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Some new exact solutions of (3 + 1)-dimensional Burgers system via Lie symmetry analysis

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Abstract

In this paper, by using the Lie symmetry analysis, all of the geometric vector fields of the (3 + 1)-Burgers system are obtained. We find the 1, 2, and 3-dimensional optimal system of the Burger system and then by applying the 3-dimensional optimal system reduce the order of the system. Also the nonclassical symmetries of the (3 + 1)-Burgers system will be found by employing nonclassical methods. Finally, the ansatz solutions of *BS* equations with the aid of the tanh method has been presented. The achieved solutions are investigated through two- and three-dimensional plots for different values of parameters. The analytical simulations are presented to ensure the efficiency of the considered technique. The behavior of the obtained results for multiple cases of symmetries is captured in the present framework. The outcomes of the present investigation show that the considered scheme is efficient and powerful to solve nonlinear differential equations that arise in the sciences and technology.

Keywords: Analysis burgers equation; Symmetry group; Optimal system; Nonclassical symmetries; Tanh method

1 Introduction

The Burgers system describes the propagation processes for nonlinear waves in fluid mechanics such as diverse non-equilibrium, nonlinear phenomena in turbulence, and interface dynamics [12]. Also, this system is used in solitary wave theory to expand integrable models with the extending of famous physical equations. The physicists and mathematicians are looking to study the (1 + 1)-dimensional and the (2 + 1)-dimensional integrable models. Because of the real physical space-time being (3 + 1)-dimensional, researchers have been attempting to detect higher-dimensional models in some ways [29, 31, 32, 51]. Researchers have used many effective techniques to discover the different solution of the Burgers system [10, 48]. One of the Sophus Lie's significant discoveries in differential equation is to indicate that transforming nonlinear conditions is possible by through infinitesimal invariants which can correspond to the generators of the symmetry group of the system [28]. Having the symmetry group of a system of differential equations has many advantages including the ability to classify the solutions of the differential equations system. This classification is such that we consider both answers in a category that can be con-

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verted by some transformational equation. Researchers interested in further reading may refer to recently published articles, including [4, 16, 24, 49]. Another use of these groups is that the differential equations can be classified according to the desired parameter or function. Ovsiannikov [43] gave the strategy for halfway invariant solutions. His technique depends on the idea of an equivalence group, which is a Lie symmetry group working in the extended space of free variables, functions and their subordinates, and saving the class of incomplete differential conditions. The investigation of the correct arrangements has a noteworthy influence in the perusing of nonlinear physical frameworks [37]. Probably the main strategies are the inverse scattering method [18], the Hirota bilinear method [19, 27], Lie symmetry analysis [11, 40], Darboux and Bäcklund transformations [27], the tanh-function method (Duffy and Parkes [15]). In [13], Chao-Qieg and Yue have obtained a new analytical solution of the (3 + 1)-dimensional Burgers system relied on the Riccati equation. And In [12] another mapping equation is utilized to find variable separation solutions of the (3 + 1)-dimensional Burgers system and three class of variable separation solutions are driven. Also in [22, 23], Ibragimovs new technique for finding conservation laws and the idea of nonlinear self-adjointness were explained that widely implemented to find conservation laws of equations (for example see [1, 5]). Especially In [3], the nonlinear self-adjointness and conservation laws of the (3 + 1)-dimensional Burgers equation has been obtained by the aid of Ibragimovs method,

$$BS : u_t = 2uu_y + 2vu_x + 2wu_z + u_{xx} + u_{yy} + u_{zz}, \qquad u_x = v_y, u_z = w_y.$$

This equation describes the propagation processes for nonlinear waves in fluid mechanics such as diverse non-equilibrium, nonlinear phenomena in turbulence and inter-face dynamics. Lie symmetries of BS will present some additional results which are also obtained from the Lie algebra structure of Lie symmetry group. We, therefore, plan to make an optimal system of 1-subalgebras of the BS which is useful for classifying of group invariant solutions and to construct an optimal system of 2-subalgebras for the BS, also for 3-subalgebra of the BS which are effectively helping to classify the group invariant solutions. After that, the reduced equation for each element of the optimal system is obtained. Then, using the method of Lie symmetry group a solution will be presented for reduced equations. Afterward, the group invariant solutions of BS is achieved. In [30], the Lie point symmetries of (3 + 1)-dimensional Burgers system is obtained and the system is reduced by them. Finally, we found the nonclassical symmetries that were first discussed in Bluman and Cole [8] in their treatment of generalized self-similar solutions of the linear heat equation [42]. This method and its correlation to direct reduction methods of Clarkson [14] and Galaktionov [17] have become the focus of much research and many applications to physically significant partial differential equations. Obviously, other related points, such as partially invariant solutions differentially partially invariant solutions of group foliation, will give rise to effective and delicate methods of treating differential equations. In [33, 50], they have been able to find a powerful solution method, By using the tanh method for the computation of traveling wave solutions. First an ansatz of power series in tanh was used to obtain solutions of traveling wave of certain nonlinear evolution equations [34]. Recently, Fan and Hon [16] introduced a generalized tanh method to investigate special types of nonlinear equations.

Hereafter, the symmetry reductions, explicit solutions, convergence analysis and conservation laws to the Chen–Lee–Liu model in nonlinear optics were been studied by the authors in [4]. Inc and co-authors surveyed the time fractional generalized Burgers–Huxley equation with Riemann–Liouville derivative via Lie symmetry analysis and the power series expansion method [24]. Tchier and et al. presented an investigation and analysis for the space-time Carleman equation (STCE) in nonlinear dynamical system [49]. Also, some researchers utilized the Lie symmetry analysis for various nonlinear PD equations. See references [2–5]. Some authors studied the stochastic influenza model with constant vaccination strategy [6], the stochastic meme epidemic model with investigate effect of threshold number [47], a reliable numerical analysis of a stochastic HIV/AIDS model in a two-sex population considering counseling and anti-retroviral therapy [46]. Some research as regards nonlinear study of equations has been made covering nonlinear vibrations of Euler–Bernoulli beams [44], parametrically excited nonlinear oscillators [45], and nonlinear free vibration analysis of tapered beams [7]. This article is organized as follows. In Sect. 2, the Lie symmetries of BS equations have been obtained. By the aid of obtained symmetries, the invariant groups of BS equations are given in Sect. 3. In Sects. 4, 5, and 6 the optimal systems of order one, two, and three of BS equations are presented, respectively. By considering the new coordinates the BS equations are reduced in Sect. 7. In Sect. 8, the nonclassical symmetries of the BS equation are driven. In Sect. 9, by implementing of tanh-function method ansatz solution of BS equations has been found and plotted. Also, in Sect. 10 the results and discussion of graphs and their behavior are given.

2 Lie symmetry of BS

The method which is conventionally applied to determine the classical symmetries of a partial differential equation is standard and is explained in [9, 40, 41] we consider an infinitesimal generator of Lie symmetry from the following form in order to get the Lie symmetry:

$$X = \xi_t \partial_t + \xi_x \partial_x + \xi_y \partial_y + \xi_z \partial_z + \eta_1 \partial_u + \eta_2 \partial_v + \eta_3 \partial_w,$$

where $\xi^t, \xi^x, \xi^y, \xi^z$ are functions of t, x, y, z, and η^u, η^v, η^w are functions of t, x, y, z, u, v, w. By using the invariant condition, such as applying $\mathrm{pr}^{(2)}X$ the second prolongation of X to BS, we have the following system of equations:

$$\eta_{zt}^3 = \eta_z^3 = \eta_z^2 = \eta_{wz}^3 = \eta_{wz}^2 = \eta_w^3 = \eta_w^2 = \eta_x^1 = \eta_y^2 = \eta_y^3 = 0,$$

$$\eta_w^1 = \xi_x^1 = \xi_x^3 = \xi_y^1 = \xi_y^2 = \xi_y^4 = \xi_z^1 = \xi_z^3 = \xi_u^1 = \xi_u^2 = \eta_v^1 = \eta_u^3 = 0,$$

$$\xi_u^3 = \xi_u^4 = \xi_v^1 = \xi_v^2 = \xi_v^3 = \xi_v^4 = \xi_w^1 = \xi_w^2 = \xi_w^3 = \xi_w^4 = \eta_z^1 = \eta_u^2 = 0,$$

$$\eta_{wt}^3 = 2\eta_z^3, \qquad \eta_{wt}^2 = -2\eta_z^2, \qquad \xi_t^1 = -2\eta_w^3, \qquad \xi_t^2 - 2\eta_w^2 w - 2\eta_w^3 v + 2\eta^2 = 0,$$

$$\xi_t^3 - 2\eta_w^3 u = 2\eta^1, \qquad \eta_t^1 = 2\eta_z^3 u = 0, \qquad \xi_t^4 + 2\eta_w^3 v = 2\eta_w^3 w - 2\eta^3,$$

$$\xi_x^2 = -\eta_w^3, \qquad \xi_y^3 = -\eta_w^3, \qquad \xi_z^4 = -\eta_w^3, \qquad \xi_z^2 = \eta_w^2, \qquad \eta_x^2 = \eta_z^2, \qquad \eta_x^3 = -\eta_z^2,$$

$$\eta_u^1 = \eta_w^3, \qquad \eta_v^2 = \eta_w^3, \qquad \eta_v^3 = -\eta_w^2, \qquad \eta_y^1 = -\eta_z^3, \qquad \eta_x^3 = -\eta_z^2.$$

Therefore, the Lie symmetry group of BS includes a Lie algebra generator in the form of the vector field υ having these functional coefficients:

$$\xi_1 = \frac{c_1}{2}t^2 + c_2 t + c_3, \qquad \eta_1 = -\frac{c_1 t + c_2}{2}u - \frac{c_1}{4}y - \frac{c_4}{2},$$

$$\xi_2 = -F_1 z + F_3 + \frac{c_1 t + c_2}{2}x, \qquad \eta_2 = -\frac{1}{2}\left(F_1' + F_3'\right) - F_1' w - \frac{c_1 t + c_2}{2}v - \frac{c_1}{4}x,$$

$$\xi_3 = \frac{c_1 y + 2c_4}{2}t + \frac{c_2}{2}y + c_5, \qquad \eta_3 = -\frac{1}{2}\left(F_1' + F_2'\right) + F_1' v + \frac{c_1 t + c_2}{2}w - \frac{c_1}{4}z,$$

$$\xi_4 = F_1 + F_2 + \frac{c_1 t + c_2}{2}z,$$

where c_i, $i = 1, \dots, 5$ are arbitrary constants and $F_i(t)$, $i = 1, 2, 3$ are arbitrary smooth functions. In the case the above system is solved, the following theorem is introduced.

Theorem 2.1 *The Lie algebra $L(G)$ of projectable Lie symmetries of BS is spanned by*

$$X_1 = \frac{t}{2}(t\partial_t + x\partial_x + y\partial_y + z\partial_z) - \frac{1}{4}(2tu + y)\partial_u - \frac{1}{4}(2tv + x)\partial_v - \frac{1}{4}(2tw + z)\partial_w,$$

$$X_2 = \partial_t, \qquad X_3 = t\partial_t + \frac{1}{2}(x\partial_x + y\partial_y + z\partial_z - u\partial_u - v\partial_v - w\partial_w),$$

$$X_4 = t\partial_y - \frac{1}{2}\partial_u, \qquad X_5 = \partial_y, \qquad A_f = f\partial_x - \frac{1}{2}f'\partial_v, \qquad B_g = g\partial_z - \frac{1}{2}g'\partial_v,$$

$$C_h = h(-z\partial_x + x\partial_z) + \frac{1}{2}\left(zh' - wh\right)\partial_w - \frac{1}{2}\left(xh' - vh\right)\partial_w,$$

where $f(t), g(t), h(t)$ are arbitrary smooth functions.

Having functional coefficients, these vector fields produce a Lie pseudo-group $L(G)$. This Lie pseudo-algebra $L(G)$ has a 5-subalgebra $\mathbf{h}$ generated by $\mathbf{v}_1, \dots, \mathbf{v}_5$, and an infinite dimensional ideal $\mathbf{i}$ generated by A_f, B_g, C_h. Therefore $L(G) \simeq \mathbf{h} \times \mathbf{i}$. To having a reduction in BS, a Lie subgroup of the above pseudo-group is chosen. For executing this chose, simpler forms for every one of the coefficients in the previously mentioned vector fields are chosen.

3 Group invariant solutions of BS

In order to have the group of transformations which are generated by vector fields $\mathbf{v}_i$, $i = 1, \dots, 5$, we need to at first solve first order system including the first order equation which is in agreement whit each of the same generators. If $\Phi_k(s)$ is the parametric group represented by $\mathbf{v}_k$, $k = 1, \dots, 5$, then $P_k = \Phi_k(s)(t, x, y, z, u, v, w)$ are, respectively,

$$P_1 = \left[at, ax, ay, az, \frac{u}{a} + \frac{sy}{4}, \frac{v}{a} + \frac{sx}{4}, \frac{w}{a} + \frac{sz}{4}\right], \qquad a = \frac{2}{st + 2};$$

$$P_2 = [t + s, x, y, z, u, v, w]; \qquad P_3 = \left[e^{s/2}t, x, y, z, e^{-s}u, e^{-s}v, e^{-s}w\right];$$

$$P_4 = \left[t, x, y + st, z, u - \frac{s}{2}, v, w\right]; \qquad P_5 = [t, x, y + s, z, u, v, w];$$

and if Ψ_h, Ξ_g, Θ_f are the 1-parameter group generated by A_h, B_g, C_f, respectively, then

$$\Psi_h = \left[t, x + sh, y, z, u, v - \frac{s}{2}h', w\right]; \qquad \Xi_g = \left[t, x, y, z + sg, u, v, w - \frac{s}{2}g'\right];$$

$$\Theta_g = \Big[t, x - szf, y + sxf - \frac{s^2}{2}zf^2, z, u,$$

$$\cos(sf)v - \sin(sf)w + \frac{g}{2f}\big(z(z+1)\sin(sf) - x\cos(sf) + x - szf\big),$$

$$\sin(sf)v + \cos(sf)w + \frac{g}{2f}\big(-x\sin(sf) + z(1-z)\cos(sf) + z^2 + z\big)\Big].$$

It should be mentioned that, generally, for each, a parameter of the subgroup of complete symmetry group's system offers a set of invariant solutions [35, 40, 41].

Theorem 3.1 *If s be a small real number, $(u(t,x,y,z), v(t,x,y,z), w(t,x,y,z))$ be a solution of the BS and $i = 1, \ldots, 5$, there are the functions $\varphi_i := (U, V, W) = \Phi_k(s)(u, v, w)$:*

$$\varphi_1 = a.(u, v, w)(at, ax, ay, az) - \frac{as}{2}.(x, y, z), \qquad a = \frac{2}{st + 2},$$

$$\varphi_2 = (u, v, w)(t - s, x, y, z), \qquad \varphi_3 = e^{-s/2}.(u, v, w)\big(te^{3s/2}, x, y, z\big),$$

$$\varphi_4 = (u, v, w)(t, x, y - st, z) + \frac{s}{2}.(1, 0, 0), \qquad \varphi_5 = (u, v, w)(t, x, y - s, z).$$

Furthermore, if Ψ_h, Ξ_g, Θ_f are the 1-parameter group generated by A_f, B_g, c_h, respectively, and $u = u(t,x,y,z)$, $v = v(t,x,y,z)$ and $w = w(t,x,y,z)$ are a solution of the BS, so are the functions $\phi_h := (U, V, W) = \Psi_h(u, v, w)$, $\xi_g := (U, V, W) = \Xi_g(u, v, w)$ and $\theta_f := (U, V, W) = \theta_f(u, v, w)$:

$$\psi_h = (u, v, w)\big(t, x + sh(t), y, z\big) + \frac{sh'(t)}{2}.(0, 1, 0),$$

$$\xi_g = (u, v, w)\big(t, x, y, z + sg(t)\big) + \frac{sg'(t)}{2}.(0, 0, 1),$$

$$\theta_f = \big(u, \cos\big(sf(t)\big)v + \sin\big(sf(t)\big)w, -\sin\big(sf(t)\big)v + \cos\big(sf(t)\big)w\big)$$

$$\left(t, x - szf(t), y + sf(t)x - \frac{s^2}{2}zf(t)^2, z\right)$$

$$+ \frac{g(t)}{2f(t)}.\big(0, -z(z+1)\sin\big(sf(t)\big) + x\cos\big(sf(t)\big)$$

$$+ x + sf(t)z, x\sin\big(sf(t)\big) + z(1-z)\cos\big(sf(t)\big) + z^2 + z\big).$$

This theorem has several useful consequences. For example, by using the command *PDETools[TWSolutions]* of MAPLE 2016© for the system BS, we find a seven parametrized set of tanh-solutions for BS as

$$u = a_3 \tanh(a_2 x + a_3 y + a_4 z + 2a_5 t + a_1) + a_6,$$

$$v = a_2 \tanh(a_2 x + a_3 y + a_4 z + 2a_5 t + a_1) + a_7, \qquad (1)$$

$$w = a_4 \tanh(a_2 x + a_3 y + a_4 z + 2a_5 t + a_1) - \frac{a_2 a_7 + a_3 a_6 - a_5}{a_4},$$

where $c_1, \ldots, c_7 \in IR$ are arbitrary numbers with $c_4 \neq 0$. Now, we find a large set of solutions of BS by using Theorem 3.1. For example, by the ψ_h of Theorem 3.1 for (1), we find

$$u = a_3 \tanh(a_2 h(t) + 2a_5 t + a_2 x + a_3 y + a_4 z + a_1) + a_6,$$

$$v = a_2 \tanh(a_2 h(t) + 2a_5 t + a_2 x + a_3 y + a_4 z + a_1) + a_7 - \frac{h'(t)}{2},$$

$$w = a_4 \tanh(a_2 h(t) + 2a_5 t + a_2 x + a_3 y + a_4 z + a_1) - \frac{a_2 a_7 + a_3 a_6 - a_5}{a_4},$$

where $c_1, \ldots, c_7 \in \mathbb{R}$ are arbitrary numbers with $c_4 \neq 0$, and h is an arbitrary smooth function of t.

4 Optimal system of 1-subalgebras

In this part, we take the advantage of symmetry group in order to obtain the OS (optimal system) of 1-subalgebras of BS. Regarding the fact that every linear combination of symmetries is a symmetry, there will be an endless number of 1-subgroups for G. Therefore, determining the subgroups which give different types of solutions is emphasized. So, we need to look symmetry transformations for invariant solutions which are unable to be to convert to each other in the full symmetry groups. This, in turn, leads to the notional of an OS of subalgebra. The problem of classifying this 1-subalgebra is identical to the problem which exists in the classification of the orbits of the adjoint representation [36, 40]. The optimal set of subalgebras is achieved by selecting just one representative from any class of equivalent subalgebras.

By using a general member in the Lie algebra and simplifying it via diverse adjoint transformations, it is possible to solve the problem of orbits classifications. The following Lie series are considered to include the adjoint representation [38]:

$$\mathrm{Ad}\big(\exp(s.X_i).X_j\big) = X_j - s.[X_i, X_j] + \frac{s^2}{2}.[X_i, [X_i, X_j]) + \cdots,$$

where s is the group parameter and $\mathrm{ad}_{X_i} X_j = [X_i, X_j]$ is the Lie algebra communicator and $i, j = 1, \ldots, 9$ An adjoint action is considered for Lie algebra $L(G)$ so we consider the following theorem,

Theorem 4.1 *The OS of* 1*-subalgebras for BS is*

(1) $\langle \mathbf{X}_4 \rangle$, (2) $\langle \mathbf{X}_1 + \mathbf{X}_2 \rangle$, (3) $\langle \mathbf{X}_2 - \mathbf{X}_4 \rangle$, (4) $\langle \mathbf{X}_1 + \mathbf{X}_5 \rangle$,

(5) $\langle \mathbf{X}_5 \rangle$, (6) $\langle \mathbf{X}_1 - \mathbf{X}_2 \rangle$, (7) $\langle \mathbf{X}_2 + \mathbf{X}_4 \rangle$, (8) $\langle \mathbf{X}_1 \rangle$,

(9) $\langle \mathbf{X}_3 \rangle$, (10) $\langle \mathbf{X}_2 \rangle$.

[20].

Proof Let $F_i^s : L(G) \to L(G)$ be the linear map, by $X \longmapsto \mathbf{Ad}(\exp(sX_i)X)$, $i = 1, \ldots, 5$. The matrix M_i^s of F_i^s, $i = 1, \ldots, 5$, with respect to the basis $\{X_1, \ldots, X_5\}$ is

$$M_1^s = I + \frac{s_1^2}{2} E_{12} - s_1 E_{13} - s_1 E_{32} + \frac{s_1}{2} E_{45}, \qquad M_4^s = I - \frac{s_4}{2} E_{43} - s_4 E_{52},$$

$$M_2^s = I + 2s_2^2 E_{22} + s_2 E_{23} + s_2 E_{31} + s_2 E_{54}, \qquad M_5^s = I + \frac{s_5}{4}(2E_{41} - s_5 E_{53}),$$

$$M_3^s = e^{-s3}(E_{11} - E_{66}) + e^{s3/2}(E_{44} - E_{55}),$$

respectively. Let $X = \sum_{i=1}^{5} a_i X_i$, then $(F_5^{s_5} \circ F_4^{s_4} \circ \cdots \circ F_1^{s_1})(X)$ is

$$\left[\left(1 + \frac{1}{4}s_1^2 s_2^2 - s_1 s_2 \right) e^{s3} a_1 + \frac{1}{2}s_1^2 e^{-s3} a_6 + \left(\frac{1}{2}s_1^2 s_2 - s_1 \right) a_3 \right] X_1 + \cdots$$

$$+ \left[((\frac{1}{2}s_4 s_2 e^{-s3/2} a_5 a_1 - e^{s6/2} s_4 a_2 + \left(-\frac{1}{2}s_2 e^{s3/2} s_4 + \frac{1}{2}e^{-s3/2} s_5 \right) a_3 \right.$$

$$\left. + s_2 e^{s3/2} a_4 + e^{-s3/2} a_5 \right] X_5.$$

We can simplify X as follows:

- If $a_5 = a_3 = a_2 = a_1 = 0$, then X is decreased to the case (1).
- If $a_5 = a_2 = a_1 = 0$ and $a_3 \neq 0$, so we can get the coefficient of X_4 vanish by using $F_4^{2a_4/a_3}$. In this case is reduced to the case (9).
- If $a_5 = a_3 = a_2 = 0$ and $a_1 \neq 0$, so we can get the coefficient of X_4 vanish by using $F_5^{2a_4/a_1}$. In this case is reduced to the case (8).
- If $a_5 = a_4 = a_3 = 0$ and $a_2 = 1$, then coefficient of X_1 can be vanished or be ± 1 by $F_3^{s_3}$ for $s_3 = -\ln|1/a_1|$. So X is reduced to the case (2, 6).
- If $a_5 = a_3 = a_1 = 0$, then we can make the coefficient of X_4 vanish or be ± 1 by using $F_3^{s_3}$ for $s_3 = (2/3)\ln|a_2/a_4|$. In this case is reduced to the case (3, 7).
- If $a_1 = a_3 = a_2 = 0$ and $a_5 \neq 0$, then we can assume that $a_5 = 1$ so we can get the coefficient of X_4 vanish by using $F_1^{2a_4}$. In this case it is reduced to the case (5).
- If $a_5 = a_2 = a_1 \neq 0$, then we can suppose that $a_1 = a_5 = 1$, so we can get the coefficient of X_4 and X_3 vanish by using $F_1^{2a_4}$ and $F_2^{-a_3}$. In this case is to the case (4).
- If $a_1 = a_3 = 0$ and $a_5 = a_2 \neq 0$, then we can make the coefficient of X_4 vanish by using $F_1^{2a_4}$, and X_5 vanish by using F_4^{1/a_2}. In this case is decreased to the case (10). □

5 OS of 2-subalgebras

Here, we get OS of 2-subalgebras for BS, choose X^1 or X^2, as an element of the OS of 2-subalgebras in Theorem 3.1 and regard $X = c_1 \mathbf{v}_1 + \cdots + c_5 \mathbf{v}_5$ as an optimal vector field where $c_i^5 s$ are smooth functions of (t, x, y, z, u, v, w). By such selection, we have $[\mathbf{v}_i, X] = \lambda X^1 + MX$. The following system (2) is a computation of both sides of these equations:

$$C_{jk}^i a_j a_k = \lambda a_i + \mu a_i, \quad i = 1, \dots, 5. \tag{2}$$

The elements of the OS of 2-subalgebras are reached by solving the system of linear equations for each choice of the OS of 1-subalgebras from the Theorem 3.1. When these elements are in the combination form, they could be simplified as a 1-dim case by acting the adjoint matrices to each of them and the following theorem is suggested.

Theorem 5.1 *An OS of 2-subalgebras from the BS is*

(1) $\langle X_1, X_2 + X_3 \rangle,$ (2) $\langle X_1, X_3 - X_4 \rangle,$ (3) $\langle X_1, X_3 + X_4 \rangle,$

(4) $\langle X_3, X_5 - X_2 \rangle,$ (5) $\langle X_3, X_1 - X_5 \rangle,$ (6) $\langle X_3, X_1 + X_5 \rangle,$

(7) $\langle X_3, X_4 + X_2 \rangle$,   (8) $\langle X_1, X_3 + X_4 \rangle$,   (9) $\langle X_2, X_3 + X_5 \rangle$,

(10) $\langle X_3, X_4 - X_2 \rangle$,   (11) $\langle X_1, X_3 - X_4 \rangle$,   (12) $\langle X_3, X_5 \rangle$,

(13) $\langle X_4, X_2 + X_5 \rangle$,   (14) $\langle X_3, X_1 + X_4 \rangle$,   (15) $\langle X_1, X_4 \rangle$,

(16) $\langle X_4, X_2 - X_5 \rangle$,   (17) $\langle X_3, X_1 - X_4 \rangle$,   (18) $\langle X_2, X_5 \rangle$,

(19) $\langle X_3, X_5 + X_2 \rangle$,   (20) $\langle X_4, X_3 \rangle$,   (21) $\langle X_4, X_5 \rangle$,

(22) $\langle X_3, X_2 \rangle$,   (23) $\langle X_3, X_1 \rangle$.

Proof Each 2-subalgebra needs two generators. By choosing one of the generators of the OS of 1-subalgebras that have explained in the previous theorem and the second one arbitrary, 2-subalgebras can be classified. let $\mathbf{h} = \mathbf{Span}\{X, Y\}$ is a 2-subalgebra where X is a 1-subalgebra that is selected from the 1-subalgebra list and Y is an arbitrary vector described by $Y = b_1\mathbf{v}_1 + \cdots b_5\mathbf{v}_5$. Now, we need to simplify $\mathbf{h}$ as much as possible by implementing various adjoint transformations on it and proceed algebraically [38].

Every adjoint transformation is a linear map $Fl_i^s : L(G) \to L(G)$ defined by $X \longmapsto \mathbf{Ad}(\exp(s\mathbf{v}_i)X)$ for $i = 1, \ldots, 5$, we only illustrate one of the cases in the following: If $X = \mathbf{v}_1 + a\mathbf{v}_2$ then

$$\mathbf{h} = \langle X, Y \rangle = \left\langle X_1 + aX_2, \sum_{i=1}^{5} b_i X_i \right\rangle$$

$$= \langle X_1 + aX_2, b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X_4 + b_5 X_5 \rangle$$

So, we have:

- If $b_3 = b_4 = b_5 = 0$ then we have $\mathbf{h} = \langle X_1, b_2 X_2 \rangle = \langle X_1, X_2 \rangle$ and $[X_1, b_2 X_2] = b_3 X_3 \neq rX_1 + s(b_2 X_2)$ for any $r, s \in RR$. Since $\mathbf{h}$, is not closed under the Lie bracket, So there are no two-dimensional subalgebras in this case.

- If $b_5 = 0$ and $b_4 \neq 0$ so we can get the coefficient of X_2 vanish by F_2^{s2}; By setting $s_3 = (-2/3)\ln|b_4/b_2|$. Then we have $\mathbf{h} = \langle X_1, b_3 X_3 + b_4 X_4 \rangle$ and $[X_1, b_3 X_3 + b_4 X_4] = -b_3 X_1$. Therefore $\mathbf{h}$ is closed under the Lie bracket and we have the case $\langle X_1, X_3 \pm X_4 \rangle$.

- If $b_5, b_4 = 0$ and $a_3 \neq 0$. Then we have $\mathbf{h} = \langle X_1, b_2 X_2 + b_3 X_3 \rangle$ and $[X_1, b_2 X_2 + b_3 X_3] = b_2 X_3 + b_1 X_1$ So $\mathbf{h}$ is closed under the Lie bracket and we have the case $\langle X_1, X_2 + X_3 \rangle$.

- If $b_3 = 0$ and $b_5 \neq 0$ and $b_5 = 1$ so we can get the coefficient of X_4 vanish by F_5^{s5}; By setting $s_5 = -2a_4$ also, we have scaling if necessary, we can assume that $b_5 = 1$. Then we have $\mathbf{h} = \langle X_1, b_2 X_2 + X_5 \rangle$ and $[X_1, b_2 X_2 + X_5] = b_2 X_3 + b_5 X_4$. So, $\mathbf{h}$ is not closed under the Lie bracket and we have no two-dimensional subalgebras in this case.

It should be noted that there is no other way to check, and in each case the $\mathbf{h}$ is simplified as much as possible. In the same way, two-dimensional algebras can be obtained for other states.   □

6 OS of 3-subalgebras

In this section, we get OS of 3-subalgebras for BS. To find 3-subalgebras, we must consider one 3-subalgebra as h = $\langle X, Y, Z \rangle$ of symmetry group, so that $Z = \sum_{i=1}^{5} c_i \mathbf{v}_i$.

Table 1 Some similarity reduced equations

OP_i	P	U_i	Similarity reduced equations
X_3, X_2, X_5	$\frac{z}{x}$	xu	$\varphi'' + \varphi'' p^2 + 2p\varphi'\eta + 4p\varphi' + 2\varphi = 2p\varphi'\psi + 2\psi\varphi$
		xv	$\varphi' p + \varphi = 0$
		xw	$\varphi' = 0$
X_1, X_3, X_4	$\frac{z}{x}$	$\frac{x(2tu+y)}{2t}$	$\varphi'' + p^2\varphi'' + 2p\varphi'\eta + 4p\varphi' + 2\varphi = 2p\varphi'\psi + 2\psi\varphi$
		$\frac{x(2tv+x)}{2t}$	$\varphi' p + \varphi = 0$
		$\frac{x(2tw+z)}{2t}$	$\varphi' = 0$

To classify 3-subalgebras, we need to be chosen two of the generators from the OS of 2-subalgebras, and another generator should be taken arbitrary, then we should check that $[X, Z]$ and $[Y, Z]$ are closed under the Lie bracket.

Theorem 6.1 *The OS of* 3*-subalgebra from the BS are*

(1) $\langle X_2, X_3, X_5 \rangle$,   (2) $\langle X_2, X_3 + X_5, X_1 \rangle$,   (3) $\langle X_2, X_3, X_4 \rangle$,

(4) $\langle X_2, X_3 + X_5, X_4 \rangle$,   (5) $\langle X_5, X_2, X_1 \rangle$,   (6) $\langle X_3, X_1, X_4 + X_5 \rangle$,

(7) $\langle X_2, X_3 + X_5, X_1 + X_4 \rangle$,   (8) $\langle X_3, X_1 + X_5, X_4 \rangle$,   (9) $\langle X_3, X_1 + X_5, X_2 \rangle$,

(10) $\langle X_1, X_3 + X_4, X_5 \rangle$,   (11) $\langle X_3, X_1, X_5 \rangle$,   (12) $\langle X_3, X_4 + X_2, X_1 \rangle$,

(13) $\langle X_3, X_1, X_2 \rangle$,   (14) $\langle X_3, X_4 + X_2, X_5 \rangle$,   (15) $\langle X_3, X_4, X_5 \rangle$,

(16) $\langle X_3, X_1 + X_4, X_2 \rangle$,   (17) $\langle X_3, X_1, X_4 \rangle$,   (18) $\langle X_3, X_1 + X_4, X_5 \rangle$,

(19) $\langle X_1, X_3 + X_4, X_2 \rangle$,   (20) $\langle X_3, X_5 + X_2, X_1 \rangle$,   (21) $\langle X_1, X_4, X_2 \rangle$,

(22) $\langle X_3, X_5 + X_2, X_4 \rangle$.

7 Similarity reduction of BS

If expressed in the new coordinates, BS is reduced. The BS is presented in the coordinates of (t, x, y, z, u, v, w) and we have to look for this equation's form in the appropriate coordinates in order to make it reduced. These new coordinates will come in hand by searching independent invariants (p, φ, ψ, η) which correspond to the generators of the symmetry group. Hence, If we use the new coordinates and apply chain derivative role into account, we can have the reduced equation. We introduce the present procedure for one of the infinitesimal generators in the OS of Theorem 6.1 and provide the list of results for some other instances.

This equation has an independent variable, named p, and three dependent variables labeled as (φ, ψ, η). Similarly, it is possible to compute all of the identically reduced equation which correspond to the infinitesimal symmetries that were mentioned in Theorem 6.1. Some of them are listed in Tables 1 and 2.

Then we can reduce some equations obtained in the previous section to ODEs,

$$\varphi'' + p^2\varphi'' + 2p\varphi'(\eta - \psi + 2) + 2\varphi(1 - \psi) = 0, \qquad p\varphi' + \varphi = 0, \qquad \varphi' = 0.$$

Therefore, $\varphi = 0$, $\psi = \psi(z/x)$, $\eta = \eta(z/x)$. We have a non-trivial solution of equation BS:

$$u = 0, \qquad v = \frac{1}{x}\psi\left(\frac{z}{x}\right), \qquad w = \frac{1}{x}\eta\left(\frac{z}{x}\right).$$

Table 2 The commutator table of $L(G)$

$[\cdot,\cdot]$	X_1	X_2	X_3	X_4	X_5	$A_{\tilde{f}}$	$B_{\tilde{g}}$	$C_{\tilde{h}}$
X_1	0	$-X_3$	$-X_1$	0	$-\frac{1}{2}X_4$	$A_{t(t\tilde{f}'-\tilde{f})/2}$	$B_{(t\tilde{g}'-\tilde{g})/2}$	$C_{t^2\tilde{h}'/2}$
X_2	X_3	0	0	X_5	0	$A_{\tilde{f}'}$	$B_{\tilde{g}'}$	$C_{\tilde{h}'}$
X_3	X_1	0	0	$\frac{1}{2}X_4$	$-\frac{1}{2}X_5$	$A_{t\tilde{f}'-\tilde{f}/2}$	$B_{t\tilde{g}'-\tilde{g}/2}$	$C_{t\tilde{h}'}$
X_4	0	$-X_5$	$-\frac{1}{2}X_4$	0	0	0	0	0
X_5	$\frac{1}{2}X_4$	0	$\frac{1}{2}X_5$	0	0	0	0	0
A_f	$-A_{(tf'-f)/2}$	$-A_{f'}$	$-A_{tf'-f/2}$	0	0	0	0	$B_{\tilde{h}f}$
B_g	$-B_{(tg'-g)/2}$	$-B_{g'}$	$-B_{tg'-g/2}$	0	0	0	0	$-A_{\tilde{h}g}$
C_h	$-C_{t^2h'/2}$	$-C_{h'}$	$-C_{th'}$	0	0	$-B_{h\tilde{f}}$	$A_{h\tilde{g}}$	0

By applying Theorem 3.1 we are able to obtain a new solution of the equation as follows.

Theorem 7.1 *Let h,g,φ,ψ are arbitrary smooth functions and s be a real number. Then, each of the sets*

(1) $\quad u = 0, \qquad v = \frac{1}{x}\varphi\left(\frac{z}{x}\right), \qquad w = \frac{1}{x}\psi\left(\frac{z}{x}\right),$

(2) $\quad u = \frac{-sy}{2st+4}, \qquad v = \frac{-sx+(4/x)\varphi(z/x)}{2st+4}, \qquad w = \frac{-sz+4(x\psi(z/x))^{-1}}{2st+4};$

(3) $\quad u = \frac{s}{2}, \qquad v = \frac{1}{x}\varphi\left(\frac{z}{x}\right), \qquad w = \frac{1}{x}\psi\left(\frac{z}{x}\right);$

(4) $\quad u = 0, \qquad v = \frac{1}{x+sh(t)}\varphi\left(\frac{z}{x+sh(t)}\right) + \frac{sh'(t)}{2}, \qquad w = \frac{1}{x}\psi\left(\frac{z+sg(t)}{x}\right) + \frac{sg'(t)}{2},$

are solutions of the equation BS.

8 Nonclassical symmetries of BS

A kth order system of differential equations is naturally treated as a submanifold $E \subset J^k$ of the kth order jet space on the space of independent and dependent variables. Consider a kth order system E of differential equations [21]:

$$\Delta_v\left(x, u, u^{(k)}\right) = 0, \quad v = 1, \dots, l, \tag{3}$$

involving $x = (x^1, \dots, x^n)$ and $u = (u^1, \dots, u^q)$ as independent and dependent variables, respectively, and with $u^{(k)}$ denoting the derivatives of the us with respect to the xs up to order k. Suppose that V is a vector field on the space $R^n \times R^q$ of independent and dependent variables:

$$V = \sum_{i=1}^{n} \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(x,u)\partial_{u^j}. \tag{4}$$

A n-dimensional submanifold of $RR^n \times RR^q$ of the space of independent and dependent variables is defined by a map of the solution

$$u^\alpha = f^\alpha(x_1, \dots, x_n) = 0, \quad \alpha = 1, \dots, q, \tag{5}$$

to the system. Then we must add the invariant surface conditions of these equations:

$$Q^\alpha\left(x, u, u^{(1)}\right) = \varphi^\alpha(x, u) - \sum_{i=1}^{n} \xi^i(x, u)\frac{\partial u^\alpha}{\partial x_i} = 0, \quad \alpha = 1, \dots, q, \tag{6}$$

where $Q = (Q^1, \dots, Q^n)$ is known as the characteristic of the vector field (5). The kth prolongation of the invariant surface condition (6) will be denoted by E_Q^k. If the kth prolongation of the $V^{(k)}$ vector field V is tangent to the intersection $E \cap E_Q^k$ then the system (3) and (6) are compatible:

$$V^k(\triangle_\nu)\,|_{E \cap E_Q^k} = 0, \quad \nu = 1, \dots, l. \tag{7}$$

The vector field (4) is called a nonclassical infinitesimal symmetry of the system (3) If Eqs. (7) are satisfied [30].

For finding the nonclassical symmetries, according to the system E of BS we do as follows.

Case 1: If we suppose that the coefficient of ∂_t in the (4) is not equal to zero, we can set coefficient of ∂t equal to 1 without changing the totality, then for the vector field $V = \partial_t + \xi\partial_x + \eta\partial_y + \zeta\partial_z + \phi\partial_u + \psi\partial_v + \theta\partial_w$, the invariant surface conditions are as follows:

$$\begin{aligned}(\phi, \psi, \theta) = (&u_t + \xi u_x + \eta u_y + \zeta u_z, \\ &v_t + \xi v_x + \eta v_y + \zeta v_z, w_t + \xi w_x + \eta w_y + \zeta w_z).\end{aligned} \tag{8}$$

According to Eq. (7), we can find the variables $u_t, \dots, v_t, \dots, w_t$ from (BS) and (4):

$$u_t = \frac{-2\zeta u w_y + 2\eta v v_y + 2\eta w w_y - 2\xi u v_y + \eta u_{xx} + \eta u_{yy} + \eta u_{zz} + 2\phi u}{\eta + 2u},$$

$$u_x = v_y, \qquad u_y = \frac{\zeta w_y + \xi v_y + 2v v_y + 2w w_y - \phi + u_{xx} + u_{yy} + u_{zz}}{\eta + 2u},$$

$$u_z = w_y, \qquad v_t = -\zeta v_z - \eta v_y - \xi v_x + \psi, \qquad w_t = -\zeta w_z - \eta w_y - \xi w_x + \theta.$$

By using nonclassical methods for the BS, we have the following determining equations:

$$\eta_x = \eta_z = \eta_u = \eta_v = \eta_w = 0, \qquad \psi_y = \psi_u = \psi_{wz} = \psi_{ww} = \psi_{zz} = 0,$$

$$\zeta_y = \zeta_u = \zeta_w = 0, \qquad \theta_y = \theta_u = \theta zz = \theta_{ww} = \theta_{wz} = 0,$$

$$\phi_x = \phi_z = \phi_v = \phi_w = 0, \qquad \xi_y = \xi_u = \xi_v = \xi_w = 0, \qquad \psi_v = \theta_w,$$

$$\psi_x = \theta_z, \qquad \theta_v = -\psi_w, \theta_x = -\psi_z, \qquad \eta_y = -\theta_w, \qquad \phi_u = \theta_w,$$

$$\phi_y = \theta_z, \qquad \xi_z = \psi_w, \qquad \xi_x = -\theta_w, \qquad \zeta_z = -\theta_w, \qquad \zeta_x = -\psi_w,$$

$$\theta_{zt} = 2\theta_w\theta_z, \qquad \theta_{wt} = 2\left(\theta_w^2 + \theta_z\right), \qquad \psi_{wt} = 2(\theta_w\psi_w - \psi_z),$$

$$\xi_t = 2(w\psi_w + \xi\theta_w + v\theta_w - \psi), \qquad \eta_t = 2(\eta\theta_w + u\theta_w - \phi),$$

$$\zeta_t = 2(\zeta\theta_w + w\theta_w - v\psi_w - \theta), \qquad \phi_t = 2(\phi\theta_w + u\theta_z).$$

Therefore we have the solutions

$$(\xi, \eta, \zeta, \phi, \psi, \theta) = \frac{1}{2t^2 + 1}\bigg(2tx - z + 1, -1 + (2y - 1)t, 2tx - z + 1,$$

$$\frac{1}{2} - 2tu - y, -2tv - x - w, -2tw - z + v\bigg).$$

Case 2: Now we suppose that the coefficient of ∂_t in (4) is equal to zero and try to obtain the infinitesimal nonclassical symmetries of the form $V = \partial_x + \eta\partial_y + \zeta\partial_z + \phi\partial_x + \psi\partial_v + \theta\partial_w$. So we have the category of solutions $\eta = 1$, $\phi = \psi = -1/2$ and $\theta = \zeta = 0$.

Case 3: Again we suppose that the coefficient of ∂_x in (4) is equal to zero and try to obtain the infinitesimal nonclassical symmetries of the form $V = \partial_y + \zeta\partial_z + \phi\partial_x + \psi\partial_v + \theta\partial_w$. So we obtain the solutions $\phi = -1/2$ and $\psi = \theta = \zeta = 0$.

Case 4: In the end, we assume that the coefficient of ∂_y in (4) is equal to zero and try to obtain the infinitesimal nonclassical symmetries of the form $V = \partial_z + \phi\partial_x + \psi\partial_v + \theta\partial_w$. Then we have $\phi = \psi = 0$, and $\theta = F(t, x, z, w)$, where F is an arbitrary function.
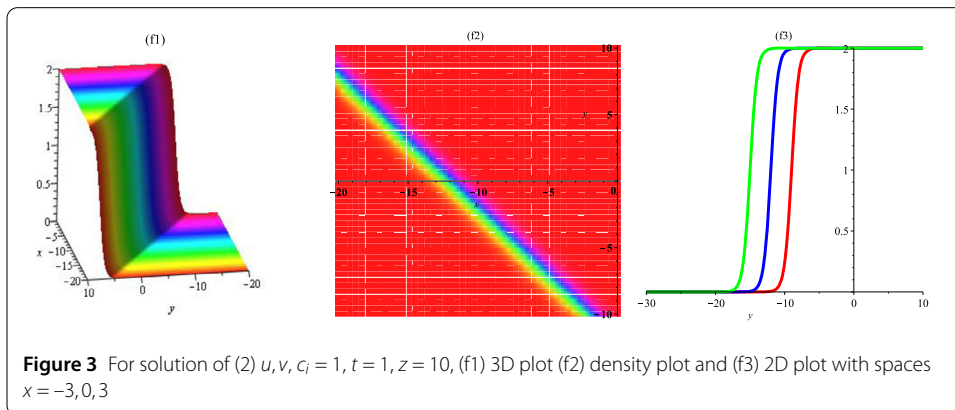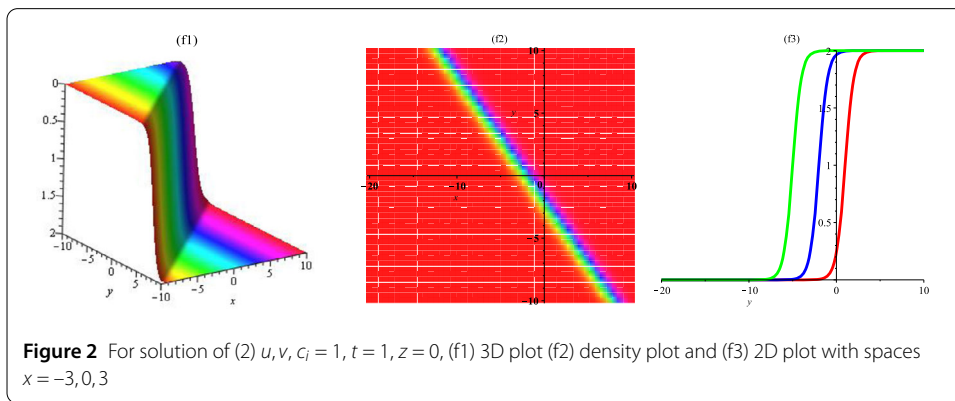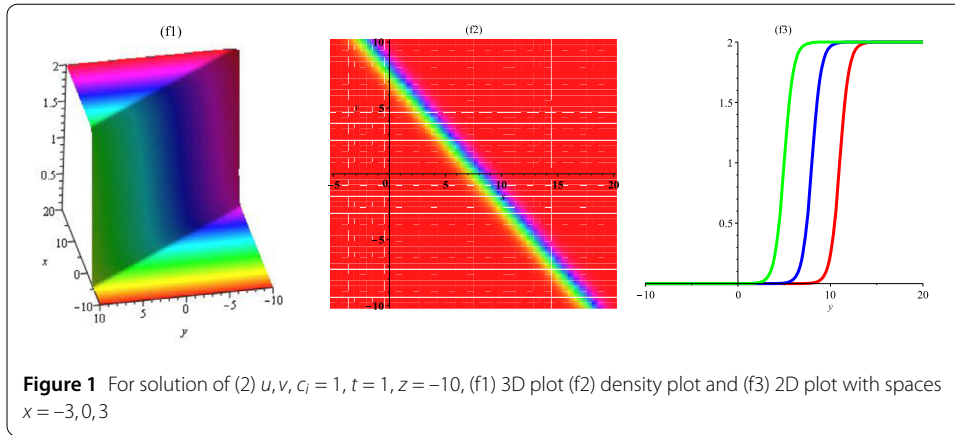
9 Ansatz solutions of BS

Now we consider the most important ansatz method (specially, the tanh-function method, [16, 34, 50]) for constructing exact traveling wave solutions of this nonlinear system of PDEs. For this, we introduce a new variable $\tau = \tanh(c_0 + c_1 t + c_2 x + c_3 y + c_4 z)$ and the ansatz $u = A_{1,0} + A_{1,1}\tau$, $v = A_{2,0} + A_{2,1}\tau$ and $w = A_{3,0} + A_{3,1}\tau$, where $A_{i,k}$ and c_i are arbitrary constants. Substituting expansions into the BS equations, we obtain the following system of algebraic equations:

$$c_3 v'(\tau) - c_2 u'(\tau) = 0, \qquad c_3 w'(\tau) - c_4 u'(\tau) = 0,$$

$$\big(c_2^2 + c_3^2 + c_4^2\big)\big(\tau^2 + 2\tau - 1\big)u''(\tau) = 0,$$

$$\big(2c_3 u(\tau) - 2c_2 v(\tau) + 2c_4 w(\tau) - c_1\big)u'(\tau) = 0.$$

Finally, we obtain three sets of exact solutions, with linear algebra and required simplifications:
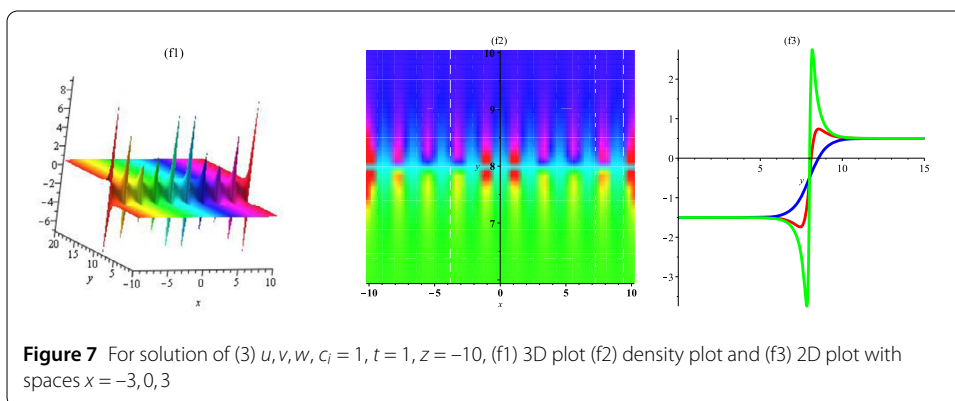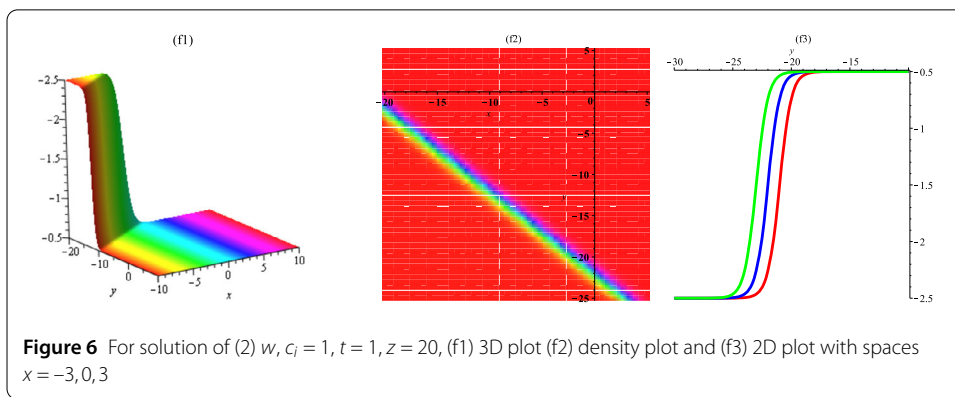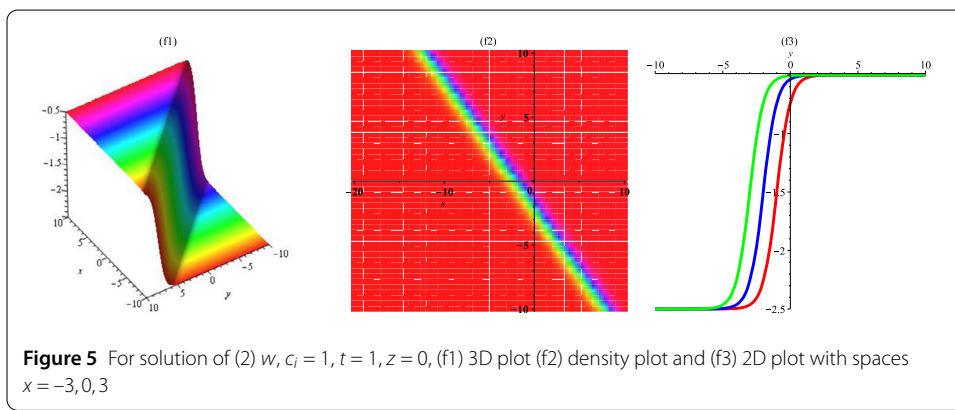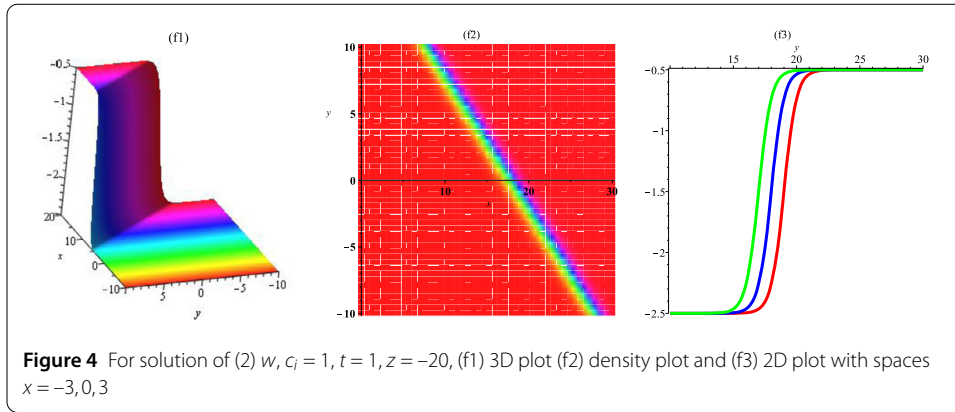
(1) $u = a_1,$ $\qquad v = a_2,$ $\qquad w = a_3,$

(2) $u = a_3 \tanh\theta + a_4 a_5,$ $\qquad v = a_2 \tanh\theta + a_4 a_7,$

$\qquad w = a_4 \tanh\theta + a_1 - a_2 a_7 - a_3 a_5,$

(3) $u = a_5 a_6 + \dfrac{a_7 \sinh(2\theta)}{\sinh^2\theta + \cos^2(\alpha x)},$ $\qquad v = a_8 + \dfrac{a_7 \alpha \sinh(2\alpha x)}{\sinh^2\theta + a_4 \cos^2(\alpha x)},$

$\qquad w = a_2 - a_4 a_6 + \dfrac{a_5 a_7 \sinh\theta}{\sinh^2\theta + a_4 \cos^2(\alpha x)},$
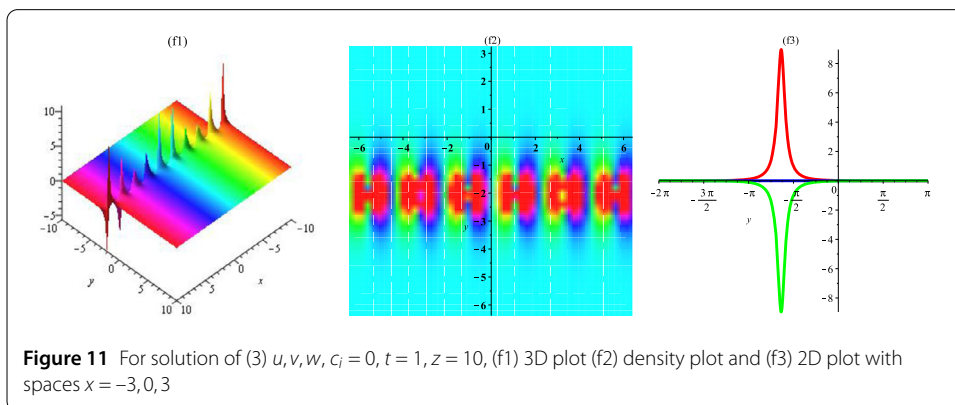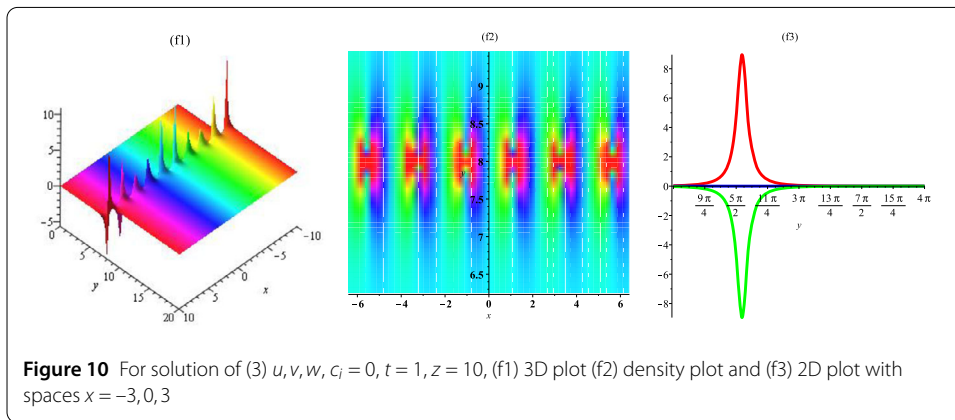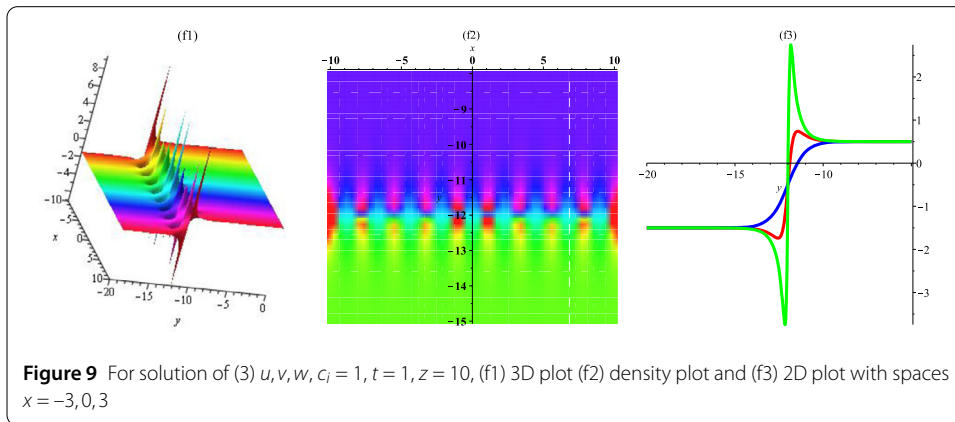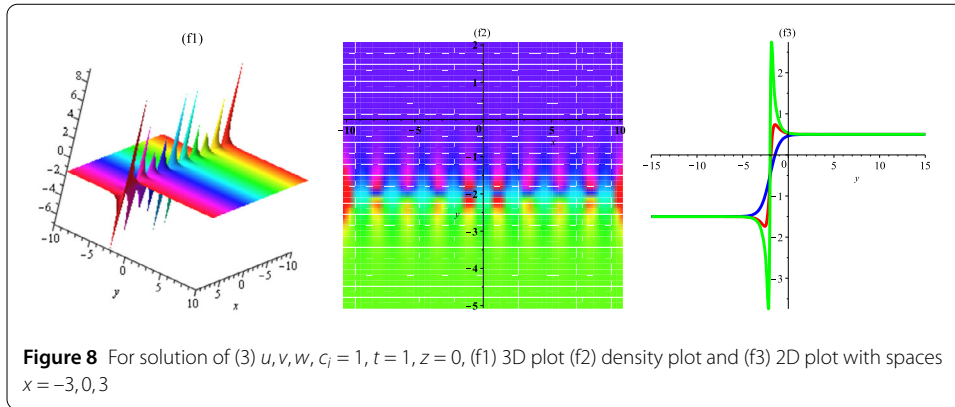
where $\alpha = \sqrt{a_4^2 + a_5^2}$, $\theta = a_1 + 2a_2 a_5 t + a_4 y + a_5 z$ and the a_i are constants.

Figure 1 For solution of (2) $u, v, c_i = 1, t = 1, z = -10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 2 For solution of (2) $u, v, c_i = 1, t = 1, z = 0$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 3 For solution of (2) $u, v, c_i = 1, t = 1, z = 10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$
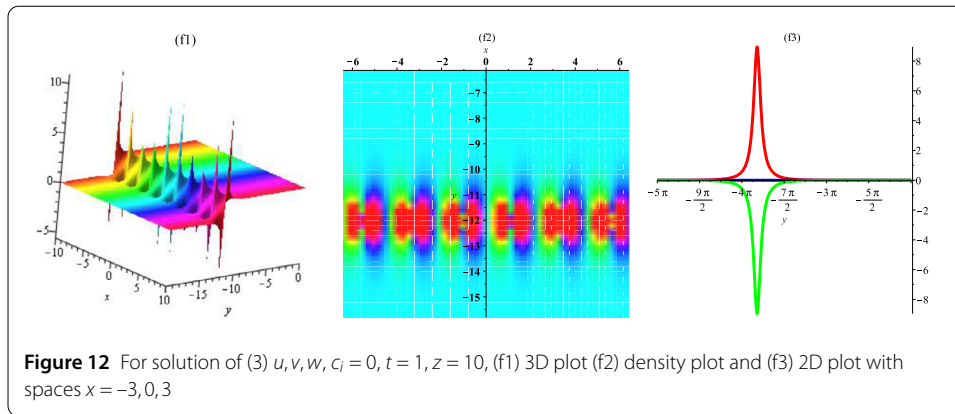
10 Results and discussion

The graphical representation of BS equations produces the substantial information to interpret the phenomena physically. This section deals with a physical interpretation of the solutions given via choosing the appropriate amounts of parameters, the graphic display of traveling wave solution is presented in Figs. 1 to 12 including 3D plot, density plot, and 2D plot when three spaces arise at spaces $z = -10$, $z = 0$, and $z = 10$ for u, v and $z = -20$, $z = 0$, and $z = 20$ for w. The solutions contain various arbitrary constants and functions and their appropriate choices are crucial to describe the significant behavior of the phenomena. The simulation is performed on MATLAB for Figs. 1–12, when $x = -3, 0, 3$.

Figure 4 For solution of (2) w, $c_i = 1$, $t = 1$, $z = -20$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 5 For solution of (2) w, $c_i = 1$, $t = 1$, $z = 0$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 6 For solution of (2) w, $c_i = 1$, $t = 1$, $z = 20$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 7 For solution of (3) u, v, w, $c_i = 1$, $t = 1$, $z = -10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 8 For solution of (3) $u, v, w, c_i = 1, t = 1, z = 0$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 9 For solution of (3) $u, v, w, c_i = 1, t = 1, z = 10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 10 For solution of (3) $u, v, w, c_i = 0, t = 1, z = 10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 11 For solution of (3) $u, v, w, c_i = 0, t = 1, z = 10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

Figure 12 For solution of (3) u, v, w, $c_i = 0$, $t = 1$, $z = 10$, (f1) 3D plot (f2) density plot and (f3) 2D plot with spaces $x = -3, 0, 3$

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