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Study of composite fractional relaxation differential equation using fractional operators with and without singular kernels and special functions

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Abstract

Our aim in this article is to solve the composite fractional relaxation differential equation by using different definitions of the non-integer order derivative operator D_t^α , more specifically we employ the definitions of Caputo, Caputo–Fabrizio and Atangana–Baleanu of non-integer order derivative operators. We apply the Laplace transform method to solve the problem and express our solutions in terms of Lorenzo and Hartley’s generalised G function. Furthermore, the effects of the parameters involved in the model are graphically highlighted.

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1 Introduction

In these days, fractional calculus (FC) modelling of dynamical setups is getting popular due to the ability of non-integer order derivatives (NIOD) to explain well the complex behaviour of several phenomena, for example the heredity and memory characteristics of materials and processes [1]. Moreover, NIOD models have the potential of adequate correlation with experimental data [2]. So, FC has attracted the scientists and investigators to renew their research for accurate modelling of dynamical systems in the framework of fractional calculus. For details, see [3–8]. It is pertinent to mention here that the most commonly used fractional derivative operators in dynamical model problems are Riemann–Liouville and Caputo derivative operators [9], but they have a singular kernel. In 2015 Caputo and Fabrizio [10], and later in 2016 Atangana and Baleanu [11], proposed modern definitions of NIOD operator with non-singular kernel. Interestingly, these nascent derivative operators hold all the properties of Caputo and Riemann–Liouville operators but have smooth kernels. The Caputo and Fabrizio derivative operator is suitable for both temporal and spatial variables, while the Atangana and Baleanu derivative operator, defined in terms of the Mittag-Leffler function [12], is useful in material and thermal

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sciences. During the past few years, the main properties of these operators have been explored and their advantages have been extensively investigated for various practical cases [13–18].

The motion of the sphere dipped in an in-compressible viscous fluid poses a classical problem, which has many practical implications in flows of geophysical and engineering interest [19, 20]. In order to linearise the Navier–Stokes equations characterising the motion of fluid, the low Reynolds number limit or slow motion assumption is considered. The problem of a sphere under the influence of gravity was first discussed independently by Boussinesq [19] in 1885 and by Basset [20] in 1888, who further introduced a special hydrodynamic force, regarding the history of the relative acceleration of the sphere, later known as Basset force. Practically, for the case when a body is immersed in a fluid, the acceleration of that body with respect to the fluid gives rise to an unsteady force that can be divided into two parts, namely the virtual mass effect and the Basset force. The Basset force deals with the viscous effects and explains the temporal-delay-in-boundary-layer-development as the relative velocity changes with time [21]. It is also acknowledged as the history term.

The fractional differential equation

$$\frac{du}{dt} + aD_t^\alpha u(t) + u(t) = f(t), \quad t \geq 0, \quad (1)$$

along with the initial conditions $u(0^+) = u_0$ is referred to as composite fractional relaxation equation (CFRE).

Here, D_t^α is a NOID operator of order $\alpha \in (0, 1)$. Moreover, in the equation a is a positive constant and $a = \left(\frac{9\rho_f}{2\rho_p + \rho_f}\right)^\alpha$, where ρ_p and ρ_f denote the densities of particle and fluid respectively, $u(t)$ is the field variable, whereas the given function $f(t)$ is presumed to be continuous.

The composite fractional relaxation equation with $\alpha = 1/2$ is called Basset problem in fluid dynamics. For $\alpha = 1/4, 3/4$, it is referred to as the generalised Basset problem [22, 23]. Furthermore, the well-posedness of Eq. (1) is discussed in [24].

Considering the NIOD operator in the sense of Caputo, this problem was discussed by Basset and was first solved by Boggio [25] for $\alpha = 1/2$ in terms of Gauss and Fresnel integrals. Further, the solutions of the Basset problem are found in [26, 27]. Moreover, Mainardi [22] solved the problem by using the Laplace transform method and expressed the solution in terms of Mittag-Leffler functions. Later on, in 2014 Anjara and Solofoniaina [28] solved the Basset problem by Adomian's method.

In solving non-integer order differential equations (DEs) using the Laplace transform method, the inverse Laplace transform is not trivial. In this regard we have to introduce some special functions. For example, Mittag-Leffler function, Robotnov and Hartley's function, Lorenzo and Hartley's generalised R function, generalised G functions etc. Such functions produce a direct solution and give important interpretations for the fundamental linear non-integer order DEs and corresponding IVPs. These functions are helpful in the solutions of the FC problems and more notably in the solution of fractional differential equations. Our aim in this article is to solve the composite fractional relaxation equation by using different definitions of NIOD operators, more specifically we employ the definitions of Caputo, Caputo–Fabrizio and Atangana–Baleanu of NIOD operators. We apply the Laplace transform method to solve the problem and express our solutions in terms

of Lorenzo and Hartley’s generalised G function. Moreover, to get some understanding regarding the influence of the two parameters α and a on the generalised as well as classical Basset problem, we present some diagrams for the particle’s velocity, analogous to the solution of (1). For the sake of clarity, we assume a diminishing initial velocity condition and $f(t) = H(t)$ and $\sin(\omega t)$.

The manuscript is structured in six sections. Following this short introductory section, Sect. 2 describes some mathematical preliminaries. Section 3 discusses the solution of the problem employing various definitions of NIOD operators, while some special cases derived from the obtained results are reported in Sect. 4. Section 5 is devoted to the parametric analysis, and finally the useful conclusions are recorded in Sect. 6.

2 Mathematical preliminaries

In this section we present some basic definitions and properties of some important special functions and Caputo, CF and AB-time fractional derivative operators [29–31].

2.1 Special functions

As already mentioned in the introduction, for solving the NIOD DEs using the Laplace transform method, the inverse Laplace transform is not trivial. In this regard we have to introduce some special functions [32]. For example, Mittag-Leffler function [33], Robotnov and Hartley’s function [34] and Lorenzo and Hartley’s functions [35]. Such functions contribute a direct solution and critical insight for the fundamental linear fractional-order DEs and associated IVPs. Moreover, they are appropriate in the solutions of the problems related to FC and more importantly in the solution of NOIDEs.

In the following, we present some special functions together with their definitions, Laplace transforms and some examples.

1. *Mittag-Leffler function.* It is a significant function and has many applications in the field of FC. Analogous to the exponential function that arises in a natural way from the solutions of differential equations of integer order, the Mittag-Leffler function plays a similar role in the solution of non-integer order differential equations [36, 37]. As a matter of fact, the exponential function is a special form of it. The Mittag-Leffler function is defined as [33]

$$E_\alpha(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\alpha + 1)}; \quad \alpha > 0.$$

It is not difficult to observe that, for $\alpha = 1$, we get

$$E_1(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k + 1)} = e^t.$$

Moreover,

$$L\{E_\alpha(-at^\alpha)\} = L\left\{\sum_{k=0}^\infty \frac{(-a)^k t^{k\alpha}}{\Gamma(k\alpha + 1)}\right\} = \frac{q^\alpha}{q(q^\alpha + a)}; \quad \alpha > 0.$$

2. *Erdelyi's function*. It is the generalisation of Mittag-Leffler function and is defined as [38]

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}; \quad \alpha, \beta > 0.$$

Setting $\beta = 1$, we have

$$E_{\alpha,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)} = E_{\alpha}(t).$$

For $\alpha = 1$ and $\beta = 2$, we have

$$E_{1,2}(t) = \frac{e^t - 1}{t}.$$

Similarly, we have

$$E_{\frac{1}{2},1}(t) = e^{t^2} \operatorname{erfc}(-t)$$

and

$$E_{2,2}(t^2) = \frac{\sinh(t)}{t},$$

where erfc denotes the complementary error function and is defined as [32]

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-u^2} du.$$

Further,

$$L\{E_{\alpha,\beta}(t)\} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k\alpha + 1)} \frac{1}{q^{k+1}}; \quad \alpha, \beta > 0.$$

3. *Robotnov and Hartley function*. This function was introduced by Hartley and Lorenzo [34] and was studied by Robotnov for application to solid mechanics. It is defined as

$$F_{\alpha}(-at) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\alpha}}{\Gamma(k\alpha + 1)}; \quad \alpha > 0.$$

As

$$E_{\alpha}(-at^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad F_{\alpha}(-at) = t^{\alpha-1} E_{\alpha}(-at^{\alpha}),$$

so

$$L\{F_{\alpha}(-at)\} = \frac{1}{q^{\alpha} + a}; \quad \alpha > 0.$$

4. *Miller and Ross' function.* This function was introduced by Miller and Ross [39]. It is defined as

$$E_t(v, a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(v + k + 1)}; \quad \text{Re}(v) > 1,$$

$$L\{E_t(v, a)\} = \frac{q^{-v}}{q - a}; \quad \text{Re}(v) > 1.$$

5. *Generalised R-function.* This function was introduced by Lorenzo and Hartley [35] and is defined as

$$R_{\alpha, \beta}(a, t) = \sum_{k=0}^{\infty} \frac{a^k t^{(k+1)\alpha - \beta - 1}}{\Gamma((k + 1)\alpha - \beta)}; \quad \text{Re}(\alpha - \beta) > 0.$$

It is easy to see that $R_{1,0}(a, t) = e^{at}$, $aR_{2,0}(-a^2, t) = \sin(at)$ and $R_{2,1}(-a^2, t) = \cos(at)$. For $a = 1$, $\beta = \alpha - 1$, we have

$$R_{\alpha, \alpha - 1}(1, t) = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha + 1)} = E_\alpha(t^\alpha).$$

Similarly, setting $a = 1$, $\beta = \alpha - v$ yields

$$R_{\alpha, \alpha - v}(1, t) = t^{v-1} \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha + v)} = t^{v-1} E_{\alpha, v}(t^\alpha).$$

Moreover,

$$L\{R_{\alpha, \beta}(a, t)\} = \frac{q^\beta}{q^\alpha - a}; \quad \text{Re}(\alpha - \beta) > 0, \text{Re}(q) > 0.$$

6. *Generalised G-function.* This function was also introduced by Lorenzo and Hartley [35], it is the generalisation of the R-function. It is defined as follows:

$$G_{\alpha, \beta, \gamma}(a, t) = \sum_{k=0}^{\infty} \frac{a^k \Gamma(\gamma + k)}{\Gamma(\gamma) \Gamma(k + 1)} \frac{t^{(\gamma+k)\alpha - \beta - 1}}{\Gamma((k + 1)\alpha - \beta)}; \quad \text{Re}(\alpha\gamma - \beta) > 0.$$

Setting $\gamma = 1$, we get

$$G_{\alpha, \beta, 1}(a, t) = \sum_{k=0}^{\infty} \frac{a^k \Gamma(1 + k)}{\Gamma(1) \Gamma(k + 1)} \frac{t^{(1+k)\alpha - \beta - 1}}{\Gamma((k + 1)\alpha - \beta)} = R_{\alpha, \beta}(a, t).$$

Also

$$\int_0^s G_{\alpha, \beta, \gamma}(a, t) dt = \sum_{k=0}^{\infty} \frac{a^k \Gamma(\gamma + k)}{\Gamma(\gamma) \Gamma(k + 1)} \frac{s^{(\gamma+k)\alpha - \beta}}{((\gamma+k)\alpha - \beta) \Gamma((k + 1)\alpha - \beta)}$$

$$= \sum_{k=0}^{\infty} \frac{a^k \Gamma(\gamma + k)}{\Gamma(\gamma) \Gamma(k + 1)} \frac{s^{(\gamma+k)\alpha - \beta}}{\Gamma((k + 1)\alpha - \beta + 1)} = G_{\alpha, \beta - 1, \gamma}(a, s).$$

Moreover,

$$L\{G_{\alpha,\beta,\gamma}(a,t)\} = \frac{q^\beta}{(q^\alpha - a)^\gamma}; \quad \text{Re}(\alpha\gamma - \beta) > 0, \text{Re}(q) > 0, \left| \frac{a}{q^\alpha} \right| < 1.$$

Definition 2.1 Let $h \in H^1(a, b)$, $a < b$, $p \in [0, 1)$, then the Caputo fractional derivative is given by [9]

$${}^C D_t^p h(t) = \begin{cases} \frac{1}{\Gamma(1-p)} \int_0^t \frac{h'(\tau) d\tau}{(t-\tau)^p}, & 0 < p < 1, \\ \frac{dh(t)}{dt}, & p = 1, \end{cases} \tag{2}$$

or

$${}^C D_t^p h(t) = k_C(p, t) * h'(t); \quad 0 < p < 1,$$

where $k_C(p, t) = \frac{1}{\Gamma(1-p)} \frac{1}{t^p}$ is the kernel of the derivative, “*” denotes the convolution, and Laplace transform of ${}^C D_t^p h(t)$ is defined as

$$L\{{}^C D_t^p h(t)\} = q^p \tilde{h}(q) - q^{p-1} h(0). \tag{3}$$

Definition 2.2 Let $h \in H^1(a, b)$, $a < b$, $p \in [0, 1)$, then the CF-fractional derivative is given by [10, 29]

$${}^{CF} D_t^p h(t) = \begin{cases} \frac{1}{(1-p)} \int_0^t h'(\tau) \exp\left(-\frac{p(t-\tau)}{1-p}\right) d\tau, & 0 < p < 1, \\ \frac{dh(t)}{dt}, & p = 1, \end{cases} \tag{4}$$

or

$${}^{CF} D_t^p \{h(t)\} = k_{CF}(p, t) * h'(t); \quad 0 < p < 1,$$

where $k_{CF}(p, t) = \exp\left(-\frac{pt}{1-p}\right)$, the kernel of the derivative.

Moreover,

$$L\{{}^{CF} D_t^p h(t)\} = \frac{q\tilde{h}(q) - h(0)}{(1-p)q + p} \tag{5}$$

and

$${}^{CF} D_t^\alpha h(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha(t-\tau)}{1-\alpha}} h'(\tau) d\tau, \tag{6}$$

$$L\{{}^{CF} D_t^\alpha h(t)\} = \frac{qH(q) - h(0)}{(1-\alpha)q + \alpha},$$

$$\lim_{\alpha \rightarrow 1} L\{{}^{CF} D_t^\alpha h(t)\} = qH(q) - h(0) = L\{h'(t)\},$$

$${}^{CF} D_t^1 h(t) = h'(t).$$

Definition 2.3 Let $h \in H^1(a, b)$, $a < b$, $p \in [0, 1]$, then the ABC-NOID is given by [11, 31]

$${}^a ABC D_t^p h(t) = \frac{N(p)}{1-p} \int_a^t h'(\tau) E_p \left[-\frac{p}{1-p} (t-\tau)^p \right] d\tau = k_{ABC}(p, t) * h'(t), \tag{7}$$

where $N(p)$ is a normalisation function fulfilling the condition $N(0) = N(1) = 1$, $k_{ABC}(p, t)$ is the kernel of the derivative

$$k_{ABC}(p, t) = \frac{N(p)}{1-p} E_p \left[-\frac{pt^p}{1-p} \right], \tag{8}$$

and $E_p(\cdot)$ is a one-parametric form of the Mittag-Leffler function [12].

Also

$$L\{ {}^0 ABC D_t^p h(t) \} = \frac{N(p)q^p \tilde{h}(q)}{(1-p)q^p + p} - \frac{N(p)q^{p-1}h(0)}{(1-p)q^p + p}.$$

Furthermore, the normalisation function can be any function fulfilling the condition $N(0) = N(1) = 1$. For example, it could be chosen as $N(p) = 1 - p + \frac{p}{\Gamma(p)}$ [40]. In the present work we choose $N(p)$ to be identically one.

Again

$$L\{ {}^0 ABC D_t^p h(t) \} = \frac{N(p)q^p L\{h(t)\}}{(1-p)q^p + p} - \frac{N(p)q^{p-1}h(0)}{(1-p)q^p + p}. \tag{9}$$

Taking the Laplace transform of (8), we get

$$L\{ k_{ABC}(p, t) \} = \frac{N(p)}{1-p} \frac{q^{p-1}}{q^p + p}, \tag{10}$$

also

$$\begin{aligned} L\{ k_{ABC}(0, t) \} &= \frac{1}{q}, & L\{ k_{ABC}(1, t) \} &= 1, \\ k_{ABC}(0, t) &= 1, & \lim_{p \rightarrow 1} k_{ABC}(p, t) &= \delta(t), \end{aligned} \tag{11}$$

where $\delta(t)$ is Dirac’s delta function.

Moreover,

$${}^a ABC D_t^1 \{h(t)\} = k_{ABC}(1, t) * h'(t) = (\delta * h')(t) = h'(t) \tag{12}$$

and

$${}^a ABC D_t^0 \{h(t)\} = k_{ABC}(0, t) * h'(t) = (I * h')(t) = \int_0^t h'(u) du = h(t) - h(0). \tag{13}$$

Equation (12) and Eq. (13) represent the relation between the ABC-fractional derivative and the classical derivative.

3 Solution of the problem

In this part, we solve Eq. (1) by customising the operator $D_t^\alpha(\cdot)$ according to Caputo, Caputo–Fabrizio and Atangana–Baleanu.

3.1 Solution of the problem using Caputo NIOD operator

Replace the operator $D_t^\alpha(\cdot)$ in Eq. (1) by using Definition 2.1, Eq. (9). Basset equation in the sense of Caputo derivative operator becomes

$$\frac{du}{dt} + a^C D_t^\alpha + u(t) = f(t), \quad u(0^+) = u_0, 0 < \alpha < 1. \tag{14}$$

Employing Laplace transform [41] to Eq. (14) and making use of the initial conditions lead to

$$\tilde{U}(q) = \frac{1 + aq^{\alpha-1}}{q + aq^\alpha + 1} u_0 + \frac{\tilde{f}(q)}{(q + aq^\alpha + 1)} \tag{15}$$

or

$$\begin{aligned} \tilde{U}(q) = & \left(\sum_{k=0}^{\infty} \frac{(-1)^k a^k q^{\alpha k}}{(1+q)^{k+1}} + a \sum_{k=0}^{\infty} \frac{(-1)^k a^k q^{\alpha k + \alpha - 1}}{(1+q)^{k+1}} \right) u_0 \\ & + \sum_{k=0}^{\infty} \frac{(-1)^k a^k q^{\alpha k}}{(1+q)^{k+1}} \tilde{f}(q). \end{aligned} \tag{16}$$

Taking an inverse Laplace transform, we get

$$\begin{aligned} u(t) = & \sum_{k=0}^{\infty} (-1)^k a^k (G_{1,\alpha k, k+1}(-1, t) + G_{1,\alpha(k+1)-1, k+1}(-1, t)) u_0 \\ & + \sum_{k=0}^{\infty} (-1)^k a^k \int_0^t f(t-\tau) G_{1,\alpha k, k+1}(-1, \tau) d\tau, \end{aligned} \tag{17}$$

where G is the Lorenzo–Hartley generalised G function [35].

3.2 Solution of the problem using CF-NIOD operator

Replacing the operator $D_t^\alpha(\cdot)$ in Eq. (1) by using Definition 2.2, Eq. (11), we get

$$\frac{du}{dt} + a^{CF} D_t^\alpha + u(t) = f(t), \quad u(0^+) = u_0, 0 < \alpha < 1. \tag{18}$$

Employing LT [41] to Eq. (18) and making use of the initial conditions, we get

$$\tilde{U}(q) = \frac{(1-\alpha)q + \alpha - a}{(1-\alpha)q^2 + (a+1)q + \alpha} u_0 + \frac{(1-\alpha)q + \alpha}{(1-\alpha)q^2 + (a+1)q + \alpha} \tilde{f}(q), \tag{19}$$

$$\tilde{U}(q) = \frac{q + \frac{\alpha-a}{1-\alpha}}{q^2 + (\frac{a+1}{1-\alpha})q + \frac{\alpha}{1-\alpha}} u_0 + \frac{q + \frac{\alpha}{1-\alpha}}{q^2 + (\frac{a+1}{1-\alpha})q + \frac{\alpha}{1-\alpha}} \tilde{f}(q), \tag{20}$$

$$\tilde{U}(q) = \frac{q + \frac{\alpha-a}{1-\alpha}}{(q+a_1)(q+a_2)} u_0 + \frac{q + \frac{\alpha}{1-\alpha}}{(q+a_1)(q+a_2)} \tilde{f}(q), \tag{21}$$

where

$$a_1 = \frac{\left(\frac{a+1}{1-\alpha}\right) + \sqrt{\left(\frac{a+1}{1-\alpha}\right)^2 - \frac{4\alpha}{1-\alpha}}}{2}, \quad a_2 = \frac{\left(\frac{a+1}{1-\alpha}\right) - \sqrt{\left(\frac{a+1}{1-\alpha}\right)^2 - \frac{4\alpha}{1-\alpha}}}{2}.$$

Employing the inverse LT, after long but straightforward computation, leads to

$$\begin{aligned}
 u(t) &= \frac{1-\alpha}{\sqrt{(a+1)^2 - 4\alpha(1-\alpha)}} \left(\left(\frac{\alpha-a}{1-\alpha} - a_2 \right) e^{-a_2 t} - \left(\frac{\alpha-a}{1-\alpha} - a_1 \right) e^{-a_1 t} \right) \\
 &+ \frac{1-\alpha}{\sqrt{(a+1)^2 - 4\alpha(1-\alpha)}} \\
 &\times \int_0^t f(t-\tau) \left(\left(\frac{\alpha-a}{1-\alpha} - a_2 \right) e^{-a_2 \tau} - \left(\frac{\alpha-a}{1-\alpha} - a_1 \right) e^{-a_1 \tau} \right) d\tau.
 \end{aligned} \tag{22}$$

3.3 Solution of the problem using ABC-NIOD operator

Replacing the operator $D_t^\alpha(\cdot)$ in Eq. (1) by using Definition 2.3, Eq. (13), we get

$$\frac{du}{dt} + \alpha^{ABC} D_t^\alpha u(t) = f(t), \quad u(0^+) = u_0, \quad 0 < \alpha < 1. \tag{23}$$

Application of LT [41] to Eq. (23) and making use of the initial conditions lead to

$$\begin{aligned}
 \tilde{U}(q) &= \frac{(1-\alpha)q^\alpha + \alpha q^{\alpha-1} + \alpha}{(1-\alpha)q^{\alpha+1} + \alpha q + (a+1-\alpha)q^\alpha + \alpha} u_0 \\
 &+ \frac{(1-\alpha)q^\alpha + 1}{q((1-\alpha)q^{\alpha+1} + \alpha q + (a+1-\alpha)q^\alpha + \alpha)}, \\
 \tilde{U}(q) &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha} \right)^{k-m} \left(\gamma^k \frac{q^{(1+\alpha)k - \alpha m + \alpha}}{(q^{\alpha+1} + \gamma)^{k+1}} \right. \\
 &+ \frac{\gamma^{k+1}}{\alpha} \frac{q^{(1+\alpha)k - \alpha m + \alpha - 1}}{(q^{\alpha+1} + \gamma)^{k+1}} + \gamma^{k+1} \frac{q^{(1+\alpha)k - \alpha m}}{(q^{\alpha+1} + \gamma)^{k+1}} \left. \right) u_0 \\
 &+ \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha} \right)^{k-m} \\
 &\times \left(\gamma^k \frac{q^{(1+\alpha)k - \alpha m + \alpha}}{(q^{\alpha+1} + \gamma)^{k+1}} + \frac{\gamma^{k+1}}{\alpha} \frac{q^{(1+\alpha)k - \alpha m}}{(q^{\alpha+1} + \gamma)^{k+1}} \right) \tilde{f}(q),
 \end{aligned} \tag{24}$$

where $\gamma = \frac{\alpha}{1-\alpha}$.

Finally, employing the inverse LT, we get

$$\begin{aligned}
 u(t) &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha} \right)^{k-m} \left(\gamma^k G_{\alpha+1, (1+\alpha)k - \alpha m, k+1}(-\gamma, t) \right. \\
 &+ \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k - \alpha m + \alpha - 1, k+1}(-\gamma, t) \\
 &\left. + \gamma^{k+1} G_{\alpha+1, (1+\alpha)k - \alpha m, k+1}(-\gamma, t) \right) u_0
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 &+ \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha}\right)^{k-m} \int_0^t f(t-\tau) \left(\gamma^k G_{\alpha+1, (1+\alpha)k-\alpha m+\alpha, k+1}(-\gamma, \tau)\right. \\
 &\left. + \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, \tau)\right) d\tau.
 \end{aligned}$$

4 Special cases

In this section, from our general solutions, we discuss two cases, namely when excitation function is $H(t)$ and $\sin(\omega t)$.

For the first case, we take $f(t) = H(t)$ in (17), (22) and (26), we get

$$\begin{aligned}
 u_C(t) &= \sum_{k=0}^{\infty} (-1)^k a^k (G_{1,\alpha k, k+1}(-1, t) + G_{1,\alpha(k+1)-1, k+1}(-1, t)) u_0 \\
 &+ \sum_{k=0}^{\infty} (-1)^k a^k G_{1,\alpha k-1, k+1}(-1, t),
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 u_{CF}(t) &= 1 + \frac{1-\alpha}{\sqrt{(a+1)^2 - 4\alpha(1-\alpha)}} \left(\frac{2\alpha - a - 1 - \sqrt{(a+1)^2 - 4\alpha(1-\alpha)}}{a+1 + \sqrt{(a+1)^2 - 4\alpha(1-\alpha)}} \right. \\
 &\left. - (2\alpha - 3a - 1 - \sqrt{(a+1)^2 - 4\alpha(1-\alpha)}) u_0 \right) e^{-a_1 t} \\
 &- \frac{1-\alpha}{\sqrt{(a+1)^2 - 4\alpha(1-\alpha)}} \left(\frac{2\alpha - a - 1 - \sqrt{(a+1)^2 - 4\alpha(1-\alpha)}}{2(1-\alpha)} \right. \\
 &\left. - (2\alpha - 3a - 1 - \sqrt{(a+1)^2 - 4\alpha(1-\alpha)}) u_0 \right) e^{-a_1 t},
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 u_{ABC}(t) &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha}\right)^{k-m} \left(\gamma^k G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, t)\right. \\
 &+ \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k-\alpha m+\alpha-1, k+1}(-\gamma, t) \\
 &\left. + \gamma^{k+1} G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, t)\right) u_0 \\
 &+ \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha}\right)^{k-m} \left(\gamma^k G_{\alpha+1, (1+\alpha)k-\alpha m+\alpha-1, k+1}(-\gamma, t)\right. \\
 &\left. + \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k-\alpha m-1, k+1}(-\gamma, t)\right),
 \end{aligned} \tag{29}$$

where u_C , u_{CF} and u_{ABC} denote the solutions obtained by employing Caputo, Caputo–Fabrizio and Atangana–Baleanu NIOD operators respectively.

Similarly, for the second case (sinusoidal excitation), we take $f(t) = \sin(\omega t)$ in (17), (22), (26) and get

$$\begin{aligned}
 u_C(t) &= \sum_{k=0}^{\infty} (-1)^k a^k (G_{1,\alpha k, k+1}(-1, t) + G_{1,\alpha(k+1)-1, k+1}(-1, t)) u_0 \\
 &+ \sum_{k=0}^{\infty} (-1)^k a^k \int_0^t \sin(\omega(t-\tau)) G_{1,\alpha k, k+1}(-1, \tau) d\tau,
 \end{aligned} \tag{30}$$

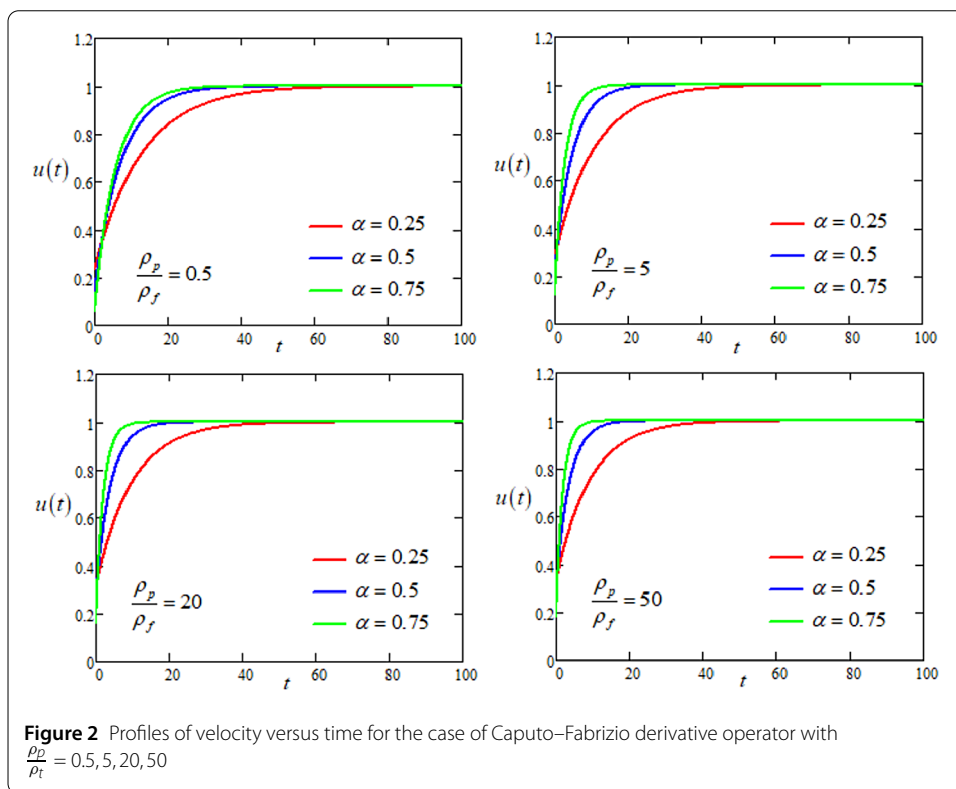
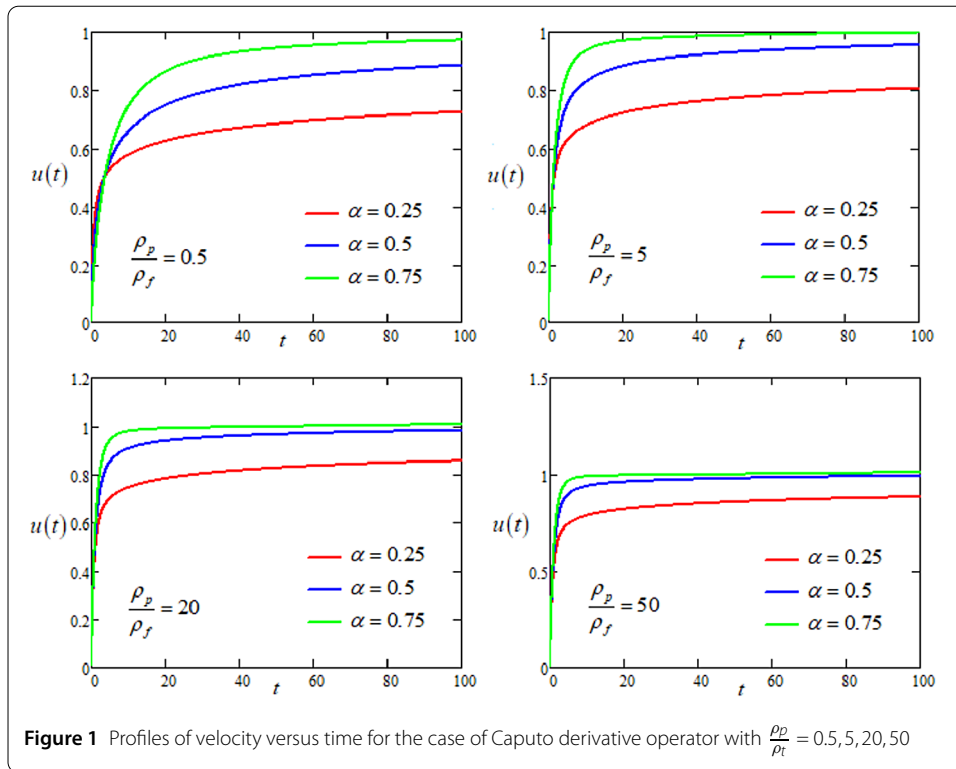
$$\begin{aligned}
 u_{CF}(t) &= \frac{1-\alpha}{\sqrt{(a+1)^2-4\alpha(1-\alpha)}} \left(\left(\frac{\alpha-a}{1-\alpha} - a_2 \right) e^{-a_2 t} - \left(\frac{\alpha-a}{1-\alpha} - a_1 \right) e^{-a_1 t} \right) \\
 &+ \frac{1-\alpha}{\sqrt{(a+1)^2-4\alpha(1-\alpha)}} \\
 &\times \int_0^t \sin(\omega(t-\tau)) \left(\left(\frac{\alpha-a}{1-\alpha} - a_2 \right) e^{-a_2 \tau} - \left(\frac{\alpha-a}{1-\alpha} - a_1 \right) e^{-a_1 \tau} \right) d\tau
 \end{aligned} \tag{31}$$

and

$$\begin{aligned}
 u(t) &= \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha} \right)^{k-m} \left(\gamma^k G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, t) \right. \\
 &+ \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k-\alpha m+\alpha-1, k+1}(-\gamma, t) \\
 &+ \left. \gamma^{k+1} G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, t) \right) u_0 \\
 &+ \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-1)^k k!}{(k-m)! m!} \left(\frac{a\gamma + \alpha}{\alpha} \right)^{k-m} \int_0^t \sin(\omega(t-\tau)) \\
 &\times \left(\gamma^k G_{\alpha+1, (1+\alpha)k-\alpha m+\alpha, k+1}(-\gamma, \tau) \right. \\
 &+ \left. \frac{\gamma^{k+1}}{\alpha} G_{\alpha+1, (1+\alpha)k-\alpha m, k+1}(-\gamma, \tau) \right) d\tau.
 \end{aligned} \tag{32}$$

5 Analysis of the influence of parameters on the Basset problem

With the aim to have proper understanding associated with the influence of the parameters α and a on the classical as well generalised Basset problem, we prepare some diagrams for the velocity of the particle, analogous to the solution of (1). Further, for the sake of clarity, we assume $u(0) = 0$, i.e. vanishing initial velocity and $f(t) = H(t)$. We will discuss three scenarios for α , namely $\alpha = 1/2$ (the classical Basset problem) and $\alpha = 1/4, 3/4$ (the generalised Basset problem). For every α , we choose these values of a relating to $\chi = \frac{\rho_p}{\rho_f} = 0.5, 5, 20, 50$. From Figs. 1–3, we compare velocities for each pair $\{\alpha, \frac{\rho_p}{\rho_f}\}$ using different definitions of NIOD operators in the composite fractional relaxation equation. From Fig. 1, when the Caputo NOID operator is used, it is observed that velocities increase for increasing values of α , and after some time they attain a constant value for large times. Moreover, time to achieve that constant value is getting small for increasing values of the ratio $\frac{\rho_p}{\rho_f}$. For the case when the Caputo–Fabrizio derivative is used in the fractional relaxation equation (see Fig. 2), the same trend is observed as in the case when the Caputo derivative operator is employed. Additionally, it is reported that all velocities converge to the same constant value showing that the influence of the fractional order parameter diminishes with time. From Fig. 3, it is noticed that when the Atangana–Baleanu definition of NIOD is used for increasing values of α , velocity increases, and the range of velocity is larger as compared to the case when Caputo or Caputo–Fabrizio derivative operators are employed. In Fig. 4, the comparison of velocities with three definitions of fractional derivatives is presented. It is noted that, for $\alpha = 1/4$ and $\alpha = 3/4$, the behaviour of velocity when Caputo and Caputo–Fabrizio derivative operators are employed is the same. On the



other hand, for $\alpha = 1/2$, the trend of velocities is significantly different for the case when the Atangana–Baleanu derivative operator is used.

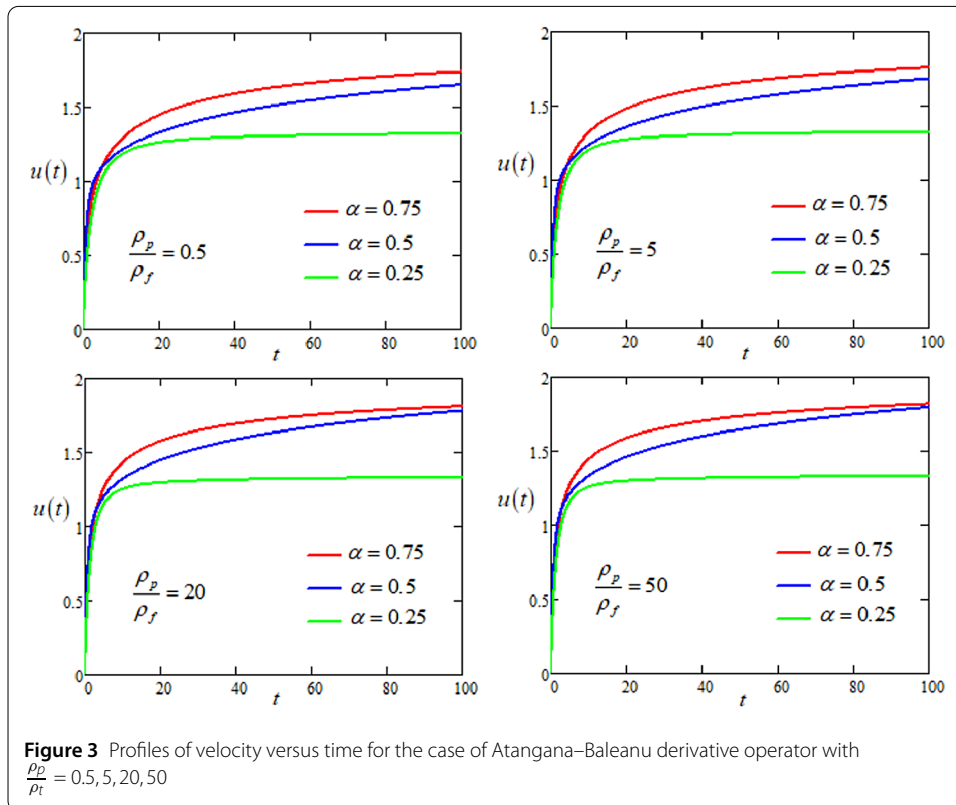


Figure 3 Profiles of velocity versus time for the case of Atangana–Baleanu derivative operator with $\frac{\rho_p}{\rho_f} = 0.5, 5, 20, 50$

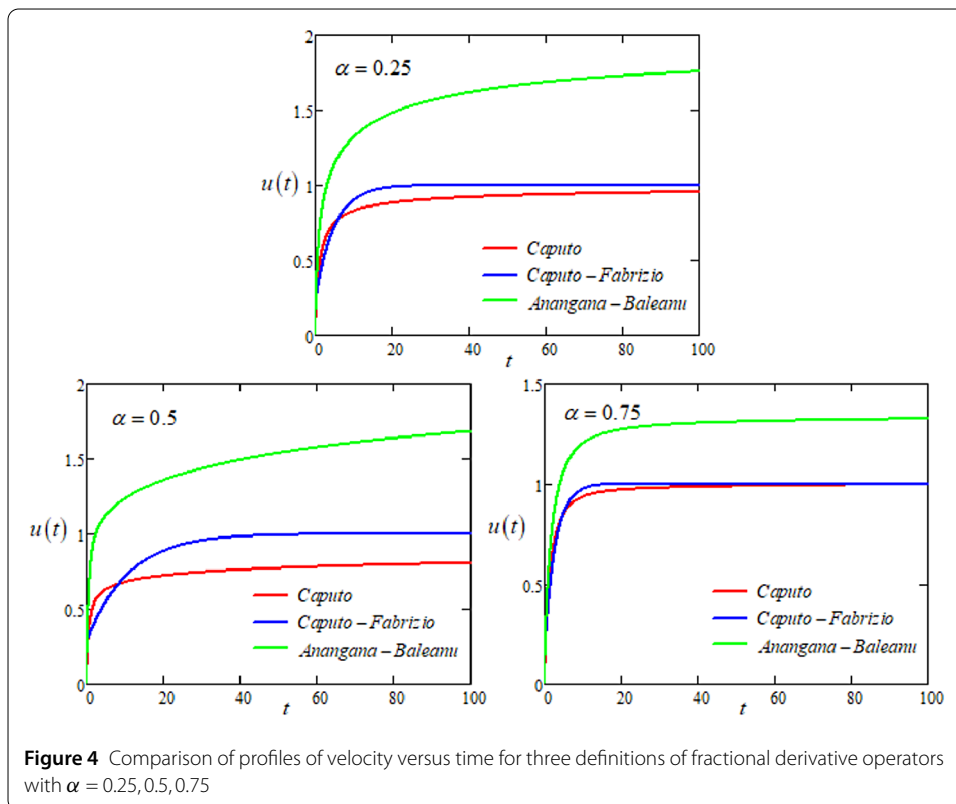


Figure 4 Comparison of profiles of velocity versus time for three definitions of fractional derivative operators with $\alpha = 0.25, 0.5, 0.75$

6 Conclusions

In this article, we have solved the composite fractional relaxation equation by using different definitions of NIOD operators. More specifically, we have employed the definitions NIOD operators proposed by Caputo, Caputo–Fabrizio and Atangana–Baleanu. The solution of the non-integer order differential equation is obtained by applying the Laplace transform method. Moreover, the solutions of the problem are expressed in terms of Lorenzo and Hartley’s generalised G -function that is the generalisation of many special functions that arise in the solution of non-integer order differential equations. Furthermore, the effects of the parameters involved in the model of generalised and classical Bassett problem are shown, and the comparison of these model in terms of different definitions of NIOD operators is done by graphical analysis. The useful conclusions are as follows:

1. When Caputo, Caputo–Fabrizio as well as Atangana–Baleanu fractional order derivative operators are used, velocities increase for increasing values of α , and after some time they attain a constant value for large times. Moreover, time to achieve that constant value is getting small for increasing values of the ratio $\frac{\rho_p}{\rho_f}$.
2. For the case when the Caputo–Fabrizio derivative is used in the fractional relaxation equation, the same trend is observed as in the case when the Caputo derivative operator is employed. Additionally, it is reported that, for the case when the Caputo–Fabrizio derivative is used, all velocities converge to the same constant value showing that the influence of the fractional order parameter diminishes with time.
3. The velocity attains a constant value after its initiation, and time to reach this constant value is smaller for a large value of $\frac{\rho_p}{\rho_f}$.
4. The velocity attains higher values when the Atangana–Baleanu derivative operator is used in comparison to the case when Caputo and Caputo–Fabrizio derivative operators are employed.
5. As the Atangana–Baleanu derivative operator has a non-singular and smooth kernel and has shown better velocity response, it is preferable to be employed in the composite fractional relaxation equation model.

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

JA: conceptualisation, supervision, project administration. AAZ: investigation, methodology, data curation. OM: software, visualisation. MBR: modelling of the problem, validation, writing—review and editing. Help in the revision to address reviewers comments and final edition to improve the quality of the paper. All authors read and approved the final manuscript.

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