= \mathcal{A}'\big(\tau, \omega(\tau), {}^cD^p y(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'\big(s, \omega(s), {}^cD^p y(s)\big)\,ds \\
&= \mathcal{A}'\big(\tau, \omega(\tau), z(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'\big(s, \omega(s), z(s)\big)\,ds.
\end{aligned}$$

System (1.4) can be transformed into a fixed point problem.

Define the operators $\mathcal{T}_r, \mathcal{T}_p : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ by

$$\mathcal{T}_r(\omega, y)(\tau) = \begin{cases}
\frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\nu(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds \\
\quad + \frac{1}{\Gamma(r)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\nu(s)\,ds \\
\quad + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\nu(s)\,ds + \frac{\mathrm{T}-\tau_1}{\Gamma(r-1)}\int_0^{\tau_1}(\tau_1-s)^{r-2}\nu(s)\,ds \\
\quad + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(\omega(\tau_1)) + \Upsilon_1(\omega(\tau_1)) + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}], \quad \tau \in \mathcal{J}_0, \\
\frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\nu(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}] \\
\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^m[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\nu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\nu(s)\,ds \\
\quad + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\nu(s)\,ds \\
\quad + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i(\omega(\tau_i)) + \Upsilon_i(\omega(\tau_i))], \quad \tau \in \mathcal{J}_i,
\end{cases}$$

and

$$\mathcal{T}_p(\omega, y)(\tau) = \begin{cases}
\frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}z(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}[\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y(s)\,ds \\
\quad + \frac{1}{\Gamma(p)}\int_{\tau_1}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}z(s)\,ds \\
\quad + \frac{1}{\Gamma(p)}\int_0^{\tau_1}(\tau_1-s)^{p-1}z(s)\,ds + \frac{\mathrm{T}-\tau_1}{\Gamma(p-1)}\int_0^{\tau_1}(\tau_1-s)^{p-2}z(s)\,ds \\
\quad + (\mathrm{T}-\tau_1)\hat{\Upsilon}_1(y(\tau_1)) + \Upsilon_1(y(\tau_1)) + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4}], \quad \tau \in \mathcal{J}_0, \\
\frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}z(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}[\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4}] \\
\quad - \frac{\tau}{\mathrm{T}}\sum_{j=1}^n[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}z(s)\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}z(s)\,ds \\
\quad + \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}z(s)\,ds \\
\quad + (\mathrm{T}-\tau_j)\hat{\Upsilon}_j(y(\tau_j)) + \Upsilon_j(y(\tau_j))], \quad \tau \in \mathcal{J}_j,
\end{cases}$$

with $\mathcal{T}(\omega, y)(\tau) = (\mathcal{T}_r(\omega, y)(\tau), \mathcal{T}_p(\omega, y)(\tau))$.

We further need the following hypotheses:

- $[\tilde{A_1}]$ there exist constants $\mathrm{M}_1 > 0$ and $\mathrm{N}_1 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{X}$, and $w, \overline{w} \in \mathbb{R}$, we have

$$\big|\mathcal{A}(\tau, u, w) - \mathcal{A}(\tau, \overline{u}, \overline{w})\big| \le \mathrm{M}_1|u - \overline{u}| + \mathrm{N}_1|w - \overline{w}|.$$

Similarly, there exist constants $\mathrm{M}_2 > 0$ and $\mathrm{N}_2 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{X}$, and $w, \overline{w} \in \mathbb{R}$, we have

$$\left|\mathcal{B}(\tau,u,w) - \mathcal{B}(\tau,\overline{u},\overline{w})\right| \le \mathrm{M}_2|u-\overline{u}| + \mathrm{N}_2|w-\overline{w}|;$$

- $[\tilde{A_2}]$ there exist constants $\mathrm{M}'_1 > 0$ and $\mathrm{N}'_1 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{Y}$, and $w, \overline{w} \in \mathbb{R}$, we have

$$\left|\mathcal{A}'(\tau,u,w) - \mathcal{A}'(\tau,\overline{u},\overline{w})\right| \le \mathrm{M}'_1|u-\overline{u}| + \mathrm{N}'_1|w-\overline{w}|.$$

Similarly, there exist constants $\mathrm{M}'_2 > 0$ and $\mathrm{N}'_2 \in (0,1)$ such that, for all $\tau \in \mathcal{J}$, $u, \overline{u} \in \mathcal{Y}$, and $w, \overline{w} \in \mathbb{R}$, we have

$$\left|\mathcal{B}'(\tau,u,w) - \mathcal{B}'(\tau,\overline{u},\overline{w})\right| \le \mathrm{M}'_2|u-\overline{u}| + \mathrm{N}'_2|w-\overline{w}|;$$

- $[\tilde{A_3}]$ for any $w, \overline{w} \in \mathcal{X} \times \mathcal{Y}$, there exist constants $A_{\Upsilon_i}, A_{\hat{\Upsilon}_i} > 0$ such that

$$\begin{aligned}
&\left|\Upsilon_i\big(w(\tau_i)\big) - \Upsilon_i\big(\overline{w}(\tau_i)\big)\right| \le A_{\Upsilon_i}\left|w(\tau_i) - \overline{w}(\tau_i)\right|;\\
&\left|\hat{\Upsilon}_i\big(w(\tau_i)\big) - \hat{\Upsilon}_i\big(\overline{w}(\tau_i)\big)\right| \le A_{\hat{\Upsilon}_i}\left|w(\tau_i) - \overline{w}(\tau_i)\right|, \quad i = 1,2,\ldots,m.
\end{aligned}$$

Similarly, for any $y, \overline{y} \in \mathcal{X} \times \mathcal{Y}$, there exist constants $A_{\Upsilon_j}, A_{\hat{\Upsilon}_j} > 0$ such that

$$\begin{aligned}
&\left|\Upsilon_j\big(w(\tau_j)\big) - \Upsilon_j\big(\overline{w}(\tau_j)\big)\right| \le A_{\Upsilon_j}\left|w(\tau_j) - \overline{w}(\tau_j)\right|;\\
&\left|\hat{\Upsilon}_j\big(w(\tau_j)\big) - \hat{\Upsilon}_j\big(\overline{w}(\tau_j)\big)\right| \le A_{\hat{\Upsilon}_j}\left|w(\tau_j) - \overline{w}(\tau_j)\right|, \quad j = 1,2,\ldots,n;
\end{aligned}$$

- $[\tilde{A_4}]$ there exist $a_1, b_1, c_1 \in \mathcal{X}$ such that

$$\left|\mathcal{A}\big(\tau,u(\tau),w(\tau)\big)\right| \le a_1(\tau) + b_1(\tau)\left|u(\tau)\right| + c_1(\tau)\left|w(\tau)\right|$$

with $a_1^* = \sup_{\tau\in\mathcal{J}} a_1(\tau), b_1^* = \sup_{\tau\in\mathcal{J}} b_1(\tau), c_1^* = \sup_{\tau\in\mathcal{J}} c_1(\tau) < 1$.
Similarly, there exist $a_2, b_2, c_2 \in \mathcal{X}$ such that

$$\left|\mathcal{B}\big(\tau,u(\tau),w(\tau)\big)\right| \le a_2(\tau) + b_2(\tau)\left|u(\tau)\right| + c_2(\tau)\left|w(\tau)\right|$$

with $a_2^* = \sup_{\tau\in\mathcal{J}} a_2(\tau), b_2^* = \sup_{\tau\in\mathcal{J}} b_2(\tau), c_2^* = \sup_{\tau\in\mathcal{J}} c_2(\tau) < 1$ with $1 - c_1^* - c_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}$;
- $[\tilde{A_5}]$ there exist $l_1, m_1, n_1 \in \mathcal{Y}$ such that

$$\left|\mathcal{A}'\big(\tau,u(\tau),w(\tau)\big)\right| \le l_1(\tau) + m_1(\tau)\left|u(\tau)\right| + n_1(\tau)\left|w(\tau)\right|$$

with $l_1^* = \sup_{\tau\in\mathcal{J}} l_1(\tau), m_1^* = \sup_{\tau\in\mathcal{J}} m_1(\tau), n_1^* = \sup_{\tau\in\mathcal{J}} n_1(\tau) < 1$.
Similarly, there exist $l_2, m_2, n_2 \in \mathcal{Y}$ such that

$$\left|\mathcal{B}'\big(\tau,u(\tau),w(\tau)\big)\right| \le l_2(\tau) + m_2(\tau)\left|u(\tau)\right| + n_2(\tau)\left|w(\tau)\right|$$

with $l_2^* = \sup_{\tau\in\mathcal{J}} l_2(\tau), m_2^* = \sup_{\tau\in\mathcal{J}} m_2(\tau), n_2^* = \sup_{\tau\in\mathcal{J}} n_2(\tau) < 1$ with $1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)} > 0$;

- $[\tilde{A_6}]$ The functions $\Upsilon_i : \mathbb{R} \to \mathbb{R}; i = 1, 2, \dots, m$, are continuous for each $u \in \mathbb{R}$. There exist constants $\mathcal{K}_{\Upsilon_i}, \mathcal{L}_{\Upsilon_i} > 0$ such that $|\Upsilon_i(u(\tau_i))| \leq \mathcal{K}_{\Upsilon_i}|u(\tau)| + \mathcal{L}_{\Upsilon_i}$.
  Similarly, the functions $\hat{\Upsilon}_i : \mathbb{R} \to \mathbb{R}; i = 1, 2, \dots, m$, are continuous for each $u \in \mathbb{R}$. There exist constants constants $\mathcal{K}'_{\hat{\Upsilon}_i}, \mathcal{L}'_{\hat{\Upsilon}_i} > 0$ such that $|\hat{\Upsilon}_i(u(\tau_i))| \leq \mathcal{K}'_{\hat{\Upsilon}_i}|u(\tau)| + \mathcal{L}'_{\hat{\Upsilon}_i}$;
- $[\tilde{A_7}]$ The functions $\Upsilon_j : \mathbb{R} \to \mathbb{R}; j = 1, 2, \dots, n$, are continuous for each $u \in \mathbb{R}$. There exist constants $\mathcal{K}_{\Upsilon_j}, \mathcal{L}_{\Upsilon_j} > 0$ such that $|\Upsilon_j(u(\tau_j))| \leq \mathcal{K}_{\Upsilon_j}|u(\tau)| + \mathcal{L}_{\Upsilon_j}$.
  Similarly, the functions $\hat{\Upsilon}_j : \mathbb{R} \to \mathbb{R}; j = 1, 2, \dots, n$, are continuous for each $u \in \mathbb{R}$. There exist constants $\mathcal{K}'_{\hat{\Upsilon}_i}, \mathcal{L}'_{\hat{\Upsilon}_i} > 0$ such that $|\hat{\Upsilon}_j(u(\tau_j))| \leq \mathcal{K}'_{\hat{\Upsilon}_i}|u(\tau)| + \mathcal{L}'_{\hat{\Upsilon}_i}$;
- $[\tilde{A_8}]$ Denote

$$\Delta_1 = \left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right) + \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A}_{\hat{\Upsilon}_i} + \mathbb{A}_{\Upsilon_i})\right] < 1 \quad \text{with } 1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)} > 0$$

and

$$\Delta_2 = \left[\left(\frac{n\mathrm{T}^p}{\Gamma(p+1)} + \frac{n\mathrm{T}^{p-1}}{\Gamma(p)}\right)\left(\frac{\mathrm{M}'_1}{1 - \mathrm{N}'_1 - \mathrm{N}'_2\frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}'_2\frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}'_1 - \mathrm{N}'_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right) + \frac{\xi_4\mathrm{T}^p}{\eta_4\Gamma(p+1)} + n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j})\right] < 1 \quad \text{with } 1 - \mathrm{N}'_1 - \mathrm{N}'_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)} > 0.$$

Now, we are in position to state the main results of this section.

Theorem 2.7 *If hypotheses* $[\tilde{A_1}]$–$[\tilde{A_4}]$ *are satisfied, then problem* (1.4) *has at least one solution.*

Proof See Appendix 2. □

Theorem 2.8 *If* $\Delta = \max(\Delta_1, \Delta_2) < 1$, *then under hypotheses* $[\tilde{A_1}]$–$[\tilde{A_7}]$, *system* (1.4) *has a unique solution.*

Proof See Appendix 2. □

3 Hyers–Ulam stability

In this section, we provide novel characterizations of the Hyers–Ulam stability for systems (1.3) and (1.4). We rely on stability notions from [21]; for various concepts of Hyers–Ulam stability, see, for example [37, 43, 46, 47].

3.1 Hyers–Ulam stability concepts for system (1.3)

For $\omega \in \mathcal{M}, \epsilon_r > 0, \phi_r \geq 0$, and a nondecreasing function $\psi_r \in C(\mathcal{J}, \mathbb{R}_+)$, the following set of inequalities are satisfied:

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau, \omega(\tau), {}^c\mathcal{D}^r\omega(\tau)) - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, \omega(s), {}^c\mathcal{D}^r\omega(s))\,ds| \leq \epsilon_r, \\ \quad \tau \in \mathcal{J}_i, i = 1, 2, \dots, m, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \leq \epsilon_r, \quad i = 1, 2, \dots, m, \end{cases} \tag{3.1}$$

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau,\omega(\tau),{}^c\mathcal{D}^r\omega(\tau)) - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s,\omega(s),{}^c\mathcal{D}^r\omega(s))\,ds| \leq \psi_r(\tau), \\ \quad \tau \in \mathcal{J}_i, i = 1,2,\ldots,m, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \leq \phi_r, \quad i = 1,2,\ldots,m, \end{cases} \tag{3.2}$$

and

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau,\omega(\tau),{}^c\mathcal{D}^r\omega(\tau)) - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s,\omega(s),{}^c\mathcal{D}^r\omega(s))\,ds| \\ \quad \leq \epsilon_r\psi_r(\tau), \quad \tau \in \mathcal{J}_i, i = 1,2,\ldots,m, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \leq \epsilon_r\phi_r, \quad i = 1,2,\ldots,m. \end{cases} \tag{3.3}$$

Recall the definitions of stability concepts from [21].

Definition 3.1 Problem (1.3) is said to be Hyers–Ulam stable if there exists $\mathcal{C}_{\mathcal{A},\mathcal{B}} > 0$ such that, for each $\epsilon_r > 0$ and any solution $\omega \in \mathcal{M}$ of inequality (3.1), there exists a unique solution $\omega^* \in \mathcal{M}$ of problem (1.3) such that

$$\|\omega(\tau) - \omega^*(\tau)\|_{\mathcal{M}} \leq \mathcal{C}_{\mathcal{A},\mathcal{B}}\epsilon_r \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.2 Problem (1.3) is said to be generalized Hyers–Ulam stable if there exists a function $\vartheta \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$ with $\vartheta(0) = 0$ such that, for each $\epsilon_r > 0$ and any solution $\omega \in \mathcal{M}$ of inequality (3.1), there exists a unique solution $\omega^* \in \mathcal{M}$ of problem (1.3) such that

$$\|\omega(\tau) - \omega^*(\tau)\|_{\mathcal{M}} \leq \mathcal{C}_{\mathcal{A},\mathcal{B}}\vartheta(\epsilon_r) \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.3 Problem (1.3) is said to be Hyers–Ulam–Rassias stable with respect to (ϕ_r, ψ_r) if there exists $\mathcal{C}_{\mathcal{A},\mathcal{B}} > 0$ such that, for each $\epsilon_r > 0$ and any solution $\omega \in \mathcal{M}$ of inequality (3.3), there exists a unique solution $\omega^* \in \mathcal{M}$ of problem (1.3) such that

$$\|\omega(\tau) - \omega^*(\tau)\|_{\mathcal{M}} \leq \mathcal{C}_{\mathcal{A},\mathcal{B}}\epsilon_r\big(\phi_r + \psi_r(\tau)\big) \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.4 Problem (1.3) is said to be generalized Hyers–Ulam–Rassias stable with respect to (ϕ_r, ψ_r) if there exists $\mathcal{C}_{\mathcal{A},\mathcal{B}} > 0$ such that, for each $\epsilon_r > 0$ and any solution $\omega \in \mathcal{M}$ of inequality (3.2), there exists a unique solution $\omega^* \in \mathcal{M}$ of problem (1.3) such that

$$\|\omega(\tau) - \omega^*(\tau)\|_{\mathcal{M}} \leq \mathcal{C}_{\mathcal{A},\mathcal{B}}\big(\phi_r + \psi_r(\tau)\big) \quad \text{for all } \tau \in \mathcal{J}.$$

Some remarks are in order.

Remark 3.5 Definition 3.1 implies Definition 3.2, and Definition 3.3 implies Definition 3.4.

Remark 3.6 A function $\omega \in \mathcal{M}$ is a solution of inequality (3.1) if there exist a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on ω) such that

(i) $|\Phi(\tau)| \leq \epsilon_r$ and $|\Phi_i| \leq \epsilon_r$ for all $\tau \in \mathcal{J}, i = 1,2,\ldots,m$;

(ii) ${}^c\mathcal{D}^r\omega(\tau) = \mathcal{A}(\tau,\omega(\tau),{}^c\mathcal{D}^r\omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s,\omega(s),{}^c\mathcal{D}^r\omega(s))\,ds + \Phi(\tau)$ for all$\tau \in \mathcal{J}$; and

(iii) $\Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)) + \Phi_i$ for all $\tau \in \mathcal{J}, i = 1,2,\ldots,m$.

Remark 3.7 A function $\omega \in \mathcal{M}$ is a solution of inequality (3.2) if there exist a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on ω) such that

(i) $|\Phi(\tau)| \leq \psi_r(\tau)$ and $|\Phi_i| \leq \phi_r$ for all $\tau \in \mathcal{J}, i = 1, 2, \ldots, m$;

(ii) ${}^c\mathcal{D}^r\omega(\tau) = \mathcal{A}(\tau, \omega(\tau), {}^c\mathcal{D}^r\omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, \omega(s), {}^c\mathcal{D}^r\omega(s))\,ds + \Phi(\tau)$ for all $\tau \in \mathcal{J}$; and

(iii) $\Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)) + \Phi_i$ for all $\tau \in \mathcal{J}, i = 1, 2, \ldots, m$.

Remark 3.8 A function $\omega \in \mathcal{M}$ is a solution of inequality (3.3) if there exist a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on ω) such that

(i) $|\Phi(\tau)| \leq \psi_r(\tau)$ and $|\Phi_i| \leq \epsilon_r\phi_r$ for all $\tau \in \mathcal{J}, i = 1, 2, \ldots, m$;

(ii) ${}^c\mathcal{D}^r\omega(\tau) = \mathcal{A}(\tau, \omega(\tau), {}^c\mathcal{D}^r\omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, \omega(s), {}^c\mathcal{D}^r\omega(s))\,ds + \Phi(\tau)$ for all $\tau \in \mathcal{J}$; and

(iii) $\Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)) + \Phi_i$ for all $\tau \in \mathcal{J}, i = 1, 2, \ldots, m$.

Definition 3.9 A function $\omega \in \mathcal{M}$ that satisfies (1.3) and its conditions on $\mathcal{J}$ is a solution of problem (1.3).

Theorem 3.10 *If $\omega \in \mathcal{M}$ is a solution of inequality* (3.1), *then ω is a solution of the inequality*

$$\left|\omega(\tau) - q(\tau)\right| \leq \left(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}}\right)\epsilon_r.$$

Proof Let ω be a solution of inequality (3.1). Then by Remark 3.6 ω is also a solution of

$$\begin{cases} {}^c\mathcal{D}^r\omega(\tau) = \mathcal{A}(\tau, \omega(\tau), {}^c\mathcal{D}^r\omega(\tau)) \\ \qquad + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, \omega(s), {}^c\mathcal{D}^r\omega(s))\,ds + \Phi(\tau), \\ \quad \tau \in \mathcal{J}, \tau \neq \tau_i, i = 1, 2, \ldots, m, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\ \eta_1\omega(0) + \xi_1 I^r\omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r\omega(\mathrm{T}) = \nu_2, \end{cases} \tag{3.4}$$

that is,

$$\begin{aligned} \omega(\tau) = {} & \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\nu(s)\,ds \\ & + \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\Phi(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\left[\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}\right. \\ & \left. + \frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\omega(s)\,ds\right] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^m\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\nu(s)\,ds\right. \\ & + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\Phi(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\nu(s)\,ds \\ & \left. + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-2}\nu(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\big(\omega(\tau_i)\big) + \Upsilon_i\big(\omega(\tau_i)\big) + \Phi_i\right]. \end{aligned} \tag{3.5}$$

For simplicity, let $q(\tau)$ denote the terms of $\omega(\tau)$ that are free from $\Phi(\tau)$, that is,

$$\begin{aligned} q(\tau) &= \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\nu(s)\,ds + \frac{\nu_1}{\eta_1} \\ &\quad - \frac{\tau}{\mathrm{T}}\left[\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} + \frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\omega(s)\,ds\right] \\ &\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\nu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\nu(s)\,ds \right. \\ &\quad \left. + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-2}\nu(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\big(\omega(\tau_i)\big) + \Upsilon_i\big(\omega(\tau_i)\big)\right]. \end{aligned}$$

Thus (3.5) can be written as

$$\begin{aligned} &\left|\omega(\tau) - q(\tau)\right| \\ &\quad \le \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\left|\Phi(s)\right|ds - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\left|\Phi(s)\right|ds + |\Phi_i|\right]. \end{aligned}$$

Using (i) from Remark 3.6, we get

$$\left|\omega(\tau) - q(\tau)\right| \le \left(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{T\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}}\right)\epsilon_r. \qquad \square$$

Theorem 3.11 *If hypothesis* $[A_1]$ *holds and*

$$\begin{aligned} &\left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right) \right. \\ &\quad \left. + \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A}+\mathbb{B})\right] < 1, \end{aligned} \tag{3.6}$$

then problem (1.3) *is Ulam–Hyers and generalized Ulam–Hyers stable.*

Proof See Appendix 3. □

Assume that

- $[A_5]$ there exist a nondecreasing function $\psi_r \in \mathcal{M}$ and a constant $\varrho_{\psi_r} > 0$ such that, for each $\tau \in \mathcal{J}$, we have

$$I^{\varrho}\psi_r(\tau) \le \varrho_{\psi_r}\psi_r(\tau).$$

From Theorem 3.11 and $[A_5]$ we obtain the following theorem.

Theorem 3.12 *Under hypotheses* $[A_1]$–$[A_5]$ *and condition* (3.6), *problem* (1.3) *is Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.*

3.2 Hyers–Ulam stability concepts for system (1.4)

Let $\epsilon_r, \epsilon_p > 0, \mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ be continuous functions, and $\psi_r, \psi_p : \mathcal{J} \to \mathbb{R}^+$ be nondecreasing functions. Consider the following inequalities:

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, y(s), {}^c\mathcal{D}^r\omega(s))\,ds| \le \epsilon_r, \quad \tau \in \mathcal{J}, \\ |{}^c\mathcal{D}^p y(\tau) - \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\,ds| \le \epsilon_p, \quad \tau \in \mathcal{J}, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \le \epsilon_r, \quad i = 1, 2, \dots, m, \\ |\Delta y(\tau_j) - \Upsilon_j(y(\tau_j))| \le \epsilon_p, \quad j = 1, 2, \dots, n, \end{cases} \tag{3.7}$$

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, y(s), {}^c\mathcal{D}^r\omega(s))\,ds| \le \psi_r, \quad \tau \in \mathcal{J}, \\ |{}^c\mathcal{D}^p y(\tau) - \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\,ds| \le \psi_p, \quad \tau \in \mathcal{J}, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \le \phi_r, \quad i = 1, 2, \dots, m, \\ |\Delta y(\tau_j) - \Upsilon_j(y(\tau_j))| \le \phi_p, \quad j = 1, 2, \dots, n, \end{cases} \tag{3.8}$$

and

$$\begin{cases} |{}^c\mathcal{D}^r\omega(\tau) - \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, y(s), {}^c\mathcal{D}^r\omega(s))\,ds| \le \epsilon_r\psi_r, \quad \tau \in \mathcal{J}, \\ |{}^c\mathcal{D}^p y(\tau) - \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) \\ \quad - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\,ds| \le \epsilon_p\psi_p, \quad \tau \in \mathcal{J}, \\ |\Delta\omega(\tau_i) - \Upsilon_i(\omega(\tau_i))| \le \epsilon_r\phi_r, \\ \quad i = 1, 2, \dots, m, \\ |\Delta y(\tau_j) - \Upsilon_j(y(\tau_j))| \le \epsilon_p\phi_p, \quad j = 1, 2, \dots, n. \end{cases} \tag{3.9}$$

Recall the appropriate definitions of stability concepts from [21].

Definition 3.13 Problem (1.4) is said to be Hyers–Ulam stable if there exists $\mathcal{C}_{r,p} = \max(\mathcal{C}_r, \mathcal{C}_p) > 0$ for some $\epsilon = (\epsilon_r, \epsilon_p)$ and for each solution $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ of (3.7), there exists a solution $(\omega^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of (1.4) with

$$\left\|(\omega, y)(\tau) - \left(\omega^*, y^*\right)(\tau)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \mathcal{C}_{r,p}\epsilon \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.14 Problem (1.4) is said to be generalized Hyers–Ulam stable if there exists a function $\Theta \in C(\mathcal{J}, \mathbb{R})$ with $\Theta(0) = 0$ such that for each solution $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ of (3.7), there exists a solution $(\omega^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of (1.4) with

$$\left\|(\omega, y)(\tau) - \left(\omega^*, y^*\right)(\tau)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \Theta(\epsilon) \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.15 Problem (1.4) is said to be Hyers–Ulam–Rassias stable with respect to $\psi_{r,p} = (\psi_r, \psi_p) \in C^1(\mathcal{J}, \mathbb{R})$ if there exists a constant $\mathcal{C}_{\psi_r,\psi_p} = \max(\mathcal{C}_{\psi_r}, \mathcal{C}_{\psi_p})$ such that, for some $\epsilon = (\epsilon_r, \epsilon_p) > 0$ and for each solution $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ of (3.8), there exists a solution $(\omega^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of (1.4) with

$$\left\|(\omega,y)(\tau) - \left(\omega^*,y^*\right)(\tau)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \mathcal{C}_{\psi_r,\psi_p}\epsilon \quad \text{for all } \tau \in \mathcal{J}.$$

Definition 3.16 Problem (1.4) is said to be generalized Hyers–Ulam–Rassias stable with respect to $\psi_{r,p} = (\psi_r, \psi_p) \in C^1(\mathcal{J}, \mathbb{R})$ if there exists a constant $\mathcal{C}_{\psi_r,\psi_p} = \max(\mathcal{C}_{\psi_r}, \mathcal{C}_{\psi_p}) > 0$ such that, for each solution $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ of (3.9), there exists a solution $(\omega^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of (1.4) with

$$\left\|(\omega,y)(\tau) - \left(\omega^*,y^*\right)(\tau)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \mathcal{C}_{\psi_r,\psi_p}\psi_{r,p} \quad \text{for all } \tau \in \mathcal{J}.$$

We have two remarks.

Remark 3.17 Definition 3.13 implies Definition 3.14, and Definition 3.15 implies Definition 3.16.

Remark 3.18 We say that $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ is a solution of (3.7) if there exist the functions $\mu_{\mathcal{A},\mathcal{B}}$, $\Lambda_{\mathcal{A}',\mathcal{B}'} \in \mathcal{X} \times \mathcal{Y}$, depending upon ω, y, respectively, such that

(i) $|\mu_{\mathcal{A},\mathcal{B}}(\tau)| \le \epsilon_r, |\Lambda_{\mathcal{A}',\mathcal{B}'}(\tau)| \le \epsilon_p$ for all $\tau \in \mathcal{J}$;

(ii)

$$\begin{aligned} {}^c\mathcal{D}^r\omega(\tau) &= \mathcal{A}\left(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)\right) \\ &\quad + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\left(s, y(s), {}^c\mathcal{D}^r\omega(s)\right) ds + \mu_{\mathcal{A},\mathcal{B}}(\tau), \\ &\quad \tau \in \mathcal{J}_i, \end{aligned}$$

and

$$\begin{aligned} {}^c\mathcal{D}^p y(\tau) &= \mathcal{A}\left(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)\right) \\ &\quad + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\left(s, \omega(s), {}^c\mathcal{D}^r y(s)\right) ds + \Lambda_{\mathcal{A}',\mathcal{B}'}(\tau), \\ &\quad \tau \in \mathcal{J}_j; \end{aligned}$$

(iii) $\Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)) + \mu_i, \tau \in \mathcal{J}_i, i = 1, 2, \ldots, m$, and $\Delta y(\tau_j) = \Upsilon_j(y(\tau_j)) + \Lambda_j$, $\tau \in \mathcal{J}_j$, $j = 1, 2, \ldots, n$.

Theorem 3.19 *Let $(\omega, y) \in \mathcal{X} \times \mathcal{Y}$ be a solution of inequality (3.7). Then we have*

$$\begin{cases} |\omega(\tau) - q(\tau)| \le (\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}})\epsilon_r, & \tau \in \mathcal{J}, \\ |y(\tau) - q'(\tau)| \le (\frac{\tau^p}{\Gamma(p+1)} - \frac{n\tau^{p+1}}{\mathrm{T}\Gamma(p+1)} - \frac{\tau n}{\mathrm{T}})\epsilon_p, & \tau \in \mathcal{J}. \end{cases}$$

Proof Let (ω, y) be a solution of inequality (3.7). Then by Remark 3.18 (ω, y) is also a solution of

$$\begin{cases} {}^c\mathcal{D}^r\omega(\tau) = \mathcal{A}(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s, y(s), {}^c\mathcal{D}^r\omega(s))\,ds + \mu_{\mathcal{A},\mathcal{B}} \\ \quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_i \text{ for } i = 1, 2, \ldots, m, \\ {}^c\mathcal{D}^p y(\tau) = \mathcal{A}'(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'(s, \omega(s), {}^c\mathcal{D}^p y(s))\,ds + \Lambda_{\mathcal{A}',\mathcal{B}'} \\ \quad \text{where } \tau \in \mathcal{J}, \tau \neq \tau_j, j = 1, 2, \ldots, n, \\ \Delta\omega(\tau_i) = \Upsilon_i(\omega(\tau_i)), \qquad \Delta\omega'(\tau_i) = \hat{\Upsilon}_i(\omega(\tau_i)), \quad i = 1, 2, \ldots, m, \\ \Delta y(\tau_j) = \Upsilon_j(y(\tau_j)), \qquad \Delta y'(\tau_j) = \hat{\Upsilon}_j(y(\tau_j)), \quad j = 1, 2, \ldots, n, \\ \eta_1\omega(0) + \xi_1 I^r\omega(0) = \nu_1, \qquad \eta_2\omega(\mathrm{T}) + \xi_2 I^r\omega(\mathrm{T}) = \nu_2, \\ \eta_3 y(0) + \xi_3 I^p y(0) = \nu_3, \qquad \eta_4 y(\mathrm{T}) + \xi_4 I^p y(\mathrm{T}) = \nu_4, \end{cases} \tag{3.10}$$

that is,

$$\begin{aligned} \omega(\tau) = {} & \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\mu(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\biggl[\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} \\ & + \frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\omega(s)\,ds\biggr] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\alpha(s)\,ds \\ & + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\mu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-1}\alpha(s)\,ds \\ & + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr) + \Upsilon_i\bigl(\omega(\tau_i)\bigr) + \mu_i\biggr] \end{aligned} \tag{3.11a}$$

and

$$\begin{aligned} y(\tau) = {} & \frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}\beta(s)\,ds + \frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}\lambda(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}\biggl[\frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4} \\ & + \frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{p-1}y(s)\,ds\biggr] - \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\biggl[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}} (\mathrm{T}-s)^{p-1}\beta(s)\,ds \\ & + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j} (\tau_j-s)^{p-1}\lambda(s)\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j} (\tau_j-s)^{p-1}\beta(s)\,ds \\ & + \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j} (\tau_j-s)^{p-2}\beta(s)\,ds + (\mathrm{T}-\tau_j)\hat{\Upsilon}_j\bigl(y(\tau_j)\bigr) + \Upsilon_j\bigl(y(\tau_j)\bigr) + \lambda_j\biggr], \end{aligned} \tag{3.11b}$$

where

$$\alpha(\tau) = \mathcal{A}\bigl(\tau, y(\tau), {}^c\mathcal{D}^r\omega(\tau)\bigr) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, y(s), {}^c\mathcal{D}^r\omega(s)\bigr)\,ds$$

and

$$\beta(\tau) = \mathcal{A}'\bigl(\tau, \omega(\tau), {}^c\mathcal{D}^p y(\tau)\bigr) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'\bigl(s, \omega(s), {}^c\mathcal{D}^p y(s)\bigr)\,ds.$$

From (3.11a) we have

$$\omega(\tau) = \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\mu(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\biggl[\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} + \frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds\biggr] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\mu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr) + \Upsilon_i\bigl(\omega(\tau_i)\bigr) + \mu_i\biggr]. \tag{3.12}$$

Thus (3.12) becomes

$$\bigl|\omega(\tau)-q(\tau)\bigr| \le \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\bigl|\mu(s)\bigr|\,ds - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\bigl|\mu(s)\bigr|\,ds + |\mu_i|\biggr],$$

where

$$q(\tau) = \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\alpha(s)\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\biggl[\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} + \frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega(s)\,ds\biggr] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\alpha(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\alpha(s)\,ds + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\alpha(s)\,ds + (\mathrm{T}-\tau_i)\hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr) + \Upsilon_i\bigl(\omega(\tau_i)\bigr)\biggr].$$

Using (i) from Remark 3.18, we obtain

$$\bigl|\omega(\tau)-q(\tau)\bigr| \le \biggl(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}}\biggr)\epsilon_r.$$

Repeating a similar procedure for (3.11b) together with (i) from Remark 3.18, we have

$$\bigl|y(\tau)-q'(\tau)\bigr| \le \biggl(\frac{\tau^p}{\Gamma(p+1)} - \frac{n\tau^{p+1}}{\mathrm{T}\Gamma(p+1)} - \frac{\tau n}{\mathrm{T}}\biggr)\epsilon_p,$$

where

$$q'(\tau) = \frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}\beta(s)\,ds + \frac{\nu_3}{\eta_3} - \frac{\tau}{\mathrm{T}}\biggl[\frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4} + \frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y(s)\,ds\biggr] - \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\biggl[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\beta(s)\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}\beta(s)\,ds + \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\beta(s)\,ds + (\mathrm{T}-\tau_j)\hat{\Upsilon}_j\bigl(y(\tau_j)\bigr) + \Upsilon_j\bigl(y(\tau_j)\bigr)\biggr].$$

Thus the proof is complete. □

Theorem 3.20 *If hypotheses* $[\tilde{A}_1]$–$[\tilde{A}_3]$ *hold with*

$$\Delta = 1 - \mathcal{Q}_r\mathcal{Q}_p > 0, \tag{3.13}$$

then system (1.4) *is stable, in the sense of Ulam–Hyers.*

Proof See Appendix 3. □

In the next section, we provide an example demonstrating how (3.13) can be computed in a specific case. We conclude this section with two remarks.

Remark 3.21 We set $\Theta(\epsilon) = C_{r,p}\epsilon$, $\Theta(0) = 0$ in (C.10). By Definition 3.14 the proposed system (1.4) is generalized Ulam–Hyers stable.

To obtain the connections between the Ulam–Hyers–Rassias stability concepts, we introduce the following hypothesis.

- $[\tilde{A_9}]$ Let $\Omega_r, \Omega_p \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ be an increasing functions. Then there exist $\Lambda_{\Omega_r}, \Lambda_{\Omega_p} > 0$ such that, for each $\tau \in \mathcal{J}$,

  $$I^r\Omega_r(\tau) \leq \Lambda_{\Omega_r}\Omega_r(\tau) \quad \text{and} \quad I^{r-1}\Omega_r(\tau) \leq \Lambda_{\Omega_r}\Omega_r(\tau)$$

  and

  $$I^p\Omega_{p(\tau)} \leq \Lambda_{\Omega_p}\Omega_p(\tau) \quad \text{and} \quad I^{p-1}\Omega_p(\tau) \leq \Lambda_{\Omega_p}\Omega_p(\tau).$$

Remark 3.22 Under hypotheses $[\tilde{A_1}]$–$[\tilde{A_9}]$, by (3.13) and Theorems 3.19 and 3.20 system (1.4) is Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.

4 Illustrative examples

We present two examples to demonstrate the existence and stability of our obtained results.

Example 4.1 Consider

$$\begin{cases} {}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau) = \frac{|\omega(\tau)| + \cos|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|}{90e^{\tau+2}(1+|\omega(\tau)|+|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|)} \\ \qquad + \frac{1}{\Gamma(\frac{5}{2})}\int_0^1(\tau-s)^{\frac{3}{2}}\frac{|\omega(s)|+\sin|{}^c\mathcal{D}^{\frac{3}{2}}\omega(s)|}{101e^{\tau+2}(1+|\omega(s)|+|{}^c\mathcal{D}^{\frac{3}{2}}\omega(s)|)}, \quad \tau \neq \frac{1}{3}, \\ \omega(0) + I^{\frac{3}{2}}\omega(0) = \frac{1}{2}, \qquad \omega(1) + I^{\frac{3}{2}}\omega(1) = \frac{1}{2}, \\ \Delta\omega(\frac{1}{3}) = \Upsilon(\omega(\frac{1}{3})), \qquad \Delta\omega'(\frac{1}{3}) = \hat{\Upsilon}(\omega(\frac{1}{3})), \end{cases} \tag{4.1}$$

where $r = \frac{3}{2}$, $\mathcal{J}_0 = [0, \frac{1}{3}]$, $\mathcal{J}_1 = (\frac{1}{3}, 1]$.

Set

$$\mathcal{A}(\tau, \omega, y) = \frac{|\omega(\tau)| + \cos|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|}{90e^{\tau+2}(1+|\omega(\tau)|+|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|)},$$

$$\mathcal{B}(\tau, \omega, y) = \frac{|\omega(\tau)| + \sin|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|}{101e^{\tau+2}(1+|\omega(\tau)|+|{}^c\mathcal{D}^{\frac{3}{2}}\omega(\tau)|)}.$$

Obviously, $\mathcal{A}$ and $\mathcal{B}$ are jointly continuous functions. Now, for all $\omega,\overline{\omega} \in \mathcal{M}$, $y,\overline{y} \in \mathbb{R}$, and $\tau \in [0,1]$, we have

$$\left|\mathcal{A}(\tau,\omega,y) - \mathcal{A}(t,\overline{\omega},\overline{y})\right| \le \frac{1}{90e^2}\left(|\omega - \overline{\omega}| + |y - \overline{y}|\right)$$

and

$$\left|\mathcal{B}(\tau,\omega,y) - \mathcal{B}(\tau,\overline{\omega},\overline{y})\right| \le \frac{1}{101e^2}\left(|\omega - \overline{\omega}| + |y - \overline{y}|\right).$$

These satisfy condition $[A_1]$ with $\mathrm{M}_1 = \mathrm{N}_1 = \frac{1}{90e^2}$ and $\mathrm{M}_2 = \mathrm{N}_2 = \frac{1}{101e^2}$.

Set

$$\Upsilon_1\left(\omega\left(\frac{1}{3}\right)\right) = \frac{|\omega(\frac{1}{3})|}{40 + |\omega(\frac{1}{3})|} \quad \text{for } \omega \in \mathcal{M}$$

and

$$\hat{\Upsilon}_1\left(\omega\left(\frac{1}{3}\right)\right) = \frac{|\omega(\frac{1}{3})|}{20 + |\omega(\frac{1}{3})|} \quad \text{for } \omega \in \mathcal{M}.$$

Then we have

$$\left|\Upsilon_1\left(\omega\left(\frac{1}{3}\right)\right) - \Upsilon_1\left(\overline{\omega}\left(\frac{1}{3}\right)\right)\right| \le \frac{1}{35}|\omega - \overline{\omega}|$$

and

$$\left|\hat{\Upsilon}_1\left(\omega\left(\frac{1}{3}\right)\right) - \hat{\Upsilon}_1\left(\overline{\omega}\left(\frac{1}{3}\right)\right)\right| \le \frac{1}{20}|\omega - \overline{\omega}|,$$

respectively. Hence $\mathbb{A} = \frac{1}{35}$ and $\mathbb{B} = \frac{1}{20}$. Thus condition $[A_2]$ is satisfied.

Also,

$$\begin{aligned}&\left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)\right.\\ &\quad\left. + \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A} + \mathbb{B})\right] \approx 0.83374 < 1\end{aligned}$$

with $m = 1, \mathrm{T} = 1, \xi_2 = \eta_2 = 1, \sigma = \delta = \frac{5}{2}, r = \frac{3}{2}, \mathrm{M}_1 = \mathrm{N}_1 = \frac{1}{90e^2}, \mathrm{M}_2 = \mathrm{N}_2 = \frac{1}{101e^2}, \mathbb{A} = \frac{1}{35}, \mathbb{B} = \frac{1}{20}$. Therefore by Theorem 2.4 problem (4.1) has a unique solution. Also, letting $\psi(\tau) = |\tau|, \tau \in [0,1]$, we have

$$I^{\frac{1}{2}}\psi(\tau) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\tau (\tau - s)^{(\frac{1}{2}-1)}|s|\,ds = \frac{4^{\frac{3}{2}}}{3\sqrt{\pi}} \le \frac{2\tau}{\sqrt{\pi}}.$$

Hence $[A_5]$ is satisfied with $\mathcal{L}_\psi = \frac{2}{\sqrt{\pi}}$. Therefore by Theorem 3.12 the given problem is Ulam–Hyers–Rassias stable and consequently generalized Ulam–Hyers–Rassias stable.

Example 4.2 Consider

$$\begin{cases} {}^c\mathcal{D}^{\frac{1}{2}}\omega(\tau) = \frac{1+|y(\tau)|+\cos|{}^c\mathcal{D}^{\frac{1}{2}}\omega(\tau)|}{104e^{\tau+5}(1+|y(\tau)|+|{}^c\mathcal{D}^{\frac{1}{2}}\omega(\tau)|)} \\ \qquad + \int_0^1 \frac{(\tau-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\frac{1+|y(s)|+\sin|{}^c\mathcal{D}^{\frac{1}{2}}\omega(s)|}{104e^{s+5}(1+|y(s)|+|{}^c\mathcal{D}^{\frac{1}{2}}\omega(s)|)}\,ds, \quad \tau\in[0,1], \tau\neq\frac{1}{3}, \\ {}^c\mathcal{D}^{\frac{1}{2}}y(\tau) = \frac{2+|\omega(\tau)|+\cos|{}^c\mathcal{D}^{\frac{1}{2}}y(\tau)|}{70e^{\tau+2}(1+|\omega(\tau)|+|{}^c\mathcal{D}^{\frac{1}{2}}y(\tau)|)} \\ \qquad + \int_0^1 \frac{(\tau-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\frac{|\omega(s)|+\cos|{}^c\mathcal{D}^{\frac{1}{2}}y(s)|}{70e^{s+2}(1+|\omega(s)|+|{}^c\mathcal{D}^{\frac{1}{2}}y(s)|)}\,ds, \quad \tau\in[0,1], \tau\neq\frac{1}{4}, \\ \omega(0)+I^{\frac{1}{2}}\omega(0)=\frac{3}{2}, \qquad \omega(1)+I^{\frac{1}{2}}\omega(1)=\frac{3}{2}, \\ y(0)+I^{\frac{1}{2}}y(0)=\frac{3}{2}, \qquad y(1)+I^{\frac{1}{2}}y(1)=\frac{3}{2}, \\ \Delta\omega(\frac{1}{3})=\Upsilon(\omega(\frac{1}{3})), \qquad \Delta\omega'(\frac{1}{3})=\hat{\Upsilon}(\omega(\frac{1}{3})), \\ \Delta y(\frac{1}{4})=\Upsilon(y(\frac{1}{4})), \qquad \Delta y'(\frac{1}{4})=\hat{\Upsilon}(y(\frac{1}{4})), \end{cases} \tag{4.2}$$

$\tau_i = \frac{1}{3}$ for $i = 1,2,3,\dots,60$, and $\tau_j = \frac{1}{4}$ for $j = 1,2,3,\dots,100$.

For any $\omega,\overline{\omega},y,\overline{y}\in\mathbb{R}$ and $\tau\in[0,1]$, we obtain

$$\bigl|\mathcal{A}(\tau,\omega,y)-\mathcal{A}(\tau,\overline{\omega},\overline{y})\bigr| \le \frac{1}{104e^5}\bigl(|\omega-\overline{\omega}|+|y-\overline{y}|\bigr)$$

and

$$\bigl|\mathcal{B}(\tau,\omega,y)-\mathcal{B}(\tau,\overline{\omega},\overline{y})\bigr| \le \frac{1}{104e^5}\bigl(|\omega-\overline{\omega}|+|y-\overline{y}|\bigr).$$

Similarly, for any $\omega,\overline{\omega},y,\overline{y}\in\mathbb{R}$, and $\tau\in[0,1]$, we obtain

$$\bigl|\mathcal{A}'(\tau,\omega,y)-\mathcal{A}'(\tau,\overline{\omega},\overline{y})\bigr| \le \frac{1}{70e^2}\bigl(|\omega-\overline{\omega}|+|y-\overline{y}|\bigr)$$

and

$$\bigl|\mathcal{B}'(\tau,\omega,y)-\mathcal{B}'(\tau,\overline{\omega},\overline{y})\bigr| \le \frac{1}{70e^2}\bigl(|\omega-\overline{\omega}|+|y-\overline{y}|\bigr).$$

These satisfy condition $[\tilde{A}_1]$ with $\mathrm{M}_1=\mathrm{M}_2=\mathrm{N}_1=\mathrm{N}_2=\frac{1}{104e^5}, \mathrm{M}_1'=\mathrm{M}_2'=\mathrm{N}_1'=\mathrm{N}_2'=\frac{1}{70e^2}$.

Set

$$\Upsilon_i\biggl(\omega\biggl(\frac{1}{3}\biggr)\biggr)=\frac{|\omega(\frac{1}{3})|}{40+|\omega(\frac{1}{3})|} \quad \text{for } \omega\in\mathcal{X}$$

and

$$\hat{\Upsilon}_1\biggl(\omega\biggl(\frac{1}{3}\biggr)\biggr)=\frac{|\omega(\frac{1}{3})|}{20+|\omega(\frac{1}{3})|} \quad \text{for } \omega\in\mathcal{X}.$$

Then for $\omega,\overline{\omega}\in\mathcal{X}$, we have

$$\biggl|\Upsilon_i\biggl(\omega\biggl(\frac{1}{3}\biggr)\biggr)-\Upsilon_i\biggl(\overline{\omega}\biggl(\frac{1}{3}\biggr)\biggr)\biggr| = \biggl|\frac{|\omega(\frac{1}{3})|}{40+|\omega(\frac{1}{3})|}-\frac{|\overline{\omega}(\frac{1}{3})|}{40+|\overline{\omega}(\frac{1}{3})|}\biggr|$$

$$\le \frac{1}{35}|\omega - \overline{\omega}|$$

and

$$\left|\hat{\Upsilon}_1\left(\omega\left(\frac{1}{3}\right)\right) - \hat{\Upsilon}_1\left(\overline{\omega}\left(\frac{1}{3}\right)\right)\right| \le \frac{1}{20}|\omega - \overline{\omega}|,$$

respectively. Hence $\mathbb{A}_{\Upsilon_i} = \frac{1}{35}$ and $\mathbb{A}_{\hat{\Upsilon}_i} = \frac{1}{20}$. Thus condition $[\tilde{A}_2]$ is satisfied. Similarly, if

$$\Upsilon_j\left(y\left(\frac{1}{4}\right)\right) = \frac{|y(\frac{1}{4})|}{50 + |y(\frac{1}{4})|} \quad \text{for } y \in \mathcal{Y},$$

then for $y, \overline{y} \in \mathcal{Y}$, we have

$$\begin{aligned}\left|\Upsilon_j\left(y\left(\frac{1}{4}\right)\right) - \Upsilon_j\left(\overline{y}\left(\frac{1}{4}\right)\right)\right| &= \left|\frac{|y(\frac{1}{4})|}{50 + |y(\frac{1}{4})|} - \frac{|\overline{y}(\frac{1}{4})|}{50 + |\overline{y}(\frac{1}{4})}\right| \\ &\le \frac{1}{50}|y - \overline{y}|,\end{aligned}$$

and if

$$\hat{\Upsilon}_j\left(y\left(\frac{1}{4}\right)\right) = \frac{|y(\frac{1}{4})|}{101 + |y(\frac{1}{4})|} \quad \text{for } y \in \mathcal{Y},$$

then for $y, \overline{y} \in \mathcal{Y}$, we have

$$\begin{aligned}\left|\hat{\Upsilon}_j\left(y\left(\frac{1}{4}\right)\right) - \hat{\Upsilon}_j\left(\overline{y}\left(\frac{1}{4}\right)\right)\right| &= \left|\frac{|y(\frac{1}{4})|}{101 + |y(\frac{1}{4})|} - \frac{|\overline{y}(\frac{1}{4})|}{101 + |\overline{y}(\frac{1}{4})}\right| \\ &\le \frac{1}{101}|y - \overline{y}|.\end{aligned}$$

Thus $\mathbb{A}_{\Upsilon_j} = \frac{1}{50}$ and $\mathbb{A}_{\hat{\Upsilon}_j} = \frac{1}{101}$ satisfy our requirement from $[\tilde{A}_3]$.

The condition

$$\begin{aligned}\Delta_1 = \Bigg[&\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right) \\ &+ \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A}_{\hat{\Upsilon}_i} + \mathbb{A}_{\Upsilon_i})\Bigg] \approx 0.83097 < 1\end{aligned}$$

is valid with $m = 1, \mathrm{T} = 1, \xi_2 = \eta_2 = 1, \sigma = \delta = \frac{5}{2}, r = \frac{1}{2}, \mathrm{M}_1 = \mathrm{N}_1 = \mathrm{M}_2 = \mathrm{N}_2 = \frac{1}{104e^5}, \mathbb{A}_{\Upsilon_i} = \frac{1}{35}, \mathbb{A}_{\hat{\Upsilon}_i} = \frac{1}{20}$.

Also,

$$\begin{aligned}\Delta_2 = \Bigg[&\left(\frac{n\mathrm{T}^p}{\Gamma(p+1)} + \frac{n\mathrm{T}^{p-1}}{\Gamma(p)}\right)\left(\frac{\mathrm{M}_1'}{1 - \mathrm{N}_1' - \mathrm{N}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1' - \mathrm{N}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right) \\ &+ \frac{\xi_4\mathrm{T}^p}{\eta_4\Gamma(p+1)} + n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j})\Bigg] \approx 0.78689 < 1\end{aligned}$$

with $n = 1, \mathrm{T} = 1, \xi_4 = \eta_4 = 1, \sigma = \delta = \frac{5}{2}, p = \frac{1}{2}, \mathrm{M}_1' = \mathrm{N}_1' = \mathrm{M}_2' = \mathrm{N}_2' = \frac{1}{70e^2}, \mathbb{A}_{\Upsilon_j} = \frac{1}{50}, \mathbb{A}_{\hat{\Upsilon}_j} = \frac{1}{101}$. Hence $\Delta = \max(\Delta_1, \Delta_2) < 1$ is also true.

It is easy to check that

$$1 - \mathcal{Q}_r \mathcal{Q}_p \approx 1.00000 > 0$$

and condition (3.13) is verified. We conclude that problem (4.2) is Ulam–Hyers stable, generalized Ulam–Hyers stable, Ulam–Hyers–Rassias stable, and generalized Ulam–Hyers–Rassias stable.

Appendix 1: Supplementary results

The following definitions are adopted from [15].

Definition A.1 The integral of a function $u \in L^1(\mathcal{J}, \mathbb{R})$ of order $r \in \mathbb{R}^+$ is defined by

$$I^r u(\tau) = \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} u(s)\, ds,$$

provided that the integral exists.

Definition A.2 The Caputo derivative of a function $u \in C^{(\rho)}((0, \infty), \mathbb{R})$ of arbitrary order r is defined by

$$^c\mathcal{D}^r u(\tau) = \frac{1}{\Gamma(\rho - r)} \int_0^\tau (\tau - s)^{\rho - r - 1} u^{(\rho)}(s)\, ds,$$

where $\rho = [r] + 1$ in which $[r]$ is the integer part of r.

Lemma A.3 *For $r > 0$, the solution of the Caputo fractional differential equation $^cD^r_{0,\tau} u(\tau) = 0$ is*

$$u(\tau) = z_0 + z_1 \tau + z_2 \tau^2 + \cdots + z_{\rho-1} \tau^{\rho-1},$$

where $z_i \in \mathbb{R}$, $i = 0, 1, \ldots, \rho - 1$, and $\rho = [r] + 1$.

Lemma A.4 *For $r > 0$, the solution of $^c\mathcal{D}^r u(\tau) = \beta(\tau)$ is given by*

$$u(\tau) = I^r \beta(\tau) + \sum_{\rho=0}^{n-1} \frac{u^{(\rho)}(0)}{\rho!} \tau^\rho,$$

where $\rho = [r] + 1$.

Theorem A.5 ([10]) *Let $\mathcal{M}$ be a Banach space, let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a completely continuous operator, and let the set $\Omega = \{\omega \in \mathcal{M} : \omega = \aleph \mathcal{T}(\omega), 0 < \aleph < 1\}$ be bounded. Then $\mathcal{T}$ has at least one fixed point in $\mathcal{M}$.*

Theorem A.6 ([10]) *Let $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ be a contraction on a nonempty closed subset of a Banach space $\mathcal{M}$. Then $\mathcal{T}$ has a unique fixed point.*

Theorem A.7 ([40]) *Let $\mathcal{H}$ be a convex, closed, and nonempty subset of Banach space $\mathcal{X} \times \mathcal{Y}$, and let $\mathcal{F},\mathcal{G}$ be the operators such that*

(i) *$\mathcal{F}\omega + \mathcal{G}y \in \mathcal{H}$ whenever $\omega, y \in \mathcal{H}$.*

(ii) *$\mathcal{F}$ is compact and continuous, and $\mathcal{G}$ is a contraction mapping.*

Then there exists $z \in \mathcal{H}$ such that $z = \mathcal{F}z + \mathcal{G}z$, where $z = (\omega, y) \in \mathcal{X} \times \mathcal{Y}$.

Appendix 2

Proof of Theorem 2.3 Consider the operator $\mathcal{T}$ defined in (2.5). We have to show that problem (1.3) has at least one solution.

We show the operator $\mathcal{T}$ is continuous. Consider the sequence $\{\omega_n\}$ such that $\omega_n \to \omega \in \mathcal{M}, \tau \in \mathcal{J}$. Then

$$\begin{aligned}
&\bigl|(\mathcal{T}\omega_n)(\tau) - (\mathcal{T}\omega)(\tau)\bigr| \\
&\quad\leq \frac{1}{\Gamma(r)}\int_0^\tau (\tau - s)^{r-1}\bigl|\nu_n(s) - \nu(s)\bigr|\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T} - s)^{r-1}\bigl|\omega_n(s) - \omega(s)\bigr|\,ds \\
&\qquad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\biggl[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T} - s)^{r-1}\bigl|\nu_n(s) - \nu(s)\bigr|\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1}\bigl|\nu_n(s) - \nu(s)\bigr|\,ds \\
&\qquad + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2}\bigl|\nu_n(s) - \nu(s)\bigr|\,ds + (\mathrm{T} - \tau_i)\bigl|\hat{\Upsilon}_i\bigl(\omega_n(\tau_i)\bigr) - \hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr)\bigr| \\
&\qquad + \bigl|\Upsilon_i\bigl(\omega_n(\tau_i)\bigr) - \Upsilon_i\bigl(\omega(\tau_i)\bigr)\bigr|\biggr],
\end{aligned} \tag{B.1}$$

where $\nu_n, \nu \in \mathcal{M}$ are given by

$$\nu_n(\tau) = \mathcal{A}\bigl(\tau, \omega_n(\tau), \nu_n(\tau)\bigr) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, \omega_n(s), \nu_n(s)\bigr)\,ds$$

and

$$\nu(\tau) = \mathcal{A}\bigl(\tau, \omega(\tau), \nu(\tau)\bigr) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, \omega(s), \nu(s)\bigr)\,ds,$$

respectively. Using hypothesis $[A_1]$, we have

$$\begin{aligned}
&\bigl|\nu_n(\tau) - \nu(\tau)\bigr| \\
&\quad= \biggl|\mathcal{A}\bigl(\tau, \omega_n(\tau), \nu_n(\tau)\bigr) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, \omega_n(s), \nu_n(s)\bigr)\,ds \\
&\qquad - \mathcal{A}\bigl(\tau, \omega(\tau), \nu(\tau)\bigr) - \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s, \omega(s), \nu(s)\bigr)\,ds\biggr| \\
&\quad\leq \bigl|\mathcal{A}\bigl(\tau, \omega_n(\tau), \nu_n(\tau)\bigr) - \mathcal{A}\bigl(\tau, \omega(\tau), \nu(\tau)\bigr)\bigr| \\
&\qquad + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\bigl|\mathcal{B}\bigl(s, \omega_n(s), \nu_n(s)\bigr) - \mathcal{B}\bigl(s, \omega(s), \nu(s)\bigr)\bigr|\,ds \\
&\quad\leq \mathrm{M}_1\bigl|\omega_n(\tau) - \omega(\tau)\bigr| + \mathrm{N}_1\bigl|\nu_n(\tau) - \nu(\tau)\bigr|
\end{aligned}$$

$$+ \frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}\big(\mathrm{M}_2\big|\omega_n(\tau)-\omega(\tau)\big| + \mathrm{N}_2\big|\nu_n(\tau)-\nu(\tau)\big|\big).$$

Then

$$\big|\nu_n(\tau)-\nu(\tau)\big| \le \left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}}\right)\big|\omega_n(\tau)-\omega(\tau)\big|. \tag{B.2}$$

Hypotheses $[A_1]$, $[A_2]$ and inequalities (B.1) and (B.2) lead to

$$\begin{aligned}
&\big|(\mathcal{T}\omega_n)(\tau)-(\mathcal{T}\omega)(\tau)\big| \\
&\quad\le \Bigg[\left(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau\mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{m\tau^r}{\mathrm{T}\Gamma(r)}\right) \\
&\qquad\times\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}}\right) - \frac{\xi_2\tau\mathrm{T}^{r-1}}{\eta_2\Gamma(r+1)} - \frac{\tau}{\mathrm{T}}m(\mathbb{A}+\mathbb{B})\Bigg] \\
&\qquad\times\big|\omega_n(\tau)-\omega(\tau)\big|.
\end{aligned}$$

For each $\tau \in \mathcal{J}$, the sequence $\omega_n \to \omega$ as $n \to \infty$, and hence by the Lebesgue dominated convergence theorem inequality (B.1) implies that

$$\big|(\mathcal{T}\omega_n)(\tau)-(\mathcal{T}\omega)(\tau)\big| \to 0 \quad \text{as } n\to\infty$$

and

$$\|\mathcal{T}\omega_n-\mathcal{T}\omega\|_{\mathcal{M}} \to 0 \quad \text{as } n\to\infty.$$

Hence $\mathcal{T}$ is continuous on $\mathcal{J}$.

Now we have to show that $\mathcal{T}$ is bounded in $\mathcal{M}$. For any $\wp > 0$, there is $\mathrm{R}_{\mathbf{E}} > 0$ such that

$$\mathbf{E} = \big\{\omega \in \mathcal{M} : \|\omega\|_{\mathcal{M}} \le \wp\big\},$$

which leads to

$$\|\mathcal{T}\omega\|_{\mathcal{M}} \le \mathrm{R}_{\mathbf{E}}.$$

For $\tau \in \mathcal{J}_i$, we obtain

$$\begin{aligned}
&\big|(\mathcal{T}\omega)(\tau)\big| \\
&\quad\le \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\big|\nu(s)\big|\,ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\left[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\omega(s)\big|\,ds\right. \\
&\qquad\left.+\frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}\right] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\nu(s)\big|\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\big|\nu(s)\big|\,ds\right. \\
&\qquad\left.+\frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\big|\nu(s)\big|\,ds + (\mathrm{T}-\tau_i)\big|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\big| + \big|\Upsilon_i\big(\omega(\tau_i)\big)\big|\right].
\end{aligned} \tag{B.3}$$

Further, using hypothesis $[A_3]$, we have

$$\begin{aligned}\left|\nu(\tau)\right| &\leq \left|\mathcal{A}\big(\tau,\omega(\tau),\nu(\tau)\big)\right| + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\big(s,\omega(s),\nu(s)\big)\right| ds \\ &\leq l_1(\tau) + m_1(\tau)\left|\omega(\tau)\right| + n_1(\tau)\left|\nu(\tau)\right| + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(l_2(\tau) + m_2(\tau)\left|\omega(\tau)\right| + n_2(\tau)\left|\nu(\tau)\right|\big) \\ &\leq l_1^* + m_1^*\|\omega\|_{\mathcal{M}} + n_1^*\|\nu\|_{\mathcal{M}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\big(l_2^* + m_2^*\|\omega\|_{\mathcal{M}} + n_2^*\|\nu\|_{\mathcal{M}}\big).\end{aligned}$$

Therefore we get

$$\left|\nu(\tau)\right| \leq \|\nu\|_{\mathcal{M}} \leq \frac{l_1^* + m_1^*\|\omega\|_{\mathcal{M}}}{1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\frac{l_2^* + m_2^*\|\omega\|_{\mathcal{M}}}{1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \tag{B.4}$$

Now by (B.4) and $[A_4]$ relation (B.3) becomes

$$\begin{aligned}\left|\mathcal{T}\omega(\tau)\right| &\leq \frac{\hbar\tau^r}{\Gamma(r+1)} + \frac{\nu_1}{\eta_1} \\ &\quad - \frac{\tau}{\mathrm{T}}\bigg[\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} + \frac{m\hbar\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\hbar\tau^r}{\Gamma(r+1)} + \frac{m\hbar\tau^{r-1}}{\Gamma(r)} \\ &\quad + m\big(\mathcal{K}'_{\hat{\Upsilon}_i} + \mathcal{K}_{\Upsilon_i}\big)\left|\omega(\tau)\right| + m\big(\mathcal{L}'_{\hat{\Upsilon}_i} + \mathcal{L}_{\Upsilon_i}\big)\bigg] \\ &= \mathbf{C}.\end{aligned}$$

Thus

$$\|\mathcal{T}\omega\|_{\mathcal{M}} \leq \mathbf{C}.$$

Similarly for $\tau \in \mathcal{J}_0$, we can verify that

$$\|\mathcal{T}\omega\|_{\mathcal{M}} \leq \mathbf{C}.$$

Now we have to show that the operator $\mathcal{T}$ is equicontinuous in $\mathbf{E}$. Let $\tau_1, \tau_2 \in \mathcal{J}_i$ be such that $0 < \tau_1 < \tau_2 < \mathrm{T}$, and let $\omega \in \mathbf{E}$. Then

$$\begin{aligned}&\left|\mathcal{T}\omega(\tau_2) - \mathcal{T}\omega(\tau_1)\right| \\ &\quad\leq \frac{1}{\Gamma(r)}\int_0^{\tau_2}(\tau_2 - s)^{r-1}\left|\nu(s)\right| ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1 - s)^{r-1}\left|\nu(s)\right| ds \\ &\qquad - \frac{\tau}{\mathrm{T}}\sum_{0<\tau_i<\tau_2-\tau_1}\bigg[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T} - s)^{r-1}\left|\nu(s)\right| ds + \frac{(\tau_{i-1} - \tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-2}\left|\nu(s)\right| ds \\ &\qquad + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-1}\left|\nu(s)\right| ds + (\mathrm{T} - \tau_i)\left|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\right| + \left|\Upsilon_i\big(\omega(\tau_i)\big)\right|\bigg] \\ &\quad\leq \frac{1}{\Gamma(r)}\int_0^{\tau_2}\big[(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}\big]\left|\nu(s)\right| ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1 - s)^{r-1}\left|\nu(s)\right| ds \\ &\qquad - \frac{\tau}{\mathrm{T}}\sum_{0<\tau_i<\tau_2-\tau_1}\bigg[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T} - s)^{r-1}\left|\nu(s)\right| ds + \frac{(\tau_{i-1} - \tau_i)}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-2}\left|\nu(s)\right| ds\end{aligned}$$

$$+ \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \left|\nu(s)\right| ds + (\mathrm{T} - \tau_i)\left|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\right| + \left|\Upsilon_i\big(\omega(\tau_i)\big)\right|\Bigg]. \tag{B.5}$$

Obviously, the right-hand side of inequality (B.5) tends to zero as $\tau_1 \to \tau_2$. Therefore

$$\left|\mathcal{T}\omega(\tau_2) - \mathcal{T}\omega(\tau_1)\right| \to 0 \quad \text{as } \tau_1 \to \tau_2.$$

Similarly, for $\tau \in \mathcal{J}_0$. Thus $\mathcal{T}$ is equicontinuous and therefore completely continuous. Further, we consider a set $\Omega \subset \mathcal{M}$ defined as

$$\Omega = \big\{\omega \in \mathcal{M} : \omega = \aleph\mathcal{T}(\omega), 0 < \aleph < 1\big\}.$$

We need to prove that the set Ω is bounded. Suppose $\omega \in \Omega$ is such that

$$\omega(\tau) = \aleph\mathcal{T}\big(\omega(\tau)\big), \quad \text{where } 0 < \aleph < 1.$$

Then for each $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned}
\left|\omega(\tau)\right| = {} & \Bigg|\frac{\aleph}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \nu(s)\, ds + \frac{\aleph \nu_1}{\eta_1} - \frac{\aleph\tau}{\mathrm{T}}\Bigg[\frac{\xi_2}{\eta_2 \Gamma(r)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \omega(s)\, ds \\
& + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}\Bigg] - \frac{\aleph\tau}{\mathrm{T}} \sum_{i=1}^m \Bigg[\frac{1}{\Gamma(r)} \int_{\tau_i}^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \nu(s)\, ds + \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \nu(s)\, ds \\
& + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2} \nu(s)\, ds + (\mathrm{T} - \tau_i)\hat{\Upsilon}_i\big(\omega(\tau_i)\big) + \Upsilon_i\big(\omega(\tau_i)\big)\Bigg]\Bigg| \\
\leq {} & \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \left|\nu(s)\right| ds + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\Bigg[\frac{\xi_2}{\eta_2 \Gamma(r)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \left|\omega(s)\right| ds \\
& + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2}\Bigg] - \frac{\tau}{\mathrm{T}} \sum_{i=1}^m \Bigg[\frac{1}{\Gamma(r)} \int_{\tau_i}^{\mathrm{T}} (\mathrm{T} - s)^{r-1} \left|\nu(s)\right| ds \\
& + \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \left|\nu(s)\right| ds \\
& + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2} \left|\nu(s)\right| ds + (\mathrm{T} - \tau_i)\left|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\right| + \left|\Upsilon_i\big(\omega(\tau_i)\big)\right|\Bigg] \\
\leq {} & \frac{\tau^r}{\Gamma(r+1)} + \frac{\nu_1}{\eta_1} - \frac{\tau}{\mathrm{T}}\Bigg[\frac{\xi_2 \mathrm{T}^r}{\eta_2 \Gamma(r+1)} + \frac{\nu_1}{\eta_1} - \frac{\nu_2}{\eta_2} + \frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\tau^r}{\Gamma(r+1)} + \frac{m\tau^{r-1}}{\Gamma(r)} \\
& + m\big(\mathcal{K}'_{\hat{\Upsilon}_i} + \mathcal{K}_{\Upsilon_i}\big)\left|\omega(\tau)\right| + m\big(\mathcal{L}'_{\hat{\Upsilon}_i} + \mathcal{L}_{\Upsilon_i}\big)\Bigg].
\end{aligned}$$

Taking the norm on both sides, we get $\|\omega\|_{\mathcal{M}} \leq \mathcal{Q}$. Also, for $\tau \in \mathcal{J}_0$, we can show that $\|\omega\|_{\mathcal{M}} \leq \mathcal{Q}$. Thus, Ω is bounded. By Schaefer's fixed point theorem we conclude that $\mathcal{T}$ has at least one fixed point. Hence, the considered problem (1.3) has at least one solution in $\mathcal{M}$. The proof is complete. □

Proof of Theorem 2.4 For $\omega, \overline{\omega} \in \mathcal{M}$ and $\tau \in \mathcal{J}_i$, we have

$$\left|(\mathcal{T}\overline{\omega})(\tau) - (\mathcal{T}\omega)(\tau)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\big|\overline{\nu}(s)-\nu(s)\big|\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\overline{\omega}(s)-\omega(s)\big|\,ds \\
&\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\bigg[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\overline{\nu}(s)-\nu(s)\big|\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\big|\overline{\nu}(s)-\nu(s)\big|\,ds \\
&\quad + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\big|\overline{\nu}(s)-\nu(s)\big|\,ds + (\mathrm{T}-\tau_i)\big|\hat{\Upsilon}_i\big(\overline{\omega}(\tau_i)\big)-\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\big| \\
&\quad + \big|\Upsilon_i\big(\overline{\omega}(\tau_i)\big)-\Upsilon_i\big(\omega(\tau_i)\big)\big|\bigg],
\end{aligned}
\tag{B.6}
$$

where $\nu,\overline{\nu}\in\mathcal{M}$ are given by

$$
\nu(\tau) = \mathcal{A}\big(\tau,\omega(\tau),\nu(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\omega(s),\nu(s)\big)\,ds
$$

and

$$
\overline{\nu}(\tau) = \mathcal{A}\big(\tau,\overline{\omega}(\tau),\overline{\nu}(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\overline{\omega}(s),\overline{\nu}(s)\big)\,ds.
$$

Using $[A_1]$, we have

$$
\begin{aligned}
&\big|\nu(\tau)-\overline{\nu}(\tau)\big| \\
&\quad = \bigg|\mathcal{A}\big(\tau,\omega(\tau),\nu(\tau)\big) + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\omega(s),\nu(s)\big)\,ds \\
&\qquad - \mathcal{A}\big(\tau,\overline{\omega}(\tau),\overline{\nu}(\tau)\big) - \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,\overline{\omega}(s),\overline{\nu}(s)\big)\,ds\bigg| \\
&\quad \leq \big|\mathcal{A}\big(t,\omega(\tau),\nu(\tau)\big)-\mathcal{A}\big(t,\overline{\omega}(\tau),\overline{\nu}(\tau)\big)\big| \\
&\qquad + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\big|\mathcal{B}\big(s,\omega(s),\nu(s)\big)-\mathcal{B}\big(s,\overline{\omega}(s),\overline{\nu}(s)\big)\big|\,ds \\
&\quad \leq \mathrm{M}_1\big|\omega(\tau)-\overline{\omega}(\tau)\big| + \mathrm{N}_1\big|\nu(\tau)-\overline{\nu}(\tau)\big| \\
&\qquad + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(\mathrm{M}_2\big|\omega(\tau)-\overline{\omega}(\tau)\big| + \mathrm{N}_2\big|\nu(\tau)-\overline{\nu}(\tau)\big|\big).
\end{aligned}
$$

Thus

$$
\big|\nu(\tau)-\overline{\nu}(\tau)\big| \leq \left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right)\big|\omega(\tau)-\overline{\omega}(\tau)\big|.
\tag{B.7}
$$

Using hypotheses $[A_1]$, $[A_2]$ and inequalities (B.7) and (B.6), we obtain

$$
\begin{aligned}
&\big|(\mathcal{T}\omega)(\tau)-(\mathcal{T}\overline{\omega})(\tau)\big| \\
&\quad \leq \bigg[\bigg(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau\mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{m\tau^r}{\mathrm{T}\Gamma(r)}\bigg) \\
&\qquad \times\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right) - \frac{\xi_2\tau\mathrm{T}^{r-1}}{\eta_2\Gamma(r+1)} - \frac{\tau}{\mathrm{T}}m(\mathbb{A}+\mathbb{B})\bigg]
\end{aligned}
$$

$$\times \left|\omega(\tau) - \overline{\omega}(\tau)\right|.$$

Now taking the norm on both sides, we have

$$\begin{aligned}
&\|\mathcal{T}\omega - \mathcal{T}\overline{\omega}\|_{\mathcal{M}} \\
&\quad\leq \left[\left(\frac{m\mathrm{T}^{r}}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}}\right)\right. \\
&\qquad\left. + \frac{\xi_2\mathrm{T}^{r}}{\eta_2\Gamma(r+1)} + m(\mathbb{A} + \mathbb{B})\right]\|\omega - \overline{\omega}\|_{\mathcal{M}}.
\end{aligned}$$

Hence, the operator $\mathcal{T}$ is a contraction. Thus $\mathcal{T}$ has a unique fixed point, so the problem (1.3) has a unique solution. □

Proof of Theorem 2.7 Construct the closed ball $\mathcal{B} = \{(\omega, y) \in \mathcal{X} \times \mathcal{Y} : \|(\omega, y)\| \leq \mathbf{R}\}$. Split the operator $\mathcal{T}$ into two parts as $\mathcal{T} = \mathcal{F} + \mathcal{G}$ with $\mathcal{F} = (\mathcal{F}_r, \mathcal{F}_p)$ and $\mathcal{G} = (\mathcal{G}_r, \mathcal{G}_p)$, where

$$\begin{aligned}
\mathcal{F}_r(\omega, y)(\tau) &= \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau - s)^{r-1}\nu(s)\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T} - s)^{r-1}\omega(s)\,ds \\
&\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T} - s)^{r-1}\nu(s)\,ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-1}\nu(s)\,ds\right. \\
&\quad \left. + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-2}\nu(s)\,ds\right], \\
\mathcal{F}_p(\omega, y)(\tau) &= \frac{1}{\Gamma(p)}\int_0^{\tau}(\tau - s)^{p-1}z(s)\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T} - s)^{p-1}y(s)\,ds \\
&\quad - \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T} - s)^{p-1}z(s)\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j - s)^{p-1}z(s)\,ds\right. \\
&\quad \left. + \frac{\mathrm{T} - \tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j - s)^{p-2}z(s)\,ds\right], \\
\mathcal{G}_r(\omega)(\tau) &= \frac{\nu_1}{\eta_1} + \frac{\tau}{\mathrm{T}}\left[\frac{\nu_2}{\eta_2} - \frac{\nu_1}{\eta_1}\right] - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[(\mathrm{T} - \tau_i)\hat{\Upsilon}_i\big(\omega(\tau_i)\big) + \Upsilon_i\big(\omega(\tau_i)\big)\right],
\end{aligned}$$

and

$$\mathcal{G}_p(y)(\tau) = \frac{\nu_3}{\eta_3} + \frac{\tau}{\mathrm{T}}\left[\frac{\nu_4}{\eta_4} - \frac{\nu_3}{\eta_3}\right] - \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[(\mathrm{T} - \tau_j)\hat{\Upsilon}_j\big(y(\tau_j)\big) + \Upsilon_j\big(y(\tau_j)\big)\right].$$

Clearly, $\mathcal{T}_r = \mathcal{F}_r + \mathcal{G}_r$ and $\mathcal{T}_p = \mathcal{F}_p + \mathcal{G}_p$.

The first step is to show that $\mathcal{T}(\omega, y)(\tau) = \mathcal{F}(\omega, y)(\tau) + \mathcal{G}(\omega, y)(\tau) \in \mathcal{B}$ for all $(\omega, y) \in \mathcal{B}$. For any $(\omega, y) \in \mathcal{B}$, consider

$$\begin{aligned}
&\left|(\mathcal{T}_r\omega)(\tau)\right| \\
&\quad\leq \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau - s)^{r-1}\left|\nu(s)\right|ds + \frac{\nu_1}{\eta_1}
\end{aligned}$$

$$- \frac{\tau}{\mathrm{T}}\left[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\omega(s)\right|ds+\frac{\nu_1}{\eta_1}-\frac{\nu_2}{\eta_2}\right]$$
$$- \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\nu(s)\right|ds+\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\left|\nu(s)\right|ds\right.$$
$$\left.+\frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\left|\nu(s)\right|ds+(\mathrm{T}-\tau_i)\left|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)\right|+\left|\Upsilon_i\big(\omega(\tau_i)\big)\right|\right]. \tag{B.8}$$

Using $[\tilde{A_4}]$ for $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned}
\left|\nu(\tau)\right| &\le \left|\mathcal{A}\big(\tau,y(\tau),\nu(\tau)\big)\right|+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\big(s,y(s),\nu(s)\big)\right|ds \\
&\le a_1(\tau)+b_1(\tau)\left|y(\tau)\right|+c_1(\tau)\left|\nu(\tau)\right|+\frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}\big(a_2(\tau)+b_2(\tau)\left|y(\tau)\right|+c_2(\tau)\left|\nu(\tau)\right|\big) \\
&\le a_1^*+b_1^*\|y\|_{\mathcal{X}}+c_1^*\|\nu\|_{\mathcal{X}}+\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}\big(a_2^*+b_2^*\|y\|_{\mathcal{X}}+c_2^*\|\nu\|_{\mathcal{X}}\big).
\end{aligned}$$

Therefore we get

$$\left|\nu(\tau)\right| \le \|\nu\|_{\mathcal{X}} \le \frac{a_1^*+b_1^*\|y\|_{\mathcal{X}}}{1-c_1^*-c_2^*\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}}+\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}\frac{a_2^*+b_2^*\|y\|_{\mathcal{X}}}{1-c_1^*-c_2^*\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}}=\hbar. \tag{B.9}$$

Using (B.9) and $[\tilde{A_6}]$, relation (B.8) becomes

$$\begin{aligned}
\left|\mathcal{T}_r\omega(\tau)\right| &\le \frac{\hbar\tau^r}{\Gamma(r+1)}+\frac{\nu_1}{\eta_1}-\frac{\xi_2\tau\mathrm{T}^{r-1}}{\Gamma(r+1)}-\frac{\tau\nu_1}{\mathrm{T}\eta_1}+\frac{\tau\nu_2}{\mathrm{T}\eta_2}-\frac{m\tau\hbar\mathrm{T}^{r-1}}{\Gamma(r+1)} \\
&\quad -\frac{m\hbar\tau^{r+1}}{\mathrm{T}\Gamma(r+1)}-\frac{m\hbar\tau^{r-1}}{\Gamma(r)} \\
&\quad -\big(\mathcal{K}'_{\hat{\Upsilon}_i}+\mathcal{K}_{\Upsilon_i}\big)\left|\omega(\tau)\right|-\big(\mathcal{L}'_{\hat{\Upsilon}_i}+\mathcal{L}_{\Upsilon_i}\big) \\
&= \mathbf{C}.
\end{aligned}$$

Thus

$$\|\mathcal{T}_r\omega\|_{\mathcal{X}} \le \mathbf{C}.$$

Similarly, for $\tau \in \mathcal{J}_0$, we can verify that

$$\|\mathcal{T}_r\omega\|_{\mathcal{X}} \le \mathbf{C}.$$

In the similar manner, we have

$$\begin{aligned}
&\left|(\mathcal{T}_r y)(\tau)\right| \\
&\quad\le \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\left|\nu(s)\right|ds+\frac{\nu_1}{\eta_1} \\
&\qquad -\frac{\tau}{\mathrm{T}}\left[\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\omega(s)\right|ds+\frac{\nu_1}{\eta_1}-\frac{\nu_2}{\eta_2}\right]
\end{aligned}$$

$$- \frac{\tau}{\mathrm{T}} \sum_{i=1}^{m} \biggl[\frac{1}{\Gamma(r)} \int_{\tau_i}^{\mathrm{T}} (\mathrm{T}-s)^{r-1} \bigl|\nu(s)\bigr| \, ds + \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \bigl|\nu(s)\bigr| \, ds$$
$$+ \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2} \bigl|\nu(s)\bigr| \, ds + (\mathrm{T}-\tau_i) \bigl|\hat{\Upsilon}_i\bigl(\omega(\tau_i)\bigr)\bigr| + \bigl|\Upsilon_i\bigl(\omega(\tau_i)\bigr)\bigr| \biggr]. \tag{B.10}$$

Using $[\tilde{A_4}]$ for $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned} \bigl|\nu(\tau)\bigr| &\le \bigl|\mathcal{A}\bigl(\tau, y(\tau), \nu(\tau)\bigr)\bigr| + \int_0^{\tau} \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)} \bigl|\mathcal{B}\bigl(s, y(s), \nu(s)\bigr)\bigr| \, ds \\ &\le a_1(\tau) + b_1(\tau)\bigl|y(\tau)\bigr| + c_1(\tau)\bigl|\nu(\tau)\bigr| + \frac{\tau^{\sigma}}{\sigma \Gamma(\delta)} \bigl(a_2(\tau) + b_2(\tau)\bigl|y(\tau)\bigr| + c_2(\tau)\bigl|\nu(\tau)\bigr|\bigr) \\ &\le a_1^* + b_1^* \|y\|_{\mathcal{X}} + c_1^* \|\nu\|_{\mathcal{X}} + \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)} \bigl(a_2^* + b_2^* \|y\|_{\mathcal{X}} + c_2^* \|\nu\|_{\mathcal{X}}\bigr). \end{aligned}$$

Therefore we get

$$\bigl|\nu(\tau)\bigr| \le \|\nu\|_{\mathcal{X}} \le \frac{a_1^* + b_1^* \|y\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)}} + \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)} \frac{a_2^* + b_2^* \|y\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)}} = \hbar. \tag{B.11}$$

Using (B.11) and $[\tilde{A_6}]$, relation (B.10) becomes

$$\begin{aligned} \bigl|\mathcal{T}_r y(\tau)\bigr| &\le \frac{\hbar \tau^r}{\Gamma(r+1)} + \frac{\nu_1}{\eta_1} - \frac{\xi_2 \tau \mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{\tau \nu_1}{\mathrm{T}\eta_1} + \frac{\tau \nu_2}{\mathrm{T}\eta_2} - \frac{\tau \hbar \mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{m\hbar \tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{m\hbar \tau^{r-1}}{\Gamma(r)} \\ &\quad - \bigl(\mathcal{K}'_{\hat{\Upsilon}_i} + \mathcal{K}_{\Upsilon_i}\bigr)\bigl|\omega(\tau)\bigr| - \bigl(\mathcal{L}'_{\hat{\Upsilon}_i} + \mathcal{L}_{\Upsilon_i}\bigr) \\ &= \mathbf{C}. \end{aligned}$$

Thus

$$\|\mathcal{T}_r y\|_{\mathcal{X}} \le \mathbf{C}.$$

Similarly, for $\tau \in \mathcal{J}_0$, we can verify that

$$\|\mathcal{T}_r y\|_{\mathcal{X}} \le \mathbf{C}.$$

Hence

$$\bigl\|\mathcal{T}_r(\omega, y)\bigr\|_{\mathcal{X}} \le \mathbf{C}.$$

Now, for any $(\omega, y) \in \mathscr{B}$, consider

$$\begin{aligned} &\bigl|(\mathcal{T}_p \omega)(\tau)\bigr| \\ &\quad\le \frac{1}{\Gamma(p)} \int_0^{\tau} (\tau - s)^{p-1} \bigl|z(s)\bigr| \, ds + \frac{\nu_3}{\eta_3} \\ &\qquad - \frac{\tau}{\mathrm{T}} \biggl[\frac{\xi_4}{\eta_4 \Gamma(p)} \int_0^{\mathrm{T}} (\mathrm{T}-s)^{p-1} \bigl|y(s)\bigr| \, ds + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4} \biggr] \end{aligned}$$

$$- \frac{\tau}{\mathrm{T}} \sum_{j=1}^{n} \biggl[\frac{1}{\Gamma(p)} \int_{\tau_j}^{\mathrm{T}} (\mathrm{T}-s)^{p-1} \bigl| z(s) \bigr| \, ds + \frac{1}{\Gamma(p)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{p-1} \bigl| z(s) \bigr| \, ds$$

$$+ \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{p-2} \bigl| z(s) \bigr| \, ds + (\mathrm{T}-\tau_j) \bigl| \hat{\Upsilon}_j \bigl(y(\tau_j) \bigr) \bigr| + \bigl| \Upsilon_j \bigl(y(\tau_j) \bigr) \bigr| \biggr]. \tag{B.12}$$

Using $[\tilde{A}_5]$ for $\tau \in \mathcal{J}_j$, we have

$$\begin{aligned} \bigl| z(\tau) \bigr| &\leq \bigl| \mathcal{A}' \bigl(\tau, \omega(\tau), y(\tau) \bigr) \bigr| + \int_0^{\tau} \frac{(\tau - s)^{\sigma - 1}}{\Gamma(\delta)} \bigl| \mathcal{B}' \bigl(s, \omega(s), y(s) \bigr) \bigr| \, ds \\ &\leq l_1(\tau) + m_1(\tau) \bigl| \omega(\tau) \bigr| + n_1(\tau) \bigl| y(\tau) \bigr| + \frac{\tau^{\sigma}}{\sigma \Gamma(\delta)} \bigl(l_2(\tau) + m_2(\tau) \bigl| \omega(\tau) \bigr| + n_2(\tau) \bigl| y(\tau) \bigr| \bigr) \\ &\leq l_1^* + m_1^* \|\omega\|_{\mathcal{Y}} + n_1^* \|y\|_{\mathcal{Y}} + \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)} \bigl(l_2^* + m_2^* \|\omega\|_{\mathcal{Y}} + n_2^* \|y\|_{\mathcal{Y}} \bigr). \end{aligned}$$

Therefore we get

$$\bigl| z(\tau) \bigr| \leq \|z\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^* \|\omega\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)}} + \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)} \frac{l_2^* + m_2^* \|\omega\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{\mathrm{T}^{\sigma}}{\sigma \Gamma(\delta)}} = \hbar^*. \tag{B.13}$$

Using (B.13) and $[\tilde{A}_7]$, relation (B.12) becomes

$$\begin{aligned} \bigl| \mathcal{T}_p \omega(\tau) \bigr| \leq{}& \frac{\hbar^* \tau^p}{\Gamma(p+1)} + \frac{\nu_3}{\eta_3} - \frac{\xi_4 \tau \mathrm{T}^{p-1}}{\Gamma(p+1)} - \frac{\tau \nu_3}{\mathrm{T} \eta_3} + \frac{\tau \nu_4}{\mathrm{T} \eta_4} - \frac{n \tau \hbar^* \mathrm{T}^{p-1}}{\Gamma(p+1)} \\ & - \frac{n \hbar^* \tau^{p+1}}{\mathrm{T} \Gamma(p+1)} - \frac{n \hbar^* \tau^{p-1}}{\Gamma(p)} \\ & - \bigl(\mathcal{K}'_{\hat{\Upsilon}_j} + \mathcal{K}_{\Upsilon_j} \bigr) \bigl| \omega(\tau) \bigr| - \bigl(\mathcal{L}'_{\hat{\Upsilon}_j} + \mathcal{L}_{\Upsilon_j} \bigr) \\ ={}& \mathbf{C}^*. \end{aligned}$$

Thus

$$\|\mathcal{T}_p \omega\|_{\mathcal{Y}} \leq \mathbf{C}^*.$$

Similarly, for $\tau \in \mathcal{J}_0$, we can verify that

$$\|\mathcal{T}_p \omega\|_{\mathcal{Y}} \leq \mathbf{C}^*.$$

In a similar manner, we have

$$\begin{aligned} & \bigl| (\mathcal{T}_p y)(\tau) \bigr| \\ & \quad \leq \frac{1}{\Gamma(p)} \int_0^{\tau} (\tau - s)^{p-1} \bigl| z(s) \bigr| \, ds + \frac{\nu_3}{\eta_3} \\ & \qquad - \frac{\tau}{\mathrm{T}} \biggl[\frac{\xi_4}{\eta_4 \Gamma(p)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{p-1} \bigl| y(s) \bigr| \, ds + \frac{\nu_3}{\eta_3} - \frac{\nu_4}{\eta_4} \biggr] \\ & \qquad - \frac{\tau}{\mathrm{T}} \sum_{j=1}^{n} \biggl[\frac{1}{\Gamma(p)} \int_{\tau_j}^{\mathrm{T}} (\mathrm{T}-s)^{p-1} \bigl| z(s) \bigr| \, ds + \frac{1}{\Gamma(p)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{p-1} \bigl| z(s) \bigr| \, ds \end{aligned}$$

$$+\frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\big|z(s)\big|\,ds+(\mathrm{T}-\tau_j)\big|\hat{\Upsilon}_j\big(y(\tau_j)\big)\big|+\big|\Upsilon_j\big(y(\tau_j)\big)\big|\Bigg]. \tag{B.14}$$

Using $[\tilde{A}_5]$ for $\tau\in\mathcal{J}_j$, we have

$$\begin{aligned}\big|z(\tau)\big|&\le\big|\mathcal{A}'\big(\tau,\omega(\tau),y(\tau)\big)\big|+\int_0^\tau\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\big|\mathcal{B}'\big(s,\omega(s),y(s)\big)\big|\,ds\\&\le l_1(\tau)+m_1(\tau)\big|\omega(\tau)\big|+n_1(\tau)\big|y(\tau)\big|+\frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(l_2(\tau)+m_2(\tau)\big|\omega(\tau)\big|+n_2(\tau)\big|y(\tau)\big|\big)\\&\le l_1^*+m_1^*\|\omega\|_{\mathcal{Y}}+n_1^*\|y\|_{\mathcal{Y}}+\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\big(l_2^*+m_2^*\|\omega\|_{\mathcal{Y}}+n_2^*\|y\|_{\mathcal{Y}}\big).\end{aligned}$$

Therefore we get

$$\big|z(\tau)\big|\le\|z\|_{\mathcal{Y}}\le\frac{l_1^*+m_1^*\|\omega\|_{\mathcal{Y}}}{1-n_1^*-n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\frac{l_2^*+m_2^*\|\omega\|_{\mathcal{Y}}}{1-n_1^*-n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}=\hbar^*. \tag{B.15}$$

Using (B.15) and $[\tilde{A}_7]$, relation (B.14) becomes

$$\begin{aligned}\big|\mathcal{T}_py(\tau)\big|\le{}&\frac{\hbar^*\tau^p}{\Gamma(p+1)}+\frac{\nu_3}{\eta_3}-\frac{\xi_4\tau\mathrm{T}^{p-1}}{\Gamma(p+1)}-\frac{\tau\nu_3}{\mathrm{T}\eta_3}+\frac{\tau\nu_4}{\mathrm{T}\eta_4}-\frac{n\tau\hbar^*\mathrm{T}^{p-1}}{\Gamma(p+1)}\\&-\frac{n\hbar^*\tau^{p+1}}{\mathrm{T}\Gamma(p+1)}-\frac{n\hbar^*\tau^{p-1}}{\Gamma(p)}\\&-\big(\mathcal{K}'_{\hat{\Upsilon}_j}+\mathcal{K}_{\Upsilon_j}\big)\big|\omega(\tau)\big|-\big(\mathcal{L}'_{\hat{\Upsilon}_j}+\mathcal{L}_{\Upsilon_j}\big)\\={}&\mathbf{C}^*.\end{aligned}$$

Thus

$$\|\mathcal{T}_py\|_{\mathcal{Y}}\le\mathbf{C}^*.$$

Similarly, for $\tau\in\mathcal{J}_0$, we can verify that

$$\|\mathcal{T}_py\|_{\mathcal{Y}}\le\mathbf{C}^*.$$

Hence

$$\big\|\mathcal{T}_p(\omega,y)\big\|_{\mathcal{Y}}\le\mathbf{C}^*,$$

and thus

$$\big\|\mathcal{T}(\omega,y)\big\|_{\mathcal{X}\times\mathcal{Y}}\le\big\|\mathcal{T}_r(\omega,y)+\mathcal{T}_p(\omega,y)\big\|_{\mathcal{X}\times\mathcal{Y}}\le\mathbf{C}+\mathbf{C}^*=\mathbf{R},$$

which implies that $\mathcal{T}(\mathcal{B})\subseteq\mathcal{B}$.

Second, we show that $\mathcal{G}$ is a contraction. For any $(\omega,y),(\overline{\omega},\overline{y})\in\mathcal{B}$, we have

$$\big|\mathcal{G}_r(\omega)-\mathcal{G}_r(\overline{\omega})\big|\le\frac{\tau}{\mathrm{T}}\sum_{i=1}^m\big[(\mathrm{T}-\tau_i)\big|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)-\hat{\Upsilon}_i\big(\overline{\omega}(\tau_i)\big)\big|+\big|\Upsilon_i\big(\omega(\tau_i)\big)-\Upsilon_i\big(\overline{\omega}(\tau_i)\big)\big|\big]$$

$$\leq m(\mathbb{A}_{\hat{\Upsilon}_i} + \mathbb{A}_{\Upsilon_i})\|\omega - \overline{\omega}\|_{\mathcal{X}}.$$

Similarly,

$$\begin{aligned}
\left|\mathcal{G}_p(y) - \mathcal{G}_p(\overline{y})\right| &\leq \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[(\mathrm{T}-\tau_j)\left|\hat{\Upsilon}_j\big(y(\tau_j)\big) - \hat{\Upsilon}_j\big(\overline{y}(\tau_j)\big)\right| + \left|\Upsilon_j\big(y(\tau_j)\big) - \Upsilon_j\big(\overline{y}(\tau_j)\big)\right|\right] \\
&\leq n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j})\|y - \overline{y}\|_{\mathcal{Y}}.
\end{aligned}$$

From the assumptions $m(\mathbb{A}_{\hat{\Upsilon}_i} + \mathbb{A}_{\Upsilon_i}) < 1$ and $n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j}) < 1$ it follows that $\mathcal{G}$ is a contraction.

Our final step is to show that $\mathcal{F} = (\mathcal{F}_r + \mathcal{F}_p)$ is compact. The continuity of $\mathcal{F}$ follows from the continuity of $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$. For $(\omega, y) \in \mathscr{B}$, we have

$$\begin{aligned}
\left|\mathcal{F}_r\omega(\tau)\right| &\leq \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau - s)^{r-1}\left|\nu(s)\right| ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\omega(s)\right| ds \\
&\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\nu(s)\right| ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-1}\left|\nu(s)\right| ds \right. \\
&\quad \left. + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-2}\left|\nu(s)\right| ds\right]. \qquad \text{(B.16)}
\end{aligned}$$

By $[\tilde{A_4}]$, for $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned}
\left|\nu(\tau)\right| &\leq \left|\mathcal{A}\big(\tau, \omega(\tau), \nu(\tau)\big)\right| + \int_0^{\tau}\frac{(\tau - \xi)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\big(s, \omega(s), \nu(s)\big)\right| ds \\
&\leq a_1(\tau) + b_1(\tau)\left|\omega(\tau)\right| + c_1(\tau)\left|\nu(\tau)\right| + \frac{\tau^{\sigma}}{\sigma\Gamma(\delta)}\big(a_2(\tau) + b_2(\tau)\left|\omega(\tau)\right| + c_2(\tau)\left|\nu(\tau)\right|\big) \\
&\leq a_1^* + b_1^*\|\omega\|_{\mathcal{X}} + c_1^*\|\nu\|_{\mathcal{X}} + \frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}\big(a_2^* + b_2^*\|\omega\|_{\mathcal{X}} + c_2^*\|\nu\|_{\mathcal{X}}\big).
\end{aligned}$$

Therefore we get

$$\left|\nu(\tau)\right| \leq \|\nu\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^*\|\omega\|_{\mathcal{X}}}{1 - c_1^* - c_2^*\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}\frac{a_2^* + b_2^*\|\omega\|_{\mathcal{X}}}{1 - c_1^* - c_2^*\frac{\mathrm{T}^{\sigma}}{\sigma\Gamma(\delta)}} = \hbar. \qquad \text{(B.17)}$$

Using (B.17) in (B.16), after simplification, we get

$$\left|\mathcal{F}_r\omega(\tau)\right| \leq \wp_1.$$

In a similar manner, we have

$$\begin{aligned}
\left|\mathcal{F}_r y(\tau)\right| &\leq \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau - s)^{r-1}\left|\nu(s)\right| ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\omega(s)\right| ds \\
&\quad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\nu(s)\right| ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-1}\left|\nu(s)\right| ds \right. \\
&\quad \left. + \frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i - s)^{r-2}\left|\nu(s)\right| ds\right]. \qquad \text{(B.18)}
\end{aligned}$$

By $[\tilde{A}_4]$, for $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned}\left|\nu(\tau)\right| &\leq \left|\mathcal{A}\big(\tau,\omega(\tau),\nu(\tau)\big)\right| + \int_0^\tau \frac{(\tau-\xi)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\big(s,\omega(s),\nu(s)\big)\right|\,ds \\ &\leq a_1(\tau) + b_1(\tau)\left|\omega(\tau)\right| + c_1(\tau)\left|\nu(\tau)\right| + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(a_2(\tau) + b_2(\tau)\left|\omega(\tau)\right| + c_2(\tau)\left|\nu(\tau)\right|\big) \\ &\leq a_1^* + b_1^*\|\omega\|_{\mathcal{X}} + c_1^*\|\nu\|_{\mathcal{X}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\big(a_2^* + b_2^*\|\omega\|_{\mathcal{X}} + c_2^*\|\nu\|_{\mathcal{X}}\big).\end{aligned}$$

Therefore we get

$$\left|\nu(\tau)\right| \leq \|\nu\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^*\|\omega\|_{\mathcal{X}}}{1 - c_1^* - c_2^*\frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\frac{a_2^* + b_2^*\|\omega\|_{\mathcal{X}}}{1 - c_1^* - c_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \tag{B.19}$$

Using (B.19) in (B.18), after simplification, we get

$$\left|\mathcal{F}_r\omega(\tau)\right| \leq \wp_1.$$

Hence

$$\left\|\mathcal{F}_r(\omega,y)\right\|_{\mathcal{X}} \leq \wp_1.$$

Now for any $(\omega,y) \in \mathscr{B}$, we have

$$\begin{aligned}\left|\mathcal{F}_p\omega(\tau)\right| \leq{}& \frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}\left|z(s)\right|\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|y(s)\right|\,ds \\ &- \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|z(s)\right|\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}\left|z(s)\right|\,ds \right.\\ &\left.+ \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\left|z(s)\right|\,ds\right].\end{aligned} \tag{B.20}$$

By $[\tilde{A}_5]$, for $\tau \in \mathcal{J}_j$, we have

$$\begin{aligned}\left|z(\tau)\right| &\leq \left|\mathcal{A}'\big(\tau,\omega(\tau),z(\tau)\big)\right| + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}'\big(s,\omega(s),z(s)\big)\right|\,ds \\ &\leq l_1(\tau) + m_1(\tau)\left|x(\tau)\right| + n_1(\tau)\left|z(\tau)\right| + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(l_2(\tau) + m_2(\tau)\left|\omega(\tau)\right| + n_2(\tau)\left|z(\tau)\right|\big) \\ &\leq l_1^* + m_1^*\|\omega\|_{\mathcal{Y}} + n_1^*\|z\|_{\mathcal{Y}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\big(l_2^* + m_2^*\|\omega\|_{\mathcal{Y}} + n_2^*\|z\|_{\mathcal{Y}}\big).\end{aligned}$$

Therefore we get

$$\left|z(\tau)\right| \leq \|z\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^*\|\omega\|_{\mathcal{Y}}}{1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\frac{l_2^* + m_2^*\|\omega\|_{\mathcal{Y}}}{1 - n_1^* - n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \tag{B.21}$$

Using (B.21) in (B.20), after simplification, we get

$$\|\mathcal{F}_p\omega\|_{\mathcal{Y}} \leq \wp_2.$$

In a similar manner, we have

$$
\begin{aligned}
\left|\mathcal{F}_p y(\tau)\right| \le{}& \frac{1}{\Gamma(p)}\int_0^\tau (\tau-s)^{p-1}\left|z(s)\right|ds - \frac{\tau}{\mathrm{T}}\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|y(s)\right|ds \\
&- \frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|z(s)\right|ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}\left|z(s)\right|ds\right. \\
&\left. + \frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\left|z(s)\right|ds\right].
\end{aligned}
\tag{B.22}
$$

By $[\tilde{A_5}]$, for $\tau\in\mathcal{J}_j$, we have

$$
\begin{aligned}
\left|z(\tau)\right| &\le \left|\mathcal{A}'\big(\tau,\omega(\tau),z(\tau)\big)\right| + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}'\big(s,\omega(s),z(s)\big)\right|ds \\
&\le l_1(\tau)+m_1(\tau)\left|x(\tau)\right| + n_1(\tau)\left|z(\tau)\right| + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(l_2(\tau)+m_2(\tau)\left|\omega(\tau)\right| + n_2(\tau)\left|z(\tau)\right|\big) \\
&\le l_1^* + m_1^*\|\omega\|_{\mathcal{Y}} + n_1^*\|z\|_{\mathcal{Y}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\big(l_2^*+m_2^*\|\omega\|_{\mathcal{Y}}+n_2^*\|z\|_{\mathcal{Y}}\big).
\end{aligned}
$$

Therefore we get

$$
\left|z(\tau)\right| \le \|z\|_{\mathcal{Y}} \le \frac{l_1^*+m_1^*\|\omega\|_{\mathcal{Y}}}{1-n_1^*-n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}\frac{l_2^*+m_2^*\|\omega\|_{\mathcal{Y}}}{1-n_1^*-n_2^*\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} = \hbar.
\tag{B.23}
$$

Using (B.23) in (B.22), after simplification, we get

$$
\|\mathcal{F}_p y\|_{\mathcal{Y}} \le \wp_2.
$$

Hence

$$
\left\|\mathcal{F}_p(\omega,y)\right\|_{\mathcal{Y}} \le \wp_2.
$$

Thus

$$
\left\|\mathcal{F}(\omega,y)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \left\|\mathcal{F}_r(\omega,y)+\mathcal{F}_p(\omega,y)\right\|_{\mathcal{X}\times\mathcal{Y}} \le \wp_1+\wp_2 = \mathbf{R}_1,
$$

which implies that $\mathcal{F}$ is uniformly bounded on $\mathcal{B}$.

Take a bounded subset $\mathbb{C}$ of $\mathcal{B}$ and $(\omega,y)\in\mathbb{C}$. Then for $\tau_1,\tau_2\in\mathcal{J}_i$ with $0\le\tau_1\le\tau_2\le 1$, we have

$$
\begin{aligned}
&\left|\mathcal{F}_r\omega(\tau_2)-\mathcal{F}_r\omega(\tau_1)\right| \\
&\quad\le \frac{1}{\Gamma(r)}\int_0^{\tau_2}(\tau_2-s)^{r-1}\left|v(s)\right|ds + \frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\left|v(s)\right|ds \\
&\qquad - \frac{\tau}{\mathrm{T}}\sum_{0<\tau_i<\tau_2-\tau_1}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|v(s)\right|ds + \frac{(\tau_{i-1}-\tau_i)}{\Gamma(r-1)}\int_{\tau_i}^{\tau_{i-1}}(\tau_i-s)^{r-2}\left|v(s)\right|ds\right. \\
&\qquad \left. - \frac{1}{\Gamma(r)}\int_{\tau_i}^{\tau_{i-1}}(\tau_i-s)^{r-1}\left|v(s)\right|ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(r)}\int_0^{\tau_2}\left[(\tau_2-s)^{r-1}-(\tau_1-s)^{r-1}\right]\left|\nu(s)\right|ds+\frac{1}{\Gamma(r)}\int_0^{\tau_1}(\tau_1-s)^{r-1}\left|\nu(s)\right|ds \\
&\quad-\frac{\tau}{\mathrm{T}}\sum_{0<\tau_i<\tau_2-\tau_1}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\left|\nu(s)\right|ds+\frac{(\tau_{i-1}-\tau_i)}{\Gamma(r-1)}\int_{\tau_i}^{\tau_{i-1}}(\tau_i-s)^{r-2}\left|\nu(s)\right|ds\right. \\
&\quad\left.-\frac{1}{\Gamma(r)}\int_{\tau_i}^{\tau_{i-1}}(\tau_i-s)^{r-1}\left|\nu(s)\right|ds\right]. \qquad (B.24)
\end{aligned}
$$

Obviously, the right-hand side of inequality (B.24) tends to zero as $\tau_1\to\tau_2$.

Therefore

$$\left|\mathcal{F}_r\omega(\tau_2)-\mathcal{F}_r\omega(\tau_1)\right|\to 0 \quad \text{as } \tau_1\to\tau_2.$$

Similarly,

$$\left|\mathcal{F}_r y(\tau_2)-\mathcal{F}_r y(\tau_1)\right|\to 0 \quad \text{as } \tau_1\to\tau_2.$$

Now for any $\tau_1,\tau_2\in\mathcal{J}_j$ with $0\leq\tau_1\leq\tau_2\leq 1$, we have

$$
\begin{aligned}
&\left|\mathcal{F}_p\omega(\tau_2)-\mathcal{F}_p\omega(\tau_1)\right| \\
&\leq \frac{1}{\Gamma(p)}\int_0^{\tau_2}(\tau_2-s)^{p-1}\left|z(s)\right|ds+\frac{1}{\Gamma(p)}\int_0^{\tau_1}(\tau_1-s)^{p-1}\left|z(s)\right|ds \\
&\quad-\frac{\tau}{\mathrm{T}}\sum_{0<\tau_j<\tau_2-\tau_1}\left[\frac{1}{\Gamma(p)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|z(s)\right|ds+\frac{(\tau_{j-1}-\tau_j)}{\Gamma(p-1)}\int_{\tau_j}^{\tau_{j-1}}(\tau_j-s)^{p-2}\left|z(s)\right|ds\right. \\
&\quad\left.-\frac{1}{\Gamma(p)}\int_{\tau_j}^{\tau_{j-1}}(\tau_j-s)^{p-1}\left|z(s)\right|ds\right] \\
&\leq \frac{1}{\Gamma(p)}\int_0^{\tau_2}\left[(\tau_2-s)^{p-1}-(\tau_1-s)^{p-1}\right]\left|z(s)\right|ds+\frac{1}{\Gamma(p)}\int_0^{\tau_1}(\tau_1-s)^{p-1}\left|z(s)\right|ds \\
&\quad-\frac{\tau}{\mathrm{T}}\sum_{0<\tau_j<\tau_2-\tau_1}\left[\frac{1}{\Gamma(p)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\left|z(s)\right|ds+\frac{(\tau_{j-1}-\tau_j)}{\Gamma(p-1)}\int_{\tau_j}^{\tau_{j-1}}(\tau_j-s)^{p-2}\left|z(s)\right|ds\right. \\
&\quad\left.-\frac{1}{\Gamma(p)}\int_{\tau_j}^{\tau_{j-1}}(\tau_j-s)^{p-1}\left|z(s)\right|ds\right]. \qquad (B.25)
\end{aligned}
$$

Obviously, the right-hand side of inequality (B.25) tends to zero as $\tau_1\to\tau_2$.

Therefore

$$\left|\mathcal{F}_p\omega(\tau_2)-\mathcal{F}_p\omega(\tau_1)\right|\to 0 \quad \text{as } \tau_1\to\tau_2.$$

Similarly,

$$\left|\mathcal{F}_p y(\tau_2)-\mathcal{F}_p y(\tau_1)\right|\to 0 \quad \text{as } \tau_1\to\tau_2.$$

Thus

$$\left|\mathcal{F}(x,y)(\tau_2)-\mathcal{F}(x,y)(\tau_1)\right|\to 0 \quad \text{as } \tau_1\to\tau_2.$$

Hence $\mathcal{F}$ is equicontinuous, and by the Arzelà–Ascoli theorem we obtain that $\mathcal{F}$ is compact. Finally, by Theorem A.7 system (1.4) has at least one solution, which completes the proof. □

Proof of Theorem 2.8 Suppose $\omega,\overline{\omega} \in \mathcal{X}$. For $\tau \in \mathcal{J}_i$, we have

$$\begin{aligned}
&\left|\mathcal{T}_r\omega(\tau) - \mathcal{T}_r\overline{\omega}(\tau)\right| \\
&\quad\leq \frac{1}{\Gamma(r)}\int_0^\tau (\tau - s)^{r-1}\left|\nu(s) - \overline{\nu}(s)\right| ds - \frac{\tau}{\mathrm{T}}\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T} - s)^{r-1}\left|\omega(s) - \overline{\omega}(s)\right| ds \\
&\qquad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T} - s)^{r-1}\left|\nu(s) - \overline{\nu}(s)\right| ds + \frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1}\left|\nu(s) - \overline{\nu}(s)\right| ds\right. \\
&\qquad + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2}\left|\nu(s) - \overline{\nu}(s)\right| ds + (\mathrm{T} - \tau_i)\left|\hat{\Upsilon}_i\big(\omega(\tau_i)\big) - \hat{\Upsilon}_i\big(\overline{\omega}(\tau_i)\big)\right| \\
&\qquad \left. + \left|\Upsilon_i\big(\omega(\tau_i)\big) - \Upsilon_i\big(\overline{\omega}(\tau_i)\big)\right|\right],
\end{aligned} \tag{B.26}$$

where

$$\nu(\tau) = \mathcal{A}\big(\tau, y(\tau), \nu(\tau)\big) + \int_0^\tau \frac{(\tau - \xi)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, y(s), \nu(s)\big)\, ds$$

and

$$\overline{\nu}(\tau) = \mathcal{A}\big(\tau, \overline{y}(\tau), \overline{\nu}(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, \overline{y}(s), \overline{\nu}(s)\big)\, ds.$$

Using $[\tilde{A}_1]$, we have

$$\begin{aligned}
&\left|\nu(\tau) - \overline{\nu}(\tau)\right| \\
&\quad= \left|\mathcal{A}\big(\tau, y(\tau), \nu(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, y(s), \nu(s)\big)\, ds\right. \\
&\qquad \left. - \mathcal{A}\big(\tau, \overline{y}(\tau), \overline{\nu}(\tau)\big) - \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s, \overline{y}(s), \overline{\nu}(s)\big)\, ds\right| \\
&\quad\leq \left|\mathcal{A}\big(\tau, y(\tau), \nu(\tau)\big) - \mathcal{A}\big(\tau, \overline{y}(\tau), \overline{\nu}(\tau)\big)\right| \\
&\qquad + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\big(s, y(s), \nu(s)\big) - \mathcal{B}\big(s, \overline{y}(s), \overline{\nu}(s)\big)\right| ds \\
&\quad\leq \mathrm{M}_1\left|y(\tau) - \overline{y}(\tau)\right| + \mathrm{N}_1\left|\nu(\tau) - \overline{\nu}(\tau)\right| \\
&\qquad + \frac{\tau^\sigma}{\sigma\Gamma(\delta)}\big(\mathrm{M}_2\left|y(\tau) - \overline{y}(\tau)\right| + \mathrm{N}_2\left|\nu(\tau) - \overline{\nu}(\tau)\right|\big).
\end{aligned}$$

Thus

$$\left|\nu(\tau) - \overline{\nu}(\tau)\right| \leq \left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right)\left|y(\tau) - \overline{y}(\tau)\right|. \tag{B.27}$$

Using hypotheses $[\tilde{A_1}]$, and $[\tilde{A_3}]$ and inequalities (B.27) and (B.26), we get

$$\begin{aligned}
&\big|(\mathcal{T}_r\omega)(\tau) - (\mathcal{T}_r\overline{\omega})(\tau)\big| \\
&\quad\leq \Bigg[\bigg(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau\mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{m\tau^r}{\mathrm{T}\Gamma(r)}\bigg) \\
&\qquad\times\bigg(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\bigg)\Bigg]\big|y(\tau)-\overline{y}(\tau)\big| \\
&\qquad-\bigg[\frac{\xi_2\tau\mathrm{T}^{r-1}}{\eta_2\Gamma(r+1)} + \frac{\tau}{\mathrm{T}}m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})\bigg]\big|\omega(\tau)-\overline{\omega}(\tau)\big|.
\end{aligned}$$

Now taking the norm on both sides, we have

$$\begin{aligned}
&\|\mathcal{T}_r\omega - \mathcal{T}_r\overline{\omega}\|_{\mathcal{X}} \\
&\quad\leq \Bigg[\bigg(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\bigg)\bigg(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\bigg)\Bigg]\|y-\overline{y}\|_{\mathcal{X}} \\
&\qquad+\bigg[\frac{\xi_2\mathrm{T}^{r}}{\eta_2\Gamma(r+1)} + m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})\bigg]\|\omega-\overline{\omega}\|_{\mathcal{X}}.
\end{aligned} \tag{B.28}$$

In the same way, we can directly verify that

$$\begin{aligned}
&\|\mathcal{T}_r y - \mathcal{T}_r\overline{y}\|_{\mathcal{X}} \\
&\quad\leq \Bigg[\bigg(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\bigg)\bigg(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\bigg)\Bigg]\|y-\overline{y}\|_{\mathcal{X}} \\
&\qquad+\bigg[\frac{\xi_2\mathrm{T}^{r}}{\eta_2\Gamma(r+1)} + m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})\bigg]\|\omega-\overline{\omega}\|_{\mathcal{X}}.
\end{aligned} \tag{B.29}$$

Therefore from (B.28) and (B.29) we get

$$\big\|\mathcal{T}_r(\omega,y) - \mathcal{T}_r(\overline{\omega},\overline{y})\big\|_{\mathcal{X}} \leq \Delta_1\big\|(\omega,y)-(\overline{\omega},\overline{y})\big\|_{\mathcal{X}}.$$

Now, suppose $\omega,\overline{\omega}\in\mathcal{Y}$. For $\tau\in\mathcal{J}_j$, we have

$$\begin{aligned}
&\big|\mathcal{T}_p\omega(\tau) - \mathcal{T}_p\overline{\omega}(\tau)\big| \\
&\quad\leq \frac{1}{\Gamma(p)}\int_0^\tau(\tau-s)^{p-1}\big|z(s)-\overline{z}(s)\big|\,ds - \frac{\tau}{\mathrm{T}}\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\big|y(s)-\overline{y}(s)\big|\,ds \\
&\qquad-\frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\bigg[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}\big|z(s)-\overline{z}(s)\big|\,ds + \frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}\big|z(s)-\overline{z}(s)\big|\,ds \\
&\qquad+\frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}\big|z(s)-\overline{z}(s)\big|\,ds + (\mathrm{T}-\tau_j)\big|\hat{\Upsilon}_j\big(y(\tau_j)\big)-\hat{\Upsilon}_j\big(\overline{y}(\tau_j)\big)\big| \\
&\qquad+\big|\Upsilon_j\big(y(\tau_j)\big)-\Upsilon_j\big(\overline{y}(\tau_j)\big)\big|\bigg],
\end{aligned} \tag{B.30}$$

where

$$z(\tau) = \mathcal{A}'\big(\tau, \omega(\tau), z(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'\big(s, \omega(s), z(s)\big)\, ds$$

and

$$\overline{z}(\tau) = \mathcal{A}'\big(\tau, \overline{\omega}(\tau), \overline{z}(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'\big(s, \overline{\omega}(s), \overline{z}(s)\big)\, ds.$$

Using $[\tilde{A_2}]$, we have

$$\begin{aligned}
&\big|z(\tau) - \overline{z}(\tau)\big| \\
&\quad = \bigg|\mathcal{A}'\big(\tau, \omega(\tau), z(\tau)\big) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'\big(s, \omega(s), z(s)\big)\, ds \\
&\qquad - \mathcal{A}'\big(\tau, \overline{\omega}(\tau), \overline{z}(\tau)\big) - \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}'\big(s, \overline{\omega}(s), \overline{z}(s)\big)\, ds\bigg| \\
&\quad \le \big|\mathcal{A}'\big(\tau, \omega(\tau), z(\tau)\big) - \mathcal{A}'\big(\tau, \overline{\omega}(\tau), \overline{z}(\tau)\big)\big| \\
&\qquad + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \big|\mathcal{B}'\big(s, \omega(s), z(s)\big) - \mathcal{B}'\big(s, \overline{\omega}(s), \overline{z}(s)\big)\big|\, ds \\
&\quad \le \mathrm{M}_1' \big|\omega(\tau) - \overline{\omega}(\tau)\big| + \mathrm{N}_1' \big|z(\tau) - \overline{z}(\tau)\big| \\
&\qquad + \frac{\tau^\sigma}{\sigma \Gamma(\delta)} \big(\mathrm{M}_2' \big|\omega(\tau) - \overline{\omega}(\tau)\big| + \mathrm{N}_2' \big|z(\tau) - \overline{z}(\tau)\big|\big).
\end{aligned}$$

Thus

$$\big|z(\tau) - \overline{z}(\tau)\big| \le \left(\frac{\mathrm{M}_1'}{1 - \mathrm{N}_1' - \mathrm{N}_2' \frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2' \frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1' - \mathrm{N}_2' \frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right) \big|\omega(\tau) - \overline{\omega}(\tau)\big|. \tag{B.31}$$

Using hypotheses $[\tilde{A_2}]$, $[\tilde{A_3}]$ and inequalities (B.31) and (B.30), we have

$$\begin{aligned}
&\big|(\mathcal{T}_p\omega)(\tau) - (\mathcal{T}_p\overline{\omega})(\tau)\big| \\
&\quad \le \bigg[\left(\frac{\tau^p}{\Gamma(p+1)} - \frac{n\tau \mathrm{T}^{p-1}}{\Gamma(p+1)} - \frac{n\tau^{p+1}}{\mathrm{T}\Gamma(p+1)} - \frac{n\tau^p}{\mathrm{T}\Gamma(p)}\right) \\
&\qquad \times \left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2 \frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right)\bigg] \big|\omega(\tau) - \overline{\omega}(\tau)\big| \\
&\qquad - \left[\frac{\xi_4 \tau \mathrm{T}^{p-1}}{\eta_4 \Gamma(p+1)} + \frac{\tau}{\mathrm{T}} n(\mathbb{A}_{\hat{\gamma}_j} + \mathbb{A}_{\gamma_j})\right] \big|y(\tau) - \overline{y}(\tau)\big|.
\end{aligned}$$

Now taking the norm on both sides, we have

$$\begin{aligned}
&\|\mathcal{T}_p\omega - \mathcal{T}_p\overline{\omega}\|_{\mathcal{Y}} \\
&\quad \le \bigg[\left(\frac{n\mathrm{T}^p}{\Gamma(p+1)} + \frac{n\mathrm{T}^{p-1}}{\Gamma(p)}\right)\left(\frac{\mathrm{M}_1'}{1 - \mathrm{N}_1' - \mathrm{N}_2' \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2' \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1 - \mathrm{N}_1' - \mathrm{N}_2' \frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)\bigg] \|\omega - \overline{\omega}\|_{\mathcal{Y}}
\end{aligned}$$

$$+\left[\frac{\xi_4 \mathrm{T}^p}{\eta_4 \Gamma(p+1)} + n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j})\right]\|y-\overline{y}\|_{\mathcal{Y}}. \tag{B.32}$$

In the same way, we can obtain

$$\begin{aligned}
&\|\mathcal{T}_p y - \mathcal{T}_p \overline{y}\|_{\mathcal{Y}} \\
&\quad \le \left[\left(\frac{n\mathrm{T}^p}{\Gamma(p+1)} + \frac{n\mathrm{T}^{p-1}}{\Gamma(p)}\right)\left(\frac{\mathrm{M}_1'}{1-\mathrm{N}_1'-\mathrm{N}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1'-\mathrm{N}_2'\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)\right]\|y-\overline{y}\|_{\mathcal{Y}} \\
&\qquad + \left[\frac{\xi_4 \mathrm{T}^p}{\eta_4 \Gamma(p+1)} + n(\mathbb{A}_{\hat{\Upsilon}_j} + \mathbb{A}_{\Upsilon_j})\right]\|\omega-\overline{\omega}\|_{\mathcal{Y}}.
\end{aligned} \tag{B.33}$$

Thus from (B.32) and (B.33) we get

$$\big\|\mathcal{T}_p(\omega,y) - \mathcal{T}_p(\overline{\omega},\overline{y})\big\|_{\mathcal{Y}} \le \Delta_2 \big\|(\omega,y)-(\overline{\omega},\overline{y})\big\|_{\mathcal{Y}}.$$

Hence it follows that

$$\big\|\mathcal{T}(\omega,y) - \mathcal{T}(\overline{\omega},\overline{y})\big\|_{\mathcal{X}\times\mathcal{Y}} \le \max(\Delta_1,\Delta_2)\big(\|\omega-\overline{\omega}\|_{\mathcal{X}\times\mathcal{Y}} + \|y-\overline{y}\|_{\mathcal{X}\times\mathcal{Y}}\big).$$

This implies that $\mathcal{T}$ is a contraction and hence has a unique fixed point. This completes the proof. □

Appendix 3

Proof of Theorem 3.11 Let $\omega \in \mathcal{M}$ be a solution of inequality (3.1), and let ω^* be a solution of the considered problem (1.3). Then

$$\begin{cases}
{}^c\mathcal{D}^r\omega^*(\tau) = \mathcal{A}(\tau,\omega^*(\tau),{}^c\mathcal{D}^r\omega^*(\tau)) \\
\qquad\qquad + \int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s,\omega^*(s),{}^c\mathcal{D}^r\omega^*(s))\,ds, \quad \tau\in\mathcal{J},\tau\neq\tau_i, i=1,2,\dots,m, \\
\Delta\omega^*(\tau_i) = \Upsilon_i(\omega^*(\tau_i)), \qquad \Delta\omega^{*'}(\tau_i) = \hat{\Upsilon}_i(\omega^*(\tau_i)), \quad i=1,2,\dots,m, \\
\eta_1\omega^*(0) + \xi_1 I^r\omega^*(0) = \nu_1, \qquad \eta_2\omega^*(\mathrm{T}) + \xi_2 I^r\omega^*(\mathrm{T}) = \nu_2.
\end{cases}$$

Using the inequality

$$\big|\omega(\tau)-\omega^*(\tau)\big| \le \big|\omega(\tau)-q(\tau)\big| + \big|q(\tau)+\omega^*(\tau)\big|, \tag{C.1}$$

by Theorem 3.10 we have

$$\begin{aligned}
&\big|\omega(\tau)-\omega^*(\tau)\big| \\
&\quad \le \left[\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}}\right]\epsilon_r + \frac{1}{\Gamma(r)}\int_0^\tau (\tau-s)^{r-1}\big|\nu(s)-\nu^*(s)\big|\,ds \\
&\qquad - \frac{\tau\xi_2}{\mathrm{T}\eta_2\Gamma(r)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\big|\omega(s)-\omega^*(s)\big|\,ds \\
&\qquad - \frac{\tau}{\mathrm{T}}\sum_{i=1}^m\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}} (\mathrm{T}-s)^{r-1}\big|\nu(s)-\nu^*(s)\big|\,ds\right.
\end{aligned}$$

$$+ \frac{1}{\Gamma(r)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-1} \left| \nu(s) - \nu^*(s) \right| ds + \frac{\mathrm{T} - \tau_i}{\Gamma(r-1)} \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{r-2} \left| \nu(s) - \nu^*(s) \right| ds$$
$$+ (\mathrm{T} - \tau_i) \left| \hat{\Upsilon}_i(\omega(\tau_i)) - \hat{\Upsilon}_i(\omega^*(\tau_i)) \right| + \left| \Upsilon_i(\omega(\tau_i)) - \Upsilon_i(\omega^*(\tau_i)) \right| \Bigg], \tag{C.2}$$

where $\nu, \nu^* \in \mathcal{M}$ are given by

$$\nu(\tau) = \mathcal{A}(\tau, \omega(\tau), \nu(\tau)) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, \omega(s), \nu(s))\, ds$$

and

$$\nu^*(\tau) = \mathcal{A}(\tau, \omega^*(\tau), \nu^*(\tau)) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, \omega^*(s), \nu^*(s))\, ds.$$

Using $[A_1]$, we have

$$\begin{aligned}
&\left| \nu(\tau) - \nu^*(\tau) \right| \\
&\quad = \left| \mathcal{A}(\tau, \omega(\tau), \nu(\tau)) + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, \omega(s), \nu(s))\, ds \right. \\
&\qquad \left. - \mathcal{A}(\tau, \omega^*(\tau), \nu^*(\tau)) - \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \mathcal{B}(s, \omega^*(s), \nu^*(s))\, ds \right| \\
&\quad \le \left| \mathcal{A}(\tau, \omega(\tau), \nu(\tau)) - \mathcal{A}(\tau, \omega^*(\tau), \nu^*(\tau)) \right| \\
&\qquad + \int_0^\tau \frac{(\tau - s)^{\sigma-1}}{\Gamma(\delta)} \left| \mathcal{B}(s, \omega(s), \nu(s)) - \mathcal{B}(s, \omega^*(s), \nu^*(s)) \right| ds \\
&\quad \le \mathrm{M}_1 \left| \omega(\tau) - \omega^*(\tau) \right| + \mathrm{N}_1 \left| \nu(\tau) - \nu^*(\tau) \right| \\
&\qquad + \frac{\tau^\sigma}{\sigma \Gamma(\delta)} \left(\mathrm{M}_2 \left| \omega(\tau) - \omega^*(\tau) \right| + \mathrm{N}_2 \left| \nu(\tau) - \nu^*(\tau) \right| \right).
\end{aligned}$$

Thus

$$\left| \nu(\tau) - \nu^*(\tau) \right| \le \left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}} + \frac{\mathrm{M}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}} \right) \left| \omega(\tau) - \omega^*(\tau) \right|. \tag{C.3}$$

Using hypothesis $[A_2]$ and (C.3), by inequality (C.2) we get

$$\begin{aligned}
&\left| \omega(\tau) - \omega^*(\tau) \right| \\
&\quad \le \left[\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{\tau m}{\mathrm{T}} \right] \epsilon_r \\
&\qquad + \left[\left(\frac{\tau^r}{\Gamma(r+1)} - \frac{m\tau \mathrm{T}^{r-1}}{\Gamma(r+1)} - \frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)} - \frac{m\tau^r}{\mathrm{T}\Gamma(r)} \right) \right. \\
&\qquad \times \left. \left(\frac{\mathrm{M}_1}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}} + \frac{\mathrm{M}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}}{1 - \mathrm{N}_1 - \mathrm{N}_2 \frac{\tau^\sigma}{\sigma \Gamma(\delta)}} \right) - \frac{\xi_2 \tau \mathrm{T}^{r-1}}{\eta_2 \Gamma(r+1)} - \frac{\tau}{\mathrm{T}} m(\mathbb{A} + \mathbb{B}) \right] \\
&\qquad \times \left| \omega(\tau) - \omega^*(\tau) \right|.
\end{aligned}$$

By taking the norm and simplifying we get

$$\begin{aligned}
&\|\omega-\omega^*\|_{\mathcal{M}} \\
&\quad\leq\left[\frac{\mathrm{T}^r}{\Gamma(r+1)}-\frac{m\mathrm{T}^r}{\Gamma(r+1)}-m\right]\epsilon_r+\left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)}+\frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\right. \\
&\quad\quad\times\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)+\left.\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)}+m(\mathbb{A}+\mathbb{B})\right] \\
&\quad\quad\times\|\omega-\omega^*\|_{\mathcal{M}},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\|\omega-\omega^*\|_{\mathcal{M}} \\
&\quad\leq\frac{[\frac{\mathrm{T}^r}{\Gamma(r+1)}-\frac{m\mathrm{T}^r}{\Gamma(r+1)}-m]\epsilon_r}{1-[(\frac{m\mathrm{T}^r}{\Gamma(r+1)}+\frac{m\mathrm{T}^{r-1}}{\Gamma(r)})(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}})+\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)}+m(\mathbb{A}+\mathbb{B})]}.
\end{aligned}$$

Thus

$$\|\omega-\omega^*\|_{\mathcal{M}}\leq\mathbf{C}_r\epsilon_r,$$

where

$$\mathbf{C}_r=\frac{[\frac{\mathrm{T}^r}{\Gamma(r+1)}-\frac{m\mathrm{T}^r}{\Gamma(r+1)}-m]}{1-[(\frac{m\mathrm{T}^r}{\Gamma(r+1)}+\frac{m\mathrm{T}^{r-1}}{\Gamma(r)})(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}})+\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)}+m(\mathbb{A}+\mathbb{B})]},$$

that is, problem (1.3) is Ulam–Hyers stable. Now putting $\vartheta(\epsilon)=\mathbf{C}_r\epsilon_r\vartheta(0)=0$ yields that problem (1.3) is generalized Ulam–Hyers stable. □

Proof of Theorem 3.20 Let $(\omega,y)\in\mathcal{X}\times\mathcal{Y}$ be a solution of inequality (3.9), and let $(\omega^*,y^*)\in\mathcal{X}\times\mathcal{Y}$ be a solution of the system

$$\begin{cases}
{}^c\mathcal{D}^r\omega^*(\tau)=\mathcal{A}(\tau,y^*(\tau),{}^c\mathcal{D}^r\omega^*(\tau))+\int_0^\tau\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}(s,y^*(s),{}^c\mathcal{D}^r\omega^*(s))\,ds\\
\quad\text{where }\tau\in\mathcal{J},\tau\neq\tau_i\text{ for }i=1,2,\ldots,m,\\
{}^c\mathcal{D}^py^*(\tau)=\mathcal{A}'(\tau,\omega^*(\tau),{}^c\mathcal{D}^py^*(\tau))+\int_0^\tau\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'(s,\omega^*(s),{}^c\mathcal{D}^py^*(s))\,ds\\
\quad\text{where }\tau\in\mathcal{J},\tau\neq\tau_j\text{ for }j=1,2,\ldots,n,\\
\Delta\omega^*(\tau_i)=\Upsilon_i(\omega^*(\tau_i)),\qquad\Delta\omega'^*(\tau_i)=\hat{\Upsilon}_i(\omega^*(\tau_i)),\quad i=1,2,\ldots,m,\\
\Delta y^*(\tau_j)=\Upsilon_j(y^*(\tau_j)),\qquad\Delta y'^*(\tau_j)=\hat{\Upsilon}_j(y^*(\tau_j)),\quad j=1,2,\ldots,n,\\
\eta_1\omega^*(0)+\xi_1I^r\omega^*(0)=\nu_1,\qquad\eta_2\omega^*(\mathrm{T})+\xi_2I^r\omega^*(\mathrm{T})=\nu_2,\\
\eta_3y^*(0)+\xi_3I^py^*(0)=\nu_3,\qquad\eta_4y^*(\mathrm{T})+\xi_4I^py^*(\mathrm{T})=\nu_4.
\end{cases}\tag{C.4}$$

Then in view of Lemma A.3, the solution of (C.4) is

$$\omega^*(\tau)=\frac{1}{\Gamma(r)}\int_0^\tau(\tau-s)^{r-1}\nu(s)\,ds+\frac{\nu_1}{\eta_1}-\frac{\tau}{\mathrm{T}}\left[\frac{\nu_1}{\eta_1}-\frac{\nu_2}{\eta_2}+\frac{\xi_2}{\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\omega^*(s)\,ds\right]$$

$$-\frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\nu(s)\,ds+\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\nu(s)\,ds\right.$$
$$\left.+\frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\nu(s)\,ds+(\mathrm{T}-\tau_i)\hat{\Upsilon}_i\big(\omega^*(\tau_i)\big)+\Upsilon_i\big(\omega^*(\tau_i)\big)\right]$$

and

$$y^*(\tau)=\frac{1}{\Gamma(p)}\int_0^{\tau}(\tau-s)^{p-1}z(s)\,ds-\frac{\tau}{\mathrm{T}}\left[\frac{\nu_3}{\eta_3}-\frac{\nu_4}{\eta_4}+\frac{\xi_4}{\eta_4\Gamma(p)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{p-1}y^*(s)\,ds\right]$$
$$-\frac{\tau}{\mathrm{T}}\sum_{j=1}^{n}\left[\frac{1}{\Gamma(p)}\int_{\tau_j}^{\mathrm{T}}(\mathrm{T}-s)^{p-1}z(s)\,ds+\frac{1}{\Gamma(p)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-1}z(s)\,ds\right.$$
$$\left.+\frac{\mathrm{T}-\tau_j}{\Gamma(p-1)}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{p-2}z(s)\,ds+(\mathrm{T}-\tau_j)\hat{\Upsilon}_j\big(y^*(\tau_j)\big)+\Upsilon_j\big(y^*(\tau_j)\big)\right],$$

where

$$\nu(\tau)=\mathcal{A}\big(\tau,y(\tau),\nu(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,y(s),\nu(s)\big)\,ds$$

and

$$z(\tau)=\mathcal{A}'\big(\tau,\omega(\tau),z(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}'\big(s,\omega(s),z(s)\big)\,ds.$$

Consider

$$\begin{aligned}
&\big|\omega(\tau)-\omega^*(\tau)\big|\\
&\quad\le\big|\omega(\tau)-q(\tau)\big|+\big|q(\tau)-\omega^*(\tau)\big|\\
&\quad\le\left[\frac{\tau^r}{\Gamma(r+1)}-\frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)}-\frac{\tau m}{\mathrm{T}}\right]\epsilon_r+\frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-s)^{r-1}\big|\nu(s)-\nu^*(s)\big|\,ds\\
&\qquad-\frac{\tau\xi_2}{\mathrm{T}\eta_2\Gamma(r)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\omega(s)-\omega^*(s)\big|\,ds\\
&\qquad-\frac{\tau}{\mathrm{T}}\sum_{i=1}^{m}\left[\frac{1}{\Gamma(r)}\int_{\tau_i}^{\mathrm{T}}(\mathrm{T}-s)^{r-1}\big|\nu(s)-\nu^*(s)\big|\,ds\right.\\
&\qquad+\frac{1}{\Gamma(r)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-1}\big|\nu(s)-\nu^*(s)\big|\,ds+\frac{\mathrm{T}-\tau_i}{\Gamma(r-1)}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{r-2}\big|\nu(s)-\nu^*(s)\big|\,ds\\
&\qquad\left.+(\mathrm{T}-\tau_i)\big|\hat{\Upsilon}_i\big(\omega(\tau_i)\big)-\hat{\Upsilon}_i\big(\omega^*(\tau_i)\big)\big|+\big|\Upsilon_i\big(\omega(\tau_i)\big)-\Upsilon_i\big(\omega^*(\tau_i)\big)\big|\right],
\end{aligned}\tag{C.5}$$

where $\nu,\nu^*\in\mathcal{X}$ are given by

$$\nu(\tau)=\mathcal{A}\big(\tau,y(\tau),\nu(\tau)\big)+\int_0^{\tau}\frac{(t-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,y(s),\nu(s)\big)\,ds$$

and

$$\nu^*(\tau)=\mathcal{A}\big(\tau,y^*(\tau),\nu^*(\tau)\big)+\int_0^{\tau}\frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\big(s,y^*(s),\nu^*(s)\big)\,ds.$$

Using $[\tilde{A}_1]$, we have

$$\begin{aligned}
&\left|v(\tau)-v^*(\tau)\right| \\
&\quad= \left|\mathcal{A}\bigl(\tau,y(\tau),v(\tau)\bigr)+\int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s,y(s),v(s)\bigr)\,ds\right. \\
&\qquad \left.-\mathcal{A}\bigl(\tau,y^*(\tau),v^*(\tau)\bigr)-\int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\mathcal{B}\bigl(s,y^*(s),v^*(s)\bigr)\,ds\right| \\
&\quad\le \left|\mathcal{A}\bigl(\tau,y(\tau),v(\tau)\bigr)-\mathcal{A}\bigl(\tau,y^*(\tau),v^*(\tau)\bigr)\right| \\
&\qquad +\int_0^\tau \frac{(\tau-s)^{\sigma-1}}{\Gamma(\delta)}\left|\mathcal{B}\bigl(s,y(s),v(s)\bigr)-\mathcal{B}\bigl(s,y^*(s),v^*(s)\bigr)\right|\,ds \\
&\quad\le \mathrm{M}_1\left|y(\tau)-y^*(\tau)\right|+\mathrm{N}_1\left|v(\tau)-v^*(\tau)\right| \\
&\qquad +\frac{\tau^\sigma}{\sigma\Gamma(\delta)}\bigl(\mathrm{M}_2\left|y(\tau)-y^*(\tau)\right|+\mathrm{N}_2\left|v(\tau)-v^*(\tau)\right|\bigr).
\end{aligned}$$

Thus

$$\left|v(\tau)-v^*(\tau)\right|\le\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right)\left|y(\tau)-y^*(\tau)\right|. \tag{C.6}$$

Using hypothesis $[\tilde{A}_3]$ and (C.6), inequality (C.5) implies

$$\begin{aligned}
&\left|\omega(\tau)-\omega^*(\tau)\right| \\
&\quad\le\left[\frac{\tau^r}{\Gamma(r+1)}-\frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)}-\frac{\tau m}{\mathrm{T}}\right]\epsilon_r \\
&\qquad+\left[\left(\frac{\tau^r}{\Gamma(r+1)}-\frac{m\tau\mathrm{T}^{r-1}}{\Gamma(r+1)}-\frac{m\tau^{r+1}}{\mathrm{T}\Gamma(r+1)}-\frac{m\tau^r}{\mathrm{T}\Gamma(r)}\right)\right. \\
&\qquad\left.\times\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\tau^\sigma}{\sigma\Gamma(\delta)}}\right)\left|y(\tau)-y^*(\tau)\right|\right] \\
&\qquad-\left[\frac{\tau\xi_2\mathrm{T}^{r-1}}{\eta_2\Gamma(r+1)}+\frac{\tau}{\mathrm{T}}m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})\right]\left|\omega(\tau)-\omega^*(\tau)\right|.
\end{aligned}$$

By taking the norm and simplifying, we get

$$\begin{aligned}
\|\omega-\omega^*\|_{\mathcal{X}}\le{}&\left[\frac{\mathrm{T}^r}{\Gamma(r+1)}-\frac{m\mathrm{T}^r}{\Gamma(r+1)}-m\right]\epsilon_r+\left[\left(\frac{m\mathrm{T}^r}{\Gamma(r+1)}+\frac{m\mathrm{T}^{r-1}}{\Gamma(r)}\right)\right. \\
&\left.\times\left(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}+\frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}\right)\|y-y^*\|_{\mathcal{X}}\right] \\
&-\left[\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)}+m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})\right]\|\omega-\omega^*\|_{\mathcal{X}}.
\end{aligned} \tag{C.7}$$

For simplicity, we consider

$$\mathcal{S}_r=\frac{\frac{\mathrm{T}^r}{\Gamma(r+1)}-\frac{m\mathrm{T}^r}{\Gamma(r+1)}-m}{1+\frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)}+m(\mathbb{A}_{\hat{\Upsilon}_i}+\mathbb{A}_{\Upsilon_i})},$$

$$\mathcal{Q}_r = \frac{(\frac{m\mathrm{T}^r}{\Gamma(r+1)} + \frac{m\mathrm{T}^{r-1}}{\Gamma(r)})(\frac{\mathrm{M}_1}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}} + \frac{\mathrm{M}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}}{1-\mathrm{N}_1-\mathrm{N}_2\frac{\mathrm{T}^\sigma}{\sigma\Gamma(\delta)}})}{1 + \frac{\xi_2\mathrm{T}^r}{\eta_2\Gamma(r+1)} + m(\mathbb{A}_{\hat{\Upsilon}_i} + \mathbb{A}_{\Upsilon_i})}.$$

Then (C.7) implies

$$\|\omega - \omega^*\|_{\mathcal{X}} \le \mathcal{S}_r\epsilon_r + \mathcal{Q}_r\|y - y^*\|_{\mathcal{X}} \tag{C.8}$$

and, similarly,

$$\|y - y^*\|_{\mathcal{Y}} \le \mathcal{S}_p\epsilon_p + \mathcal{Q}_p\|\omega - \omega^*\|_{\mathcal{Y}}. \tag{C.9}$$

From (C.8) and (C.9) we write

$$\begin{aligned}
&\|\omega - \omega^*\|_{\mathcal{X}} - \mathcal{Q}_r\|y - y^*\|_{\mathcal{X}} \le \mathcal{S}_r\epsilon_r, \\
&\|y - y^*\|_{\mathcal{Y}} - \mathcal{Q}_p\|\omega - \omega^*\|_{\mathcal{Y}} \le \mathcal{S}_p\epsilon_p, \\
&\begin{bmatrix} 1 & -\mathcal{Q}_r \\ -\mathcal{Q}_p & 1 \end{bmatrix} \begin{bmatrix} \|\omega - \omega^*\|_{\mathcal{X}\times\mathcal{Y}} \\ \|y - y^*\|_{\mathcal{X}\times\mathcal{Y}} \end{bmatrix} \le \begin{bmatrix} \mathcal{S}_r\epsilon_r \\ \mathcal{S}_p\epsilon_p \end{bmatrix}.
\end{aligned}$$

Solving the last inequality, we have

$$\begin{bmatrix} \|\omega - \omega^*\|_{\mathcal{X}\times\mathcal{Y}} \\ \|y - y^*\|_{\mathcal{X}\times\mathcal{Y}} \end{bmatrix} \le \begin{bmatrix} \frac{1}{\Delta} & \frac{\mathcal{Q}_r}{\Delta} \\ \frac{\mathcal{Q}_p}{\Delta} & \frac{1}{\Delta} \end{bmatrix} \begin{bmatrix} \mathcal{S}_r\epsilon_r \\ \mathcal{S}_p\epsilon_p \end{bmatrix},$$

where

$$\Delta = 1 - \mathcal{Q}_r\mathcal{Q}_p > 0.$$

Further simplification gives

$$\begin{aligned}
&\|\omega - \omega^*\|_{\mathcal{X}\times\mathcal{Y}} \le \frac{\mathcal{S}_r\epsilon_r}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_p\epsilon_p}{\Delta}, \\
&\|y - y^*\|_{\mathcal{X}\times\mathcal{Y}} \le \frac{\mathcal{S}_p\epsilon_p}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_r\epsilon_r}{\Delta},
\end{aligned}$$

from which we have

$$\|\omega - \omega^*\|_{\mathcal{X}\times\mathcal{Y}} + \|y - y^*\|_{\mathcal{X}\times\mathcal{Y}} \le \frac{\mathcal{S}_r\epsilon_r}{\Delta} + \frac{\mathcal{S}_p\epsilon_p}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_p\epsilon_p}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_r\epsilon_r}{\Delta}. \tag{C.10}$$

Let $\max\{\epsilon_r, \epsilon_p\} = \epsilon$. Then from (C.10) we get

$$\|(\omega, y) - (\omega^*, y^*)\|_{\mathcal{X}\times\mathcal{Y}} \le C_{r,p}\epsilon,$$

where

$$C_{r,p} = \left[\frac{\mathcal{S}_r}{\Delta} + \frac{\mathcal{S}_p}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_p}{\Delta} + \frac{\mathcal{Q}_r\mathcal{S}_r}{\Delta}\right].$$

This completes the proof. □

Acknowledgements
The research of the second author is supported by Prince Sultan University through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM), group number RG-DES-2017-01-17.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
On behalf of all authors, the corresponding author declares that they have no conflict of interest.

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All authors have equally and significantly contributed to the contents of this manuscript. All authors read and approved the final manuscript.

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Received: 1 November 2019 Accepted: 29 January 2020 Published online: 07 February 2020

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