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Mixed nonlocal boundary value problem for implicit fractional integro-differential equations via ψ-Hilfer fractional derivative

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Abstract

In this paper, we investigate the existence and uniqueness of a solution for a class of ψ-Hilfer implicit fractional integro-differential equations with mixed nonlocal conditions. The arguments are based on Banach's, Schaefer's, and Krasnoselskii's fixed point theorems. Further, applying the techniques of nonlinear functional analysis, we establish various kinds of the Ulam stability results for the analyzed problem, that is, the Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. Finally, we provide some examples to illustrate the applicability of our results.

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1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration of arbitrary order, which can be noninteger. Differential equations of fractional order have attracted the attention of several researchers; see the monographs [1–8] and references therein. In the literature, there exist several definitions of fractional integrals and derivatives, from the most popular Riemann–Liouville and Caputo-type fractional derivatives to the other ones such as Hadamard fractional derivative, the Erdélyi–Kober fractional derivative, and so forth. A generalization of both Riemann–Liouville and Caputo derivatives was given by Hilfer [9], which is known as *the Hilfer fractional derivative $D^{\alpha,\beta}x(t)$* of order α and type $\beta \in [0,1]$. The Hilfer fractional derivative interpolates between the Riemann–Liouville and Caputo derivatives as it reduces to the Riemann–Liouville and Caputo fractional derivatives for $\beta = 0$ and $\beta = 1$, respectively. The Hilfer fractional derivative is used in theoretical simulation of dielectric relaxation in glass-forming materials and in fractional diffusion equations; see [10, 11]. Some properties and applications of the Hilfer derivative can be found in [12–16] and references therein.

The fractional derivative with another function, in the Hilfer sense, called the *ψ-Hilfer fractional derivative* and introduced in [17], generalizes the Hilfer fractional derivative [9].

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The ψ-Hilfer fractional derivative is defined with respect to another function and unifies several definitions of fractional derivatives available in the literature. Thus the ψ-Hilfer fractional derivative covers a wide class of fractional derivatives and provides a platform to obtain a particular one by fixing the function ψ; see Remark 2.4. For some recent results on the existence and uniqueness of solutions of initial value problems and on the Ulam–Hyers–Rassias stability, see [10, 11, 18–27] and references therein.

Nonlocal boundary value problems have become a rapidly growing area of research. The study of this type of problems is driven not only by theoretical interest, but also by the fact that several phenomena in engineering, physics, and life sciences can be modeled in this way. The idea of nonlocal conditions dates back to the work of Hilb [28]. However, the systematic investigation of a certain class of spatial nonlocal problems was carried out by Bitsadze and Samarskii [29]. We refer the reader to [30, 31] and references therein for a motivation regarding nonlocal conditions.

In [32] the authors considered fractional differential equations with mixed nonlocal fractional derivatives, integrals, and multipoint conditions of the form

$$\begin{cases} {}^c\mathfrak{D}^\alpha x(t) = f(t, x(t)), & t \in (0, T], \\ \sum_{i=1}^m \gamma_i x(\eta_i) + \sum_{j=1}^n \lambda_j {}^c\mathfrak{D}^{\beta_j} x(\xi_i) + \sum_{r=1}^k \sigma_r \mathcal{I}^{\delta_r} x(\phi_r) = A, \end{cases} \tag{1.1}$$

where $x \in C^1([0, T], \mathbb{R})$, ${}^c\mathfrak{D}^\alpha$ and ${}^c\mathfrak{D}^{\beta_j}$ denote the Caputo fractional derivatives of orders α and β_j, respectively, $0 < \beta_j \leq \alpha \leq 1$ for $j = 1, 2, \ldots, n$, $\mathcal{I}^{\delta_r}$ is the Riemann–Liouville fractional integral operator of order $\delta_r > 0$ for $r = 1, 2, \ldots, k$, $\gamma_i, \lambda_j, \sigma_r, A \in \mathbb{R}$, $\eta_i, \xi_j, \phi_r \in [0, T]$, $i = 1, 2, \ldots, m$, and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. The existence and uniqueness results were obtained by applying Schaefer's fixed point theorem and Banach's contraction mapping principle. In addition, the authors established different kinds of Ulam stability for the problem.

In [33] the authors studied the existence, uniqueness, and Ulam–Hyers–Rassias stability for a class of ψ-Hilfer fractional differential equations described by

$$\begin{cases} {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t)), & t \in J = (a, T], \\ \mathcal{I}_{a^+}^{1-\gamma;\psi} x(a) = x_a, & \alpha \leq \gamma = \alpha + \rho - \alpha\rho, T > a, \end{cases} \tag{1.2}$$

where ${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0, 1]$ and type $\rho \in [0, 1]$, $\mathcal{I}_{a^+}^{1-\gamma;\psi}$ is the Riemann–Liouville fractional integral of order $1 - \gamma$ with respect to the function $\psi, f \in C(J \times \mathbb{R}^2, \mathbb{R})$, and $x_a \in \mathbb{R}$.

Harikrishman et al. [34] discussed the existence and uniqueness of nonlocal initial value problems for Pantograph equations with ψ-Hilfer fractional derivative of the form

$$\begin{cases} {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), x(\lambda t)), & t \in J = (a, b], 0 < \lambda < 1, \\ \mathcal{I}_{a^+}^{1-\gamma;\psi} x(a) = \sum_{i=1}^k c_i x(\tau_i), & \tau_i \in (a, b], \alpha \leq \gamma = \alpha + \rho - \alpha\rho, \end{cases} \tag{1.3}$$

where ${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0, 1)$ and type $\rho \in [0, 1]$, $\mathcal{I}_{a^+}^{1-\gamma;\psi}$ is the Riemann–Liouville fractional integral of order $1 - \gamma$ with respect to the continuous function ψ such that $\psi' > 0$, and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$.

In [35] the authors established existence, uniqueness and Ulam–Hyers stability of implicit Pantograph fractional differential equations involving ψ-Hilfer fractional derivatives

of the form

$$
\begin{cases}
{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), x(\lambda t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(\lambda t)), \quad t \in J = (0, T], T > 0, 0 < \lambda < 1, \\
\mathcal{I}_{0^+}^{1-\gamma;\psi} x(0^+) = \sum_{i=1}^{m} b_i \mathcal{I}_{0^+}^{\beta;\psi} x(\xi_i), \quad \xi_i \in J, \alpha \le \gamma = \alpha + \rho - \alpha\rho,
\end{cases}
\tag{1.4}
$$

where ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\rho \in [0,1]$, $\mathcal{I}_{0^+}^{1-\gamma;\psi}$ and $\mathcal{I}_{0^+}^{\beta;\psi}$ are the ψ-Riemann–Liouville fractional integrals of orders $1-\gamma$ and $\beta > 0$, respectively, with respect to the continuous function ψ such that $\psi' \ne 0, f \in C(J \times \mathbb{R}^3, \mathbb{R})$, $b_i \in \mathbb{R}$, and $0 < \xi_1 \le \xi_2 \le \cdots \le \xi_m < T$.

Motivated by papers [33–35] and some familiar results on fractional integro-differential equations, we establish the existence and uniqueness results and different types of Ulam stability, such as Ulam–Hyers (UH), generalized Ulam–Hyers (GUH), Ulam–Hyers–Rassias (UHR), and generalized Ulam–Hyers–Rassias (GUHR) stability for a class of ψ-Hilfer implicit fractional integro-differential equations with mixed nonlocal boundary conditions of the form

$$
\begin{cases}
{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t), \mathcal{I}_{0^+}^{\alpha;\psi} x(t)), \quad t \in (0, T], \\
\sum_{i=1}^{m} \omega_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j {}^{H}\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\zeta_j) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} x(\theta_r) = A,
\end{cases}
\tag{1.5}
$$

where ${}^{H}\mathfrak{D}_{0^+}^{u,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $u = \{\alpha, \beta_j\}$ with $0 < \alpha, \beta_j \le 1$, $\alpha \ge \beta_j + \rho(1 - \beta_j), j = 1, \ldots, n$, and $0 \le \rho \le 1$, $\mathcal{I}_{0^+}^{\alpha;\psi}$ and $\mathcal{I}_{0^+}^{\delta_r;\psi}$ are the ψ-Riemann–Liouville fractional integrals of orders α and $\delta_r > 0$, respectively, $\omega_i, \kappa_j, \sigma_r, A \in \mathbb{R}$, $\eta_i, \zeta_j, \theta_r \in J$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, r = 1, 2, \ldots, k, f : J \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, and $J := [0, T], T > 0$. We emphasize that *the mixed nonlocal boundary conditions include multipoint, fractional derivative multiorder, and fractional integral multiorder boundary conditions.*

The paper is organized as follows: In Sect. 2, we recall some basic and essential definitions and lemmas. In Sect. 3, we obtain the existence and uniqueness results for problem (1.5) via Banach's, Schaefer's, and Krasnosel'skii's fixed point theorems. In Sect. 4, we discuss the Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability results. Finally, in Sect. 5, we give some examples to illustrate the benefit of our main results.

2 Background material and auxiliary results

In this section, we introduce some notation, spaces, definitions, and some useful fundamental lemmas.

We denote $\mathcal{C}[J, \mathbb{R}]$ the Banach space of all continuous functions from an interval J into $\mathbb{R}$ with the norm defined by

$$
\|f\| = \sup_{t \in J} \{|f(t)|\}.
$$

The weighted space $\mathcal{C}_{\gamma,\psi}[J, \mathbb{R}]$ of continuous functions f on J is defined by

$$
\mathcal{C}_{\gamma,\psi}[J, \mathbb{R}] = \{f(t) : (0, T] : (\psi(t) - \psi(0))^\gamma f(t) \in \mathcal{C}[J, \mathbb{R}]\}
$$

with the norm

$$\|f\|_{\mathcal{C}_{\gamma,\psi}[J,\mathbb{R}]} = \left\| \left(\psi(t) - \psi(0) \right)^{\gamma} f(t) \right\| = \sup_{t \in J} \left\{ \left| \left(\psi(t) - \psi(0) \right)^{\gamma} f(t) \right| \right\}.$$

Definition 2.1 ([2]) Let $(0, b]$ be a finite or infinite interval on the half-axis $\mathbb{R}^+$, and let $\alpha \in \mathbb{R}^+$. Also, let $\psi(x)$ be an increasing positive function on $(0, b]$, having a continuous derivative $\psi'(x)$ on $(0, b)$. The ψ-Riemann–Liouville fractional integral of a function f with respect to another function ψ on $[0, b]$ is defined by

$$\mathcal{I}_{0^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\tau) \big(\psi(t) - \psi(\tau) \big)^{\alpha-1} f(\tau) \, d\tau, \quad t > 0, \tag{2.1}$$

where Γ is the gamma function.

Definition 2.2 ([2]) Let $\psi'(x) \neq 0$, $\alpha > 0$, and $n \in \mathbb{N}$. The Riemann–Liouville fractional derivatives of a function f with respect to another function ψ of order α is defined by

$$\mathfrak{D}_{0^+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathcal{I}_{0^+}^{n-\alpha;\psi} f(t), \tag{2.2}$$

where $n = [\alpha] + 1$, and $[\alpha]$ represents the integer part of the real number α.

Definition 2.3 ([17]) Let $f, \psi \in \mathcal{C}^n(J, \mathbb{R})$ be two functions such that $\psi \geq 0$ and $\psi'(t) \neq 0$ for all $t \in J$ and $n - 1 < \alpha < n$ with $n \in \mathbb{N}$. The ψ-Hilfer fractional derivative of a function f of order α and type $0 \leq \rho \leq 1$ is defined by

$$^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} f(t) = \mathcal{I}_{0^+}^{\rho(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathcal{I}_{0^+}^{(1-\rho)(n-\alpha);\psi} f(t), \tag{2.3}$$

where $n = [\alpha] + 1$, and $[\alpha]$ represents the integer part of the real number α.

Remark 2.4 The operator $^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is reduced to the fractional derivative of Hilfer when $\psi(t) \to t$ [9], of Hilfer–Hadamard when $\psi(t) \to \log t$ [36], of Hilfer–Katugampola when $\psi(t) \to t^\rho$, $\rho > 0$ [37], of Riemann–Liouville when $\psi(t) \to t$, $\beta \to 0$ [2], of Caputo type when $\psi(t) \to t$, $\beta \to 1$ [2], of generalized Riemann–Liouville when $\beta \to 0$ [2], and of generalized Caputo when $\beta \to 1$ [38].

The following lemma presents the semigroup properties of the ψ-Hilfer fractional integral and derivative.

Lemma 2.5 ([2]) *Let $\alpha \geq 0$, $0 \leq \rho < 1$, and $f \in L^1[J, \mathbb{R}]$. Then*

$$\mathcal{I}_{0^+}^{\alpha;\psi} \mathcal{I}_{0^+}^{\beta;\psi} f(t) = \mathcal{I}_{0^+}^{\alpha+\beta;\psi} f(t) \quad \text{for a.e. } t \in J. \tag{2.4}$$

In particular, if $f \in \mathcal{C}_{\gamma,\psi}[J, \mathbb{R}]$ and $f \in \mathcal{C}[J, \mathbb{R}]$, then

$$\mathcal{I}_{0^+}^{\alpha;\psi} \mathcal{I}_{0^+}^{\beta;\psi} f(t) = \mathcal{I}_{0^+}^{\alpha+\beta;\psi} f(t) \quad \text{for all } t \in (0, T],$$

and

$$^{H}\mathfrak{D}_{0^+}^{\alpha;\psi}\mathcal{I}_{0^+}^{\alpha;\psi}f(t) = f(t) \quad \textit{for all } t \in J.$$

The composition of the ψ-Hilfer fractional integral and derivative operators is given by the following lemma.

Lemma 2.6 ([2]) *Let $0 < \alpha \leq 1$, $0 \leq \rho < 1$, and $\gamma = \alpha + \rho - \alpha\rho$. If $f(t) \in \mathcal{C}_{1-\gamma}^{\gamma}[J, \mathbb{R}]$, then*

$$\mathcal{I}_{0^+}^{\gamma;\psi}{}^{H}\mathfrak{D}_{0^+}^{\gamma;\psi}f(t) = \mathcal{I}_{0^+}^{\alpha;\psi}{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}f(t), \quad \textit{and} \quad {}^{H}\mathfrak{D}_{0^+}^{\gamma;\psi}\mathcal{I}_{0^+}^{\alpha;\psi}f(t) = {}^{H}\mathfrak{D}_{0^+}^{\rho(1-\alpha);\psi}f(t).$$

Next, we take into account some important properties of the ψ-fractional derivative and integral operators.

Proposition 2.7 ([2, 17]) *Let $\alpha \geq 0$, $\upsilon > 0$, and $t > 0$. Then the ψ-fractional integral and derivative of a power function are given by*

(i) $\mathcal{I}_{0^+}^{\alpha;\psi}(\psi(s) - \psi(0))^{\upsilon-1}(t) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}(\psi(t) - \psi(0))^{\upsilon+\alpha-1}$;

(ii) $\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}(\psi(s) - \psi(0))^{\upsilon-1}(t) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)}(\psi(t) - \psi(0))^{\upsilon-\alpha-1}$;

(iii) ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}(\psi(s) - \psi(0))^{\upsilon-1}(t) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)}(\psi(t) - \psi(0))^{\upsilon-\alpha-1}$.

In particular, for $n \leq k \in \mathbb{N}$ and $\upsilon > n$,

$$^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}\left(\psi(s) - \psi(0)\right)^{\upsilon-1}(t) = \frac{k!}{\Gamma(k+1-\alpha)}\left(\psi(t) - \psi(0)\right)^{k-\alpha}.$$

On the other hand, for $n \geq k$,

$$^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}\left(\psi(s) - \psi(0)\right)^{k}(t) = 0.$$

Lemma 2.8 *Let $0 < \alpha, \beta \leq 1$, $0 \leq \rho \leq 1$, and $\alpha \geq \beta + \rho(1-\beta)$. If $f \in \mathcal{C}_{1-\gamma,\psi}[0, T]$, then*

$$^{H}\mathfrak{D}_{0^+}^{\beta,\rho;\psi}\mathcal{I}_{0^+}^{\alpha;\psi}f(t) = \mathcal{I}_{0^+}^{\alpha-\beta;\psi}f(t). \tag{2.5}$$

Proof Letting $\lambda = \beta + \rho(1-\beta)$, we get

$$\begin{aligned}
{}^{H}\mathfrak{D}_{0^+}^{\beta,\rho;\psi}\left(\mathcal{I}_{0^+}^{\alpha;\psi}f(t)\right) &= \mathcal{I}_{0^+}^{\lambda-\beta;\psi}\mathfrak{D}_{0^+}^{\lambda;\psi}\left(\mathcal{I}_{0^+}^{\alpha;\psi}f(t)\right) \\
&= \mathcal{I}_{0^+}^{\lambda-\beta;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathcal{I}_{0^+}^{1-\lambda;\psi}\left(\mathcal{I}_{0^+}^{\alpha;\psi}f(t)\right) \\
&= \mathcal{I}_{0^+}^{\lambda-\beta;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathcal{I}_{0^+}^{1-\lambda+\alpha;\psi}f(t).
\end{aligned}$$

By Definition 2.1 we obtain

$$\begin{aligned}
&\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathcal{I}_{0^+}^{1-\lambda+\alpha;\psi}f(t) \\
&= \frac{1}{\psi'(t)\Gamma(1-\lambda+\alpha)}\frac{d}{dt}\int_0^t \psi'(\tau)\left(\psi(t) - \psi(\tau)\right)^{\alpha-\lambda}f(\tau)\,d\tau
\end{aligned}$$

$$= \frac{1}{\psi'(t)\Gamma(1-\lambda+\alpha)} \int_0^t (\alpha-\lambda)\psi'(t)\psi'(\tau)\big(\psi(t)-\psi(\tau)\big)^{\alpha-\lambda-1} f(\tau)\, d\tau$$

$$= \frac{1}{\Gamma(\alpha-\lambda)} \int_0^t \psi'(\tau)\big(\psi(t)-\psi(\tau)\big)^{\alpha-\lambda-1} f(\tau)\, d\tau$$

$$= I_{0^+}^{\alpha-\lambda;\psi} f(t).$$

Then

$${}^{H}\mathfrak{D}_{0^+}^{\beta,\rho;\psi} \mathcal{I}_{0^+}^{\alpha;\psi} f(t) = \mathcal{I}_{0^+}^{\alpha-\beta;\psi} f(t). \qquad\qquad\qquad \square$$

Lemma 2.9 ([17]) *If $f \in \mathcal{C}^n[J,\mathbb{R}]$, $n-1 < \alpha < n$, $0 \le \beta \le 1$, and $\gamma = \alpha + \rho(n-\alpha)$, then*

$$\mathcal{I}_{0^+}^{\alpha;\psi}\,{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(0))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \mathcal{I}_{0^+}^{(1-\rho)(n-\alpha);\psi} f(0) \qquad (2.6)$$

for all $t \in J$, where $f_{\psi}^{[n]}f(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$. Moreover, if $0 < \alpha < 1$, then

$$\mathcal{I}_{0^+}^{\alpha;\psi}\,{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} f(t) = f(t) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{I}_{0^+}^{(1-\rho)(1-\alpha);\psi} f(0) \qquad (2.7)$$

for all $0 < \gamma < 1$ and $t \in J$.

In addition, if $f \in C_{1-\gamma;\psi}[J,\mathbb{R}]$ and $\mathcal{I}_{0^+}^{1-\gamma;\psi} f \in C_{1-\gamma;\psi}^1[J,\mathbb{R}]$, then

$$\mathcal{I}_{0^+}^{\gamma;\psi}\,{}^{H}\mathfrak{D}_{0^+}^{\gamma;\psi} f(t) = f(t) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{I}_{0^+}^{(1-\gamma);\psi} f(0)$$

for all $0 < \gamma < 1$ and $t \in J$.

To transform problem (1.5) into a fixed point problem, problem (1.5) must be converted to an equivalent Volterra integral equation. We provide the following lemma, which is important in our main results and concerns a linear variant of problem (1.5).

Lemma 2.10 *Let $h \in \mathcal{C}(J,\mathbb{R})$, $\alpha \in (0,1]$, $\rho \in [0,1)$, $\gamma = \alpha + \rho(1-\alpha)$, $\beta_j \in (0,1]$, $\alpha \ge \beta_j + \rho(1-\beta_j)$, $j = 1,2,\ldots,n$, $\delta_r > 0$, $r = 1,2,\ldots,k$, and $\Omega \ne 0$. Then the function $x \in \mathcal{C}_{1-\gamma,\psi}(J,\mathbb{R})$ is a solution of the linear ψ-Hilfer fractional differential equation equipped with mixed nonlocal conditions*

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t) = h(t), & t \in (0,T], \\ \sum_{i=1}^{m} \omega_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j\, {}^{H}\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\zeta_j) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} x(\theta_r) = A, \end{cases} \qquad (2.8)$$

if and only if x satisfies the integral equation

$$x(t) = \mathcal{I}_{0^+}^{\alpha;\psi} h(t) + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Omega}\Bigg[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r)$$

$$- \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_j) \Bigg], \qquad (2.9)$$

where

$$\Omega = \sum_{i=1}^{m} \omega_i \left(\psi(\eta_i) - \psi(0) \right)^{\gamma-1} + \sum_{r=1}^{k} \frac{\sigma_r \Gamma(\gamma)(\psi(\theta_r) - \psi(0))^{\gamma+\delta_r-1}}{\Gamma(\gamma + \delta_r)}$$

$$+ \sum_{j=1}^{n} \frac{\kappa_j \Gamma(\gamma)(\psi(\zeta_j) - \psi(0))^{\gamma-\beta_j-1}}{\Gamma(\gamma - \beta_j)}. \tag{2.10}$$

Proof Let x be a solution of problem (2.8). By Lemma 2.9 we have

$$x(t) = \mathcal{I}_{0^+}^{\alpha;\psi} h(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} c_1, \tag{2.11}$$

where $c_1 \in \mathbb{R}$ is an arbitrary constant.

Taking the operators ${}^H\mathfrak{D}_{0^+}^{\beta_j,\rho_j;\psi}$ and $\mathcal{I}_{0^+}^{\delta_r;\psi}$ on (2.11), we obtain

$${}^H\mathfrak{D}_{0^+}^{\beta_j,\rho_j;\psi} x(t) = \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(t) + c_1 \frac{(\psi(t) - \psi(0))^{\gamma-\beta_j-1}}{\Gamma(\gamma - \beta_j)},$$

$$\mathcal{I}_{0^+}^{\delta_r;\psi} x(t) = \mathcal{I}_{0^+}^{\delta_r+\alpha;\psi} h(t) + c_1 \frac{(\psi(t) - \psi(0))^{\gamma+\delta_r-1}}{\Gamma(\gamma + \delta_r)}.$$

Applying the given boundary condition in (2.8), we get

$$A = \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) + \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_j)$$

$$+ c_1 \left[\sum_{i=1}^{m} \frac{\omega_i(\psi(\eta_i) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} + \sum_{r=1}^{k} \frac{\sigma_r(\psi(\theta_r) - \psi(0))^{\gamma+\delta_r-1}}{\Gamma(\gamma + \delta_r)} \right.$$

$$+ \left. \sum_{j=1}^{n} \frac{\kappa_j(\psi(\zeta_j) - \psi(0))^{\gamma-\beta_j-1}}{\Gamma(\gamma - \beta_j)} \right],$$

from which we get

$$c_1 = \frac{\Gamma(\gamma)}{\Omega} \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_j) \right],$$

where Ω is defined by (2.10). Inserting this value of c_1 into (2.11), we get (2.9).

Conversely, suppose that x is a solution of problem (2.8). Taking the ψ-Hilfer fractional derivative ${}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}$ into both sides of the Volterra integral equation (2.9) and using Proposition 2.7 with Lemma 2.8, it follows that

$${}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t) = {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} \mathcal{I}_{0^+}^{\alpha;\psi} h(t) + \frac{1}{\Omega} \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) \right.$$

$$\left. - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_i) \right] {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} \left(\psi(t) - \psi(0) \right)^{\gamma-1}$$

$$= h(t),$$

for $t \in J$, where $0 < \gamma = \alpha + \rho(1 - \alpha) \le 1$. Next, we show that x satisfies the boundary conditions. Applying the operator ${}^H\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi}$ and $\mathcal{I}_{0^+}^{\delta_r;\psi}$ to both sides of (2.9) with Lemma 2.8 and Proposition 2.7, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $r = 1, 2, \ldots, k$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{m} \omega_i x(\eta_i) = {} & \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) + \sum_{i=1}^{m} \frac{\omega_i(\psi(\eta_i) - \psi(0))^{\gamma-1}}{\Omega\Gamma(\gamma)} \\
& \times \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_j) \right],
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=1}^{n} \kappa_j {}^H\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\zeta_j) = {} & \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_j) + \sum_{j=1}^{n} \frac{\kappa_j \Gamma(\gamma)(\psi(\zeta_j) - \psi(0))^{\gamma-\beta_j-1}}{\Omega\Gamma(\gamma - \beta_j)} \\
& \times \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) \right. \\
& \left. - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_i) \right],
\end{aligned}
$$

$$
\begin{aligned}
\sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} x(\theta_r) = {} & \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) + \sum_{r=1}^{k} \frac{\sigma_r \Gamma(\gamma)(\psi(\theta_r) - \psi(0))^{\gamma+\delta_r-1}}{\Omega\Gamma(\gamma + \delta_r)} \\
& \times \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} h(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} h(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} h(\zeta_i) \right],
\end{aligned}
$$

where Ω is given by (2.10). Therefore

$$
\sum_{i=1}^{m} \omega_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j {}^H\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\zeta_j) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} x(\theta_r) = A.
$$

The lemma is proved. □

Fixed point theorems play a major role in establishing the existence theory for problem (1.5). We collect here some well-known fixed point theorems used in this paper.

Lemma 2.11 (Banach contraction principle [39]) *Let D be a nonempty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Lemma 2.12 (Schaefer's fixed point theorem [39]) *Let $\mathbb{M}$ be a Banach space, let $T : \mathbb{M} \to \mathbb{M}$ be a completely continuous operator, and let the set $D = \{x \in \mathbb{M} : x = \kappa Tx, 0 < \kappa \le 1\}$ be bounded. Then T has a fixed point in $\mathbb{M}$.*

Lemma 2.13 (Krasnosel'skiĭ's fixed point theorem [40]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space. Let A, B be the operators such that (i) $Ax + By \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in \mathcal{M}$ such that $z = Az + Bz$.*

3 Existence results

In this section, we present results on the existence of a solution of problem (1.5).

For simplicity, we set

$$F_x(t) = f\big(t, x(t), {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t), \mathcal{I}_{0^+}^{\alpha;\psi} x(t)\big), \quad t \in J.$$

In this paper, the expression $\mathcal{I}_{0^+}^{\alpha,\rho} F_x(s)(c)$ means that

$$\mathcal{I}_{0^+}^{q;\psi} F_x(s)(c) = \frac{1}{\Gamma(q)} \int_0^c \psi'(s)\big(\psi(c) - \psi(s)\big)^{q-1} F_x(s)\, ds,$$

where $q = \{\alpha, \alpha - \beta_j, \alpha + \delta_r\}$ and $c = \{t, \eta_i, \zeta_j, \theta_r\}$, $i = 1, 2, \dots, m, j = 1, 2, \dots, n, r = 1, 2, \dots, k$.

In view of Lemma 2.10, the operator $\mathcal{Q}: C_{1-\gamma,\psi}[J, \mathbb{R}] \to C_{1-\gamma,\psi}[J, \mathbb{R}]$ is defined by

$$(\mathcal{Q}x)(t) = \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega}\left[A - \sum_{i=1}^m \omega_i I_{0^+}^{\alpha;\psi} F_x(s)(\eta_i) \right.$$

$$\left. - \sum_{r=1}^k \sigma_r I_{0^+}^{\alpha+\delta_r;\psi} F_x(s)(\theta_r) - \sum_{j=1}^n \kappa_j I_{0^+}^{\alpha-\beta_j;\psi} F_x(s)(\zeta_j) \right]. \tag{3.1}$$

Note that problem (1.5) has solutions if and only if the operator $\mathcal{Q}$ has fixed points. In the following subsection, we establish the existence of solutions for the problem (1.5) by applying Banach's, Schaefer's, and Krasnosel'skiĭ's fixed point theorems.

We list here the necessary assumptions to prove our main results.

(H_1) There exist constants $L_1, L_3 > 0$, and $0 < L_2 < 1$ such that

$$\big|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)\big| \le L_1|u_1 - u_2| + L_2|v_1 - v_2| + L_3|w_1 - w_2|$$

for any $u_i, v_i, w_i \in \mathbb{R}$, $i = 1, 2$, and $t \in J$.

(H_2) There exist nonnegative continuous functions $h_1, h_2, h_3, h_4 \in \mathbb{R}$ such that

$$\big|f(t, u, v, w)\big| \le h_1(t) + h_2(t)|u| + h_3(t)|v| + h_4(t)|w|, \quad u, v, w \in \mathbb{R}, t \in J,$$

with $h_1^* = \sup_{t \in J} h_1(t)$, $h_2^* = \sup_{t \in J} h_2(t)$, $h_3^* = \sup_{t \in J} h_3(t)$, $h_4^* = \sup_{t \in J} h_4(t)$.

(H_3) $f(t, u, v, w) \le q(t), (t, u, v, w) \in J \times \mathbb{R}^3$, where $q \in \mathcal{C}(J, \mathbb{R}^+)$.

For computational convenience, we use of the following notations:

$$\Psi(u, B) = \frac{(\psi(B) - \psi(0))^{u+\gamma-1}}{\Gamma(u + \gamma)}, \tag{3.2}$$

$$\Lambda_1(u, B) = \frac{\Gamma(\gamma)}{1 - L_2}\big[L_1\Psi(u, B) + L_3\Psi(\alpha + u, B)\big], \tag{3.3}$$

$$\Lambda_2(u, B) = \frac{\Gamma(\gamma)}{1 - h_3^*}\big[h_2^*\Psi(u, B) + h_4^*\Psi(\alpha + u, B)\big], \tag{3.4}$$

$$\Theta_\varepsilon = \frac{1}{|\Omega|}\left[\sum_{i=1}^m |\omega_i|\Lambda_\varepsilon(\alpha, \eta_i) + \sum_{r=1}^k |\sigma_r|\Lambda_\varepsilon(\alpha + \delta_r, \theta_r) + \sum_{j=1}^n |\kappa_j|\Lambda_\varepsilon(\alpha - \beta_j, \zeta_j) \right], \tag{3.5}$$

$$\Phi_\varrho = \big(\psi(T) - \psi(0)\big)^{1-\gamma}\Lambda_\varrho(\alpha, T) + \Theta_\varrho, \tag{3.6}$$

$$\Xi = \left(\psi(T) - \psi(0)\right)^{1-\gamma} \Psi(\alpha - \gamma + 1, T) + \frac{1}{|\Omega|}\left[\sum_{i=1}^{m} |\omega_i| \Psi(\alpha - \gamma + 1, \eta_i)\right.$$

$$\left. + \sum_{r=1}^{k} |\sigma_r| \Psi(\alpha + \delta_r - \gamma + 1, \theta_r) + \sum_{j=1}^{n} |\kappa_j| \Psi(\alpha - \beta_j - \gamma + 1, \zeta_j)\right], \tag{3.7}$$

where $\varepsilon = \{1, 2\}$ and $\varrho = \{1, 2\}$.

3.1 Existence and uniqueness via Banach contraction mapping principle

We will first prove the existence and uniqueness of a solution for problem (1.5) by using the Banach contraction mapping principle (Banach's fixed point theorem).

Theorem 3.1 *Let $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (H_1). If*

$$\Phi_1 < 1, \tag{3.8}$$

where Φ_1 is given by (3.6), then problem (1.5) has a unique solution $x \in \mathcal{C}_{1-\gamma,\psi}$ on J.

Proof Firstly, we transform problem (1.5) into a fixed point problem, $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined as in (3.1). It is clear that the fixed points of the operator $\mathcal{Q}$ are solutions of problem (1.5). Applying the Banach contraction mapping principle, we will show that the operator $\mathcal{Q}$ has a fixed point, which is a unique solution of problem (1.5).

Let $\sup_{t \in [0,T]} |f(t, 0, 0, 0)| := M_1 < \infty$. Next, we set $B_{\Upsilon_1} := \{x \in \mathcal{C}_{1-\gamma,\psi} : \|x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq \Upsilon_1\}$ with

$$\Upsilon_1 \geq \frac{1}{1 - \Phi_1}\left(\frac{M_1 \Xi}{1 - L_2}\right), \tag{3.9}$$

where Ω, Φ_1, and Ξ are given by (2.10), (3.6), and (3.7), respectively. Observe that B_{Υ_1} is a bounded, closed, and convex subset of $\mathcal{C}_{1-\gamma,\psi}$. The proof is divided into two steps.

Step I. We show that $\mathcal{Q}B_{\Upsilon_1} \subset B_{\Upsilon_1}$.

For any $x \in B_{\Upsilon_1}$, we have

$$\left|\left(\psi(t) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t)\right|$$

$$\leq \left(\psi(T) - \psi(0)\right)^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} |F_x(s)|(T) + \frac{1}{|\Omega|}\left(|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha;\psi} |F_x(s)|(\eta_i)\right.$$

$$\left. + \sum_{r=1}^{k} |\sigma_r| \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} |F_x(s)|(\theta_r) + \sum_{j=1}^{n} |\kappa_j| \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} |F_x(s)|(\zeta_j)\right).$$

Consider

$$\mathcal{I}_{0^+}^{\alpha;\psi} |x(\tau)|(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \psi'(\tau)\left(\psi(s) - \psi(\tau)\right)^{\alpha-1} |x(\tau)| \, d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \psi'(\tau)\left(\psi(s) - \psi(\tau)\right)^{\alpha-1}\left(\psi(\tau) - \psi(0)\right)^{\gamma-1} \|x\|_{\mathcal{C}_{1-\gamma,\psi}} \, d\tau$$

$$= \frac{\Gamma(\gamma)(\psi(s) - \psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha + \gamma)} \|x\|_{\mathcal{C}_{1-\gamma,\psi}}.$$

It follows from condition (H_1) that

$$\left|F_x(t)\right| \leq \left|f\left(t, x(t), {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t), \mathcal{I}_{0^+}^{\alpha;\psi} x(t)\right) - f(t, 0, 0, 0)\right| + \left|f(t, 0, 0, 0)\right|$$

$$\leq L_1\left|x(t)\right| + L_2\left|{}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t)\right| + L_3\left|\mathcal{I}_{0^+}^{\alpha;\psi} x(t)\right| + M_1.$$

Then

$$\left|F_x(t)\right| \leq \frac{1}{1 - L_2}\left(L_1\left(\psi(t) - \psi(0)\right)^{\gamma-1} + \frac{L_3\Gamma(\gamma)(\psi(t) - \psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha + \gamma)}\right)\|x\|_{\mathcal{C}_{1-\gamma,\psi}} + \frac{M_1}{1 - L_2}.$$

Thus we get

$$\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)\right|(T) \leq \Lambda_1(\alpha, T)\|x\|_{\mathcal{C}_{1-\gamma,\psi}} + \frac{M_1(\psi(T) - \psi(0))^{\alpha}}{(1 - L_2)\Gamma(\alpha + 1)}, \tag{3.10}$$

$$\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s)\right|(\eta_i) \leq \Lambda_1(\alpha, \eta_i)\|x\|_{\mathcal{C}_{1-\gamma,\psi}} + \frac{M_1(\psi(\eta_i) - \psi(0))^{\alpha}}{(1 - L_2)\Gamma(\alpha + 1)}, \tag{3.11}$$

$$\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}\left|F_x(s)\right|(\theta_r) \leq \Lambda_1(\alpha + \delta_r, \theta_r)\|x\|_{\mathcal{C}_{1-\gamma,\psi}} + \frac{M_1(\psi(\theta_r) - \psi(0))^{\alpha+\delta_r}}{(1 - L_2)\Gamma(\alpha + \delta_r + 1)}, \tag{3.12}$$

$$\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}\left|F_x(s)\right|(\zeta_j) \leq \Lambda_1(\alpha - \beta_j, \zeta_j)\|x\|_{\mathcal{C}_{1-\gamma,\psi}} + \frac{M_1(\psi(\zeta_j) - \psi(0))^{\alpha-\beta_j}}{(1 - L_2)\Gamma(\alpha - \beta_j + 1)}. \tag{3.13}$$

From (3.10)–(3.13) we obtain

$$\left|\left(\psi(t) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t)\right| \leq \Phi_1\Upsilon_1 + \frac{M_1\Xi}{1 - L_2},$$

which implies that $\|\mathcal{Q}x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq \Upsilon_1$. Therefore $\mathcal{Q}B_{\Upsilon_1} \subset B_{\Upsilon_1}$.

Step II. We show that the operator $\mathcal{Q} : \mathcal{C}_{1-\gamma,\psi} \to \mathcal{C}_{1-\gamma,\psi}$ is a contraction.

For all $x, y \in \mathcal{C}_{1-\gamma,\psi}$ and $t \in J$, we have

$$\left|\left(\psi(t) - \psi(0)\right)^{1-\gamma}\left((\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)\right)\right|$$

$$\leq \left(\psi(T) - \psi(0)\right)^{1-\gamma}\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s) - F_y(s)\right|(T) + \frac{1}{|\Omega|}\left(\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha;\psi}\left|F_x(s) - F_y(s)\right|(\eta_i)\right.$$

$$\left. + \sum_{r=1}^{k}|\sigma_r|\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}\left|F_x(s) - F_y(s)\right|(\theta_r) + \sum_{j=1}^{n}|\kappa_j|\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}\left|F_x(s) - F_y(s)\right|(\zeta_j)\right). \tag{3.14}$$

From (H_1) we obtain

$$\left|F_x(t) - F_y(t)\right| = \left|f\left(t, x(t), F_x(t), \mathcal{I}_{0^+}^{\alpha;\psi} x(t)\right) - f\left(t, y(t), F_y(t), \mathcal{I}_{0^+}^{\alpha;\psi} y(t)\right)\right|$$

$$\leq L_1\left|x(t) - y(t)\right| + L_2\left|F_x(t) - F_y(t)\right| + L_3\left|\mathcal{I}_{0^+}^{\alpha;\psi} x(t) - \mathcal{I}_{0^+}^{\alpha;\psi} y(t)\right|,$$

and thus

$$\left|F_x(t) - F_y(t)\right|$$

$$\leq \frac{\|x - y\|_{\mathcal{C}_{1-\gamma,\psi}}}{1 - L_2}\left(L_1\left(\psi(t) - \psi(0)\right)^{\gamma-1} + \frac{L_3\Gamma(\gamma)(\psi(t) - \psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha + \gamma)}\right). \tag{3.15}$$

Then by substituting (3.15) into (3.14) we get

$$
\left| \left(\psi(t) - \psi(0) \right)^{1-\gamma} \left((\mathcal{Q}x)(t) - (\mathcal{Q}y)(t) \right) \right|
$$

$$
\leq \left\{ \left(\psi(T) - \psi(0) \right)^{1-\gamma} \Lambda_1(\alpha, T) + \frac{1}{|\Omega|} \left[\sum_{i=1}^{m} |\omega_i| \Lambda_1(\alpha, \eta_i) + \sum_{r=1}^{k} |\sigma_r| \Lambda_1(\alpha + \delta_r, \theta_r) \right. \right.
$$

$$
\left. \left. + \sum_{j=1}^{n} |\kappa_j| \Lambda_1(\alpha - \beta_j, \zeta_j) \right] \right\} \|x - y\|_{\mathcal{C}_{1-\gamma,\psi}},
$$

which implies that $\|\mathcal{Q}x - \mathcal{Q}y\|_{\mathcal{C}_{1-\gamma,\psi}} \leq \Phi_1 \|x - y\|_{\mathcal{C}_{1-\gamma,\psi}}$. Since $\Phi_1 < 1$, the operator $\mathcal{Q}$ is a contraction map. Therefore by the Banach contraction mapping principle (Lemma 2.11) problem (1.5) has a unique solution in $\mathcal{C}_{1-\gamma,\psi}$. The proof is completed. □

3.2 Existence result via Schaefer's fixed point theorem
The next existence result is based on Schaefer's fixed point theorem.

Theorem 3.2 *Let $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (H_2). Then problem (1.5) has at least one solution on J.*

Proof We show that the operator $\mathcal{Q}$ defined in (3.1) has at least one fixed point in $\mathcal{C}_{1-\gamma,\psi}$. The proof is divided into four steps.

Step I. The operator $\mathcal{Q}$ is continuous.

Let x_n be a sequence such that $x_n \to x$ in $\mathcal{C}_{1-\gamma,\psi}$. Then for each $t \in J$, we obtain

$$
\left| \left(\psi(t) - \psi(0) \right)^{1-\gamma} \left((\mathcal{Q}x_n)(t) - (\mathcal{Q}x)(t) \right) \right|
$$

$$
\leq \left(\psi(T) - \psi(0) \right)^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} \left| F_{x_n}(s) - F_x(s) \right| (T) + \frac{1}{|\Omega|} \left[\sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha;\psi} \left| F_{x_n}(s) - F_x(s) \right| (\eta_i) \right.
$$

$$
\left. + \sum_{r=1}^{k} |\sigma_r| \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} \left| F_{x_n}(s) - F_x(s) \right| (\theta_r) + \sum_{j=1}^{n} |\kappa_j| \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} \left| F_{x_n}(s) - F_x(s) \right| (\zeta_j) \right]
$$

$$
\leq \left[\left(\psi(T) - \psi(0) \right)^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} \left(\psi(s) - \psi(0) \right)^{\gamma-1} (T) \right.
$$

$$
+ \frac{1}{|\Omega|} \left(\sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha;\psi} \left(\psi(s) - \psi(0) \right)^{\gamma-1} (\eta_i) \right.
$$

$$
\left. \left. + \sum_{r=1}^{k} |\sigma_r| \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} \left(\psi(s) - \psi(0) \right)^{\gamma-1} (\theta_r) + \sum_{j=1}^{n} |\kappa_j| \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} \left(\psi(s) - \psi(0) \right)^{\gamma-1} (\zeta_j) \right) \right]
$$

$$
\times \|F_{x_n} - F_x\|_{\mathcal{C}_{1-\gamma,\psi}}.
$$

Since f is a continuous, this implies that F_x is also continuous. Hence we obtain

$$
\|\mathcal{Q}x_n - \mathcal{Q}x\|_{\mathcal{C}_{1-\gamma,\psi}} \to 0 \quad \text{as } n \to \infty.
$$

Step II. The operator $\mathcal{Q}$ maps bounded sets into bounded sets in $\mathcal{C}_{1-\gamma,\psi}$.

For $\Upsilon_2 > 0$, there exists a constant $\mu > 0$ such that, for each $x \in \bar{B}_{\Upsilon_2} = \{x \in \mathcal{C}_{1-\gamma,\psi} : \|x\|_{\mathcal{C}_{1-\gamma,\psi}} \le \Upsilon_2\}$, we have $\|\mathcal{Q}x\|_{\mathcal{C}_{1-\gamma,\psi}} \le \mu$.

Indeed, for any $t \in J$ and $x \in \bar{B}_{\Upsilon_2}$, we have

$$
\begin{aligned}
&\left|\left(\psi(t) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t)\right| \\
&\le \left(\psi(T) - \psi(0)\right)^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi}|F_x(s)|(T) + \frac{1}{|\Omega|}\left[|A| + \sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha;\psi}|F_x(s)|(\eta_i) \right. \\
&\left. + \sum_{r=1}^{k}|\sigma_r|\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}|F_x(s)|(\theta_r) + \sum_{j=1}^{n}|\kappa_j|\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}|F_x(s)|(\zeta_j)\right].
\end{aligned} \tag{3.16}
$$

It follows from condition (H_2) that

$$
\left|F_x(t)\right| \le h_1^* + h_2^*\left|x(t)\right| + h_3^*\left|F_x(t)\right| + h_4^*\left|\mathcal{I}_{0^+}^{\alpha;\psi}x(t)\right|,
$$

and thus

$$
\left|F_x(t)\right| \le \frac{1}{1-h_3^*}\left[h_1^* + \left(h_2^*\left(\psi(t) - \psi(0)\right)^{\gamma-1} + \frac{h_4^*\Gamma(\gamma)\left(\psi(t) - \psi(0)\right)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right)\|x\|_{\mathcal{C}_{1-\gamma,\psi}}\right]. \tag{3.17}
$$

Then by substituting (3.17) into (3.16) we get

$$
\begin{aligned}
&\left|\left(\psi(t) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t)\right| \\
&\le \|x\|_{1-\gamma,\psi}\left[\left(\psi(T) - \psi(0)\right)^{1-\gamma}\Lambda_2(\alpha, T) + \frac{1}{|\Omega|}\left(\sum_{i=1}^{m}|\omega_i|\Lambda_2(\alpha, \eta_i)\right.\right. \\
&\left.\left. + \sum_{r=1}^{k}|\sigma_r|\Lambda_2(\alpha + \delta_r, \theta_r) + \sum_{j=1}^{n}|\kappa_j|\Lambda_2(\alpha - \beta_j, \zeta_j)\right)\right] + \frac{h_1^*\Xi}{1-h_3^*},
\end{aligned}
$$

from which we get

$$
\|\mathcal{Q}x\|_{\mathcal{C}_{1-\gamma,\psi}} \le \Phi_2\Upsilon_2 + \frac{h_1^*\Xi}{1-h_3^*} := \mu.
$$

Step III. The operator $\mathcal{Q}$ maps bounded sets into equicontinuous sets of $\mathcal{C}_{1-\gamma,\psi}$.

For $0 \le t_1 < t_2 \le T$ and $x \in \bar{B}_{\Upsilon_2}$ where $\bar{B}_{\Upsilon_2}$ is as defined in Step II, since f is bounded on the compact set $J \times \bar{B}_{\Upsilon_2}^3$, we have

$$
\begin{aligned}
&\left|\left(\psi(t_2) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t_2) - \left(\psi(t_1) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t_1)\right| \\
&= \left|\left(\psi(t_2) - \psi(0)\right)^{1-\gamma}\mathcal{I}_{0^+}^{\alpha;\psi}F_x(t_2) - \left(\psi(t_1) - \psi(0)\right)^{1-\gamma}\mathcal{I}_{0^+}^{\alpha;\psi}F_x(t_1)\right| \\
&= \left|\left(\psi(t_2) - \psi(0)\right)^{1-\gamma}\int_0^{t_2}\psi'(\tau)\left(\psi(t_2) - \psi(\tau)\right)^{\alpha-1}F_x(\tau)\,d\tau \right. \\
&\left. - \left(\psi(t_1) - \psi(0)\right)^{1-\gamma}\int_0^{t_1}\psi'(\tau)\left(\psi(t_1) - \psi(\tau)\right)^{\alpha-1}F_x(\tau)\,d\tau\right|.
\end{aligned}
$$

Set $\sup_{(t,u,v,w)\in J\times B_{\Upsilon_2}^3}|f(t,u,v,w)| = \widehat{f} < \infty$. Since $(\psi(t)-\psi(0))^{1-\gamma}(\psi(t)-\psi(\tau))^{\alpha-1}$ is a decreasing function on $t \in (0,T)$, it follows that

$$\left|\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t_2) - \left(\psi(t_1)-\psi(0)\right)^{1-\gamma}(\mathcal{Q}x)(t_1)\right|$$

$$\leq \frac{\widehat{f}}{\Gamma(\alpha)}\left|\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\int_{t_1}^{t_2}\psi'(\tau)\left(\psi(t_2)-\psi(\tau)\right)^{\alpha-1}d\tau\right.$$

$$+ \left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\int_0^{t_1}\psi'(\tau)\left(\psi(t_1)-\psi(\tau)\right)^{\alpha-1}d\tau$$

$$\left.- \left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\int_0^{t_1}\psi'(\tau)\left(\psi(t_2)-\psi(\tau)\right)^{\alpha-1}d\tau\right|$$

$$= \frac{\widehat{f}}{\Gamma(\alpha+1)}\Big[\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\left(\psi(t_2)-\psi(t_1)\right)^{\alpha} + \left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\left(\psi(t_1)-\psi(0)\right)^{\alpha}$$

$$+ \left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\left\{\left(\psi(t_2)-\psi(t_1)\right)^{\alpha} - \left(\psi(t_2)-\psi(0)\right)^{\alpha}\right\}\Big].$$

This inequality is independent of x and tends to zero as $t_2 \to t_1$, which implies that $\|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\|_{\mathcal{C}_{1-\gamma,\psi}} \to 0$ as $t_2 \to t_1$. Thus, Steps I to III, together with the Arzelá–Ascoli theorem, we conclude that the operator $\mathcal{Q}$ is completely continuous.

Step IV. The set $\mathbb{E} = \{x \in \mathcal{C}_{1-\gamma,\psi} : x = \varrho\mathcal{Q}x, 0 < \varrho \leq 1\}$ is bounded (a priori bounds).

Let $x \in \mathbb{E}$. Then $x = \varrho\mathcal{Q}x$ for some $0 < \varrho \leq 1$. From (H_2), for each $t \in J$, we can get the estimate

$$x(t) = \varrho\left(\mathcal{I}_{0^+}^{\alpha;\psi}F_x(t) + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Omega}\left[A - \sum_{i=1}^m \omega_i\mathcal{I}_{0^+}^{\alpha;\psi}(F_x(s))(\eta_i)\right.\right.$$

$$\left.\left.- \sum_{r=1}^k \sigma_r\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}(F_x(s))(\theta_r) - \sum_{j=1}^n \kappa_j\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}(F_x(s))(\zeta_j)\right]\right).$$

It follows from Step II that for each $t \in J$, $\|\mathcal{Q}x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq \mu < \infty$. This implies that the set $\mathcal{C}_{1-\gamma,\psi}$ is bounded.

By all hypotheses of Theorem 3.2 we conclude that there exists a positive constant N such that $\|x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq N < \infty$. By Schaefer's fixed point theorem (Lemma 2.12) the operator $\mathcal{Q}$ has at least one fixed point, which is a solution of problem (1.5). This completes the proof. □

3.3 Existence result via Krasnosel'skiĭ's fixed point theorem

By using Krasnosel'skiĭ's fixed point theorem, we obtain the final existence theorem.

Theorem 3.3 *Let $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (H_1) and (H_3). If*

$$\Theta_1 < 1, \tag{3.18}$$

where Θ_1 is defined by (3.5), then problem (1.5) has at least one solution on J.

Proof Let $\sup_{t \in J}|q(t)| = q^*$. By choosing a suitable $B_{\Upsilon_3} := \{x \in \mathcal{C}_{1-\gamma,\psi} : \|x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq \Upsilon_3\}$, where $\Upsilon_3 \geq \frac{|A|}{|\Omega|} + q^*\Xi$, we define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_{Υ_3} by

$$(\mathcal{Q}_1x)(t) = \mathcal{I}_{0^+}^{\alpha;\psi}F_x(s)(t), \quad t \in J,$$

$$(\mathcal{Q}_2 x)(t) = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left[A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(\eta_i) \right.$$
$$\left. - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_x(s)(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_x(s)(\zeta_j) \right], \quad t \in J.$$

Note that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. For any $x, y \in B_{\Upsilon_3}$, we have

$$\left| (\psi(t) - \psi(0))^{1-\gamma} ((\mathcal{Q}_1 x)(t) + (\mathcal{Q}_2 y)(t)) \right|$$

$$\leq (\psi(T) - \psi(0))^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} |F_x(s)|(T) + \frac{1}{|\Omega|} \left[|A| + \sum_{i=1}^{m} |\omega_i| \mathcal{I}_{0^+}^{\alpha;\psi} |F_y(s)|(\eta_i) \right.$$

$$\left. + \sum_{r=1}^{k} |\sigma_r| \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} |F_y(s)|(\theta_r) + \sum_{j=1}^{n} |\kappa_j| \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} |F_y(s)|(\zeta_j) \right]$$

$$\leq q^* (\psi(T) - \psi(0))^{1-\gamma} \Psi(\alpha - \gamma + 1, T) + \frac{1}{|\Omega|} \left[|A| + q^* \left(\sum_{i=1}^{m} |\omega_i| \Psi(\alpha - \gamma + 1, \eta_i) \right. \right.$$

$$\left. \left. + \sum_{r=1}^{k} |\sigma_r| \Psi(\alpha + \delta_r - \gamma + 1, \theta_r) + \sum_{j=1}^{n} |\kappa_j| \Psi(\alpha - \beta_j - \gamma + 1, \zeta_j) \right) \right]$$

$$\leq \frac{|A|}{|\Omega|} + q^* \left[(\psi(T) - \psi(0))^{1-\gamma} \Psi(\alpha - \gamma + 1, T) + \frac{1}{|\Omega|} \left(\sum_{i=1}^{m} |\omega_i| \Psi(\alpha - \gamma + 1, \eta_i) \right. \right.$$

$$\left. \left. + \sum_{r=1}^{k} |\sigma_r| \Psi(\alpha + \delta_r - \gamma + 1, \theta_r) + \sum_{j=1}^{n} |\kappa_j| \Psi(\alpha - \beta_j - \gamma + 1, \zeta_j) \right) \right]$$

$$\leq \frac{|A|}{|\Omega|} + q^* \Xi \leq \Upsilon_3.$$

This implies that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_{\Upsilon_3}$, which satisfies assumption (i) of Lemma 2.13.

We now show that assumption (ii) of Lemma 2.13 is satisfied.

Let x_n be a sequence such that $x_n \to x$ in $\mathcal{C}_{1-\gamma,\psi}$. Then for each $t \in J$, we have

$$\left| (\psi(t) - \psi(0))^{1-\gamma} ((\mathcal{Q}_1 x_n)(t) - (\mathcal{Q}_1 x)(t)) \right|$$

$$\leq (\psi(T) - \psi(0))^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} |F_{x_n}(s) - F_x(s)|(T)$$

$$\leq (\psi(T) - \psi(0))^{1-\gamma} \mathcal{I}_{0^+}^{\alpha;\psi} (\psi(s) - \psi(0))^{\gamma-1}(T) \|F_{x_n} - Fx\|_{\mathcal{C}_{1-\gamma,\psi}}.$$

Since f is continuous, this implies that the operator F_x is also continuous. Hence we obtain

$$\|F_{x_n} - F_x\|_{\mathcal{C}_{1-\gamma,\psi}} \to 0 \quad \text{as } n \to \infty.$$

This shows that the operator $\mathcal{Q}_1 x$ is continuous, since $\|\mathcal{Q}_1 x_n - \mathcal{Q}_1 x\|_{1-\gamma,\psi} \to 0$ as $n \to \infty$. Also, the set $\mathcal{Q}_1 B_{\Upsilon_3}$ is uniformly bounded as

$$\|\mathcal{Q}_1 x\|_{\mathcal{C}_{1-\gamma,\psi}} \leq q^* (\psi(T) - \psi(0))^{1-\gamma} \Psi(\alpha - \gamma + 1, T).$$

Next, we prove the compactness of $\mathcal{Q}_1$. For each $t_1, t_2 \in J$ with $0 \le t_1 < t_2 \le T$, we have (see Step III of Theorem 3.2)

$$\left|\left(\psi(t_1) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}_1 x)(t_1) - \left(\psi(t_2) - \psi(0)\right)^{1-\gamma}(\mathcal{Q}_1 x)(t_2)\right|$$
$$\le \frac{q^*}{\Gamma(\alpha+1)}\Big[\left(\psi(t_2) - \psi(0)\right)^{1-\gamma}\left(\psi(t_2) - \psi(t_1)\right)^{\alpha} + \left(\psi(t_1) - \psi(0)\right)^{1-\gamma}\left(\psi(t_1) - \psi(0)\right)^{\alpha}$$
$$+ \left(\psi(t_2) - \psi(0)\right)^{1-\gamma}\left\{\left(\psi(t_2) - \psi(t_1)\right)^{\alpha} - \left(\psi(t_2) - \psi(0)\right)^{\alpha}\right\}\Big].$$

Obviously, the right-hand side in this inequality is independent of x and tends to zero as $t_2 \to t_1$. Therefore the operator $\mathcal{Q}_1$ is equicontinuous, and so by the Arzelà–Ascoli theorem, $\mathcal{Q}_1$ is relatively compact.

Moreover, it is easy to prove using condition (3.18) that the operator $\mathcal{Q}_2$ is a contraction mapping, and thus assumption (iii) of Lemma 2.13 holds. Thus all the assumptions of Lemma 2.13 are satisfied. So the conclusion of Lemma 2.13 implies that the boundary value problem (1.5) has at least one solution on J. The proof is completed. □

4 Stability analysis

In this section, we develop some sufficient conditions under which the concerned problem (1.5) satisfies the hypotheses of different types of Ulam stability such as the Ulam–Hyers stability (UH), generalized Ulam–Hyers stability (GUH), Ulam–Hyers–Rassias stability (UHR), and generalized Ulam–Hyers–Rassias stability (GUHR).

Before stating the main theorem, we need the following definitions. Let $\epsilon > 0$, and let $\mathcal{B}: [0, T] \to [0, \infty)$ be a continuous function. We consider the following inequalities:

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) - f\left(t, z(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t)\right)\right| \le \epsilon, \tag{4.1}$$

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) - f\left(t, z(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t)\right)\right| \le \epsilon\mathcal{B}(t), \tag{4.2}$$

$$\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) - f\left(t, z(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t)\right)\right| \le \mathcal{B}(t). \tag{4.3}$$

Definition 4.1 ([41]) Problem (1.5) is said to be UH stable if there exists a constant $\tau > 0$ such that for $\epsilon > 0$ and each solution $z \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ of inequality (4.1), there exists a solution $x \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ of problem (1.5) with

$$\left|z(t) - x(t)\right| \le \tau\epsilon, \quad t \in J. \tag{4.4}$$

Definition 4.2 ([41]) Problem (1.5) is said to be GUH stable if there exists a function $\mathcal{B} \in \mathcal{C}_{1-\gamma,\psi}(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathcal{B}(0) = 0$ such that for each solution $z \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ of inequality (4.2), there exists a solution $x \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ of problem (1.5) with

$$\left|z(t) - x(t)\right| \le \mathcal{B}(\epsilon), \quad t \in J. \tag{4.5}$$

Definition 4.3 ([41]) Problem (1.5) is said to be UHR stable with respect to $\mathcal{B} \in \mathcal{C}_{1-\gamma,\psi}(J, \mathbb{R}^+)$ if there exists a real number $m_{f,\mathcal{B}} > 0$ such that for each solution $z \in \mathcal{C}_{1-\gamma,\psi}^1[J, \mathbb{R}]$ of inequality (4.2), there exists a solution $x \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ of problem (1.5) with

$$\left|z(t) - x(t)\right| \le m_{f,\mathcal{B}}\epsilon\mathcal{B}(t), \quad t \in J. \tag{4.6}$$

Definition 4.4 ([41]) Problem (1.5) is said to be GUHR stable with respect to $\mathcal{B} \in \mathcal{C}_{1-\gamma,\psi}(J,\mathbb{R}^+)$ if there exists a real number $m_{f,\mathcal{B}} > 0$ such that for each solution $z \in \mathcal{C}_{1-\gamma,\psi}^1(J,\mathbb{R})$ of inequality (4.3), there exists a solution $x \in \mathcal{C}_{1-\gamma,\psi}^1(J,\mathbb{R})$ of problem (1.5) with

$$\left| z(t) - x(t) \right| \le m_{f,\mathcal{B}} \mathcal{B}(t), \quad t \in J. \tag{4.7}$$

Remark 4.5 It is clear that (i) Definition 4.1 $\Rightarrow$ Definition 4.2; (ii) Definition 4.3 $\Rightarrow$ Definition 4.4; (iii) Definition 4.3 for $\mathcal{B}(\cdot) = 1 \Rightarrow$ Definition 4.1.

Remark 4.6 A function $z \in \mathcal{C}_{1-\gamma,\psi}^1(J,\mathbb{R})$ is a solution of inequality (4.1) if and only if there exists a function $w \in \mathcal{C}_{1-\gamma,\psi}(J,\mathbb{R})$ (dependent on z) such that:

  (i) $|w(t)| \le \epsilon, t \in J$.
  (ii) ${}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) = f(t, z(t), {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t)) + w(t), t \in J$.

Firstly, we present an important lemma that will be used in the proofs of UH and GUH stability.

Lemma 4.7 *Let $\alpha \in (0,1]$, $\rho \in [0,1)$. If $z \in \mathcal{C}_{1-\gamma,\psi}^1(J,\mathbb{R})$ is a solution of inequality (4.1), then z is a solution of the inequality*

$$\left| z(t) - \mathcal{H}_z(t) \right| \le \left(\psi(T) - \psi(0) \right)^{\gamma-1} \Xi \epsilon, \tag{4.8}$$

where

$$\mathcal{H}_z(t) = \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(A - \sum_{i=1}^m \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(\eta_i) \right.$$
$$\left. - \sum_{r=1}^k \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_z(s)(\theta_r) - \sum_{j=1}^n \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_z(s)(\zeta_j) \right),$$

and Ξ is given by (3.7).

Proof Let z be a solution of inequality (4.1). So, in view of Remark 4.6(ii) and Lemma 2.10, we have

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) = f(t, z(t), {}^H\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t)) + \omega(t), \quad t \in (0,T], \\ \sum_{i=1}^m \omega_i z(\eta_i) + \sum_{j=1}^n \kappa_j {}^H\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} z(\zeta_j) + \sum_{r=1}^k \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} z(\theta_r) = A. \end{cases} \tag{4.9}$$

Thus the solution of (4.9) is of the form

$$z(t) = \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(A - \sum_{i=1}^m \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(\eta_i) - \sum_{r=1}^k \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_z(s)(\theta_r) \right.$$
$$\left. - \sum_{j=1}^n \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_z(s)(\zeta_j) \right) + \mathcal{I}_{0^+}^{\alpha;\psi} w(s)(t) - \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(\sum_{i=1}^m \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} w(s)(\eta_i) \right.$$
$$\left. + \sum_{r=1}^k \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} w(s)(\theta_r) + \sum_{j=1}^n \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} w(s)(\zeta_j) \right).$$

Then by using (i) of Remark 4.6 it follows that

$$
\begin{aligned}
\left| z(t) - \mathcal{H}_z(t) \right| &= \left| \mathcal{I}_{0^+}^{\alpha;\psi} w(s)(t) - \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(\sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} w(s)(\eta_i) \right. \right. \\
&\quad \left. \left. + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} w(s)(\theta_r) + \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} w(s)(\zeta_j) \right) \right| \\
&\leq \left(\Psi(\alpha - \gamma + 1, T) + \frac{(\psi(T) - \psi(0))^{\gamma-1}}{|\Omega|} \left(\sum_{i=1}^{m} |\omega_i| \Psi(\alpha - \gamma + 1, \eta_i) \right. \right. \\
&\quad \left. \left. + \sum_{r=1}^{k} |\sigma_r| \Psi(\alpha - \gamma + \delta_r + 1, \theta_r) + \sum_{j=1}^{n} |\kappa_j| \Psi(\alpha - \gamma - \beta_j + 1, \zeta_j) \right) \right) \epsilon \\
&= (\psi(T) - \psi(0))^{\gamma-1} \Xi \epsilon,
\end{aligned}
$$

from which we obtain inequality (4.8). The proof is completed. $\qquad \square$

Now we present the UH and GUH stability results.

Theorem 4.8 *Let $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. If (H_1) is satisfied with $\Phi_1 < 1$, where Φ_1 is given by (3.6), then problem (1.5) is UH stable and GUH stable on J.*

Proof Suppose that $\epsilon > 0$ and $z \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ is any solution of inequality (4.1), that is,

$$
\left| {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t) - f\left(t, z(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} z(t), \mathcal{I}_{0^+}^{\alpha;\psi} z(t) \right) \right| \leq \epsilon, \quad t \in J.
$$

Let $x \in \mathcal{C}_{1-\gamma,\psi}^1(J, \mathbb{R})$ be the unique solution of problem (1.5). Then we have

$$
\begin{cases}
{}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi} x(t), \mathcal{I}_{0^+}^{\alpha;\psi} x(t)), \quad t \in J, \\
\sum_{i=1}^{m} \omega_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j {}^{H}\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\zeta_j) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\delta_r;\psi} x(\theta_r) = A.
\end{cases}
$$

By applying the triangle inequality we get

$$
\left| z(t) - x(t) \right| = \left| z(t) - \mathcal{H}_z(t) + \mathcal{H}_z(t) - x(t) \right| \leq \left| z(t) - \mathcal{H}_z(t) \right| + \left| \mathcal{H}_z(t) - x(t) \right|. \tag{4.10}
$$

By using Lemma 4.7 with (4.10) we obtain

$$
\begin{aligned}
\left| z(t) - x(t) \right| &\leq (\psi(T) - \psi(0))^{\gamma-1} \Xi \epsilon \\
&\quad + \left| \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(\eta_i) \right. \right. \\
&\quad \left. - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_z(s)(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_z(s)(\zeta_j) \right) - \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(t) \\
&\quad - \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \left(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_x(s)(\theta_r) \right.
\end{aligned}
$$

$$-\sum_{j=1}^{n}\kappa_j\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}F_x(s)(\zeta_j)\Bigg)\Bigg|$$

$$\leq \big(\psi(T)-\psi(0)\big)^{\gamma-1}\Xi\epsilon + \mathcal{I}_{0^+}^{\alpha;\psi}\big|F_z(s)-F_x(s)\big|(t)$$

$$+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\omega_i|\mathcal{I}_{0^+}^{\alpha;\psi}\big|F_z(s)-F_x(s)\big|(\eta_i)$$

$$+\sum_{r=1}^{k}|\sigma_r|\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}\big|F_z(s)-F_x(s)\big|(\theta_r) + \sum_{j=1}^{n}|\kappa_j|\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}\big|F_z(s)-F_x(s)\big|(\zeta_j)\Bigg)$$

$$\leq \big(\psi(T)-\psi(0)\big)^{\gamma-1}\Xi\epsilon + \Phi_1\big|z(t)-x(t)\big|.$$

This implies that

$$\big|z(t)-x(t)\big| \leq \tau\epsilon,$$

where

$$\tau = \frac{(\psi(T)-\psi(0))^{\gamma-1}\Xi}{1-\Phi_1}.$$

Hence problem (1.5) is UH stable. Now setting $\mathcal{B}=\tau\epsilon$ such that $\mathcal{B}(0)=0$ yields that problem (1.5) is GUH stable. The proof is completed. □

Remark 4.9 Let $\mathcal{B}\in\mathcal{C}_{1-\gamma,\psi}(J,\mathbb{R}^+)$ be an increasing function. Then there exists $\lambda_\mathcal{B}>0$ such that for each $t\in J$, we have the integral inequality

$$\mathcal{I}_{0^+}^{\alpha;\psi}\mathcal{B}(t)\leq\lambda_\mathcal{B}\mathcal{B}(t). \tag{4.11}$$

Lemma 4.10 *Let $\alpha\in(0,1]$ and $\rho\in[0,1)$. If $z\in\mathcal{C}_{1-\gamma,\psi}^1(J,\mathbb{R})$ is a solution of inequality (4.2), then z is a solution of the inequality*

$$\big|z(t)-\mathcal{H}_z(t)\big| \leq \mathcal{K}\lambda_\mathcal{B}\mathcal{B}(t)\epsilon, \tag{4.12}$$

where

$$\mathcal{K} = 1 - \frac{(\psi(T)-\psi(0))^{\gamma-1}}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\omega_i| + \sum_{r=1}^{k}|\sigma_r| + \sum_{j=1}^{n}|\kappa_j|\Bigg). \tag{4.13}$$

Proof From Lemma 4.7, using Remarks 4.6(i) and 4.9, we obtain

$$\big|z(t)-\mathcal{H}_z(t)\big| = \Bigg|\mathcal{I}_{0^+}^{\alpha;\psi}w(s)(t) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Omega}\Bigg(\sum_{i=1}^{m}\omega_i\mathcal{I}_{0^+}^{\alpha;\psi}w(s)(\eta_i)$$

$$+\sum_{r=1}^{k}\sigma_r\mathcal{I}_{0^+}^{\alpha+\delta_r;\psi}w(s)(\theta_r) + \sum_{j=1}^{n}\kappa_j\mathcal{I}_{0^+}^{\alpha-\beta_j;\psi}w(s)(\zeta_j)\Bigg)\Bigg|$$

$$\leq \left[1 - \frac{(\psi(T) - \psi(0))^{\gamma-1}}{|\Omega|} \left(\sum_{i=1}^{m} |\omega_i| + \sum_{r=1}^{k} |\sigma_r| + \sum_{j=1}^{n} |\kappa_j| \right) \right] \lambda_{\mathcal{B}} \epsilon \mathcal{B}(t)$$

$$= \mathcal{K} \lambda_{\mathcal{B}} \mathcal{B}(t) \epsilon,$$

from which we obtain inequality (4.12). This completes the proof. $\qquad\square$

Next, we are ready to prove UIHR and GUHR stability results.

Theorem 4.11 *Let $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, and let (H_1) and (4.11) be satisfied. If $\Phi_1 < 1$, where Φ_1 is given by (3.6), then problem (1.5) is UHR stable and GUHR stable on J.*

Proof Let $z \in \mathcal{C}^1_{1-\gamma,\psi}(J,\mathbb{R})$ be a solution of inequality (4.2), and let x be the unique solution of problem (1.5). By applying the triangle inequality and Lemma 4.10 we get

$$\begin{aligned}
\left| z(t) - x(t) \right| &= \left| z(t) - \mathcal{H}_z(t) + \mathcal{H}_z(t) - x(t) \right| \\
&\leq \left| z(t) - \mathcal{H}_z(t) \right| + \left| \mathcal{H}_z(t) - x(t) \right| \\
&\leq \mathcal{K} \lambda_{\mathcal{B}} \mathcal{B}(t) \epsilon + \Bigg| \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \Bigg(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_z(s)(\eta_i) \\
&\quad - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_z(s)(\theta_r) - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_z(s)(\zeta_j) \Bigg) - \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(t) \\
&\quad - \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Omega} \Bigg(A - \sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} F_x(s)(\eta_i) - \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} F_x(s)(\theta_r) \\
&\quad - \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} F_x(s)(\zeta_j) \Bigg) \Bigg| \\
&\leq \mathcal{K} \lambda_{\mathcal{B}} \mathcal{B}(t) \epsilon + \mathcal{I}_{0^+}^{\alpha;\psi} \left| F_z(s) - F_x(s) \right|(t) + \frac{(\psi(t) - \psi(0))^{\gamma-1}}{|\Omega|} \\
&\quad \times \Bigg(\sum_{i=1}^{m} \omega_i \mathcal{I}_{0^+}^{\alpha;\psi} \left| F_z(s) - F_x(s) \right|(\eta_i) + \sum_{r=1}^{k} \sigma_r \mathcal{I}_{0^+}^{\alpha+\delta_r;\psi} \left| F_z(s) - F_x(s) \right|(\theta_r) \\
&\quad + \sum_{j=1}^{n} \kappa_j \mathcal{I}_{0^+}^{\alpha-\beta_j;\psi} \left| F_z(s) - F_x(s) \right|(\zeta_j) \Bigg) \\
&\leq \mathcal{K} \lambda_{\mathcal{B}} \mathcal{B}(t) \epsilon + \Phi_1 \left| z(t) - x(t) \right|,
\end{aligned}$$

where Φ_1 is defined by (3.6), which implies that

$$\left| z(t) - x(t) \right| \leq \frac{\mathcal{K} \lambda_{\mathcal{B}}}{1 - \Phi_1} \mathcal{B}(t) \epsilon.$$

By setting

$$m_{f,\mathcal{B}} = \frac{\mathcal{K} \lambda_{\mathcal{B}}}{1 - \Phi_1}$$

we get the inequality

$$\left|z(t) - x(t)\right| \le m_{f,\mathcal{B}}\mathcal{B}(t)\epsilon. \tag{4.14}$$

Hence problem (1.5) is UHR stable. Moreover, if we set $\epsilon = 1$ in (4.14) with $\mathcal{B}(0) = 0$, then problem (1.5) is GUHR stable. The proof is completed. $\qquad\square$

5 Examples

In this section, we present two examples, which illustrate the validity and applicability of main results.

Example 5.1 Consider the nonlocal boundary problem

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\frac{3}{10},\frac{1}{4};e^{\frac{t}{2}}} x(t) = \frac{10t+2}{(5-\sin^2 \pi t)^2} \cdot \frac{|x(t)|}{4+|x(t)|} + \frac{5\cos \pi t - 2}{5} \cdot \frac{|{}^{H}\mathfrak{D}_{0^+}^{\frac{3}{10},\frac{1}{4};e^{\frac{t}{2}}} x(t)|}{6+|{}^{H}\mathfrak{D}_{0^+}^{\frac{3}{10},\frac{1}{4};e^{\frac{t}{2}}} x(t)|} \\ \qquad\qquad + (2t-1) \cdot \frac{|\mathcal{I}_{0^+}^{\frac{3}{10};e^{\frac{t}{2}}} x(t)|}{5+|\mathcal{I}_{0^+}^{\frac{3}{10};e^{\frac{t}{2}}} x(t)|}, \quad t \in [0, 4/5], \\ \sum_{i=1}^{m} \left(\frac{i+1}{10}\right) x\left(\frac{i}{10}\right) + \sum_{j=1}^{n} \left(\frac{4-j}{10}\right) {}^{H}\mathfrak{D}_{0^+}^{\frac{14+j}{100},\frac{1}{4};e^{\frac{t}{2}}} x\left(\frac{3}{25}j\right) \\ \qquad + \sum_{r=1}^{k} \left(\frac{r+1}{r+4}\right) \mathcal{I}_{0^+}^{\frac{r+2}{r+9};e^{\frac{t}{2}}} x\left(\frac{r}{5}\right) = 0, \end{cases} \tag{5.1}$$

where $\alpha = 3/10$, $\rho = 1/4$, $T = 4/5$, $m = 2$, $n = 3$, $k = 1$, $\omega_i = (i+1)/10$, $\kappa_j = (4-j)/10$, $\sigma_r = (r+1)/(r+4)$, $\eta_i = i/10$, $\zeta_j = 3j/25$, $\theta_r = r/5$, $\beta_j = (14+j)/100$, $\delta_r = (r+2)/(r+9)$, and $A = 0$. From the given data we obtain that $\Omega \approx 3.593684625 \ne 0$ and

$$f(t, u, v, w) = \frac{10t+2}{(5-\sin^2 \pi t)^2} \cdot \frac{|u|}{4+|u|} + \frac{5\cos \pi t - 2}{5} \cdot \frac{|v|}{6+|v|} + (2t-1) \cdot \frac{|w|}{5+|w|}.$$

For $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [0, 4/5]$, we have

$$\left|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\right| \le \frac{1}{10}\left(|x_1 - x_2| + |y_1 - y_2|\right) + \frac{3}{25}|z_1 - z_2|.$$

Assumption (H_1) is satisfied with $L_1 = \frac{1}{10}$, $L_2 = \frac{1}{10}$, $L_3 = \frac{3}{25}$. Hence

$$\Phi_1 \approx 0.5220929551 < 1.$$

Since all the assumptions of Theorem 3.1 are satisfied, problem (5.1) has a unique solution on $[0, 4/5]$. Furthermore, we can also compute that

$$\tau := \frac{(\psi(T) - \psi(0))^{\gamma-1}\Xi}{1 - \Phi_1} \approx 2.590447084 > 0.$$

Hence by Theorem 4.8 problem (5.1) is both UH and GUH stable.

Example 5.2 Consider the nonlocal boundary problem

$$
\begin{cases}
{}^{H}\mathfrak{D}_{0^+}^{\frac{3}{5},\frac{1}{2};\sqrt[3]{t}}x(t) = \dfrac{4t-3}{5^{3t+2}\left(1+|x(t)|+{}^{H}\mathfrak{D}_{0^+}^{\frac{3}{5},\frac{1}{2};\sqrt[3]{t}}x(t)|+|\mathcal{I}_{0^+}^{\frac{3}{5};\sqrt[3]{t}}x(t)|\right)}, & t \in [0, 3/2], \\[4mm]
\sum_{i=1}^{m}\frac{i}{5}x(\frac{i}{i+4}) + \sum_{j=1}^{n}(\frac{4-j}{10}){}^{H}\mathfrak{D}_{0^+}^{\frac{3j}{20},\frac{1}{2};\sqrt[3]{t}}x(\frac{j+1}{20}) + \sum_{r=1}^{k}(\frac{6-r}{50})\mathcal{I}_{0^+}^{\frac{r+1}{5};\sqrt[3]{t}}x(\frac{r}{5}) = \frac{\sqrt{\pi}}{2},
\end{cases}
\tag{5.2}
$$

where $\alpha = 3/5$, $\rho = 1/2$, $T = 3/5$, $m = 1$, $n = 2$, $k = 2$, $\omega_i = i/5$, $\kappa_j = (4-j)/10$, $\sigma_r = (6-r)/50$, $\eta_i = i/(i+4)$, $\zeta_j = (j+1)/20$, $\theta_r = r/5$, $\beta_j = 3j/20$, $\delta_r = (r+1)/5$, and $A = \sqrt{\pi}/2$. From the given data we obtain that $\Omega \approx 0.8384823809 \neq 0$ and

$$
f(t, u, v, w) = \frac{4t-3}{5^{3t+2}(1+|u|+|v|+|w|)}.
$$

For $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [0, 3/2]$, we have

$$
\left| f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \right| \le \frac{3}{25}\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \right).
$$

Assumption (H_1) is satisfied with $L_1 = L_2 = L_3 = \frac{3}{25}$. Hence

$$
\Phi_1 \approx 0.6225720515 < 1, \quad \text{and} \quad \mathcal{K} \approx 2.021525739.
$$

Set $\mathcal{B}(t) = \psi(t) - \psi(0)$. By using Proposition 2.7(i), we can simply calculate that

$$
\mathcal{I}_{0^+}^{\frac{3}{5};\sqrt[3]{t}}\mathcal{B}(t) = \frac{1}{\Gamma(\frac{13}{5})}\left[\psi(t) - \psi(0)\right]^{\frac{3}{5}}\mathcal{B}(t) \le \frac{\sqrt[5]{1.5}}{\Gamma(\frac{13}{5})}\mathcal{B}(t).
$$

Thus inequality (4.11) is satisfied with $\lambda_{\mathcal{B}} = \frac{\sqrt[5]{1.5}}{\Gamma(\frac{13}{5})} > 0$. It follows that

$$
m_{f,\mathcal{B}} = \frac{\mathcal{K}\lambda_{\mathcal{B}}}{1 - \Phi_1} \approx 4.062949933 > 0.
$$

Hence by Theorem 4.11 problem (5.2) is both UHR and GUHR stable.

6 Conclusion

We have proved the existence and uniqueness of a solution for a class of ψ-Hilfer fractional integro-differential equations with mixed nonlocal conditions. We emphasize that the nonlocal boundary condition is very general, including multipoint, fractional derivative multiorder, and fractional integral multiorder conditions. We used the fixed point theorems of Banach, Schaefer, and Krasnosel'ski to investigate the existence and uniqueness of solutions. Our results are not only new in the given setting but also provide some new special cases by fixing the parameters involved in the problem at hand. For instance, by fixing $\omega_j = 0$, $\lambda_k = 0$ for all $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r$ our results correspond to boundary value problems for ψ-Hilfer nonlinear fractional integro-differential equations supplemented with multipoint boundary conditions. In case we take $\delta_i = 0$, $\lambda_k = 0$ for all $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, r$, we obtain results for boundary value problems for ψ-Hilfer nonlinear fractional integro-differential equations equipped with multiterm integral boundary conditions.

We also investigated different kinds of Ulam stability, such as the Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. The obtained results are well illustrated by examples.

The work accomplished in this paper is new and enriches the literature on boundary value problems for nonlinear ψ-Hilfer fractional differential equations.

Availability of data and materials
The authors declare that all data and materials in this paper are available and veritable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
We declare that all authors have contributed equally and significantly to the contents of the paper. All authors have read and agreed to the published version of the manuscript.

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