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Modified different nonlinearities for weakly coupled systems of semilinear effectively damped waves with different time-dependent coefficients in the dissipation terms

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Abstract

We prove the global existence of small data solution in all spaces of all dimensions $n \geq 1$ for weakly coupled systems of semilinear effectively damped wave, with different time-dependent coefficients in the dissipation terms. Moreover, we assume that the nonlinearity terms $f(t,u)$ and $g(t,v)$ satisfy some properties of parabolic equations. We study the problem in several classes of regularity.

Keywords: Weakly coupled hyperbolic systems; Damped wave equations; Cauchy problem; Global existence; L^2-decay; Effective dissipation; Small data solutions

1 Introduction

Let us consider the Cauchy problem for the semilinear classical damped wave equation with power nonlinearity

$$u_{tt} - \Delta u + u_t = f(u), \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \tag{1}$$

where $t \in [0,\infty), x \in \mathbb{R}^n$, and

$$f(0) = 0, \qquad \big|f(u) - f(\tilde{u})\big| \lesssim |u - \tilde{u}|\big(|u| + |\tilde{u}|\big)^{p-1}. \tag{2}$$

Having the estimates proved in [17] for the corresponding homogeneous problem, for given compactly supported initial data $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and for $p \leq p_{GN}(n) := \frac{n}{n-2}$ if $n \geq 3$, the authors in [22] proved the local (in time) existence of energy solutions $u \in \mathcal{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T), L^2(\mathbb{R}^n))$. Moreover, they proved the global (in time) existence of small data solutions by using the technique of "potential well" and "modified potential well". The Cauchy problem (1) was also studied in [7, 12, 13, 27, 30], where the Fujita exponent $p_{Fuj}(n) := 1 + \frac{2}{n}$ has an important role as the critical exponent. The critical exponent means that we have the global (in time) existence of small data weak solutions for $p > p_{Fuj}(n)$, whereas the local (in time) existence for $p > 1$ and large data can be only expected.

Assuming a time-dependent coefficient in the dissipation term, we first consider the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x).$$

Among other classifications of the dissipation term $b(t)u_t$ introduced in [28] and [29], we are interested in the effective case, where $b = b(t)$ satisfies the following properties:

- b is a positive and monotonic function with $tb(t) \to \infty$ as $t \to \infty$,
- $((1+t)^2 b(t))^{-1} \in L^1(0,\infty)$,
- $b \in \mathcal{C}^3[0,\infty)$ and $|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}$ for $k = 1,2,3$,
- $\frac{1}{b} \notin L^1(0,\infty)$, and there exists a constant $a \in [0,1)$ such that $tb'(t) \le ab(t)$.

Examples of functions belonging to this class are the followings with $r \in (-1,1)$:

- $b(t) = \frac{\mu}{(1+t)^r}$ for some $\mu > 0$, $b(t) = \frac{\mu}{(1+t)^r}(\log(e+t))^\gamma$ for some $\mu > 0$ and $\gamma > 0$, and $b(t) = \frac{\mu}{(1+t)^r(\log(e+t))^\gamma}$ for some $\mu > 0$ and $\gamma > 0$.

In [5] the authors derived such estimates for solutions to the family of parameter-dependent Cauchy problems

$$u_{tt} - \Delta u + b(t)u_t = 0, \qquad v(\tau,x) = 0, \qquad v_t(\tau,x) = f(u)(\tau,x).$$

Using theses estimates together with Duhamel's principle, in the same paper the authors proved the global existence of small data solutions to the following semilinear Cauchy problem:

$$u_{tt} - \Delta u + b(t)u_t = f(u), \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

where $f(u)$ satisfies condition (2).

In 2013, D'Abbicco [3] proved the global existence of small data solution for low space dimensions and derived decay estimates for solutions to the Cauchy problem

$$u_{tt} - \Delta u + b(t)u_t = f(t,u), \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

where

$$f(t,0) = 0 \quad \text{and} \quad \big|f(t,v) - f(t,\tilde{v})\big| \lesssim \left(1 + \int_0^t \frac{1}{b(r)}\,dr\right)^\gamma |v - \tilde{v}|\big(|v| + |\tilde{v}|\big)^{p-1}.$$

Weakly coupled systems can be an interesting problem, treated and improved in [16] and [1]. In this paper, we study in all space dimensions the Cauchy problem of weakly coupled system of semilinear effectively damped waves

$$\begin{aligned} &u_{tt} - \Delta u + b_1(t)u_t = f(t,v), \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \\ &v_{tt} - \Delta v + b_2(t)v_t = g(t,u), \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x), \end{aligned} \tag{3}$$

where

$$\big(1+B_1(t,0)\big)^{\beta} \lesssim \big(1+B_2(t,0)\big) \lesssim \big(1+B_1(t,0)\big)^{\frac{1}{\alpha}}, \tag{4}$$

$$f(t,0)=0, \qquad \big|f(t,v)-f(t,\tilde{v})\big| \lesssim \big(1+B_1(t,0)\big)^{\gamma_1}|v-\tilde{v}|\big(|v|+|\tilde{v}|\big)^{p-1}, \tag{5}$$

$$g(t,0)=0, \qquad \big|g(t,u)-g(t,\tilde{u})\big| \lesssim \big(1+B_2(t,0)\big)^{\gamma_2}|u-\tilde{u}|\big(|u|+|\tilde{u}|\big)^{q-1}, \tag{6}$$

for $B_1(t,\tau)=\int_\tau^t \frac{1}{b_1(r)}\,dr, B_2(t,\tau)=\int_\tau^t \frac{1}{b_2(r)}\,dr, \alpha,\beta \in \mathbb{R}_+^*$, and $\gamma_1,\gamma_2 \in [-1,\infty)$. If we take $\gamma_1 < -1$ or $\gamma_2 < -1$, then we will get an empty admissible range for p or q (see the table in Remark 2.3).

Recently, Nishihara and Wakasugi [23] studied the particular case of (3), where $b_1(t) = b_2(t) = 1, f(t,v) = |v|^p$, and $g(t,u) = |u|^q$. Using the weighted energy method, they proved the global (in time) existence if the inequality

$$\frac{\max\{p;q\}+1}{pq-1} < \frac{n}{2} \tag{7}$$

is satisfied. Using an additional regularity $L^m(\mathbb{R}^n)$ for data, we conclude the so-called modified Fujita exponent $p_{Fuj,m} := 1+\frac{2m}{n}$; this new exponent implies a modified condition corresponding to (7), $\frac{\max\{p;q\}+1}{pq-1} < \frac{n}{2m}$. In [20] and [18] the authors studied the above system with the same nonlinearities assumed in [23] by taking the equivalent coefficients $b_1 = b_1(t)$ and $b_2 = b_2(t)$ or, in other words, $\alpha = \beta = 1$. The global (in time) existence of small initial data solutions was proved assuming different classes of regularity of data and for all space dimensions. Considering (3) in [21], the authors proved a global existence result for a particular case from the set of effective dissipation terms $b_1(t) = \frac{\mu}{(1+t)^{r_1}}, r_1, r_2 \in (-1,1)$, and $b_2(t) = \frac{\mu}{(1+t)^{r_2}}$ with the nonlinearities $f(t,v) = |v|^p$ and $f(t,u) = |u|^q$.

1.1 Notations

For $s > 0$ and $m \in [1,2)$, we introduce the function space

$$\mathcal{A}_{m,s} := \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^{s-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big)$$

with the norm

$$\big\|(u,v)\big\|_{\mathcal{A}_{m,s}} := \|u\|_{H^s} + \|u\|_{L^m} + \|v\|_{H^{s-1}} + \|v\|_{L^m}.$$

We denote by $\tilde{p}$ and $\tilde{q}$ the modified exponents of the exponents p and q in the power nonlinearities appearing in (5) and (6). Then

$$\tilde{p} = \begin{cases} (p-1)\beta + 1 & \text{if } \beta \geq 1, \\ (p-\frac{m}{2})\beta + \frac{m}{2} & \text{if } 0 < \beta < 1, \end{cases} \tag{8}$$

and

$$\tilde{q} = \begin{cases} (q-1)\alpha + 1 & \text{if } \alpha \geq 1, \\ (q-\frac{m}{2})\alpha + \frac{m}{2} & \text{if } 0 < \alpha < 1. \end{cases} \tag{9}$$

Remark 1.1 If $\alpha = \beta = 1$, then $(1 + B_1(t, 0)) \approx (1 + B_2(t, 0))$. This case was studied in previous papers. In this work, we restrict ourselves to the remaining cases.

2 Main results

We study the Cauchy problem (3) in several cases with respect to the regularity of the data to cover all space dimensions and the modified exponents of power nonlinearities $\tilde{p}, \tilde{q}$ and parameters $\alpha, \beta, \gamma_1, \gamma_2$. Therefore we introduce the following classification of regularity: Data from energy space $s = 1$, data from Sobolev spaces with suitable regularity $s \in (1, \frac{n}{2} + 1]$, and, finally, large regular data $s > \frac{n}{2} + 1$.

2.1 Data from the energy space

In this section, we are interested in system (3), where the data are taken from the function space $\mathcal{A}_{m,1}$. In Theorem 2.1, we treat the case where both modified exponents power $\tilde{p}$ and $\tilde{q}$ are above the modified Fujita exponents

$$p_{Fuj,m,\gamma_1} := 1 + \frac{2m(\gamma_1 + 1)}{n} \quad \text{and} \quad q_{Fuj,m,\gamma_2} := 1 + \frac{2m(\gamma_2 + 1)}{n},$$

respectively.

Theorem 2.1 *Let the data $(u_0, u_1), (v_0, v_1)$ belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$ for $m \in [1, 2)$. Moreover, let the modified exponents satisfy*

$$\tilde{p} > p_{Fuj,m,\gamma_1}, \qquad \tilde{q} > p_{Fuj,m,\gamma_2}, \tag{10}$$

and let the exponents p and q of the power nonlinearities satisfy

$$\begin{aligned} &\frac{2}{m} \leq \min\{p; q\} \leq \max\{p; q\} < \infty \quad \text{if } n \leq 2, \\ &\frac{2}{m} \leq \min\{p; q\} \leq \max\{p; q\} \leq p_{GN}(n) \quad \text{if } n > 2. \end{aligned} \tag{11}$$

Then there exists a constant ϵ_0 such that if

$$\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined global (in time) energy solution to (3) in

$$\left(\mathcal{C}([0, \infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0, \infty), L^2(\mathbb{R}^n))\right)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned} \|\nabla^j \partial_t^l u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-l} (1 + B_1(t, 0))^{-\frac{n}{2}(\frac{1}{m} - \frac{1}{2}) - \frac{j}{2} - l} \\ &\quad \times \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \\ \|\nabla^j \partial_t^l v(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-l} (1 + B_2(t, 0))^{-\frac{n}{2}(\frac{1}{m} - \frac{1}{2}) - \frac{j}{2} - l} \\ &\quad \times \left(\|(u_0, u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0, v_1)\|_{\mathcal{A}_{m,1}}\right), \end{aligned}$$

where $j + l = 0, 1$.

Remark 2.2 We remark that for $\gamma_1 = \gamma_2 = 0$, system (3) behaves in this case like one single equation because the modified power nonlinearities $\tilde{p}$ and $\tilde{q}$ are influenced separately only by the modified Fujita exponent $p_{Fuj,m}(n) = \frac{2m}{n} + 1$. Then we cannot feel in an optimal way the interplay between the powers of nonlinearities in the existence conditions.

Remark 2.3 The final admissible ranges for the exponents p and q of power nonlinearities can be fixed using several parameters such as α, β, the exponents γ_1, γ_2, the space dimension n, and the parameter of additional regularity m. As an example for the dimension $n = 1$, if we take $0 < \beta < 1$, then $\tilde{p} < p$. We distinguish two cases:

- If $\gamma_1 \geq -\frac{1}{2}$, then $p \geq \frac{2}{m}$ for $\tilde{p} > p_{Fuj,m,\gamma_1}$ which is equivalent to $p > \frac{1}{\beta}(2m(\gamma_1 + 1) - \frac{m}{2} + 1) + \frac{m}{2}$.
- If $\gamma_1 \in [-1, -\frac{1}{2})$, then the solution exists for

$$p > \max\left\{\frac{1}{\beta}\left(2m(\gamma_1 + 1) - \frac{m}{2} + 1\right) + \frac{m}{2}; \frac{2}{m}\right\}.$$

The general case for the admissible ranges from below can be summarized as follows:

β	Nonlinearity parameter γ_1	Admissible range for p
$0 < \beta < 1$	$\gamma_1 \geq -1 + \frac{n}{2}$	$p > \frac{1}{\beta} + \frac{2m(\gamma_1+1)}{n\beta} - \frac{m}{2\beta} + \frac{m}{2}$
	$\gamma_1 \in [-1, -1 + \frac{n}{2})$	$p > \max\left\{\frac{1}{\beta} + \frac{2m(\gamma_1+1)}{n\beta} - \frac{m}{2\beta} + \frac{m}{2}; \frac{2}{m}\right\}$
$\beta \geq 1$	$\gamma_1 \geq -1 + \frac{n\beta}{2}$	$p > \frac{2m(\gamma_1+1)}{n\beta} + 1$
	$\gamma_1 \in [-1, -1 + \frac{n\beta}{2})$	$p > \max\left\{\frac{2m(\gamma_1+1)}{n\beta} + 1; \frac{2}{m}\right\}$

In the same way, we can get the admissible range for q with respect to the parameters α and γ_2.

Example 2.4 Let us choose the space dimension $n = 2$, the parameters $\gamma_1 = -1, \gamma_2 = -\frac{1}{3}$, and the coefficients of the dissipation terms $b_1(t) = (1 + t)^{-\frac{1}{2}}$ and $b_2(t) = (1 + t)^{\frac{1}{2}}$, which implies $\beta = \frac{1}{\alpha} = 3$. Using (10) from the previous theorem for $m = 2$, we get $\tilde{p} > 1, \tilde{q} > \frac{7}{3}$. Theses conditions together with (11) after applying (8) and (9) imply the following admissible range for the exponents of power nonlinearities:

$$p > 1, \qquad q > 5.$$

The case where one exponent $\tilde{p}$ or $\tilde{q}$ is below the modified Fujita exponent, we distinguish four cases with respect to the values of α and β:

1. $\tilde{p} \leq 1 + \frac{2m(\gamma_1+1)}{n}, \tilde{q} > 1 + \frac{2m(\gamma_2+1)}{n}$ with $\min\{\alpha; \beta\} \geq 1$ or $\min\{\alpha; \beta\} \leq 1 \leq \max\{\alpha; \beta\}$.
2. $\tilde{p} > 1 + \frac{2m(\gamma_1+1)}{n}, \tilde{q} \leq 1 + \frac{2m(\gamma_2+1)}{n}$ with $\min\{\alpha; \beta\} \geq 1$ or $\min\{\alpha; \beta\} \leq 1 \leq \max\{\alpha; \beta\}$.

Theorem 2.5 *Let $m \in [1, 2), \alpha \geq 1$, and $\beta > 0$. The data $(u_0, u_1), (v_0, v_1)$ are assumed to belong to $\mathcal{A}_{m,1} \times \mathcal{A}_{m,1}$. Moreover, let the modified exponents satisfy*

$$\begin{aligned} \tilde{p} &< \frac{2m(\gamma_1 + 1)}{n} + 1, \\ \tilde{q} &> \frac{2m(\gamma_2 + 1)}{n} + 1. \end{aligned} \tag{12}$$

Moreover, we assume that

$$\frac{n}{2} > m\left(\frac{\tilde{q} + \alpha + \gamma_1\tilde{q} + \gamma_1(\alpha - 1) + \gamma_2}{\tilde{p}\tilde{q} - 1 + (\alpha - 1)(\tilde{p} - 1)}\right) \tag{13}$$

and the exponents p and q of the power nonlinearities satisfy

$$\frac{2}{m} \le \min\{p;q\} \le \max\{p;q\} < \infty \quad \text{if } n \le 2,$$
$$\frac{2}{m} \le \min\{p;q\} \le \max\{p;q\} \le p_{GN}(n) \quad \text{if } n > 2.$$

Then there exists a constant ϵ_0 *such that if*

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \le \epsilon_0,$$

then there exists a uniquely determined global (*in time*) *energy solution to* (3) *in*

$$\left(\mathcal{C}\left([0,\infty),H^1(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty),L^2(\mathbb{R}^n)\right)\right)^2.$$

Furthermore, the solution satisfies the following decay estimates:

$$\begin{aligned}
&\left\|\nabla^j \partial_t^l u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim b_1(t)^{-l}\left(1 + B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l+\kappa(\tilde{p})}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right), \\
&\left\|\nabla^j \partial_t^l v(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim b_2(t)^{-l}\left(1 + B_2(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l}\left(\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}}\right),
\end{aligned}$$

where $j + l = 0, 1$, *and*

$$\kappa(\tilde{p}) = \gamma_1 - \frac{n}{2m}(\tilde{p} - 1) + 1$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution u of the linear Cauchy problem with vanishing right-hand side.

Remark 2.6 Choosing $\tilde{p} = p_{Fuj,m}(n)$ in condition (12), we get an arbitrarily small loss of decay $\kappa(\tilde{p}) = \varepsilon$.

We summarize the remaining results for all cases with respect to $\alpha, \beta, \tilde{p}$, and $\tilde{q}$ as follows:

- If we assume in the statement of the previous theorem that $\alpha < 1$ and $\beta \ge 1$, then, instead of (13), we get the condition

$$\frac{n}{2} > m\left(\frac{\tilde{q} + 1 + \gamma_1\tilde{q} + \gamma_2 + \frac{m}{2}(\alpha - 1)(\gamma_1 + 1)}{\tilde{p}\tilde{q} - 1 + \frac{m}{2}(\alpha - 1)(\tilde{p} - 1)}\right).$$

- If $\tilde{p} > \frac{2m(\gamma_1+1)}{n} + 1, \tilde{q} \le \frac{2m(\gamma_2+1)}{n} + 1$, then, instead of (13), we have to assume that

$$\frac{n}{2} > m\left(\frac{\tilde{p} + \beta + \gamma_2\tilde{p} + \gamma_2(\beta - 1) + \gamma_1}{\tilde{p}\tilde{q} - 1 + (\beta - 1)(\tilde{q} - 1)}\right) \quad \text{for } \alpha > 0, \beta \ge 1,$$

$$\frac{n}{2} > m\left(\frac{\tilde{p}+1+\gamma_2\tilde{p}+\gamma_1+\frac{m}{2}(\beta-1)(\gamma_2+1)}{\tilde{p}\tilde{q}-1+\frac{m}{2}(\beta-1)(\tilde{q}-1)}\right) \quad \text{for } \alpha \geq 1, \beta < 1.$$

2.2 Data from Sobolev spaces with suitable regularity

In this section the regularity of data has a strong influence on the admissible range of the modified exponents or the exponents of power nonlinearities, respectively. For this reason, we assume that the data have a different suitable regularity, that is,

$$(u_0,u_1) \in H^{s_1}(\mathbb{R}^n) \times H^{s_1-1}(\mathbb{R}^n), \quad s_1 \in \left(1, 1+\frac{n}{2}\right],$$
$$(v_0,v_1) \in H^{s_2}(\mathbb{R}^n) \times H^{s_2-1}(\mathbb{R}^n), \quad s_2 \in \left(1, 1+\frac{n}{2}\right],$$

with an additional regularity $L^m(\mathbb{R}^n)$, $m \in [1,2)$. In this section, we use a generalized (fractional) Gagliardo–Nirenberg inequality used in [11] and [25]. Furthermore, we use a fractional Leibniz rule and a fractional chain rule, which are explained in the Appendix.

Theorem 2.7 *Let $n \geq 4$, $s_1 \in (3+2\gamma_1, \frac{n}{2}+1]$, $s_2 \in (3+2\gamma_2, \frac{n}{2}+1]$, $0 < s_2 - s_1 < 1$, and $\lceil s_1 \rceil \neq \lceil s_2 \rceil$. The data (u_0,u_1), (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2}$ with $m \in [1,2)$. Furthermore, we assume that*

$$\tilde{p} > \frac{2m}{n}\left(\frac{s_1+1+2\gamma_1}{2}\right)+1, \qquad \tilde{q} > \frac{2m}{n}\left(\frac{s_2+1+2\gamma_2}{2}\right)+1. \tag{14}$$

and that the exponents p and q of the power nonlinearities satisfy the conditions

$$\begin{aligned} &\lceil s_1 \rceil < p, \qquad \lceil s_2 \rceil < q \quad \text{if } n \leq 2s_1, \\ &\lceil s_1 \rceil < p, \qquad \lceil s_2 \rceil < q \leq 1+\frac{2}{n-2s_1} \quad \text{if } 2s_1 < n \leq 2s_2, \\ &\lceil s_1 \rceil < p \leq 1+\frac{2}{n-2s_2}, \qquad \lceil s_2 \rceil < q \leq 1+\frac{2}{n-2s_1} \quad \text{if } n > 2s_2. \end{aligned} \tag{15}$$

Then there exists a constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (3) in

$$\begin{aligned} &\left(\mathcal{C}([0,\infty), H^{s_1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_1-1}(\mathbb{R}^n))\right) \\ &\quad \times \left(\mathcal{C}([0,\infty), H^{s_2}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{s_2-1}(\mathbb{R}^n))\right). \end{aligned}$$

Furthermore, for $l = 0,1$, the solution satisfies the estimates

$$\begin{aligned} \||D|^{s_1-l}\partial_t^l u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_1(t)^{-l}(1+B_1(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-l-\frac{s_1-l}{2}} \\ &\quad \times \left(\|(u_0,u_1)\|_{\mathcal{A}_{m,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,s_2}}\right), \\ \||D|^{s_2-l}\partial_t^l v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim b_2(t)^{-l}(1+B_2(t,0))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-l-\frac{s_2-l}{2}} \end{aligned}$$

$$\times \left(\left\|(u_0,u_1)\right\|_{\mathcal{A}_{m,s_1}} + \left\|(v_0,v_1)\right\|_{\mathcal{A}_{m,s_2}}\right).$$

Particular cases:

- If $\beta \geq 1$ and $s_1 \geq 3 + 2\gamma_1$, then under the assumptions of Theorem 2.7, the condition $p > \lceil s_1 \rceil$ implies $\tilde{p} > \frac{2m}{n}(\frac{s_1+1+2\gamma_1}{2}) + 1$.
- If $\alpha \geq 1$ and $s_2 \geq 3 + 2\gamma_2$, then under the assumptions of Theorem 2.7, the condition $p > \lceil s_2 \rceil$ implies $\tilde{q} > \frac{2m}{n}(\frac{s_2+1+2\gamma_2}{2}) + 1$.

2.3 Large regular data

This case has been classified to benefit from the embedding in $L^\infty(\mathbb{R}^n)$, where the data are supposed to have a high regularity, which means that

$$(u_0,u_1) \in H^{s_1}(\mathbb{R}^n) \times H^{s_1-1}(\mathbb{R}^n), \quad s_1 > \frac{n}{2} + 1,$$
$$(v_0,v_1) \in H^{s_2}(\mathbb{R}^n) \times H^{s_2-1}(\mathbb{R}^n), \quad s_2 > \frac{n}{2} + 1.$$

Theorem 2.8 *Let $n \geq 4$, $(u_0,u_1),(v_0,v_1) \in \mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2}$, $m \in [1,2)$, $\min\{s_2;s_1\} > \frac{n}{2} + 1$, and $s_1 - s_2 \in (-1,1)$. Moreover, let*

$$p > s_1, \qquad q > s_2,$$

and

$$\tilde{p} > \frac{2m}{n}\left(\frac{s_1 + 1 + 2\gamma_1}{2}\right) + 1, \qquad \tilde{q} > \frac{2m}{n}\left(\frac{s_2 + 1 + 2\gamma_2}{2}\right) + 1. \tag{16}$$

Then there exists a constant ϵ_0 such that if

$$\left\|(u_0,u_1)\right\|_{\mathcal{A}_{m,s_1}} + \left\|(v_0,v_1)\right\|_{\mathcal{A}_{m,s_2}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (3) in

$$\left(\mathcal{C}\left([0,\infty),H^{s_1}(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty),H^{s_1-1}(\mathbb{R}^n)\right)\right)$$
$$\times \left(\mathcal{C}\left([0,\infty),H^{s_2}(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,t],H^{s_2-1}(\mathbb{R}^n)\right)\right).$$

Furthermore, the solution satisfies for $l = 0,1$ the estimates

$$\left\||D|^{s_1-l}\partial_t^l u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim b_1(t)^{-l}\left(1 + B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-l-\frac{s_1-l}{2}}$$
$$\times \left(\left\|(u_0,u_1)\right\|_{\mathcal{A}_{m,s_1}} + \left\|(v_0,v_1)\right\|_{\mathcal{A}_{m,s_2}}\right),$$
$$\left\||D|^{s_2-l}\partial_t^l v(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim b_2(t)^{-l}\left(1 + B_2(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-l-\frac{s_2-l}{2}}$$
$$\times \left(\left\|(u_0,u_1)\right\|_{\mathcal{A}_{m,s_1}} + \left\|(v_0,v_1)\right\|_{\mathcal{A}_{m,s_2}}\right).$$

3 Philosophy of our approach and proofs

3.1 Some tools

First, we recall the following result from [5].

Lemma 3.1 *The primitive* $B = B(t,\tau)$ *of* $\frac{1}{b}$ *satisfies the following properties*:

$$B(t,\tau) \approx B(t,0) \quad \text{for all } \tau \in \left[0, \frac{t}{2}\right], \tag{17}$$

$$B(\tau,0) \approx B(t,0) \quad \text{for all } \tau \in \left[\frac{t}{2}, t\right], \tag{18}$$

$$\int_{\frac{t}{2}}^{t} \frac{1}{b(\tau)} \big(1 + B(t,\tau)\big)^{-\frac{j}{2}-l} \, d\tau \lesssim \big(1 + B(t,0)\big)^{1-\frac{j}{2}-l} \log\big(1 + B(t,0)\big)^{l} \quad \text{for } j + l = 0, 1.$$

To use Duhamel's principle, we need the following results in the proofs of our main results.

Theorem 3.2 *The Sobolev solutions to the Cauchy problem*

$$u_{tt} - \Delta u + b(t) u_t = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)$$

satisfy the following estimates for $t > 0$:

For data from the energy space ($s = 1$),

$$\big\| \nabla^j \partial_t^l u(t,\cdot) \big\|_{L^2} \lesssim \big(b(t)\big)^{-l} \big(1 + B(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l} \big\| (u_0, u_1) \big\|_{\mathcal{A}_{m,1}},$$

where $j + l = 0, 1$;

for high regular data ($s > 1$),

$$\big\| u(t,\cdot) \big\|_{L^2} \lesssim \big(1 + B(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})} \big\| (u_0, u_1) \big\|_{\mathcal{A}_{m,s}},$$

$$\big\| u_t(t,\cdot) \big\|_{L^2} \lesssim b(t)^{-1} \big(1 + B(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1} \big\| (u_0, u_1) \big\|_{\mathcal{A}_{m,s}},$$

$$\big\| |D|^s u(t,\cdot) \big\|_{L^2} \lesssim \big(1 + B(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}} \big\| (u_0, u_1) \big\|_{\mathcal{A}_{m,s}},$$

$$\big\| |D|^{s-1} u_t(t,\cdot) \big\|_{L^2} \lesssim b(t)^{-1} \big(1 + B(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s-1}{2}-1} \big\| (u_0, u_1) \big\|_{\mathcal{A}_{m,s}}.$$

The proof of this theorem follows from [28] and [29].

Theorem 3.3 *The Sobolev solutions to the parameter-dependent family of Cauchy problems*

$$v_{tt} - \Delta v + b(t) v_t = 0, \qquad v(\tau,x) = 0, \qquad v_t(\tau,x) = v_1(x)$$

satisfy the following estimates for $t > \tau, \tau \geq 0$:

For data from the energy space ($s = 1$),

$$\big\| \nabla^j \partial_t^l v(t,\cdot) \big\|_{L^2} \lesssim b(t)^{-1} b(\tau)^{-l} \big(1 + B(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l} \| v_1 \|_{L^2 \cap L^m}, \tag{19}$$

where $j + l = 0, 1$;

For high regular data ($s > 1$),

$$\begin{aligned}
&\big\|v(t,\cdot)\big\|_{L^2} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|v_1\|_{H^{s-1}\cap L^m},\\
&\big\|v_t(t,\cdot)\big\|_{L^2} \lesssim b(\tau)^{-1}b(t)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\|v_1\|_{H^{s-1}\cap L^m},\\
&\big\||D|^s v(t,\cdot)\big\|_{L^2} \lesssim b(\tau)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s}{2}}\|v_1\|_{H^{s-1}\cap L^m},\\
&\big\||D|^{s-1} v_t(t,\cdot)\big\|_{L^2} \lesssim b(\tau)^{-1}b(t)^{-1}\big(1+B(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s-1}{2}-1}\\
&\qquad\qquad \times \|v_1\|_{H^{s-1}\cap L^m}.
\end{aligned} \tag{20}$$

The proof of this theorem follows from [5] and [19].

3.2 Proofs

We define the norm of the solution space $X(t)$ by

$$\big\|(u,v)\big\|_{X(t)} = \sup_{\tau\in[0,t]}\big\{M_1(\tau,u)+M_2(\tau,v)\big\},$$

where we will choose $M_1(\tau,u)$ and $M_2(\tau,v)$ with respect to the goals of each theorem.

Let N be the mapping on $X(t)$ defined by

$$N:(u,v)\in X(t)\to N(u,v)=\big(u^{ln}+u^{nl},v^{ln}+v^{nl}\big),$$

where

$$\begin{aligned}
&u^{ln}(t,x) := E_{1,0}(t,0,x) *_{(x)} u_0(x) + E_{1,1}(t,0,x) *_{(x)} u_1(x),\\
&u^{nl}(t,x) := \int_0^t E_{1,1}(t,\tau,x) *_{(x)} f(\tau,v)\,d\tau,\\
&v^{ln}(t,x) := E_{2,0}(t,0,x) *_{(x)} v_0(x) + E_{2,1}(t,0,x) *_{(x)} v_1(x),\\
&v^{nl}(t,x) := \int_0^t E_{2,1}(t,\tau,x) *_{(x)} g(\tau,u)\,d\tau.
\end{aligned}$$

We denote by $E_{1,0} = E_{1,0}(t,0,x)$ and $E_{1,1} = E_{1,1}(t,0,x)$ the fundamental solutions to the Cauchy problem

$$u_{tt}-\Delta u+b_1(t)u_t=0,\qquad u(0,x)=u_0(x),\qquad u_t(0,x)=u_1(x),$$

and by $E_{2,0} = E_{2,0}(t,0,x)$ and $E_{2,1} = E_{2,1}(t,0,x)$ the fundamental solutions to the the Cauchy problem

$$v_{tt}-\Delta v+b_2(t)v_t=0,\qquad v(0,x)=v_0(x),\qquad v_t(0,x)=v_1(x).$$

Our aim is to prove the estimates

$$\begin{aligned}
&\big\|N(u,v)\big\|_{X(t)}\\
&\quad\lesssim \big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s_1}}+\big\|(v_0,v_1)\big\|_{\mathcal{A}_{m,s_2}}+\big\|(u,v)\big\|_{X(t)}^p+\big\|(u,v)\big\|_{X(t)}^q,
\end{aligned} \tag{21}$$

$$\begin{aligned}\big\|N(u,v)-N(\tilde u,\tilde v)\big\|_{X(t)} \lesssim{}& \big\|(u,v)-(\tilde u,\tilde v)\big\|_{X(t)}\\ &\times\big(\big\|(u,v)\big\|^{p-1}_{X(t)}+\big\|(\tilde u,\tilde v)\big\|^{p-1}_{X(t)}+\big\|(u,v)\big\|^{q-1}_{X(t)}+\big\|(\tilde u,\tilde v)\big\|^{q-1}_{X(t)}\big).\end{aligned} \tag{22}$$

We can immediately obtain from the introduced norm of the solution space $X(t)$, which will be fixed for each case, the following inequality:

$$\big\|\big(u^{ln},v^{ln}\big)\big\|_{X(t)} \lesssim \big\|(u_0,u_1)\big\|_{\mathcal{A}_{m,s_1}}+\big\|(v_0,v_1)\big\|_{\mathcal{A}_{m,s_2}}.$$

We complete the proof of all results separately by showing (22) with the inequality

$$\big\|\big(u^{nl},v^{nl}\big)\big\|_{X(t)} \lesssim \big\|(u,v)\big\|^{p}_{X(t)}+\big\|(u,v)\big\|^{q}_{X(t)}, \tag{23}$$

which leads to (21).

Proof of Theorem 2.1 We choose the space of energy solutions

$$X(t)=\big(\mathcal{C}\big([0,t],H^1\big)\cap\mathcal{C}^1\big([0,t],L^2\big)\big)^2$$

with the following norms for $\tau\in(0,t]$:

$$\begin{aligned}M_1(\tau,u)={}&\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\big\|u(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\ &+\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{1}{2}}\big\|\nabla u(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\ &+b_1(\tau)\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+1}\big\|u_t(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)},\\ M_2(\tau,v)={}&\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\big\|v(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\ &+\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{1}{2}}\big\|\nabla v(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}\\ &+b_2(\tau)\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+1}\big\|v_t(\tau,\cdot)\big\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

To prove (23), we need to estimate all terms appearing in $\|(u^{nl},v^{nl})\|_{X(t)}$. Let us begin to estimate $\|u^{nl}_t(t,\cdot)\|_{L^2}$. Using (19) with $m=2$ for $\tau\in[\frac{t}{2},t]$, we get

$$\begin{aligned}\big\|u^{nl}_t(t,\cdot)\big\|_{L^2}\lesssim{}&\int_0^{\frac{t}{2}} b_1(t)^{-1}b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\big\|f(\tau,v)\big\|_{L^m\cap L^2}\,d\tau\\ &+\int_{\frac{t}{2}}^{t} b_1(t)^{-1}b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1}\big\|f(\tau,v)\big\|_{L^2}\,d\tau.\end{aligned} \tag{24}$$

By a fractional version of the Gagliardo–Nirenberg inequality (see Proposition 4.1) and (5) we obtain

$$\big\|f(\tau,v)\big\|_{L^2}\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}}\big\|(u,v)\big\|^{p}_{X(t)}, \tag{25}$$

$$\big\|f(\tau,v)\big\|_{L^m}\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{2m}}\big\|(u,v)\big\|^{p}_{X(t)}, \tag{26}$$

where we use condition (11). Plugging the last estimates into (24) and using (4), (17), and (18), we get

$$\begin{aligned}
\|u_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \|(u,v)\|_{X(t)}^p \int_0^{\frac{t}{2}} b_1(t)^{-1} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1} \\
&\quad \times \big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{2m}}\,d\tau \\
&\quad + \|(u,v)\|_{X(t)}^p \int_{\frac{t}{2}}^{t} b_1(t)^{-1} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1} \\
&\quad \times \big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}}\,d\tau \\
&\lesssim \|(u,v)\|_{X(t)}^p \int_0^{\frac{t}{2}} b_1(t)^{-1} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1} \\
&\quad \times \big(1+B_1(\tau,0)\big)^{(-\frac{n}{2m}p+\frac{n}{2m})\beta+\gamma_1}\,d\tau \\
&\quad + \|(u,v)\|_{X(t)}^p \int_{\frac{t}{2}}^{t} b_1(t)^{-1} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1} \\
&\quad \times \big(1+B_1(\tau,0)\big)^{(-\frac{n}{2m}p+\frac{n}{4})\beta+\gamma_1}\,d\tau \\
&\lesssim \|(u,v)\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1} \\
&\quad \times \int_0^{\frac{t}{2}} b_1(\tau)^{-1}\big(1+B_1(\tau,0)\big)^{(-\frac{n}{2m}p+\frac{n}{2m})\beta+\gamma_1}\,d\tau \\
&\quad + \|(u,v)\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{(-\frac{n}{2m}p+\frac{n}{4})\beta+\gamma_1} \\
&\quad \times \int_{\frac{t}{2}}^{t} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1}\,d\tau.
\end{aligned}$$

The last integral can be obtained from the definition of $B_1(t,\tau)$; indeed,

$$\int_{\frac{t}{2}}^{t} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1}\,d\tau \lesssim \log\big(1+B_1(t,0)\big) \approx \big(1+B_1(t,\tau)\big)^{\nu},$$

where ν sufficiently small.

We distinguish two cases with respect to the value of β. If $\beta \geq 1$, then we get

$$\begin{aligned}
\|u_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \|(u,v)\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1} \\
&\quad \times \int_0^{\frac{t}{2}} b_1(\tau)^{-1}\big(1+B_1(\tau,0)\big)^{-\frac{n}{2m}(\tilde{p}-1)+\gamma_1}\,d\tau \\
&\quad + \|(u,v)\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2m}(\tilde{p}-1)-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})\beta+\gamma_1} \\
&\quad \times \int_{\frac{t}{2}}^{t} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-1}\,d\tau.
\end{aligned}$$

We can conclude from $-\frac{n}{2m}(\tilde{p}-1)+\gamma_1<-1$, which is equivalent to $\tilde{p}>\frac{2m(\gamma_1+1)}{n}+1$, that $\int_0^{\frac{t}{2}} b_1(\tau)^{-1}(1+B_1(\tau,0))^{-\frac{n}{2m}(\tilde{p}-1)+\gamma_1}\,d\tau$ is bounded. Hence

$$\left\|u_t^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left\|(u,v)\right\|_{X(t)}^{p}b_1(t)^{-1}\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}.$$

If $0<\beta<1$, then we get

$$\begin{aligned}\left\|u_t^{nl}(t,\cdot)\right\|_{L^2}&\lesssim\left\|(u,v)\right\|_{X(t)}^{p}b_1(t)^{-1}\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\\&\quad\times\int_0^{\frac{t}{2}}b_1(\tau)^{-1}\left(1+B_1(\tau,0)\right)^{-\frac{n}{2m}(\tilde{p}-1)-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})(1-\beta)+\gamma_1}\,d\tau\\&\quad+\left\|(u,v)\right\|_{X(t)}^{p}b_1(t)^{-1}\left(1+B_1(t,0)\right)^{-\frac{n}{2m}\tilde{p}+\frac{n}{4}+\gamma_1}\\&\quad\times\int_{\frac{t}{2}}^{t}b_1(\tau)^{-1}\left(1+B_1(t,\tau)\right)^{-1}\,d\tau\\&\lesssim\left\|(u,v)\right\|_{X(t)}^{p}b_1(t)^{-1}\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\end{aligned}$$

for $\tilde{p}>\frac{2m(\gamma_1+1)}{n}+1$. Finally, we obtain

$$\left\|u_t^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left\|(u,v)\right\|_{X(t)}^{p}b_1(t)^{-1}\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}.\tag{27}$$

Analogously, we can prove that

$$\left\|\nabla u^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}}\left\|(u,v)\right\|_{X(t)}^{p},\tag{28}$$

$$\left\|u^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left(1+B_1(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\left\|(u,v)\right\|_{X(t)}^{p}.\tag{29}$$

For the second component v^{nl}, using the Gagliardo–Nirenberg inequality, from Proposition 4.1 we get for $\tau\in(0,t]$ the following estimates:

$$\begin{aligned}\left\|g(\tau,u)\right\|_{L^2}&\lesssim\left(1+B_2(\tau,0)\right)^{\gamma_2}\left(1+B_1(\tau,0)\right)^{-\frac{n}{2m}q+\frac{n}{4}}\left\|(u,v)\right\|_{X(t)}^{q},\\\left\|g(\tau,u)\right\|_{L^m}&\lesssim\left(1+B_2(\tau,0)\right)^{\gamma_2}\left(1+B_1(\tau,0)\right)^{-\frac{n}{2m}q+\frac{n}{2m}}\left\|(u,v)\right\|_{X(t)}^{q}.\end{aligned}$$

Taking into account the last estimates, we can prove, similarly to (27)–(29), the estimates

$$\left\|v_t^{nl}(t,\cdot)\right\|_{L^2}\lesssim b_2(t)^{-1}\left(1+B_2(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\left\|(u,v)\right\|_{X(t)}^{q},\tag{30}$$

$$\left\|\nabla v^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left(1+B_2(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}}\left\|(u,v)\right\|_{X(t)}^{q},\tag{31}$$

$$\left\|v^{nl}(t,\cdot)\right\|_{L^2}\lesssim\left(1+B_2(t,0)\right)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\left\|(u,v)\right\|_{X(t)}^{q},\tag{32}$$

for $\tilde{q}>\frac{2m(\gamma_2+1)}{n}+1$. Finally, (27)–(32) imply (23).

The proof of (22) is completely analogous to that of (21). In this way, we complete the proof of Theorem 2.1. □

Proof of Theorem 2.5 We choose the same space of energy solutions $X(t)$ with the norm $M_2(\tau,v)$ used in the proof of Theorem 2.5. We modify the norm $M_1(\tau,v)$ as follows:

$$\begin{aligned}M_1(\tau,u) &= \bigl(1+B_1(\tau,0)\bigr)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\kappa(\tilde{p})}\bigl\|u(\tau,\cdot)\bigr\|_{L^2(\mathbb{R}^n)}\\ &\quad+\bigl(1+B_1(\tau,0)\bigr)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{1}{2}-\kappa(\tilde{p})}\bigl\|\nabla u(\tau,\cdot)\bigr\|_{L^2(\mathbb{R}^n)}\\ &\quad+b_1(\tau)\bigl(1+B_1(\tau,0)\bigr)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+1-\kappa(\tilde{p})}\bigl\|u_t(\tau,\cdot)\bigr\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where $\kappa(\tilde{p})=\gamma_1-\frac{n}{2m}(\tilde{p}-1)+1$. We begin the proof of (23) by estimating the norm $\|u_t^{nl}(t,\cdot)\|_{L^2}$. Using (19) with $m=2$ for $\tau\in[\frac{t}{2},t]$ together with the Gagliardo–Nirenberg inequality and following the same steps of the proof of (27), we get

$$\begin{aligned}\bigl\|u_t^{nl}(t,\cdot)\bigr\|_{L^2} &\lesssim \bigl\|(u,v)\bigr\|_{X(t)}^p b_1(t)^{-1}\bigl(1+B_1(t,0)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\\ &\quad\times\int_0^{\frac{t}{2}} b_1(\tau)^{-1}\bigl(1+B_1(\tau,0)\bigr)^{(-\frac{n}{2m}p+\frac{n}{2m})\beta+\gamma_1}\,d\tau\\ &\quad+\bigl\|(u,v)\bigr\|_{X(t)}^p b_1(t)^{-1}\bigl(1+B_1(t,0)\bigr)^{(-\frac{n}{2m}p+\frac{n}{4})\beta+\gamma_1}\\ &\quad\times\int_{\frac{t}{2}}^{t} b_1(\tau)^{-1}\bigl(1+B_1(t,\tau)\bigr)^{-1}\,d\tau\\ &\lesssim \bigl\|(u,v)\bigr\|_{X(t)}^p b_1(t)^{-1}\bigl(1+B_1(t,0)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1+\kappa(\tilde{p})}\end{aligned}$$

for $\beta>0$. Then we have

$$\bigl\|u_t^{nl}(t,\cdot)\bigr\|_{L^2}\lesssim\bigl\|(u,v)\bigr\|_{X(t)}^p b_1(t)^{-1}\bigl(1+B_1(t,\tau)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1+\kappa(\tilde{p})}. \tag{33}$$

In the same way, we can prove

$$\bigl\|\nabla u^{nl}(t,\cdot)\bigr\|_{L^2}\lesssim\bigl(1+B_1(t,0)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}+\kappa(\tilde{p})}\bigl\|(u,v)\bigr\|_{X(t)}^p, \tag{34}$$

$$\bigl\|u^{nl}(t,\cdot)\bigr\|_{L^2}\lesssim\bigl(1+B_1(t,0)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\kappa(\tilde{p})}\bigl\|(u,v)\bigr\|_{X(t)}^p. \tag{35}$$

Now for v^{nl}, using the Gagliardo–Nirenberg inequality and the definition of the solution space $X(t)$, we can prove the following estimates:

$$\begin{aligned}\bigl\|g(\tau,u)\bigr\|_{L^2}&\lesssim\bigl(1+B_2(\tau,0)\bigr)^{\gamma_2}\bigl(1+B_1(\tau,0)\bigr)^{-\frac{n}{2m}q+\frac{n}{4}+\kappa(\tilde{p})q}\bigl\|(u,v)\bigr\|_{X(t)}^q,\\ \bigl\|g(\tau,u)\bigr\|_{L^m}&\lesssim\bigl(1+B_2(\tau,0)\bigr)^{\gamma_2}\bigl(1+B_1(\tau,0)\bigr)^{-\frac{n}{2m}q+\frac{n}{2m}+\kappa(\tilde{p})q}\bigl\|(u,v)\bigr\|_{X(t)}^q.\end{aligned}$$

Taking into account the last estimates together with (19), we obtain

$$\begin{aligned}\bigl\|v_t^{nl}(t,\cdot)\bigr\|_{L^2}&\lesssim\bigl\|(u,v)\bigr\|_{X(t)}^q b_2(t)^{-1}\bigl(1+B_2(t,0)\bigr)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\\ &\quad\times\int_0^{\frac{t}{2}} b_2(\tau)^{-1}\bigl(1+B_2(\tau,0)\bigr)^{(-\frac{n}{2m}q+\frac{n}{2m}+\kappa(\tilde{p})q)\alpha+\gamma_2}\,d\tau\\ &\quad+\bigl\|(u,v)\bigr\|_{X(t)}^q b_2(t)^{-1}\bigl(1+B_2(t,0)\bigr)^{(-\frac{n}{2m}p+\frac{n}{4}+\kappa(\tilde{p})q)\alpha+\gamma_2}\end{aligned}$$

$$\times \int_{\frac{t}{2}}^{t} b_2(\tau)^{-1}\big(1+B_2(t,\tau)\big)^{-1}\,d\tau$$
$$\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\big\|(u,v)\big\|_{X(t)}^{q},$$

where we use the condition

$$\gamma_2-\frac{n}{2m}(\tilde{q}-1)+\kappa(\tilde{p})q\alpha+\varepsilon<-1,$$

which is equivalent to condition (13). Then

$$\big\|v_t^{nl}(t,\cdot)\big\|_{L^2}\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\big\|(u,v)\big\|_{X(t)}^{q}. \tag{36}$$

Analogously, we can prove

$$\big\|\nabla v^{nl}(t,\cdot)\big\|_{L^2}\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}}\big\|(u,v)\big\|_{X(t)}^{q}, \tag{37}$$
$$\big\|v^{nl}(t,\cdot)\big\|_{L^2}\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\big\|(u,v)\big\|_{X(t)}^{q}. \tag{38}$$

Consequently, (33)–(38) imply (23).

To prove (22), we suppose the existence of (u,v) and $(\tilde{u},\tilde{v})$ belonging to the space of solution $X(t)$. Then we have

$$\begin{aligned}N(u,v)-N(\tilde{u},\tilde{v})&=\big(u^{nl}(t,x)-\tilde{u}^{nl}(t,x),v^{nl}(t,x)-\tilde{v}^{nl}(t,x)\big)\\&=\bigg(\int_0^t E_1(t,\tau,x)*_{(x)}\big(f(\tau,v)-f(\tau,\tilde{v})\big)\,d\tau,\\&\qquad\int_0^t E_1(t,\tau,x)*_{(x)}\big(g(\tau,u)-g(\tau,\tilde{u})\big)\,d\tau\bigg).\end{aligned}$$

Similarly to (25) and (26), using (5) and (6), we can prove the following estimates:

$$\begin{aligned}\big\|f(\tau,v)-f(\tau,\tilde{v})\big\|_{L^2}&\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}}\\&\quad\times\|v-\widetilde{v}\|_{X(t)}\big(\|v\|_{X(t)}^{p-1}+\|\widetilde{v}\|_{X(t)}^{p-1}\big),\end{aligned}\tag{39}$$
$$\begin{aligned}\big\|f(\tau,v)-f(\tau,\tilde{v})\big\|_{L^m}&\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{2m}}\\&\quad\times\|v-\widetilde{v}\|_{X(t)}\big(\|v\|_{X(t)}^{p-1}+\|\widetilde{v}\|_{X(t)}^{p-1}\big),\end{aligned}\tag{40}$$
$$\begin{aligned}\big\|g(\tau,u)-f(\tau,\tilde{u})\big\|_{L^2}&\lesssim\big(1+B_2(\tau,0)\big)^{\gamma_2}\big(1+B_1(\tau,0)\big)^{-\frac{n}{2m}q+\frac{n}{4}+\kappa(\tilde{p})q}\\&\quad\times\|u-\widetilde{u}\|_{X(t)}\big(\|u\|_{X(t)}^{q-1}+\|\widetilde{u}\|_{X(t)}^{q-1}\big),\end{aligned}\tag{41}$$
$$\begin{aligned}\big\|g(\tau,u)-f(\tau,\tilde{u})\big\|_{L^m}&\lesssim\big(1+B_2(\tau,0)\big)^{\gamma_2}\big(1+B_1(\tau,0)\big)^{-\frac{n}{2m}q+\frac{n}{2m}+\kappa(\tilde{p})q}\\&\quad\times\|u-\widetilde{u}\|_{X(t)}\big(\|u\|_{X(t)}^{q-1}+\|\widetilde{u}\|_{X(t)}^{q-1}\big).\end{aligned}\tag{42}$$

Analogously to (33)–(38), using (39)–(42), we can get

$$\bigg\|\nabla^j\partial_t^l\int_0^t E_1(t,\tau,x)*_{(x)}\big(f(\tau,v)-f(\tau,\widetilde{v})\big)\,d\tau\bigg\|_{L^2}$$

$$\lesssim b(t)^{-l}\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l}$$
$$\times \sup_{\tau\in[0,t]} M_2(\tau,v-\widetilde{v})\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\big), \tag{43}$$

$$\left\|\nabla^j\partial_t^l\int_0^t E_1(t,\tau,x)*_{(x)}\big(g(\tau,u)-g(\tau,\widetilde{u})\big)\,d\tau\right\|_{L^2}$$
$$\lesssim b(t)^{-l}\big(1+B_2(\tau,0)\big)^{\gamma_2}\big(1+B^1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{j}{2}-l+\kappa(\tilde{p})q}$$
$$\times \sup_{\tau\in[0,t]} M_1(\tau,u-\widetilde{u})\big(\|(u,v)\|_{X(t)}^{p-1}+\|(\tilde{u},\tilde{v})\|_{X(t)}^{p-1}\big), \tag{44}$$

where $j+l\leq 1$. The proof is completed. □

Proof of Theorem 2.7 Let us choose the space of energy solutions with suitable regularity

$$X(t)=\big(\mathcal{C}\big([0,t],H^{s_1}\big)\cap\mathcal{C}^1\big([0,t],H^{s_1-1}\big)\big)\times\big(\mathcal{C}\big([0,t],H^{s_2}\big)\cap\mathcal{C}^1\big([0,t],H^{s_2-1}\big)\big)$$

with the norm

$$\|(u,v)\|_{X(t)}=\sup_{\tau\in[0,t]}\big\{M_1(\tau,u)+M_2(\tau,v)\big\},$$

where

$$\begin{aligned}M_1(\tau,u)&=\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+b_1(\tau)\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+1}\|u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+b_1(\tau)\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{s_1-1}{2}+1}\||D|^{s_1-1}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+\big(1+B_1(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{s_1}{2}}\||D|^{s_1}u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}M_2(\tau,v)&=\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+b_2(\tau)\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+1}\|v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+b_2(\tau)\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{s_2-1}{2}+1}\||D|^{s_2-1}v_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\&\quad+\big(1+B_2(\tau,0)\big)^{\frac{n}{2}(\frac{1}{m}-\frac{1}{2})+\frac{s_2}{2}}\||D|^{s_2}v(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

To prove (23), we show how to estimate the norms $\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ and $\||D|^{s_2-1}\times v_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$. From estimate (20) it follows that

$$\||D|^{s_1-1}u_t^{nl}(t,\cdot)\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim\int_0^{\frac{t}{2}} b_1(\tau)^{-1}b_1(t)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1}$$
$$\times\|f(\tau,v)\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap\dot{H}^{s_1-1}(\mathbb{R}^n)}\,d\tau$$

$$+\int_{\frac{t}{2}}^{t} b_1(\tau)^{-1} b_1(t)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1} \times \big\|f(\tau,v)\big\|_{L^m(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\cap \dot{H}^{s_1-1}(\mathbb{R}^n)}\,d\tau.$$

Under the assumptions of Theorem 2.7 and the choice of the above introduced norm, for $0\le\tau\le t$, the inequalities (25) and (26) remain true. We calculate the norm

$$\big\|f(\tau,v)\big\|_{\dot{H}^{s_1-1}}.$$

Using (56) and (57), for $p>\lceil s_1-1\rceil$ and $0\le\tau\le t$, we get the following estimate:

$$\begin{aligned}\big\|f(\tau,v)\big\|_{\dot{H}^{s_1-1}} &\lesssim \big(1+B_1(\tau,0)\big)^{\gamma_1}\big\|v(\tau,\cdot)\big\|_{L^{q_1}}^{p-1}\big\||D|^{s_1-1}(\tau,\cdot)\big\|_{L^{q_2}}\\ &\lesssim \big(1+B_1(\tau,0)\big)^{\gamma_1}\big\|v(\tau,\cdot)\big\|_{L^2}^{(p-1)(1-\theta_1)}\\ &\quad\times \big\||D|^{s_2}v(\tau,\cdot)\big\|_{L^2}^{(p-1)\theta_1}\big\|v(\tau,\cdot)\big\|_{L^2}^{1-\theta_2}\big\||D|^{s_2}v(\tau,\cdot)\big\|_{L^2}^{\theta_2}\\ &\lesssim \big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_1-1}{2}}\big\|(u,v)\big\|_{X(t)}^p,\end{aligned}$$

where

$$\frac{p-1}{q_1}+\frac{1}{q_2}=\frac{1}{2},\qquad \theta_1=\frac{n}{s}\left(\frac{1}{2}-\frac{1}{q_1}\right)\in[0,1],$$
$$\theta_2=\frac{n}{s_2}\left(\frac{1}{2}-\frac{1}{q_2}\right)+\frac{s_1-1}{s_2}\in\left[\frac{s_1-1}{s_2},1\right].$$

To satisfy the last conditions for the parameters θ_1 and θ_2, we choose $q_2=\frac{2n}{n-2}$ and $q_1=n(p-1)$. This choice implies the condition

$$1+\frac{2}{n}\le p\le 1+\frac{2}{n-2s_2}.$$

Consequently, for $\tau\in(0,t]$, we obtain the estimate

$$\big\|f(\tau,v)|_{\dot{H}^{s_1-1}}\lesssim \big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_2-1}{2}}\big\|(u,v)\|_{X(t)}^p. \tag{45}$$

Summarizing all estimates implies

$$\begin{aligned}\big\||D|^{s_1-1}u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} &\lesssim \big\|(u,v)\big\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1}\\ &\quad\times\int_0^{\frac{t}{2}} b_1(\tau)^{-1}\big(1+B_1(\tau,0)\big)^{(-\frac{n}{2m}p+\frac{n}{2m})\beta+\gamma_1}\,d\tau\\ &\quad+\big\|(u,v)\big\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{(-\frac{n}{2m}p+\frac{n}{4})\beta+\gamma_1}\\ &\quad\times\int_{\frac{t}{2}}^{t} b_1(\tau)^{-1}\big(1+B_1(t,\tau)\big)^{-\frac{s_1-1}{2}-1}\,d\tau\\ &\lesssim \big\|(u,v)\big\|_{X(t)}^p b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1},\end{aligned}$$

where $\tilde{p}>\frac{2m}{n}(\frac{s_1+1+2\gamma_1}{2})+1$.

Then

$$\big\||D|^{s_1-1}u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1-1}{2}-1}\big\|(u,v)\big\|^p_{X(t)}. \tag{46}$$

Under the first condition of (14), in the same way, we can prove the following estimates:

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\big\|(u,v)\big\|^p_{X(t)}, \tag{47}$$

$$\big\|u_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b_1(t)^{-1}\big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\big\|(u,v)\big\|^p_{X(t)}, \tag{48}$$

$$\big\||D|^{s_1}u^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim \big(1+B_1(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_1}{2}}\big\|(u,v)\big\|^p_{X(t)}. \tag{49}$$

Using the second condition of (14), we get

$$\big\||D|^{s_2-1}v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_2-1}{2}-1}\big\|(u,v)\big\|^p_{X(t)}, \tag{50}$$

$$\big\|v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\big\|(u,v)\big\|^p_{X(t)}, \tag{51}$$

$$\big\|v_t^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim b_2(t)^{-1}\big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\big\|(u,v)\big\|^p_{X(t)}, \tag{52}$$

$$\big\||D|^{s_2}v^{nl}(t,\cdot)\big\|_{L^2(\mathbb{R}^n)}\lesssim \big(1+B_2(t,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{s_2}{2}}\big\|(u,v)\big\|^p_{X(t)}. \tag{53}$$

From (46)–(53) we get (23), which completes the proof of (21).

To prove (22), we use the same steps used in the previous proof. Indeed, from the fractional Leibniz rule (see Proposition 4.2) and the fractional chain rule (see Proposition 4.3) we may conclude for $0\le\tau\le t$ the following estimates:

$$\begin{aligned}\big\|f(\tau,v)-f(\tau,\tilde{v})\big\|_{\dot{H}^{s_1-1}}&\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2m}p+\frac{n}{4}-\frac{s_2-1}{2}}\\&\quad\times\sup_{\tau\in[0,t]}M_2(\tau,v-\widetilde{v})\big(\big\|(u,v)\big\|^{p-1}_{X(t)}+\big\|(\tilde{u},\tilde{v})\big\|^{p-1}_{X(t)}\big),\end{aligned} \tag{54}$$

and

$$\begin{aligned}\big\|g(\tau,u)-g(\tau,\tilde{u})\big\|_{\dot{H}^{s_2-1}}&\lesssim\big(1+B_2(\tau,0)\big)^{\gamma_2}\big(1+B_1(\tau,0)\big)^{-\frac{n}{2m}q+\frac{n}{4}-\frac{s_1-1}{2}}\\&\quad\times\sup_{\tau\in[0,t]}M_1(\tau,u-\widetilde{u})\big(\big\|(u,v)\big\|^{q-1}_{X(t)}+\big\|(\tilde{u},\tilde{v})\big\|^{q-1}_{X(t)}\big),\end{aligned} \tag{55}$$

where we use condition (15). From (39)–(42) without loss of decay and (54)–(55) we can complete the proof. □

Remark 3.4 Theorem 2.8 can be proved by using a similar approach as in the proof of Theorem 2.7, with modifications in the estimates of some terms. Then using Proposition 4.4, Corollary 4.5, and Lemma 4.6, we can obtain the estimates

$$\big\|f(\tau,v)\big\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}\lesssim\big(1+B_1(\tau,0)\big)^{\gamma_1}\big(1+B_2(\tau,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})p-\frac{s_1-1}{2}-\frac{s^*}{2}(p-1)}\big\|(u,v)\big\|^p_{X(t)},$$

$$\big\|g(\tau,u)\big\|_{\dot{H}^{\tilde{s}_2-1}(\mathbb{R}^n)}\lesssim\big(1+B_2(\tau,0)\big)^{\gamma_2}\big(1+B_1(\tau,0)\big)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})p-\frac{\tilde{s}_2-1}{2}-\frac{s^*}{2}(q-1)}\big\|(u,v)\big\|^q_{X(t)},$$

$$
\begin{aligned}
&\left\|f(\tau,v)-f(\tau,\tilde{v})\right\|_{\dot{H}^{s_1-1}(\mathbb{R}^n)}\\
&\quad\lesssim \left(1+B_2(\tau,0)\right)^{-\frac{n}{2m}p+\frac{n}{4}(p-1)-\frac{s_1-1}{2}-\frac{s^*}{2}(p-1)}\\
&\qquad\times \left(1+B_1(\tau,0)\right)^{\gamma_1}\left\|(u,v)-(\tilde{u},\tilde{v})\right\|_{X(t)}\left(\left\|(u,v)\right\|_{X(t)}^{p-1}+\left\|(\tilde{u},\tilde{v})\right\|_{X(t)}^{p-1}\right),\\
&\left\|g(\tau,u)-g(\tau,\tilde{u})\right\|_{\dot{H}^{\tilde{s}_2-1}(\mathbb{R}^n)}\\
&\quad\lesssim \left(1+B_1(\tau,0)\right)^{-\frac{n}{2m}p+\frac{n}{4}(q-1)-\frac{\tilde{s}_2-1}{2}-\frac{s^*}{2}(q-1)}\\
&\qquad\times \left(1+B_2(\tau,0)\right)^{\gamma_2}\left\|(u,v)-(\tilde{u},\tilde{v})\right\|_{X(t)}\left(\left\|(u,v)\right\|_{X(t)}^{q-1}+\left\|(\tilde{u},\tilde{v})\right\|_{X(t)}^{q-1}\right).
\end{aligned}
$$

Using these estimates, provided that condition (16) is satisfied, we can follow steps in the proof of Theorem 2.7 to complete our proof.

Appendix

Here we state some inequalities, which come into play in our proofs.

Proposition 4.1 *Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$, and $s \in [0,\sigma)$. Then the following fractional Gagliardo–Nirenberg inequality holds for all $u \in L^{p_0} \cap \dot{H}^{\sigma}_{p_1}$:*

$$\|u\|_{\dot{H}^s_p} \lesssim \|u\|_{L^{p_0}}^{1-\theta}\|u\|_{\dot{H}^{\sigma}_{p_1}}^{\theta}, \tag{56}$$

where

$$\theta = \theta_{s,\sigma} := \frac{\frac{1}{p_0}-\frac{1}{p}+\frac{s}{n}}{\frac{1}{p_0}-\frac{1}{p_1}+\frac{\sigma}{n}} \quad \text{and} \quad \frac{s}{\sigma} \le \theta \le 1.$$

For the proof, see [11] and [2, 8–10, 14, 15].

Proposition 4.2 *Let $s > 0$, $1 \le r \le \infty$, and $1 < p_1, p_2, q_1, q_2 \le \infty$ satisfy the relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then we have the following fractional Leibniz rule:

$$\left\||D|^s(fg)\right\|_{L^r} \lesssim \left\||D|^s f\right\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\left\||D|^s g\right\|_{L^{q_2}}$$

for all $f \in \dot{H}^s_{p_1} \cap L^{q_1}$ and $g \in \dot{H}^s_{q_2} \cap L^{p_2}$.

For more details concerning fractional Leibniz rule, see [8].

Proposition 4.3 *Let us choose $s > 0, p > \lceil s\rceil$, and $1 < r, r_1, r_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Let $F(u)$ be one of the functions $|u|^p, \pm|u|^{p-1}u$. Then we have the following fractional chain rule:

$$\big\| |D|^s F(u) \big\|_{L^r} \lesssim \|u\|_{L^{r_1}}^{p-1} \big\| |D|^s u \big\|_{L^{r_2}}. \tag{57}$$

For the proof, see [24].

Proposition 4.4 *Let $p > 1$ and $u \in H_m^s$, where $s \in (\frac{n}{m}, p)$. Then we have the following estimates*:

$$\big\| |u|^p \big\|_{H_m^s} \lesssim \|u\|_{H_m^s} \|u\|_{L^\infty}^{p-1},$$
$$\big\| u|u|^{p-1} \big\|_{H_m^s} \lesssim \|u\|_{H_m^s} \|u\|_{L^\infty}^{p-1}.$$

For the proof, see [26].

From Proposition 4.4 we can derive the following corollary.

Corollary 4.5 *Under the assumptions of Proposition 4.4, we have*

$$\big\| |u|^p \big\|_{\dot{H}_m^s} \lesssim \|u\|_{\dot{H}_m^s} \|u\|_{L^\infty}^{p-1},$$
$$\big\| u|u|^{p-1} \big\|_{\dot{H}_m^s} \lesssim \|u\|_{\dot{H}_m^s} \|u\|_{L^\infty}^{p-1}.$$

For the proof, see [6] and [25].

Lemma 4.6 *Let $0 < 2s^* < n < 2s$. Then for any function $f \in \dot{H}^{s^*} \cap \dot{H}^s$, we have the estimate*

$$\|f\|_{L^\infty} \le \|f\|_{\dot{H}^{s^*}} + \|f\|_{\dot{H}^s}.$$

For the proof, see [4].

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Author's contributions

The author provided the problem and gave the proof of the main results. Author read and approved the final manuscript.

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References

1. Abbas, S., Benchohra, M., Zhou, Y.: Weak solutions for a coupled system of Pettis–Hadamard fractional differential equations. Adv. Differ. Equ. **2017**, 332 (2017)
2. Christ, F., Weinstein, M.: Dispersion of small-amplitude solutions of the generalized Korteweg–de Vries equation. J. Funct. Anal. **100**, 87–109 (1991)
3. D'Abbicco, M.: Small data solutions for semilinear wave equations with effective damping. Discrete Contin. Dyn. Syst. **2013**, 183–191 (2013)
4. D'Abbicco, M.: The threshold of effective damping for semilinear wave equations. Math. Methods Appl. Sci. **38**, 1032–1045 (2015)
5. D'Abbicco, M., Lucente, S., Reissig, M.: Semi-linear wave equations with effective damping. Chin. Ann. Math. **34B**(3), 345–380 (2013)
6. Ebert, M.R., Reissig, M.: Methods for Partial Differential Equations. Birkhäuser, Basel (2018)
7. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t + \Delta u + u^{1+\alpha}$. J. Fac. Sci., Univ. Tokyo **13**, 109–124 (1966)
8. Grafakos, L.: Classical and Modern Fourier Analysis. Prentice Hall, New York (2004)
9. Grafakos, L., Oh, S.: The Kato–Ponce inequality. Commun. Partial Differ. Equ. **39**(6), 1128–1157 (2014)
10. Gulisashvili, A., Kon, M.: Exact smoothing properties of Schrödinger semigroups. Am. J. Math. **118**, 1215–1248 (1996)
11. Hajaiej, H., Molinet, L., Ozawa, T., Wang, B.: Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations. Harmon. Anal. Nonlinear Partial Differ. Equ. **B26**, 159–175 (2011)
12. Ikehata, R., Miyaoka, Y., Nakatake, T.: Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^n$ with lower power nonlinearities. J. Math. Soc. Jpn. **56**, 365–373 (2004)
13. Ikehata, R., Tanizawa, K.: Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^n$ with noncompactly supported initial data. Nonlinear Anal. **61**, 1189–1208 (2005)
14. Kato, T., Ponce, G.: Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle. Commun. Pure Appl. Math. **46**(4), 527–620 (1993)
15. Kenig, C.E., Ponce, G., Vega, L.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. **41**, 891–907 (1988)
16. Liu, Y., Ahmad, B., Agarwal, R.P.: Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line. Adv. Differ. Equ. **2013**, 46 (2013)
17. Matsumura, A.: On the asymptotic behavior of solutions of semi-linear wave equations. Publ. RIMS **12**, 169–189 (1976)
18. Mohammed Djaouti, A.: On the benefit of different additional regularity for the weakly coupled systems of semilinear effectively damped waves. Mediterr. J. Math. **15**, 115 (2018). https://doi.org/10.1007/s00009-018-1173-1
19. Mohammed Djaouti, A.: Semilinear systems of weakly coupled damped waves. PhD Thesis, TU Bergakademie Freiberg (2018)
20. Mohammed Djaouti, A., Reissig, M.: Weakly coupled systems of semilinear effectively damped waves with time-dependent coefficient, different power nonlinearities and different regularity of the data. Nonlinear Anal. **175**, 28–55 (2018)
21. Mohammed Djaouti, A., Reissig, M.: Weakly coupled systems of semilinear effectively damped waves with different time-dependent coefficients in the dissipation terms and different power nonlinearities. In: D'Abbicco, M., Ebert, M., Georgiev, V., Ozawa, T. (eds.) New Tools for Nonlinear PDEs and Application. Trends in Mathematics. Birkhäuser, Cham (2019)
22. Nakao, M., Ono, K.: Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations. Math. Z. **214**, 325–342 (1993)
23. Nishihara, K., Wakasugi, Y.: Critical exponent for the Cauchy problem to the weakly coupled wave system. Nonlinear Anal. **108**, 249–259 (2014)
24. Palmieri, A., Reissig, M.: Semi-linear wave models with power non-linearity and scale invariant time-dependent mass and dissipation II. Math. Nachr. **291**, 1859–1892 (2018)
25. Pham, D.T., Kainane Mezadek, M., Reissig, M.: Global existence for semilinear structurally damped σ-evolution models. J. Math. Anal. Appl. **431**, 569–596 (2015)
26. Runst, T., Sickel, W.: Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations. De Gruyter Series in Nonlinear Analysis and Applications. de Gruyter, Berlin (1996)
27. Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Differ. Equ. **174**, 464–489 (2001)
28. Wirth, J.: Asymptotic properties of solutions to wave equations with time-dependent dissipation. PhD Thesis, TU Bergakademie Freiberg (2004)
29. Wirth, J.: Wave equations with time-dependent dissipation II, effective dissipation. J. Differ. Equ. **232**, 74–103 (2007)
30. Zhang, Q.S.: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci. Paris, Ser. I **333**, 109–114 (2001)