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Continuous stage stochastic Runge–Kutta methods

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Abstract

In this work, a version of continuous stage stochastic Runge–Kutta (CSSRK) methods is developed for stochastic differential equations (SDEs). First, a general order theory of these methods is established by the theory of stochastic B-series and multicolored rooted tree. Then the proposed CSSRK methods are applied to three special kinds of SDEs and the corresponding order conditions are derived. In particular, for the single integrand SDEs and SDEs with additive noise, we construct some specific CSSRK methods of high order. Moreover, it is proved that with the help of different numerical quadrature formulas, CSSRK methods can generate corresponding stochastic Runge–Kutta (SRK) methods which have the same order. Thus, some efficient SRK methods are induced. Finally, some numerical experiments are presented to demonstrate those theoretical results.

Keywords: Stochastic differential equations; Continuous stage stochastic Runge–Kutta methods; B-series

1 Introduction

Stochastic differential equations (SDEs) have wide applications in many disciplines like biology, economics, medicine, engineering and finance (see, e.g., [1–3]). However, most SDEs arising in practice are nonlinear, and cannot be solved explicitly. There has been tremendous interests in developing effective and reliable numerical methods for SDEs during the last few decades, for example see [4–14]. Runge–Kutta (RK) methods with continuous stage were firstly presented by Butcher in 1970s [15], and they have been investigated and discussed by several authors recently because of the great advantages in conserving symplecticity [16], preserving energy [17] and so on. So constructing continuous stage stochastic Runge–Kutta (CSSRK) methods for SDEs is a valuable task. This paper mainly aims to construct CSSRK methods for SDEs which can be written in integral form as

$$\mathrm{d}x(t) = g_0\big(x(t)\big)\,\mathrm{d}t + \sum_{k=1}^{d} g_k\big(x(t)\big) \circ \mathrm{d}W_k(t), \quad x(0) = x_0 \in \mathbb{R}^m, \qquad (1.1)$$

with a d-dimensional Wiener process $(W(t))_{t\geq 0} = ((W_1(t),\ldots,W_d(t))^T)_{t\geq 0}$. Here $g_k : \mathbb{R}^m \mapsto \mathbb{R}^m$, $k = 1,\ldots,d$, are Borel measurable, satisfying Lipschitz condition and linear growth condition, $x(t)$ is the stochastic process which denotes the solution of (1.1).

In engineering, the convergence order of applied numerical methods plays an important role in efficient operation. And B-series appear as a fundamental tool to do error analysis on a wide range of problems. Therefore, a general order theory for CSSRK methods based on stochastic B-series and multicolored rooted tree is developed. Then, for the above SDE (1.1) with $d = 1$, we discuss the order conditions, and construct relevant CSSRK methods of order 1.0. Furthermore, CSSRK methods are also constructed for problems with some special structures, such as single integrand SDEs, SDEs with additive noise.

Single integrand SDEs are given by

$$\mathrm{d}x(t) = \lambda g(x)\,\mathrm{d}t + \sum_{k=1}^{d} \sigma_k g(x) \circ \mathrm{d}W_k(t), \quad x(0) = x_0 \in \mathbb{R}^m, \tag{1.2}$$

where $\lambda \in \{0,1\}$ and $\sigma_k \in \mathbb{R}, k = 1,2,\ldots,d$, are given constants. Some well known examples for single integrand SDEs are the SDE describing fatigue cracking [18], the stochastic Van der Pol equation [19]. A theory for the derivation of B-series of the numerical approximation by CSSRK methods is given. Due to the single integrant, it is similar to the ordinary differential equations (ODEs) case [20]. The deterministic continuous stage Runge–Kutta (DCSRK) methods of deterministic order p_d are corresponding with the CSSRK methods of mean square $p_\mu = \lfloor p_d/2 \rfloor$, which can be viewed as the stochastic generalization of the DCSRK methods.

Concerning the specific problems with additive noise, thus

$$\mathrm{d}x(t) = f\big(x(t)\big)\,\mathrm{d}t + \sum_{k=1}^{d} \sigma_k \circ \mathrm{d}W_k(t), \quad x(0) = x_0 \in \mathbb{R}^m, \tag{1.3}$$

where $\sigma_k, i = 1,2,\ldots,d$, are constants. Based on stochastic B-series, we propose a class of CSSRK methods, totally derivative-free for (1.3), which are able to reach mean-square order 1.5. Then we focus on the special second-order systems with additive noise of the following form:

$$\ddot{x}(t) = f(x) + \sum_{k=1}^{d} \sigma_k \circ \dot{W}_k(t), \quad x(0) = x_0 \in \mathbb{R}^m, \dot{x}(0) = y_0 \in \mathbb{R}^m, \tag{1.4}$$

the corresponding CSSRK methods are able to reach mean-square order 2.0.

Interestingly, it is verified that a great deal of SRK methods could be derived from the CSSRK methods by means of the adoption of different numerical quadrature formulas. Furthermore, the conditions on algebraic precision of the numerical quadrature formulas and the degree of the coefficient polynomials of CSSRK methods are analyzed to guarantee that SRK methods share the same order with the CSSRK one. Thus, some efficient SRK methods are obtained.

The outline of this paper is organized as follows. In the next section, a family of CSSRK methods is proposed. Based on the multicolored rooted tree and stochastic B-series, the general order conditions are calculated. Up to strong global order 1.0 of SRK methods are

proposed in Sect. 3. In Sect. 4, we consider a particular class of Stratonovich SDEs, single integrand SDEs. Sections 5 is devoted to discuss the CSSRK methods solving SDEs with additive noise. Finally, numerical experiments are reported in Sect. 6.

2 CSSRK methods and their order conditions

In this section, we first introduce the definition of CSSRK methods, then show some results which are useful in receiving the order theory.

2.1 CSSRK methods

We consider the following methods given by $y_0 = x_0$ and the step size $h > 0$, let $A_{\tau,\xi}^{(i,k)}$ be a polynomial in τ and ξ, $B_\tau^{(i,k)}$ be a polynomial in τ. The one-step method $\Phi_h : y_0 \to y_1$ given by

$$Y_\tau = y_0 + \sum_{k=0}^{d} \int_0^1 \sum_{i=0}^{\hat{r}} \bar{\phi}_{i,k} A_{\tau,\xi}^{(i,k)} g_k(Y_\xi) \, d\xi, \tag{2.1a}$$

$$y_1 = y_0 + \sum_{k=0}^{d} \int_0^1 \sum_{i=0}^{\hat{r}} \bar{\phi}_{i,k} B_\tau^{(i,k)} g_k(Y_\tau) \, d\tau, \tag{2.1b}$$

with $\bar{\phi}_{0,0} = h$, $\bar{\phi}_{i,k}$, $k = 0,\dots,d$, $i = 0,\dots,\hat{r}$, are random variables. Invoking $Z_{\tau,\xi}^{(k)} = \sum_{i=0}^{\hat{r}} \bar{\phi}_{i,k} A_{\tau,\xi}^{(i,k)}$, $z_\tau^{(k)} = \sum_{i=0}^{\hat{r}} \bar{\phi}_{i,k} B_\tau^{(i,k)}$, then another presentation can be obtained,

$$Y_\tau = y_0 + \sum_{k=0}^{d} \int_0^1 Z_{\tau,\xi}^{(k)} g_k(Y_\xi) \, d\xi, \tag{2.2a}$$

$$y_1 = y_0 + \sum_{k=0}^{d} \int_0^1 z_\tau^{(k)} g_k(Y_\tau) \, d\tau, \tag{2.2b}$$

the coefficients $Z_{\tau,\xi}^{(k)}$ and $z_\tau^{(k)}$ include random variables that depend on the stepsize h. Then it defines a m-dimensional approximation process y_n with $y_n \approx x_n$ which called CSSRK methods.

2.2 Order theory

The basic tool of constructing our numerical methods is the multicolored rooted tree theory in [21]. Thus, we briefly list some definitions and theorems used in constructing numerical schemes later. Our first goal is to find B-series representations of (2.2a)–(2.2b) which can be written as a B-series,

$$B(\phi, x_0; h) = \sum_{t \in T} \alpha(t)\phi(t)(h)F(t)(x_0),$$

where T is the set of multicolored rooted tree, $\alpha(t)$ are combination terms, the elementary weight functions $\phi(t)(h)$ are stochastic integrals or random variables, and $F(t)(x_0)$ are the elementary differentials.

Definition 2.1 (Trees and combinatorial coefficients) The set of multicolored rooted trees

$$T = \{\emptyset\} \cup T_0 \cup T_1 \cup \cdots \cup T_d,$$

is recursively defined as follows:

   (a) The graph $\bullet_k = [\emptyset]_k$ with only one vertex of color k belongs to T_k. Let $t = [t_1, t_2, \ldots, t_l]_k$ be the tree formed by joining the subtrees $t_1, t_2, \ldots, t_l$ each by a single branch to a common root of color k.

   (b) If $t_1, t_2, \ldots, t_l \in T$, then $t = [t_1, t_2, \ldots, t_l]_k \in T_k$.

Thus, T_k is the set of trees with a k-colored root, and T is the union of these sets. Further, we define $\alpha(t)$ as

$$\alpha(\emptyset) = 1, \qquad \alpha(\bullet_k) = 1, \qquad \alpha(\circ) = 1, \qquad \alpha\big(t = [t_1, t_2, \ldots, t_l]_k\big) = \frac{1}{r_1! r_2! \cdots r_q!} \prod_{j=1}^{l} \alpha(t_j),$$

where $r_1, r_2, \ldots, r_q$ count equal trees among $t_1, t_2, \ldots, t_l$.

Definition 2.2 (Elementary differentials) For a tree $t \in T$, the elementary differential is a mapping $F(t): \mathbb{R}^m \to \mathbb{R}^m$ defined recursively by

   (a) $F(\emptyset)(x_0) = x_0$.

   (b) $F([\emptyset_k])(x_0) = g_k(x_0)$.

   (c) If $t = [t_1, t_2, \ldots, t_l]_k \in T_k$, then $F(t)(x_0) = g_k^{(l)}(F(t_1)(x_0), F(t_2)(x_0), \ldots, F(t_l)(x_0))$, $t \in T_k$.

The next lemma proves that, if $Y(h)$ can be written as a B-series, then $f(Y(h))$ can be written as a similar series. The lemma is fundamental for deriving B-series for the exact and the numerical solution.

Lemma 2.1 ([21]) *Assume that $Y(h) = B(\phi, x_0; h)$ is stochastic B-series, and $f \in C^\infty(R^m, R^{\hat{m}})$, then $f(Y(h))$ can be indicated*

$$f\big(Y(h)\big) = \sum_{u \in U_f} \beta(u) \Psi_{\phi(u)}(h) G(u)(x_0), \tag{2.3}$$

where U_f are the collection derived from T.

   (a) *$[\emptyset]_f \in U_f$, and if $t_1, t_2, \ldots, t_l \in T$, then $[t_1, t_2, \ldots, t_l]_f \in U_f$;*

   (b) *$G([\emptyset]_f)(x_0) = f(x_0)$, $G(u = [t_1, t_2, \ldots, t_l]_f)(x_0) = f^{(l)}(x_0)(F(t_l)(x_0), \ldots, F(t_l)(x_0))$;*

   (c) *$\beta(u = [t_1, t_2, \ldots, t_l]_f) = \frac{1}{r_1! r_2! \cdots r_q!} \prod_{i=1}^{l} \alpha(t_i)$;*

   (d) *$\Psi_\phi([\emptyset]_f)(h) = 1$, $\Psi_\phi(u = [t_1, t_2, \ldots, t_l]_f)(h) = \prod_{i=1}^{l} \phi(t_i)(h)$.*

Lemma 2.2 ([21]) *The exact solution $x(h)$ of (1.1) can be written as a B-series $x(h) = B(\phi, x_0, h)$ with*

$$\phi(\emptyset)(h) = 1, \qquad \phi(\bullet_k)(h) = W_k(h),$$

$$\phi\big(t = [t_1, t_2, \ldots, t_l]_k\big)(h) = \int_0^h \prod_{i=1}^{l} \phi(t_i)(s) \circ dW_k(s).$$

A similar result can be found for the numerical solution of (1.1) by the CSSRK methods (2.2a)–(2.2b).

Theorem 2.1 *The numerical solution y_1 as well as the continuous stage values can be written in terms of B-series*

$$Y_\tau = B(\Phi_\tau, y_0; h), \qquad y_1 = B(\varphi, y_0; h),$$

for $\tau \in [0, 1]$, with

$$\Phi_\tau(\emptyset)(h) = 1, \qquad \Phi_\tau(\bullet_k)(h) = \int_0^1 Z_{\tau,\xi}^{(k)} \, d\xi,$$

$$\Phi_\tau\big(t = [t_1, t_2, \ldots, t_l]_k\big)(h) = \int_0^1 Z_{\tau,\xi}^{(k)} \prod_{i=1}^l \Phi_\xi(t_i)(h) \, d\xi,$$

(2.4)

$$\varphi(\emptyset)(h) = 1, \qquad \varphi(\bullet_k)(h) = \int_0^1 z_\tau^{(k)} \, d\xi,$$

$$\varphi\big(t = [t_1, t_2, \ldots, t_l]_k\big)(h) = \int_0^1 z_\tau^{(k)} \prod_{i=1}^l \Phi_\tau(t_i)(h) \, d\tau.$$

(2.5)

Proof Write Y_τ as a B-series, that is,

$$Y(\tau) = \sum_{t \in T} \alpha(t) \Phi_\tau(t)(h) F(t)(y_0),$$

for $\tau \in [0, 1]$, use the definition of the method (2.2a) and the term by term comparison yielding (2.2a) to obtain

$$
\begin{aligned}
Y_\tau &= y_0 + \sum_{k=0}^d \int_0^1 Z_{\tau,\xi}^{(k)} g_k(Y_\xi) \, d\xi \\
&= y_0 + \sum_{k=0}^d \int_0^1 Z_{\tau,\xi}^{(k)} g_k\big(B(\Phi_\xi, y_0; h)\big) \, d\xi \\
&= y_0 + \sum_{k=0}^d \int_0^1 Z_{\tau,\xi}^{(k)} \sum_{t \in T_k} \alpha(t) \Phi'_{\xi,k}(t)(h) F(t)(y_0) \, d\xi \\
&= y_0 + \sum_{k=0}^d \int_0^1 \sum_{t \in T_k} \alpha(t) \big(Z_{\tau,\xi}^{(k)} \Phi'_{\xi,k}(t)(h)\big) \, d\xi F(t)(y_0) \\
&= y_0 + \sum_{t \in T/\{\emptyset\}} \alpha(t) \left(\int_0^1 Z_{\tau,\xi}^{(k)} \prod_{i=1}^l \Phi_\xi(t_i)(h) \, d\xi\right) F(t)(y_0) \\
&= \sum_{t \in T} \alpha(t) \Phi_\tau(t)(h) F(t)(y_0),
\end{aligned}
$$

where $t = [t_1, t_2, \ldots, t_l]_k \in T_k$. The proof of (2.5) is similar. $\qquad\square$

Now, the local order of accuracy of the CSSRK methods can be decided by comparing the B-series of the exact and the numerical solution. First, we need to define the tree order.

Definition 2.3 The order of a tree $t \in T$ is defined by

$$\rho(\emptyset) = 0, \qquad \rho\big(t = [t_1, t_2, \ldots, t_l]_k\big) = \sum_{j=1}^{l} \rho(t_j) + \begin{cases} 1, & k = 0, \\ \frac{1}{2}, & k \neq 0. \end{cases}$$

The following lemma relates the global order of accuracy to the local order. Here, we assume that (2.5) is constructed such that $\varphi(t)(h) = O(h^{\rho(t)})$ for all $t \in T$ for mean square convergence.

Lemma 2.3 ([21]) *The method has mean square global order p if*

$$\big(E\|\varphi(t)(h) - \phi(t)(h)\|^2\big)^{\frac{1}{2}} = O\big(h^{p+\frac{1}{2}}\big), \quad \forall t \in T, \rho(t) \leq p, \tag{2.6}$$

$$E\big(\varphi(t)(h)\big) = E\big(\phi(t)(h)\big) + O\big(h^{p+1}\big), \quad \forall t \in T, \rho(t) \leq p + \frac{1}{2}. \tag{2.7}$$

Here, the $O(\cdot)$-notation refers to $h \to 0$ and, especially in (2.6), to the L^2-norm.

3 Construction of CSSRK methods for SDEs with one random variable

Without loss of generality, we restrict consideration to autonomous systems (the coefficients of SDEs do not depend on t explicitly) in this part. We write

$$dx(t) = f\big(x(t)\big)\, dt + g\big(x(t)\big) \circ dW(t), \quad x(0) = x_0, \tag{3.1}$$

then the CSSRK methods for system (3.1) are given by

$$Y_\tau = y_0 + h \int_0^1 A_{\tau,\xi}^{(0)} f(Y_\xi)\, d\xi + J_1 \int_0^1 A_{\tau,\xi}^{(1)} g(Y_\xi)\, d\xi, \tag{3.2a}$$

$$y_1 = y_0 + h \int_0^1 B_\tau^{(0)} f(Y_\tau)\, d\tau + J_1 \int_0^1 B_\tau^{(1)} g(Y_\tau)\, d\tau, \tag{3.2b}$$

where $J_1 = \triangle W_n = W(t_{n+1}) - W(t_n)$ is an independent $N(0, h)$-distributed Gaussian random variable. Based on the multicolored rooted tree theory in Sect. 2, we are able to obtain a set of order conditions guaranteeing that CSSRK methods (3.2a)–(3.2b) of mean-square order 1.0 as detailed below. A list of all trees with $\rho(t) \leq 1.5$ and their corresponding functions are given in Table 1.

Above all, the following conditions need to be satisfied:

(a) $\int_0^1 z_\tau^{(0)}\, d\tau = h + O(h^{\frac{3}{2}})$;
(b) $\int_0^1 z_\tau^{(1)}\, d\tau = W_1(h) + O(h^{\frac{3}{2}})$;
(c) $\int_0^1 z_\tau^{(1)} \int_0^1 Z_{\tau,\xi}^{(1)}\, d\xi\, d\tau = \int_0^h W_1(s) \circ dW_1(s) + O(h^{\frac{3}{2}})$;
(d) $E(\int_0^1 z_\tau^{(0)}\, d\tau) = E(h) + O(h^2)$;
(e) $E(\int_0^1 z_\tau^{(1)}\, d\tau) = E(W_1(h)) + O(h^2)$;
(f) $E(\int_0^1 z_\tau^{(1)} \int_0^1 Z_{\tau,\xi}^{(1)}\, d\xi\, d\tau) = E(\int_0^h W_1(s) \circ dW_1(s)) + O(h^2)$.

Through calculation, if the coefficients of CSSRK methods (3.2a)–(3.2b) satisfy the conditions

$$\int_0^1 B_\tau^{(0)}\, d\tau = 1, \qquad \int_0^1 B_\tau^{(1)}\, d\tau = 1, \qquad \int_0^1 B_\tau^{(1)} \int_0^1 A_{\tau,\xi}^{(1)}\, d\xi\, d\tau = \frac{1}{2}, \tag{3.3}$$

Table 1 Multicolored rooted trees for (3.2a)–(3.2b) with order less than or equal to 1.5

No.	t	$\rho(t)$	$\phi(t)$	$\varphi(t)$
1	$\bullet_i$	0.5	$W_i(h)$	$\int_0^1 z_\tau^{(i)}\, d\tau$
2	$\bullet\,0$	1	$W_0(h)=h$	$\int_0^1 z_\tau^{(0)}\, d\tau$
3	(tree j–i)	1	$\int_0^h W_j(s)\circ dW_i(s)$	$\int_0^1 z_\tau^{(i)}\int_0^1 Z_{\tau,\xi}^{(j)}\, d\xi\, d\tau$
4	(tree j,k–i)	1.5	$\int_0^1 W_j(s)W_k(s)\circ dW_i(s)$	$\int_0^1 z_\tau^{(i)}\int_0^1 Z_{\tau,\xi}^{(j)}\, d\xi \int_0^1 Z_{\tau,\varsigma}^{(k)}\, d\varsigma\, d\tau$
5	(tree k–j–i)	1.5	$\int_0^h\int_0^s W_i(s_1)\circ dW_j(s_1)\circ dW_k(s)$	$\int_0^1 z_\tau^{(i)}\int_0^1 Z_{\tau,\xi}^{(j)}\int_0^1 Z_{\xi,\varsigma}^{(k)}\, d\varsigma\, d\xi\, d\tau$
6	(tree 0–i)	1.5	$\int_0^h s\circ dW_i(s)$	$\int_0^1 z_\tau^{(i)}\int Z_{\tau,\xi}^{(0)}\, d\xi\, d\tau$
7	(tree i–0)	1.5	$\int_0^h W_i(s)\, ds$	$\int_0^1 z_\tau^{(0)}\int_0^1 Z_{\tau,\xi}^{(i)}\, d\xi\, d\tau$

then it is of mean-square order 1.0. We can choose some polynomials satisfying (3.3), for example

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad A_{\tau,\xi}^{(0)} = \gamma^{(0)}, \qquad A_{\tau,\xi}^{(1)} = \frac{1}{2}, \tag{3.4}$$

by $B_\tau^{(i)} = \lambda^{(i)}, A_{\tau,\xi}^{(i)} = \gamma^{(i)}, i = 0,1$,

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad A_{\tau,\xi}^{(0)} = \gamma_1^{(0)}\tau + \gamma_2^{(0)}\xi + \gamma_3^{(0)},$$
$$A_{\tau,\xi}^{(1)} = \gamma_1^{(1)}\left(\tau - \frac{1}{2}\right) + \gamma_2^{(1)}\left(\xi - \frac{1}{2}\right) + \frac{1}{2}, \tag{3.5}$$

by $B_\tau^{(i)} = \lambda^{(i)}, A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}, i = 0,1$,

$$B_\tau^{(0)} = \lambda_1^{(0)}\left(\tau - \frac{1}{2}\right) + 1, \qquad B_\tau^{(1)} = \lambda_1^{(1)}\left(\tau - \frac{1}{2}\right) + 1, \qquad A_{\tau,\xi}^{(0)} = \gamma^{(0)}, \qquad A_{\tau,\xi}^{(1)} = \frac{1}{2}, \tag{3.6}$$

by $B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}, A_{\tau,\xi}^{(i)} = \gamma^{(i)}, i = 0,1$,

$$B_\tau^{(0)} = \lambda_1^{(0)}\left(\tau - \frac{1}{2}\right) + 1, \qquad B_\tau^{(1)} = \lambda_1^{(1)}\left(\tau - \frac{1}{2}\right) + 1,$$
$$A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}, \qquad A_{\tau,\xi}^{(1)} = \gamma_1^{(1)}\left(\tau - \frac{2}{3}\right) + \gamma_2^{(1)}\left(\xi - \frac{1}{2}\right) + \frac{1}{2}, \tag{3.7}$$

by $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}, B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}, i = 0,1$.

It is almost mandatory that the practical implementation of the CSSRK methods (3.2a)–(3.2b) uses a numerical quadrature formula. By applying the quadrature formula $(b_i, c_i)_{i=1}^s$

to (3.2a)–(3.2b), we use of the notations $Y_i = Y_{c_i}$, deriving the SRK methods

$$Y_i = y_0 + h \sum_{j=1}^{s} b_j A^{(0)}_{c_i,c_j} f(Y_j) + J_1 \sum_{j=1}^{s} b_j A^{(1)}_{c_i,c_j} g(Y_j), \quad i = 1, 2, \ldots, s, \tag{3.8a}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i B^{(0)}_{c_i} f(Y_i) + J_1 \sum_{i=1}^{s} b_i B^{(1)}_{c_i} g(Y_i), \tag{3.8b}$$

where

$$A^{(0)}_{c_i,c_j} = A^{(0)}_{\tau,\xi}|_{\tau=c_i,\xi=c_j}, \qquad A^{(1)}_{c_i,c_j} = A^{(1)}_{\tau,\xi}|_{\tau=c_i,\xi=c_j},$$

$$B^{(0)}_{c_i} = B^{(0)}_{\tau}|_{\tau=c_i}, \qquad B^{(1)}_{c_i} = B^{(1)}_{\tau}|_{\tau=c_i},$$

which can be formulated by the following Butcher tableau:

$$
\begin{array}{c|cc}
& (b_j A^{(0)}_{c_i,c_j})_{s\times s} & (b_j A^{(1)}_{c_i,c_j})_{s\times s} \\
\hline
& (b_i B^{(0)}_{c_i})_{1\times s} & (b_i B^{(1)}_{c_i})_{1\times s}
\end{array}. \tag{3.9}
$$

The following result implies that we can construct SRK methods with the same order of CSSRK methods with the help of suitable numerical quadrature formulas.

Theorem 3.1 *If the CSSRK methods* (3.2a)–(3.2b) *satisfy the order conditions* (3.3)*, and* $A^{(1)}_{\tau,\xi}$ *is a* m_1*-order polynomial of* τ *and* ξ, $B^{(0)}_{\tau}$, $B^{(1)}_{\tau}$ *are* m_2-*,* m_3*-order polynomials of* τ, *respectively. Then the associated SRK methods* (3.9) *derived by using the quadrature formulas* $(b_i, c_i)_{i=1}^{s}$ *with accuracy up to degree* $\max\{m_2, m_1 + m_3\}$ *are also of order* 1.0.

Proof Recall that $A^{(1)}_{\tau,\xi}$ is m_1-order polynomial, $B^{(0)}_{\tau}$, $B^{(1)}_{\tau}$ are m_2-, m_3-order polynomial, based on algebra analysis, the numerical quadrature formulas $(b_i, c_i)_{i=1}^{s}$ with accuracy up to degree $\max\{m_2, m_1 + m_3\}$ are exactly established for (3.3). Then we have

$$\sum_{i=1}^{s} b_i B^{(0)}_{c_i} = 1, \qquad \sum_{i=1}^{s} b_i B^{(1)}_{c_i} = 1, \qquad \sum_{i=1}^{s} \sum_{j=1}^{s} b_i b_j B^{(1)}_{c_i} A^{(1)}_{c_i,c_j} = \frac{1}{2},$$

thus, the associated SRK methods (3.9) are of order 1.0. □

As an application, for the constructed CSSRK methods of mean-square order 1.0, if we use more quadrature points then we can get lots of new SRK methods which are not found in the literature. For example, it gives the SRK schemes shown in the following.

A SRK method of order 1.0 derived by $B^{(i)}_{\tau} = \lambda^{(i)}$, $A^{(i)}_{\tau,\xi} = \gamma^{(i)}$, $i = 0, 1$, using the left rectangle quadrature,

$$
\begin{array}{c|cc}
& \gamma^{(0)} & \frac{1}{2} \\
\hline
& 1 & 1
\end{array}. \tag{3.10}
$$

A SRK method of order 1.0 derived by $B_\tau^{(i)} = \lambda^{(i)}$, $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $i = 0, 1$, using the trapezium quadrature formula,

$$
\begin{array}{c|cc}
& \frac{1}{2}\gamma_3^{(0)} & \frac{1}{2}(\gamma_2^{(0)} + \gamma_3^{(0)}) \\
& \frac{1}{2}(\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{2}(\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\
\hline
& \frac{1}{2} & \frac{1}{2} \\
\hline
& \frac{1}{2}(\frac{1}{2} - \frac{1}{2}\gamma_1^{(1)} - \frac{1}{2}\gamma_2^{(1)}) & \frac{1}{2} - \frac{1}{2}\gamma_1^{(1)} \\
& \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\gamma_1^{(1)} - \frac{1}{2}\gamma_2^{(1)}) & \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\gamma_1^{(1)} + \frac{1}{2}\gamma_2^{(1)}) \\
\hline
& \frac{1}{2} & \frac{1}{2}
\end{array} \quad . \tag{3.11}
$$

A SRK method of order 1.0 derived by $B_\tau^{(i)} = \lambda^{(i)}$, $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $i = 0, 1$, using the Lobatto quadrature formula,

$$
\begin{array}{c|ccc}
& \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_2^{(0)} + \gamma_3^{(0)}) \\
& \frac{1}{12}\gamma_1^{(0)} + \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \frac{1}{2}\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\
& \frac{1}{6}(\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\
\hline
& \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\hline
& \frac{1}{12} & -\frac{1}{3}\gamma_1^{(1)} + \frac{1}{3} & -\frac{1}{12}\gamma_1^{(1)} + \frac{1}{12}\gamma_2^{(1)} + \frac{1}{12} \\
& -\frac{1}{12}\gamma_2^{(1)} + \frac{1}{12} & \frac{1}{3} & \frac{1}{12}\gamma_2^{(1)} + \frac{1}{12} \\
& \frac{1}{12}\gamma_1^{(1)} - \frac{1}{12}\gamma_2^{(1)} + \frac{1}{12} & \frac{1}{3}\gamma_1^{(1)} + \frac{1}{3} & \frac{1}{12}\gamma_1^{(1)} + \frac{1}{12}\gamma_2^{(1)} + \frac{1}{12} \\
\hline
& \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array} \quad . \tag{3.12}
$$

A SRK method of order 1.0 derived by $B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}$, $A_{\tau,\xi}^{(i)} = \gamma^{(i)}$, $i = 0, 1$, using the trapezium quadrature formula,

$$
\begin{array}{c|cccc}
& \frac{1}{2}\gamma^{(0)} & \frac{1}{2}\gamma^{(0)} & \frac{1}{4} & \frac{1}{4} \\
& \frac{1}{2}\gamma^{(0)} & \frac{1}{2}\gamma^{(0)} & \frac{1}{4} & \frac{1}{4} \\
\hline
& \frac{1}{2}(1 - \frac{1}{2}\lambda_1^{(0)}) & \frac{1}{2}(1 + \frac{1}{2}\lambda_1^{(0)}) & \frac{1}{2}(1 - \frac{1}{2}\lambda_1^{(1)}) & \frac{1}{2}(1 + \frac{1}{2}\lambda_1^{(1)})
\end{array} \quad . \tag{3.13}
$$

A SRK method of order 1.0 derived by $B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}$, $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $i = 0, 1$, using the Gaussian quadrature formula,

$$
\begin{array}{c|cc}
& \frac{1}{2}(1 - \frac{1}{2\sqrt{3}}(\gamma_1^{(0)} + \gamma_2^{(0)}) + \gamma_3^{(0)}) & \frac{1}{2}(1 - \frac{1}{2\sqrt{3}}(\gamma_1^{(0)} - \gamma_2^{(0)}) + \gamma_3^{(0)}) \\
& \frac{1}{2}(1 + \frac{1}{2\sqrt{3}}(\gamma_1^{(0)} - \gamma_2^{(0)}) + \gamma_3^{(0)}) & \frac{1}{2}(1 + \frac{1}{2\sqrt{3}}(\gamma_1^{(0)} + \gamma_2^{(0)}) + \gamma_3^{(0)}) \\
\hline
& \frac{1}{2}(1 - \frac{1}{2\sqrt{3}}\lambda_1^{(0)}) & \frac{1}{2}(1 + \frac{1}{2\sqrt{3}}\lambda_1^{(0)}) \\
\hline
& \frac{1}{4}(1 - \frac{1}{6}\gamma_1^{(1)}\lambda_1^{(1)} - \frac{1}{\sqrt{3}}(\gamma_1^{(1)} + \gamma_2^{(1)})) & \frac{1}{4}(1 - \frac{1}{6}\gamma_1^{(1)}\lambda_1^{(1)} - \frac{1}{\sqrt{3}}(\gamma_1^{(1)} - \gamma_2^{(1)})) \\
& \frac{1}{4}(1 - \frac{1}{6}\gamma_1^{(1)}\lambda_1^{(1)} + \frac{1}{\sqrt{3}}(\gamma_1^{(1)} - \gamma_2^{(1)})) & \frac{1}{4}(1 - \frac{1}{6}\gamma_1^{(1)}\lambda_1^{(1)} + \frac{1}{\sqrt{3}}(\gamma_1^{(1)} + \gamma_2^{(1)})) \\
\hline
& \frac{1}{2}(1 - \frac{1}{2\sqrt{3}}\lambda_1^{(1)}) & \frac{1}{2}(1 + \frac{1}{2\sqrt{3}}\lambda_1^{(1)})
\end{array} \quad . \tag{3.14}
$$

A SRK method of order 1.0 derived by $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}$, $i = 0, 1$, using the Lobatto quadrature formula,

$$
\begin{array}{c|ccc}
& \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_2^{(0)} + \gamma_3^{(0)}) \\
& \frac{1}{12}\gamma_1^{(0)} + \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \frac{1}{2}\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\
& \frac{1}{6}(\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\
\hline
& \frac{1}{6}(1 - \frac{1}{2}\lambda_1^{(0)}) & \frac{2}{3} & \frac{1}{6}(1 + \frac{1}{2}\lambda_1^{(0)}) \\
\hline
& \frac{1}{6}\gamma_3^{(1)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\gamma_2^{(1)} + \gamma_3^{(1)}) \\
& \frac{1}{12}\gamma_1^{(1)} + \frac{1}{6}\gamma_3^{(1)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \frac{1}{2}\gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(1)} + \gamma_2^{(1)} + \gamma_3^{(1)}) \\
& \frac{1}{6}(\gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\gamma_1^{(1} + \gamma_2^{(1)} + \gamma_3^{(1)}) \\
\hline
& \frac{1}{6}(1 - \frac{1}{2}\lambda_1^{(1)}) & \frac{2}{3} & \frac{1}{6}(1 + \frac{1}{2}\lambda_1^{(1)})
\end{array}
\tag{3.15}
$$

4 High order CSSRK methods for single integrand SDEs

In this section, we consider a particular class of Stratonovich SDEs, single integrand SDEs (1.2). All results in this section can straightforwardly be considered as the case of one-dimensional Wiener process,

$$
dx(t) = \lambda g(x)\,dt + \sigma g(x) \circ dW(t), \quad x(0) = x_0 \in \mathbb{R},
\tag{4.1}
$$

with

$$
W(t) := \frac{1}{\sigma}\sum_{k=1}^{d}\sigma_k W_k(t), \quad \sigma = \sqrt{\sum_{k=1}^{d}\sigma_k^2}.
$$

Then (4.1) is solved by (3.2a)–(3.2b), we can obtain

$$
Y_\tau = y_0 + \triangle\mu\int_0^1 A_{\tau,\xi}g(Y_\xi)\,d\xi,
\tag{4.2a}
$$

$$
y_1 = y_0 + \triangle\mu\int_0^1 B_\tau g(Y_\tau)\,d\tau,
\tag{4.2b}
$$

where $\triangle\mu = \mu(t + h) - \mu(t) = \lambda h + \sigma(W(t + h) - W(t))$, $h > 0$.

Due to the following lemmas, the main result is that the B-series of the exact and the numerical solution are exactly as in the ODEs case, with the exception that integration is now performed with respect to μ instead of h.

Lemma 4.1 [20] $n \in \mathbb{N}$, *for all $h > 0$, we have*

$$
\int_0^h \mu^n \circ d\mu = \frac{1}{1+n}\mu(h)^{n+1}.
$$

Lemma 4.2 ([20]) *For $n \in \mathbb{N}$, then we have*

$$
E\mu(h)^n =
\begin{cases}
O(h^{n/2}), & \text{if n is even,} \\
O(h^{(n+1)/2}), & \text{if n is odd.}
\end{cases}
$$

Lemma 4.3 ([20]) *Let $\gamma : T \to \mathbb{R}$ be given by*

$$\gamma(\emptyset) = 1, \qquad \gamma(\bullet) = 1, \qquad \gamma\big([t_1, t_2, \ldots, t_l]\big) = \hat{\rho}\big([t_1, t_2, \ldots, t_l]\big) \prod_{i=1}^{l} \gamma(t_i),$$

with $\hat{\rho}(t)$ denotes the number of nodes in a tree t, then the solution of (4.1) can be written as B-series $B(\phi, x_0; h)$

$$\phi(t)(h) = \frac{\mu(h)^{\hat{\rho}(t)}}{\gamma(t)}, \quad t \in T.$$

For the numerical approximation (4.2a)–(4.2b), the following results hold.

Theorem 4.1 *The numerical solution y_1 as well as the continuous stage values Y_τ can be written in terms of B-series*

$$Y_\tau = B(\Phi_\tau, y_0; h), \qquad y_1 = B(\varphi, y_0; h),$$

where $t = [t_1, t_2, \ldots, t_l] \in T$, and

$$\Phi_\tau(t) = \triangle \mu^{\hat{\rho}(t)} \hat{\Phi}_\tau(t), \qquad \hat{\Phi}_\tau(\emptyset) = 1, \qquad \varphi(t) = \triangle \mu^{\hat{\rho}(t)}, \qquad \hat{\varphi}(\emptyset) = 1,$$

$$\hat{\Phi}_\tau(\bullet) = \int_0^1 A_{\tau,\xi}\, d\xi, \qquad \hat{\Phi}_\tau(t) = \int_0^1 A_{\tau,\xi} \prod_{i=1}^{l} \hat{\Phi}_\xi(t_i)\, d\xi,$$

$$\hat{\varphi}(\bullet) = \int_0^1 B_\tau\, d\tau, \qquad \hat{\varphi}(t) \int_0^1 B_\tau \prod_{i=1}^{l} \hat{\Phi}_\tau(t_i)\, d\tau.$$

Proof Write Y_τ as a B-series, that is,

$$Y_\tau = B(\Phi_\tau, x_0; h) = \sum_{t \in T} \alpha(t) \Phi_\tau(t)(h) F(t)(y_0).$$

Use (4.2a) to obtain

$$
\begin{aligned}
Y_\tau &= y_0 + \triangle \mu \int_0^1 A_{\tau,\xi} g(Y_\xi)\, d\xi \\
&= y_0 + \triangle \mu \int_0^1 A_{\tau,\xi} \sum_{t \in T_g} \alpha(t) \Phi'_\xi(t)(h) F(t)(y_0)\, d\xi \\
&= y_0 + \sum_{t \in T_g} \alpha(t) \left(\triangle \mu \int_0^1 A_{\tau,\xi} \Phi'_\xi(t)(h)\, d\xi \right) F(t)(y_0),
\end{aligned}
$$

where

$$\Phi'_\xi(\bullet) = 1, \qquad \Phi_\tau(\emptyset) = 1, \qquad \Phi'_\xi(t)(h) = \prod_{i=1}^{l} \Phi_\xi(t_i)(h),$$

$$\Phi_\tau(t)(h) = \triangle \mu \int_0^1 A_{\tau,\xi} \Phi'_\xi(t)(h)\, d\xi = \triangle \mu \int_0^1 A_{\tau,\xi} \prod_{i=1}^{l} \Phi_\xi(t_i)(h)\, d\xi,$$

$$\hat{\Phi}_\tau(t)(h) = \int_0^1 A_{\tau,\xi} \prod_{i=1}^l \hat{\Phi}_\xi(t_i)(h)\,d\xi, \quad t = [t_1, t_2, \ldots, t_l]_g \in T_g \in T,$$

thus $\Phi_\tau(t)(h) = \triangle\mu^{\hat{\rho}(t)}\hat{\Phi}_\tau(t)(h)$. Analogous to y_1.   □

In the following, the main results of this section are presented.

Theorem 4.2 *The DCSRK methods of deterministic order p_d is corresponding with the CSSRK methods* (4.2a)–(4.2b) *of mean square $p_\mu = \lfloor p_d/2 \rfloor$, under the condition that $\triangle\mu$ is either sampled from the exact distribution or chosen such that at least the first $2p_\mu + 1$ moments are considered with the ones of the exact distribution.*

Proof With all the B-series in place, we can now present the order conditions for the mean square global order p_μ if and only if

$$E\big(\phi(t)(h) - \varphi(t)(h)\big)^2 = O\big(h^{2p_\mu+1}\big), \quad t \in T, \tag{4.3}$$

$$E\big(\phi(t)(h) - \varphi(t)(h)\big) = O\big(h^{p_\mu+1}\big), \quad t \in T, \tag{4.4}$$

and all elementary differentials $F(t)$ fulfill the linear growth condition. Assume that $\triangle\mu$ is either sampled from the exact distribution or chosen such that at least the first $2p_\mu + 1$ moments coincided with the ones of the exact distribution. Due to $E(\mu(h)^{2\hat{\rho}(t)}) = O(h^{\hat{\rho}(t)})$ by Lemma 4.3, (4.3) is then by Lemma 4.2 and Theorem 4.1 automatically fulfilled for all $t \in T$ with $\hat{\rho}(t) \geq 2p_\mu + 1$, and satisfying the remaining trees if and only if

$$\varphi(t) = \frac{1}{\gamma(t)}, \quad t \in T, \qquad \hat{\rho}(t) \leq 2p_\mu. \tag{4.5}$$

Note that (4.5) is just the condition that for the order p_d of the DCSRK methods applied to a deterministic system ($\sigma = 0$) with $p_d = 2p_\mu$. Similarly, (4.4) is automatically fulfilled for all $t \in T$ with

$$\hat{\rho}(t) \geq \begin{cases} 2p_\mu + 2, & \text{if } \hat{\rho}(t) \text{ is even,} \\ 2p_\mu, & \text{if } \hat{\rho}(t) \text{ is odd,} \end{cases}$$

satisfying the remaining trees if and only if

$$\varphi(t) = \frac{1}{\gamma(t)}, \quad t \in T, \qquad \hat{\rho}(t) \leq \begin{cases} 2p_\mu + 1, & \text{if } \hat{\rho}(t) \text{ is even,} \\ 2p_\mu, & \text{if } \hat{\rho}(t) \text{ is odd.} \end{cases}$$

Thus, the methods will be mean-square consistent of order p_μ if the deterministic order is

$$p_d = \begin{cases} 2p_\mu, & p_\mu \in \mathrm{N}, \\ 2p_\mu + 1, & p_\mu + \frac{1}{2} \in \mathrm{N}, \end{cases}$$

or, vice versa, the methods of deterministic order p_d will converge with mean-square order of $\lfloor \frac{p_d}{2} \rfloor$.   □

As is well known, when solving the deterministic differential equation, the DCSRK method with the polynomial

$$A_{\tau,\xi} = \tau + \frac{\sqrt{3}}{2}\big(6\tau^2 - 6\tau + 1\big)(2\tau - 1) - \xi + \frac{1}{2}, \qquad B_\tau = 1, \tag{4.6}$$

is of mean-square order 4.0 [22]. Hence, the corresponding CSSRK method is of order 2.0. We can also choose multiple numerical quadrature formulas to get different SRK methods of high order.

5 CSSRK methods for SDEs with additive noise

Concerning the specific problems with additive noise, thus (1.3) is solved by (2.1a)–(2.1b) defined by

$$
Y_\tau = y_0 + h \int_0^1 A_{\tau,\xi}^{(0)} f(Y_\xi)\, d\xi + \sum_{k=1}^d J_k \int_0^1 A_{\tau,\xi}^{(1)} \sigma_k\, d\xi
$$

$$
+ \sum_{k=1}^d \frac{J_{k0}}{h} \int_0^1 A_{\tau,\xi}^{(2)} \sigma_k\, d\xi, \tag{5.1a}
$$

$$
y_1 = y_0 + h \int_0^1 B_\tau^{(0)} f(Y_\tau)\tau + \sum_{k=1}^d J_k \int_0^1 B_\tau^{(1)} \sigma_k\, d\tau
$$

$$
+ \sum_{k=1}^d \frac{J_{k0}}{h} \int_0^1 B_\tau^{(2)} \sigma_k\, d\tau, \tag{5.1b}
$$

where $J_1 = \triangle W_k$, $J_{k0} = \int_0^h \int_0^s dW_k(s)\,ds$. Because the noise terms are additive, we find that a elementary differential vanishes if its multicolored rooted tree contains a node following a stochastic node directly except if the deterministic node t_0 is the only succeeding end node. Table 2 consists of multicolored rooted trees whose elementary differentials are nonzero. We are able to obtain a set of order conditions guaranteeing that the CSSRK methods (5.1a)–(5.1b) obtained mean-square order 1.5 as detailed below.

Table 2 Multicolored rooted trees for (5.1a)–(5.1b) with order less than or equal to 2.0

No.	t	$\rho(t)$	$\phi(t)$	$\varphi(t)$
1	$\bullet_i$	0.5	J_i	$\int_0^1 J_i B_\tau^{(1)}\, d\tau + \int_0^1 \frac{J_{i0}}{h} B_\tau^{(2)}\, d\tau$
2	$\bullet_0$	1	J_0	$\int_0^1 h B_\tau^{(0)}\, d\tau$
3		1.5	$\int_0^h s \circ dW_i(s)$	$\int_0^1 h B_\tau^{(0)} \big(\int_0^1 J_i A_{\tau,\xi}^{(1)}\, d\xi + \int_0^1 \frac{J_{i0}}{h} A_{\tau,\xi}^{(2)}\, d\xi\big)\, d\tau$
4		2	$\int_0^1 s\, ds$	$\int_0^1 h B_\tau^{(0)} \int_0^1 h A_{\tau,\xi}^{(0)}\, d\xi\, d\tau$
5		2	$\int_0^h W_i(s)W_j(s)\, ds$	$\int_0^1 h B_\tau^{(0)} \big(\int_0^1 J_i A_{\tau,\xi}^{(1)}\, d\xi + \int_0^1 \frac{J_{i0}}{h} A_{\tau,\xi}^{(2)}\, d\tau\big)\big(\int_0^1 J_j A_{\tau,\xi}^{(1)}\, d\xi + \int_0^1 \frac{J_{j0}}{h} A_{\tau,\xi}^{(2)}\, d\tau\big)\, d\tau$
6		2	$\int_0^h W_i(s)^2\, ds$	$\int_0^1 h B_\tau^{(0)} \big(\int_0^1 J_i A_{\tau,\xi}^{(1)}\, d\xi + \int_0^1 \frac{J_{i0}}{h} A_{\tau,\xi}^{(2)}\, d\xi\big)^2\, d\tau$

Theorem 5.1 *Suppose that* (1.3) *with d independent additive noises is approximated by the CSSRK methods* (5.1a)–(5.1b). *If coefficients* $B_\tau^{(0)}, B_\tau^{(1)}, B_\tau^{(2)}, A_{\tau,\xi}^{(0)}, A_{\tau,\xi}^{(1)}, A_{\tau,\xi}^{(2)}$ *of the CSSRK methods* (5.1a)–(5.1b) *satisfy conditions*

$$\int_0^1 B_\tau^{(0)}\,\mathrm{d}\tau = 1, \qquad \int_0^1 B_\tau^{(1)}\,\mathrm{d}\tau = 1, \qquad \int_0^1 B_\tau^{(2)} = 0,$$

$$\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(0)}\,\mathrm{d}\xi\,\mathrm{d}\tau = \frac{1}{2},$$

$$\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi\,\mathrm{d}\tau = 0, \tag{5.2}$$

$$\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi\,\mathrm{d}\tau = 1,$$

$$\int_0^1 B_\tau^{(0)}\left(\left(\int_0^1 A_{\tau,\xi}^{(1)}\,\mathrm{d}\tau\right)^2 + \frac{1}{3}\left(\int_0^1 A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi\right)^2 + \int_0^1 A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi \int_0^1 A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi\right)\mathrm{d}\tau = \frac{1}{2},$$

then they are of mean-square order 1.5.

Proof First, we have these facts for the increments:

$$E(J_i) = E(J_{i0}) = 0, \qquad E(J_i^2) = h,$$

$$E\left(J_{i0}^2 = \frac{h^3}{3}\right), \qquad E(J_i J_{i0}) = \frac{h^2}{2}, \qquad E(J_i J_{j0}) = 0, \quad i \neq j.$$

In order to attain mean-square order 1.5, we need $\phi(t) = \varphi(t)$ for trees 1–3 ($\rho(t) \leq 1.5$), and $E(\phi(t)) = E(\varphi(t))$ for trees 4–6 ($\rho(t) = 2$), respectively, in Table 2. To be specific,

  (a) $J_i = \int_0^1 J_i B_\tau^{(1)}\,\mathrm{d}\tau + \int_0^1 \frac{J_{i0}}{h} B_\tau^{(2)}\,\mathrm{d}\tau$, we have $\int_0^1 B_\tau^{(1)}\,\mathrm{d}\tau = 1$, $\int_0^1 B_\tau^{(2)}\,\mathrm{d}\tau = 0$;
  (b) $J_0 = h\int_0^1 B_\tau^{(0)}\,\mathrm{d}\tau$, we have $\int_0^1 B_\tau^{(0)} = 1$;
  (c) $\int_0^1 s \circ \mathrm{d}W_i(s) = \int_0^1 B_\tau^{(0)}(\int_0^1 J_i A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi + \int_0^1 \frac{J_{i0}}{h} A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi)\,\mathrm{d}\tau$, we have
      $\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi\,\mathrm{d}\tau = 0$;
  (d) $\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(0)}\,\mathrm{d}\xi\,\mathrm{d}\tau = \frac{1}{2}$;
  (e) $E(\int_0^1 W_i(s)W_j(s)\,\mathrm{d}s) = 0$;
  (f) $E(\int_0^1 W_i(s)^2\,\mathrm{d}s) = \frac{1}{2}h^2 = \int_0^1 hB_\tau^{(0)}(\int_0^1 J_i A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi + \int_0^1 \frac{J_{i0}}{h} A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi)^2\,\mathrm{d}\tau$.

The proof is completed. $\qquad\qquad\square$

We can choose some polynomials satisfying (5.2), for example

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad B_\tau^{(2)} = 0,$$

$$A_{\tau,\xi}^{(0)} = \gamma_1^{(0)}\left(\tau - \frac{1}{2}\right) + \gamma_2^{(0)}\left(\xi - \frac{1}{2}\right) + \frac{1}{2}, \qquad A_{\tau,\xi}^{(1)} = \gamma_1^{(1)}\left(\tau - \frac{1}{2}\right) + \gamma_2^{(1)}\left(\xi - \frac{1}{2}\right), \tag{5.3}$$

$$A_{\tau,\xi}^{(2)} = \left(\frac{-3}{2}\gamma_1^{(1)} \pm 18\sqrt{\left(\frac{1}{54} - \frac{(\gamma_1^{(1)})^2}{432}\right)}\right)\left(\tau - \frac{1}{2}\right) + \gamma_2^{(2)}\xi - \frac{1}{2}\gamma_2^{(2)} + 1,$$

by $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $B_\tau^{(i)} = \lambda_1^{(i)}\tau + \lambda_2^{(i)}$, $i = 0, 1, 2$.

Similarly, by applying the quadrature formula $(b_i, c_i)_{i=1}^s$ to (5.1a)–(5.1b), utilizing the notations $Y_i = Y_{c_i}$, $i = 1, 2, \ldots, s$, we derive the corresponding SRK methods,

$$Y_i = y_0 + h \sum_{j=1}^s b_j A_{c_i,c_j}^{(0)} f(Y_j) + \sum_{k=1}^d J_k \sum_{j=1}^s b_j \sigma_k + \sum_{k=1}^d \frac{J_{k0}}{h} \sum_{j=1}^s b_j A_{c_i,c_j}^{(2)} \sigma_k, \qquad (5.4a)$$

$$y_1 = y_0 + h \sum_{i=1}^s b_i B_{c_i}^{(0)} f(Y_i) + \sum_{k=1}^d J_k \sum_{i=1}^s b_i B_{c_i}^{(1)} \sigma_k + \sum_{k=1}^d \frac{J_{k0}}{h} \sum_{i=1}^s b_i B_{c_i}^{(2)} \sigma_k, \qquad (5.4b)$$

where

$$A_{c_i,c_j}^{(0)} = A_{\tau,\xi}^{(0)}|_{\tau=c_i,\xi=c_j}, \qquad A_{c_i,c_j}^{(1)} = A_{\tau,\xi}^{(1)}|_{\tau=c_i,\xi=c_j}, \qquad A_{c_i,c_j}^{(2)} = A_{\tau,\xi}^{(2)}|_{\tau=c_i,\xi=c_j},$$

$$B_{c_i}^{(0)} = B_\tau^{(0)}|_{\tau=c_i}, \qquad B_{c_i}^{(1)} = B_\tau^{(1)}|_{\tau=c_i}, \qquad B_{c_i}^{(2)} = B_\tau^{(2)}|_{\tau=c_i},$$

which can be formulated by the following Butcher tableau:

$$\begin{array}{c|ccc} & (b_j A_{c_i,c_j}^{(0)})_{s\times s} & (b_j A_{c_i,c_j}^{(1)})_{s\times s} & (b_j A_{c_i,c_j}^{(2)})_{s\times s} \\ \hline & (b_i B_{c_i}^{(0)})_{1\times s} & (b_i B_{c_i}^{(1)})_{1\times s} & (b_i B_{c_i}^{(2)})_{1\times s} \end{array}. \qquad (5.5)$$

The following result implies that we can also construct SRK methods of the same order via quadrature formulas.

Theorem 5.2 *If the CSSRK methods* (5.1a)–(5.1b) *satisfy the order conditions* (5.2), *and* $B_\tau^{(0)}, B_\tau^{(1)}, B_\tau^{(2)}, A_{\tau,\xi}^{(0)}, A_{\tau,\xi}^{(1)}, A_{\tau,\xi}^{(2)}$ *are, respectively,* $m_1, m_2, m_3, m_4, m_5, m_6$-*order polynomials, then the associated SRK methods* (5.5) *derived by using quadrature formulas* $(b_i, c_i)_{i=1}^s$ *with accuracy up to degree* $\max\{m_5, m_6, m_4 + m_1, 2m_2 + m_4, 2m_3 + m_4, m_2 + m_3 + m_4\}$ *are also of order* 1.5.

Proof Similar to Theorem 3.1.   □

By using polynomials with different orders and any numerical quadrature formula with order satisfied Theorem 5.2, we can get the classical SRK methods of order 1.5.

A SRK method of order 1.5 derived by $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $B_\tau^{(i)} = \lambda^{(i)}$, $i = 0, 1, 2$, using the Simpson quadrature,

$$\begin{array}{c|ccc} & \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_2^{(0)} + \gamma_3^{(0)}) \\ & \frac{1}{12}\gamma_1^{(0)} + \frac{1}{6}\gamma_3^{(0)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \frac{1}{2}\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\ & \frac{1}{6}(\gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{2}{3}(\frac{1}{2}\gamma_2^{(0)} + \gamma_1^{(0)} + \gamma_3^{(0)}) & \frac{1}{6}(\gamma_1^{(0)} + \gamma_2^{(0)} + \gamma_3^{(0)}) \\ \hline & \frac{1}{6}(1 - \frac{1}{2}\lambda^{(0)}) & \frac{2}{3} & \frac{1}{6}(1 + \frac{1}{2}\lambda^{(0)}) \\ & \frac{1}{6}\gamma_3^{(1)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\gamma_2^{(1)} + \gamma_3^{(1)}) \\ & \frac{1}{12}\gamma_1^{(1)} + \frac{1}{6}\gamma_3^{(1)} & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \frac{1}{2}\gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(1)} + \gamma_2^{(1)} + \gamma_3^{(1)}) \\ & \frac{1}{6}(\gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{2}{3}(\frac{1}{2}\gamma_2^{(1)} + \gamma_1^{(1)} + \gamma_3^{(1)}) & \frac{1}{6}(\gamma_1^{(1)} + \gamma_2^{(1)} + \gamma_3^{(1)}) \\ \hline & \frac{1}{6}(1 - \frac{1}{2}\lambda^{(1)}) & \frac{2}{3} & \frac{1}{6}(1 + \frac{1}{2}\lambda^{(1)}) \end{array}. \qquad (5.6)$$

$$\left.\begin{array}{ccc}
\frac{1}{6}(1 - \frac{1}{2}\gamma_1^{(2)} - \frac{1}{2}\gamma_2^{(2)}) & \frac{2}{3}(1 - \frac{1}{2}\gamma_1^{(2)}) & \frac{1}{6}(-\frac{1}{2}\gamma_1^{(2)} + \frac{1}{2}\gamma_2^{(2)} + 1) \\
\frac{1}{6}(-\frac{1}{2}\gamma_2^{(2)} + 1) & \frac{2}{3} & \frac{1}{6}(\frac{1}{2}\gamma_2^{(2)} + 1) \\
\frac{1}{6}(1 + \frac{1}{2}\gamma_1^{(2)} - \frac{1}{2}\gamma_2^{(2)}) & \frac{2}{3}(\frac{1}{2}\gamma_1^{(2)} + 1) & \frac{1}{6}(\frac{1}{2}\gamma_1^{(2)} + \frac{1}{2}\gamma_2^{(2)} + 1) \\
\hline
0 & 0 & 0
\end{array}\right.$$

A SRK method of order 1.5 derived by $A_{\tau,\xi}^{(i)} = \gamma_1^{(i)}\tau + \gamma_2^{(i)}\xi + \gamma_3^{(i)}$, $B_\tau^{(i)} = \lambda^{(i)}$, $i = 0, 1, 2$ with $\gamma_1^{(2)} = -3/2\gamma_1^{(1)} \pm 18(1/54 - (\gamma_1^{(1)})^2/432)^{1/2}$ using Gaussian quadrature with 2 nodes,

$$\left.\begin{array}{cc}
\frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(0)} - \frac{1}{2\sqrt{3}}\gamma_2^{(0)} + \frac{1}{2}) & \frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(0)} + \frac{1}{2\sqrt{3}}\gamma_2^{(0)} + \frac{1}{2}) \\
\frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(0)} - \frac{1}{2\sqrt{3}}\gamma_2^{(0)} + \frac{1}{2}) & \frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(0)} + \frac{1}{2\sqrt{3}}\gamma_2^{(0)} + \frac{1}{2}) \\
\hline
\frac{1}{2} & \frac{1}{2} \\[2pt]
\frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(1)} - \frac{1}{2\sqrt{3}}\gamma_2^{(1)}) & \frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(1)} + \frac{1}{2\sqrt{3}}\gamma_2^{(1)}) \\
\frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(1)} - \frac{1}{2\sqrt{3}}\gamma_2^{(1)}) & \frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(1)} + \frac{1}{2\sqrt{3}}\gamma_2^{(1)}) \\
\hline
\frac{1}{2} & \frac{1}{2} \\[2pt]
\frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(2)} - \frac{1}{2\sqrt{3}}\gamma_2^{(2)} + 1) & \frac{1}{2}(-\frac{1}{2\sqrt{3}}\gamma_1^{(2)} + \frac{1}{2\sqrt{3}}\gamma_2^{(2)} + 1) \\
\frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(2)} - \frac{1}{2\sqrt{3}}\gamma_2^{(2)} + 1) & \frac{1}{2}(\frac{1}{2\sqrt{3}}\gamma_1^{(2)} + \frac{1}{2\sqrt{3}}\gamma_2^{(2)} + 1) \\
\hline
0 & 0
\end{array}\right. \tag{5.7}$$

The special second-order systems with additive noise (1.4) can be regarded as a $2m$-dimensional system with $y(t) = \dot{x}(t)$, $y(0) = \dot{x}(0)$:

$$\begin{cases}
dx(t) = y(t)\,dt, & x(0) = x_0 \in \mathrm{R}^m, \\
dy(t) = f(x)\,dt + \sum_{k=1}^d \sigma_k \circ dW_k(t), & y(0) = y_0 \in \mathrm{R}^m.
\end{cases} \tag{5.8}$$

Obviously, this is a special example of (1.3), and the mean-square order conditions of the foregoing CSSRK methods are also available here. However, according to the specific features of (5.8), we are able to simplify some conditions and get a higher mean-square order without more effort.

Theorem 5.3 *If the CSSRK methods* (5.1a)–(5.1b) *for the second-order systems* (5.8) *possess coefficients satisfying*

$$\int_0^1 B_\tau^{(0)}\,d\tau = 1, \qquad \int_0^1 B_\tau^{(1)}\,d\tau = 1,$$
$$\int_0^1 B_\tau^{(2)} = 0, \qquad \int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(0)}\,d\xi\,d\tau = \frac{1}{2}, \tag{5.9}$$
$$\int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(1)}\,d\xi\,d\tau = 0, \qquad \int_0^1 B_\tau^{(0)} \int_0^1 A_{\tau,\xi}^{(2)}\,d\xi\,d\tau = 1,$$

then they are mean-square order 2.0.

Proof We find that conditions of the theorem above are similar to Theorem 5.1 except that condition f in Theorem 5.1 is unnecessary here. In order to obtain the mean-square order 2.0, we need to analyze two additional colored rooted trees ($\rho(t) = 2.5$) listed in Table 3.

Table 3 Colored rooted tree for (5.8) with order equal to 2.5

No.	t	$\rho(t)$	$\phi(t)$	$\varphi(t)$
7	$\bullet_i$	2.5	$\int_0^h sW_i(s)\,\mathrm{d}s$	$\int_0^1 hB_\tau^{(0)} \int_0^1 hA_{\tau,\xi}^{(0)}\,\mathrm{d}\xi \int_0^1 J_i A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi\,\mathrm{d}\tau + \int_0^1 hB_\tau^{(0)} \int_0^1 hA_{\tau,\xi}^{(0)}\,\mathrm{d}\xi \int_0^1 \frac{J_i 0}{h} A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi\,\mathrm{d}\tau$
8	$\bullet_0$	2.5	$\int_0^1 \int_0^1 s\,\mathrm{d}s \circ \mathrm{d}W_i(s)$	$\int_0^1 hB_\tau^{(0)} \int_0^1 hA_{\tau,\xi}^{(0)} \int_0^1 J_i A_{\tau,\xi}^{(1)}\,\mathrm{d}\xi\,\mathrm{d}\xi\,\mathrm{d}\tau + \int_0^1 hB_\tau^{(0)} \int_0^1 hA_{\tau,\xi}^{(0)} \int_0^1 \frac{J_i 0}{h} A_{\tau,\xi}^{(2)}\,\mathrm{d}\xi\,\mathrm{d}\xi\,\mathrm{d}\tau$

The condition (f) in Theorem 5.1 is derived from the tree 6 in Table 2, which is essential for general system. However, for system (5.8), the corresponding elemental differential of tree 6 is

$$F(t)(x) = f^{(2)}(x)\big(F(t_i)(x), F(t_i)(x)\big) = 0,$$

because the nth component of it is

$$\big(F(t)(x)\big)_n = \begin{cases} \sum_{j_1,j_2=1}^m \frac{\partial^2 (f(x)_n)}{\partial x^{j_1} \partial x^{j_2}}(0,0), & n = 1,2,\ldots,\frac{m}{2}, \\ \sum_{j_1,j_2=1}^m \frac{\partial^2 ((y)_n)}{\partial y_1^j \partial y_2^j}(\sigma_{j_1}, \sigma_{j_2}), & n = \frac{m}{2}+1,\ldots,m \end{cases} = 0,$$

where $(f(x))_n$ means the nth element of the vector function $f(x)$. Thus tree 6 is unnecessary in this situation. With the same analysis, the elementary differential of tree 5 in Table 2 vanishes as well. Moreover, for the additional trees 7, 8 in Table 3, we can check that

$$\phi(t) = \varphi(t), \qquad \rho(t) \le 2, \qquad E\big(\phi(t)\big) = E\big(\varphi(t)\big), \qquad \rho(t) = 2.5. \qquad \square$$

6 Numerical experiments

In this section, we perform numerical tests to verify the mean-square convergence order proposed in Sects. 3, 4 and 5. The first group consists of method (3.4)–(3.7) and (3.11), (3.14) of different free parameters. The second group of methods consists of the methods (4.6) of deterministic order $p_d = 4.0$ for solving deterministic differential equation, in which case the predicted stochastic order $p_\mu = 2$. Finally, we consider methods (5.9) for SDEs with additive noise. In each example, the solution is approximated with step sizes $2^{-5},\ldots,2^{-9}$ and the sample average of $M = 2000$ independent simulated realizations of the absolute error at the terminal time $T = 1$ is calculated in order to estimate the expectation, accordingly.

6.1 Linear stochastic oscillator

The one-dimensional Linear stochastic oscillator in the sense of Stratonovich SDE is described by

$$\mathrm{d}x(t) = ax(t)\,\mathrm{d}t + bx(t) \circ \mathrm{d}W(t), \quad x(0) = x_0. \tag{6.1}$$

Here a and b are constants. The exact solution of (6.1) is given by

$$x(t) = x_0 \exp\big(at + bW(t)\big).$$

In this experiment, we select $\gamma^{(0)} = 0$ in (3.4), $\gamma_2^{(0)} = \gamma_3^{(0)} = \gamma_2^{(1)} = 0$, $\gamma_1^{(0)} = \gamma_1^{(1)} = 1$ in (3.5), $\lambda_1^{(0)} = \lambda_1^{(1)} = 2$, $\gamma^{(0)} = 0$ in (3.6), $\lambda_1^{(0)} = \lambda_1^{(1)} = 2$, $\gamma_1^{(0)} = \gamma_1^{(1)} = 3$, $\gamma_2^{(0)} = \gamma_2^{(1)} = -3$, $\gamma_3^{(0)} = 1$ in (3.7),

and $\gamma_1^{(0)} = \gamma_1^{(1)} = \gamma_2^{(1)} = \gamma_2^{(0)} = 1$, $\gamma_3^{(0)} = 0$ in (3.11), $\gamma_1^{(0)} = \gamma_1^{(1)} = 1$, $\gamma_2^{(0)} = 0$, $\gamma_3^{(0)} = 1/(2\sqrt{3})$, $\gamma_2^{(1)} = -1$, $\lambda_1^{(0)} = \lambda_1^{(1)} = 2\sqrt{3}$ in (3.14), then we have the methods, respectively, of

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad A_{\tau,\xi}^{(0)} = 0, \qquad A_{\tau,\xi}^{(1)} = \frac{1}{2}, \tag{6.2}$$

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad A_{\tau,\xi}^{(0)} = \tau, \qquad A_{\tau,\xi}^{(1)} = \tau, \tag{6.3}$$

$$B_\tau^{(0)} = 2\tau, \qquad B_\tau^{(1)} = 2\tau, \qquad A_{\tau,\xi}^{(0)} = 0, \qquad A_{\tau,\xi}^{(1)} = \frac{1}{2}, \tag{6.4}$$

$$B_\tau^{(0)} = 2\tau, \qquad B_\tau^{(1)} = 2\tau, \qquad A_{\tau,\xi}^{(i)} = 3\tau - 3\xi + 1, \qquad A_{\tau,\xi}^{(1)} = 3\tau - 3\xi, \tag{6.5}$$

$$\begin{array}{c|ccc} 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}, \tag{6.6}$$

$$\begin{array}{c|cccc} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \frac{1}{4\sqrt{3}} & \frac{1}{4} - \frac{\sqrt{3}}{4} \\ & \frac{1}{4} + \frac{1}{2\sqrt{3}} & \frac{1}{4} + \frac{1}{2\sqrt{3}} & \frac{1}{4} + \frac{1}{4\sqrt{3}} & \frac{1}{4} + \frac{1}{4\sqrt{3}} \\ \hline & 0 & 1 & 0 & 1 \end{array}. \tag{6.7}$$

We choose the coefficients of (6.1) as $a = 0.5$, $b = 1$ with initial value $x_0 = 1$. We present the average sample errors $\sum_{i=1}^{2000} \sqrt{|x_N(\omega_i) - x(1, \omega_i)|^2/2000}$ for method (6.2)–(6.7) in Table 4. Figure 1 shows the results of Table 4 in a log-log plot.

(a) Method (6.2)–(6.5).   (b) Method (6.6), (6.7).

Figure 1 The convergence rates of methods
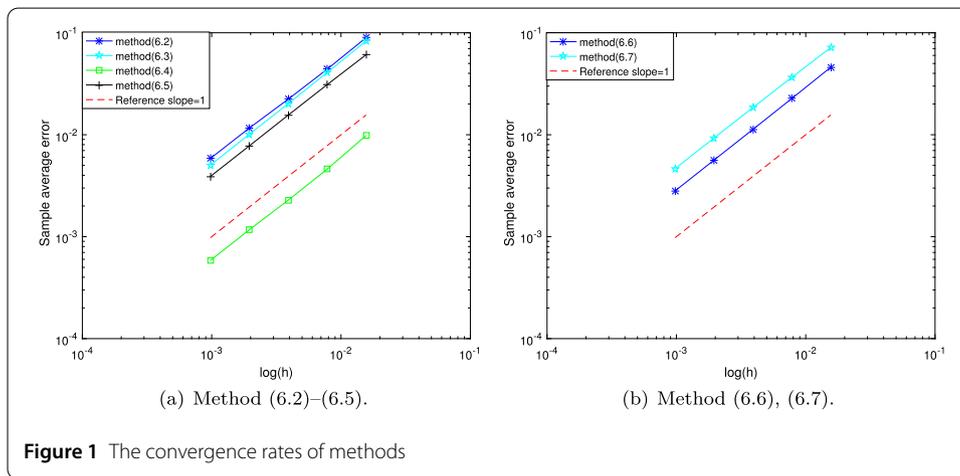
Table 4 The endpoint average sample errors of method (6.2)–(6.7) for solving (6.1)

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
method (6.2)	0.08955355	0.04394097	0.02231704	0.01162317	0.00584895
method (6.3)	0.08332414	0.04070979	0.02018808	0.01008743	0.00503145
method (6.4)	0.00981554	0.00459343	0.00228241	0.00116797	0.00058433
method (6.5)	0.06090075	0.03099001	0.01557490	0.00777261	0.00389402
method (6.6)	0.07178380	0.03655633	0.01846776	0.00928875	0.00465286
method (6.7)	0.04287447	0.02123069	0.01053777	0.00526784	0.00263307

6.2 Kubo stochastic oscillator

The Kubo stochastic oscillator in the sense of Stratonovich SDE is described by

$$
\begin{cases}
\mathrm{d}p = -aq\,\mathrm{d}t - bq \circ \mathrm{d}B(t), & t \in [0,1], \\
\mathrm{d}q = ap\,\mathrm{d}t + bp \circ \mathrm{d}B(t), & t \in [0,1], \\
p(0) = p_0 \in \mathbb{R}, & q(0) = q_0 \in \mathbb{R}.
\end{cases}
\tag{6.8}
$$

Here a and b are constants. The general solution of (6.8) is given by

$$
p(t) = p_0 \cos\big(at + bB(t)\big) - q_0 \sin\big(at + bB(t)\big),
$$
$$
q(t) = p_0 \sin\big(at + bB(t)\big) + q_0 \cos\big(at + bB(t)\big).
$$

We choose the coefficients of (6.8) as $a = 0$, $b = 1$ with initial value $p_0 = 0.5$, $q_0 = 0$. By using appropriate numerical quadrature formula, we can get the classical SRK methods of high order. For (4.6), we can choose Gaussian quadrature with 2 nodes to get the following method of order 2.0:

$$
\begin{array}{c|cccccc}
\frac{5}{36} - \frac{\sqrt{15}}{90} & \frac{2}{9} - \frac{2\sqrt{15}}{45} & \frac{5}{36} - \frac{2\sqrt{15}}{45} & \frac{5}{36} - \frac{\sqrt{15}}{90} & \frac{2}{9} - \frac{2\sqrt{15}}{45} & \frac{5}{36} - \frac{2\sqrt{15}}{45} \\
\frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} & \frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} \\
\frac{5}{36} + \frac{2\sqrt{15}}{90} & \frac{2}{9} + \frac{2\sqrt{15}}{45} & \frac{2}{9} + \frac{2\sqrt{15}}{45} & \frac{5}{36} + \frac{2\sqrt{15}}{90} & \frac{2}{9} + \frac{2\sqrt{15}}{45} & \frac{2}{9} + \frac{2\sqrt{15}}{45} \\
\hline
\frac{5}{18} & \frac{4}{9} & \frac{5}{18} & \frac{5}{18} & \frac{4}{9} & \frac{5}{18}
\end{array}.
\tag{6.9}
$$

The average sample errors at terminal time $T = 1$ are expressed by

$$
\sum_{i=1}^{2000} \sqrt{\big|p(1,\omega_i) - p_N(\omega_i)\big|^2 + \big|q(1,\omega_i) - q_N(\omega_i)\big|^2}/2000
$$

for methods in Table 5. Figure 2 shows the results of Table 5 in a log-log plot.

6.3 Linear stochastic oscillator with additive noise

Then we consider the following linear stochastic oscillator with additive noise:

$$
\begin{cases}
\mathrm{d}y(t) = -x(t)\,\mathrm{d}t + \mathrm{d}B(t), & t \geq 0, \\
\mathrm{d}x(t) = y(t)\,\mathrm{d}t, & t \geq 0, \\
y(0) = y_0 \in \mathbb{R}, & x(0) = x_0 \in \mathbb{R},
\end{cases}
\tag{6.10}
$$

where x is the position and y is the velocity of a particle under the simple harmonic restoring force. Since (6.10) is a SDE with additive noise, its Itô and Stratonovich form are identical. We consider it as a SDE of Stratonovich type. Choose the free parameters of (5.3)

Table 5 The endpoint average sample errors of method (4.6) and (6.9) for solving (6.8)

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
method (4.6)	0.00810481	0.00205308	0.00050988	0.00012790	0.00003198
method (6.9)	0.00005789	0.00001095	0.00000244	0.00000056	0.00000013

(a) Method (4.6).   (b) Method (6.9).

Figure 2 The convergence rates of methods
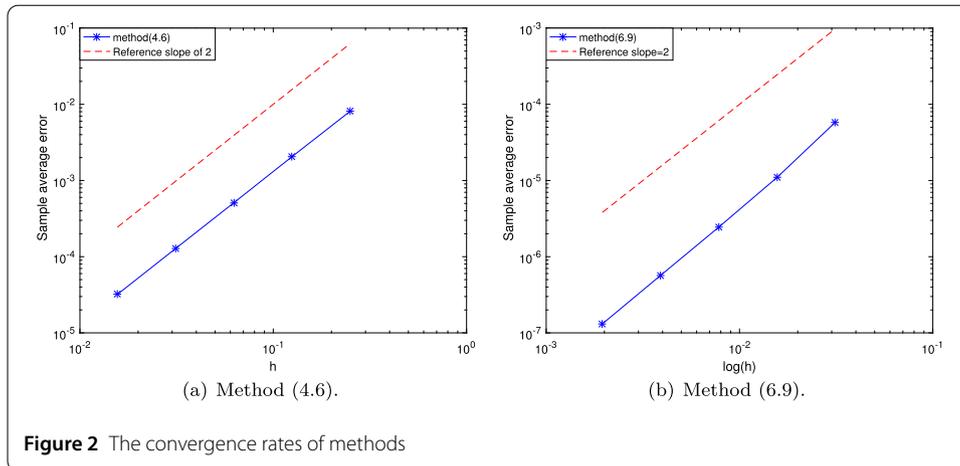
Table 6 The endpoint average sample errors of method (6.11) and (6.12) for solving (6.10)

h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
method (6.11)	0.02533476	0.00624569	0.00151650	0.00040765	0.00012035
method (6.12)	0.03889943	0.00976329	0.00241310	0.00060665	0.00015058

(a) Method (6.11).   (b) Method (6.12).

Figure 3 The convergence rates of methods
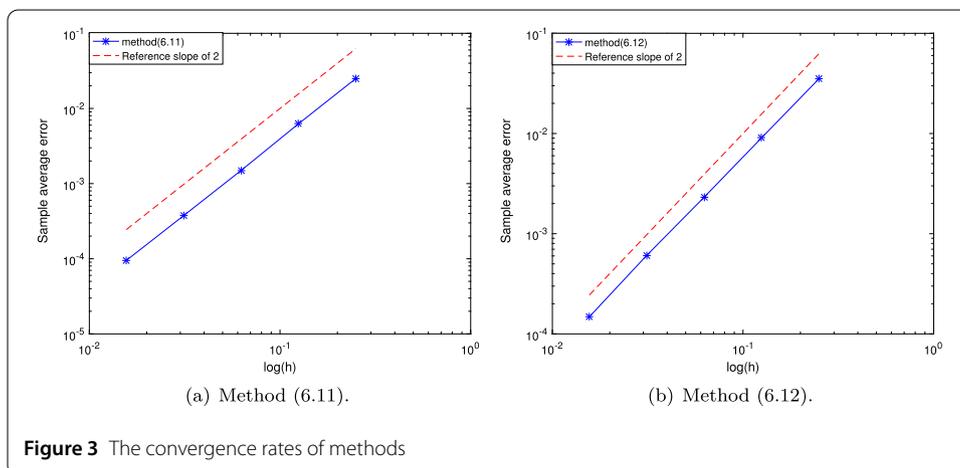
and method (5.7) as $\gamma_1^{(0)} = \gamma_1^{(1)} = \gamma_2^{(0)} = \gamma_2^{(1)} = \gamma_2^{(2)} = 0$, other choices are also promoted. Then we have the methods

$$B_\tau^{(0)} = 1, \qquad B_\tau^{(1)} = 1, \qquad B_\tau^{(2)} = 0,$$
$$A_{\tau,\xi}^{(0)} = \frac{1}{2}, \qquad A_{\tau,\xi}^{(1)} = 0, \qquad A_{\tau,\xi}^{(2)} = \left(\pm 18\sqrt{\frac{1}{54}} \right)\left(\tau - \frac{1}{2} \right) + 1, \tag{6.11}$$

$$
\begin{array}{cccc|cc}
\frac{1}{4} & \frac{1}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} + \frac{1}{2} & -\frac{\sqrt{2}}{4} + \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{\sqrt{2}}{4} + \frac{1}{2} & \frac{\sqrt{2}}{4} + \frac{1}{2} \\
\hline
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0
\end{array}
. \tag{6.12}
$$

To check the order of method (6.11), (6.12), the average sample errors at terminal time $T = 1$ (i.e., $\sum_{i=1}^{2000} \sqrt{|x(1, \omega_i) - x_N(\omega_i)|^2 + |y(1, \omega_i) - y_N(\omega_i)|^2}/2000$) for method (6.11), (6.12)

in Table 6. Figure 3 shows the results of Table 6 in a log-log plot and the slope of the reference line is 2.0.

7 Conclusions

This paper presents the extension work of DCSRK methods which are upgraded to the stochastic counterpart. For general autonomous SDEs in the Stratonovich sense, a class of CSSRK methods are proposed. The general order conditions are obtained by the use of colored rooted tree and stochastic B-series. The order conditions up to strong global order 2.0 of the CSSRK methods for solving single integrand SDEs are simplified. And for solving SDEs with additive noise, the CSSRK methods of strong global order 1.5 and 2.0 are given. It can be proved that a series of SRK methods of the same order with one CSSRK method can be obtained by the rational application of numerical quadrature formulas. The numerical experiments are examined to verify the results of our theoretical analysis and show that the schemes are useful in very long-time numerical simulation of SDEs. The properties of more numerical SRK methods can be obtained by the CSSRK methods with all of these numerical experiments demonstrating significant order convergence of the numerical methods.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. They all read and approved the final version of the manuscript.

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