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On nonlinear pantograph fractional differential equations with Atangana–Baleanu–Caputo derivative

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Abstract

In this paper, we obtain sufficient conditions for the existence and uniqueness results of the pantograph fractional differential equations (FDEs) with nonlocal conditions involving Atangana–Baleanu–Caputo (ABC) derivative operator with fractional orders. Our approach is based on the reduction of FDEs to fractional integral equations and on some fixed point theorems such as Banach's contraction principle and the fixed point theorem of Krasnoselskii. Further, Gronwall's inequality in the frame of the Atangana–Baleanu fractional integral operator is applied to develop adequate results for different kinds of Ulam–Hyers stabilities. Lastly, the paper includes an example to substantiate the validity of the results.

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1 Introduction

Fractional calculus (FC) has been growing quicker during the most recent few years, and numerous phenomena having the power-law impact have been described precisely with fractional models [1–9]. Numerous outstanding results of the fractional models have been acquired in different fields of science and engineering. One of the specificities of the FC is that we have numerous fractional derivatives (FDs) that offer the authors the chance to pick the specific FD which coincides better with a given real-world problem. The description of phenomena with memory effect is as yet a major test for the specialists. Along these lines, new tools and methods ought to be made to have the option to show a better improvement description of real-world phenomena and the existing models. In this regard, it appears that there is a need for new FDs with the nonsingular kernel. For the nonlocal FDs with the nonsingular exponential kernel, we allude to [10, 11], and for other local approaches of the FDs, we allude to the recent works [12, 13]. Probably the best competitor among the current kernels is the one dependent on Mittag-Leffler functions (MLF) [14]. In view of this, very lately a novel FD [14] (ABC fractional operators) was structured

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and applied to sundry real-world problems [15, 16]. Then, in [17, 18], the authors deliberated the discrete versions of those new operators. For the modeling and important applications in the frame of ABC fractional operator, see [19–26]. Recent investigations of the existence and uniqueness of solutions for fractional differential equations (FDEs) of the impulsive, evolution, and functional problems with initial or boundary conditions can be found within the following research series [27–30] and the references therein. Recent contributions on FDEs involving ABC-FDs can be found in the articles [13, 31–38].

On the other hand, the pantograph is an apparatus employed in electric trains to collect electric currents from the overload lines. This type of equation was designed by Ockendon and Tayler [39]. Pantograph equations play a pivotal role in pure and applied mathematics and physics. Motivated by their significance, a ton of scientists generalized these equations into different types and presented the solvability aspect of such problems both numerically and theoretically; for additional subtleties, see [40–46] and the references therein. Besides, some authors applied various kinds of fractional derivatives and studied the existence and stability of Ulam–Hyers, which can be found in [47–51]. However, not many works have been proposed for pantograph FDEs, especially those involving ABC fractional operator and nonlocal conditions.

Motivated by the above argumentations, the intent of this work is to investigate the ABC-type pantograph FDEs with nonlocal conditions described by

$$\begin{cases} {}^{ABC}\mathbb{D}_{a^+}^\vartheta \zeta(\tau) = f(\tau, \zeta(\tau), \zeta(\gamma\tau)), & \tau \in [a, T], 0 < \vartheta \leq 1, \\ \zeta(a) = \sum_{k=1}^m c_k \zeta(\tau_k), & \tau_k \in (a, T), \end{cases} \tag{1.1}$$

where $0 < \gamma < 1$, ${}^{ABC}\mathbb{D}_{a^+}^\vartheta$ is the AB-Caputo FD of order ϑ , $f : [a, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $f(a, \zeta(a), \zeta(\gamma a)) = 0$, and the constant c_k satisfies the condition $\sum_{k=1}^m c_k \neq 1$. Note that, if $\gamma = 1$, then our problem reduces to

$$\begin{cases} {}^{ABC}\mathbb{D}_{a^+}^\vartheta \zeta(\tau) = f(\tau, \zeta(\tau)), & \tau \in [a, T], 0 < \vartheta \leq 1, \\ \zeta(a) = \sum_{k=1}^m c_k \zeta(\tau_k), & \tau_k \in (a, T). \end{cases} \tag{1.2}$$

Therefore, if $\gamma = 1$, the results acquired in the present paper are also true for ABC-type pantograph FDEs (1.2).

Some fixed point theorems are applied to establish the existence and uniqueness of solution. The Ulam–Hyers stabilities are proved via Gronwall’s inequality in the frame of AB fractional integral operator. The proposed problems are more generalized, also the obtained results are recent studies and an extension of the development of FDEs involving this new operator. Moreover, the analysis of the results was limited to the minimum assumptions.

This paper is formatted as follows. Section 2 provides the background materials and preliminaries required for our analysis. Section 3 is devoted to obtaining a formula of solution to the ABC type pantograph FDEs (1.1). In Sect. 4, we prove the existence and uniqueness of solution to problems at hand by means of some techniques of FPTs. In Sect. 5, the Ulam–Hyers and generalized Ulam–Hyers stability of the pantograph ABC-FDEs (1.1) is discussed via Gronwall’s inequality in the frame of AB fractional integral operator. Finally, an illustrative example is offered in Sect. 6.

2 Background materials and preliminaries

Here, we recollect some requisite definitions and preliminary concepts related to our work.

Let $\mathfrak{Z} = [a, T]$, $\mathfrak{Z}' = (a, T) \subset \mathbb{R}$, and $\mathfrak{D} = C(\mathfrak{Z}, \mathbb{R})$ be the space of continuous functions $\zeta : \mathfrak{Z} \rightarrow \mathbb{R}$ with the norm

$$\|\zeta\| = \max\{|\zeta(\tau)| : \tau \in \mathfrak{Z}\}.$$

Clearly, \mathfrak{D} is a Banach space with the norm $\|\zeta\|$.

Definition 2.1 ([14]) Let $\vartheta \in (0, 1]$ and $p \in H^1(\mathfrak{Z})$. Then the AB-Caputo and AB-Riemann–Liouville FDs of order ϑ for a function p are described by

$${}^{ABC}\mathbb{D}_{a^+}^\vartheta p(\tau) = \frac{\mathfrak{N}(\vartheta)}{1 - \vartheta} \int_a^\tau \mathbb{E}_\vartheta \left(\frac{-\vartheta}{\vartheta - 1} (\tau - s)^\vartheta \right) p'(s) ds, \quad \tau > a,$$

and

$${}^{ABR}\mathbb{D}_{a^+}^\vartheta p(\tau) = \frac{\mathfrak{N}(\vartheta)}{1 - \vartheta} \frac{d}{d\tau} \int_a^\tau \mathbb{E}_\vartheta \left(\frac{-\vartheta}{\vartheta - 1} (\tau - s)^\vartheta \right) p(s) ds, \quad \tau > a,$$

respectively, where $\mathfrak{N}(\vartheta) > 0$ is a normalization function complying with $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$, and \mathbb{E}_ϑ is called the Mittag-Leffler function described by

$$\mathbb{E}_\vartheta(p) = \sum_{k=0}^\infty \frac{p^k}{\Gamma(k\vartheta + 1)}, \quad Re(\vartheta) > 0, p \in \mathbb{C}.$$

The associated AB fractional integral is specified by

$${}^{AB}\mathbb{I}_{a^+}^\vartheta p(\tau) = \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} p(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta p(\tau),$$

where

$$I_{a^+}^\vartheta p(\tau) = \frac{1}{\Gamma(\vartheta)} \int_a^\tau (\tau - s)^{\vartheta-1} p(s) ds.$$

Lemma 2.1 ([17]) Let $\vartheta \in (0, 1]$ and $p \in H^1(\mathfrak{Z})$, if AB-Caputo FD exists, then we have

$${}^{AB}\mathbb{I}_{a^+}^\vartheta {}^{ABC}\mathbb{D}_{a^+}^\vartheta p(\tau) = p(\tau) - p(a).$$

Definition 2.2 ([31]) The relation between the AB-Riemann–Liouville and AB-Caputo FDs is given by

$${}^{ABC}\mathbb{D}_{a^+}^\vartheta p(\tau) = {}^{ABR}\mathbb{D}_{a^+}^\vartheta p(\tau) - \frac{\mathfrak{N}(\vartheta)}{1 - \vartheta} p(a) \mathbb{E}_\vartheta \left(\frac{-\vartheta}{\vartheta - 1} (\tau - a)^\vartheta \right). \tag{2.1}$$

Remark 2.1 Replacing $p(\tau)$ with ${}^{AB}\mathbb{I}_{a^+}^\vartheta p(\tau)$ in Definition 2.2 and using Lemma 2.1, it can be shown that

$${}^{ABC}\mathbb{D}_{a^+}^\vartheta {}^{AB}\mathbb{I}_{a^+}^\vartheta p(\tau) = p(\tau) - p(a) \mathbb{E}_\vartheta \left(\frac{-\vartheta}{\vartheta - 1} (\tau - a)^\vartheta \right).$$

Hence, under the condition that $p(a) = 0$, we get the identity

$${}^{ABC}D_{a^+}^\vartheta {}^{AB}I_{a^+}^\vartheta p(\tau) = p(\tau).$$

Lemma 2.2 ([14]) *Let $\vartheta > 0$. Then ${}^{AB}I_{a^+}^\vartheta$ is bounded from \mathfrak{D} into \mathfrak{D} .*

Lemma 2.3 ([13, 17]) *Let $\vartheta \in (0, 1]$ and $\varpi \in \mathfrak{D}$ with $\varpi(a) = 0$. Then the solution of the following problem*

$${}^{ABC}D_{a^+}^\vartheta p(\tau) = \varpi(\tau), \quad \tau \in \mathfrak{J},$$

$$p(a) = p_a,$$

is given by

$$p(\tau) = p_a + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \varpi(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^\tau (\tau - s)^{\vartheta-1} \varpi(s) ds.$$

Theorem 2.1 ([52], Banach’s contraction principle) *Let \mathcal{J} be a Banach space, and \mathfrak{K} be a nonempty closed subset of \mathcal{J} . If $\mathfrak{B} : \mathfrak{K} \rightarrow \mathfrak{K}$ is a contraction, then there exists a unique fixed point of \mathfrak{B} .*

Theorem 2.2 ([52], Krasnoselskii’s fixed point theorem) *Let \mathfrak{K} be a nonempty, closed, convex subset of a Banach space \mathcal{J} . Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two operators such that (i) $\mathfrak{B}_1 u + \mathfrak{B}_2 v \in \mathfrak{K}, \forall u, v \in \mathfrak{K}$; (ii) \mathfrak{B}_1 is compact and continuous; (iii) \mathfrak{B}_2 is a contraction mapping. Then there exists $w \in \mathfrak{K}$ such that $\mathfrak{B}_1 w + \mathfrak{B}_2 w = w$.*

Theorem 2.3 ([13], Generalized Gronwall’s inequality) *Suppose that $0 < \vartheta \leq 1$, $a(\tau)(1 - \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} b(\tau))^{-1}$ is a nonnegative, nondecreasing, and locally integrable function on $[c, d]$, $\frac{\vartheta b(\tau)}{\mathfrak{N}(\vartheta)}(1 - \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} b(\tau))^{-1}$ is nonnegative and bounded on $[c, d]$, and $\sigma(\tau)$ is nonnegative and locally integrable on $[c, d]$ with*

$$\sigma(\tau) \leq a(\tau) + b(\tau)({}^{AB}I_{a^+}^\vartheta \sigma)(\tau).$$

Then

$$\sigma(\tau) \leq \frac{a(\tau)\mathfrak{N}(\vartheta)}{\mathfrak{N}(\vartheta) - (1 - \vartheta)b(\tau)} \mathbb{E}_\vartheta \left(\frac{\vartheta b(\tau)(\tau - a)^\vartheta}{\mathfrak{N}(\vartheta) - (1 - \vartheta)b(\tau)} \right).$$

3 Formulas of solution

This section is devoted to obtaining formulas of solution to linear problems corresponding to (1.1).

Theorem 3.1 *Let $0 < \vartheta \leq 1, \sum_{k=1}^m c_k \neq 1$, and let $\varpi \in \mathfrak{D}$ with $\varpi(a) = 0$. A function $\varsigma \in \mathfrak{D}$ is a solution of the fractional integral equation (FIE)*

$$\begin{aligned} \varsigma(\tau) = & \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \\ & + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \varpi(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta \varpi(\tau) \end{aligned} \tag{3.1}$$

if and only if ζ is a solution of the ABC-problem

$$\begin{aligned} {}^{ABC}\mathbb{D}_{a^+}^\vartheta \zeta(\tau) &= \varpi(\tau), \quad \tau \in \mathfrak{J}, \\ \zeta(a) &= \sum_{k=1}^m c_k \zeta(\tau_k), \quad \tau_k \in \mathfrak{J}'. \end{aligned} \tag{3.2}$$

Proof Assume that ζ satisfies the first equation of (3.2). From Lemma 2.3, we have

$$\zeta(\tau) = \zeta(a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \varpi(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^\tau (\tau-s)^{\vartheta-1} \varpi(s) ds. \tag{3.3}$$

Now, if we replace $\tau = \tau_k$ and multiply both sides by c_k in (3.3), we get

$$c_k \zeta(\tau_k) = c_k \zeta(a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} c_k \varpi(\tau_k) + \frac{\vartheta c_k}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_k} (\tau_k-s)^{\vartheta-1} \varpi(s) ds. \tag{3.4}$$

From the nonlocal condition, we get

$$\begin{aligned} \zeta(a) &= \sum_{k=1}^m c_k \zeta(\tau_k) \\ &= \sum_{k=1}^m c_k \zeta(a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k), \end{aligned}$$

which implies

$$\zeta(a) = \frac{1}{1-\sum_{k=1}^m c_k} \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right). \tag{3.5}$$

By matching the two equations (3.3) and (3.5), we get

$$\begin{aligned} \zeta(\tau) &= \frac{1}{1-\sum_{k=1}^m c_k} \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \\ &\quad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \varpi(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta \varpi(\tau). \end{aligned}$$

Thus (3.1) is satisfied.

Conversely, suppose that ζ satisfies equation (3.1). Applying ${}^{ABC}\mathbb{D}_{a^+}^\vartheta$ on both sides of (3.1), then using Remark 2.1, and from the fact ${}^{ABC}\mathbb{D}_{a^+}^\vartheta(k) = 0$, for $k = \text{constant}$, we find that

$$\begin{aligned} {}^{ABC}\mathbb{D}_{a^+}^\vartheta \zeta(\tau) &= {}^{ABC}\mathbb{D}_{a^+}^\vartheta \left[\frac{1}{1-\sum_{k=1}^m c_k} \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \right] \\ &\quad + {}^{ABC}\mathbb{D}_{a^+}^\vartheta \left[\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \varpi(\tau) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta \varpi(\tau) \right] \\ &= {}^{ABC}\mathbb{D}_{a^+}^\vartheta {}^{AB}\mathbb{I}_{a^+}^\vartheta \varpi(\tau) \\ &= \varpi(\tau). \end{aligned}$$

On the other hand, by taking $\tau \rightarrow a$ on both sides of (3.1), then using the fact that $\varpi(a) = 0$ and $\lim_{\tau \rightarrow a} I_{a^+}^\vartheta \varpi(\tau) = I_{a^+}^\vartheta \varpi(a) = 0$, we get

$$\begin{aligned} \zeta(a) &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \\ &\quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \varpi(a) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta \varpi(a) \\ &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right). \end{aligned} \tag{3.6}$$

Substitute $\tau = \tau_k$ and multiply by c_k in (3.1). Then we derive

$$\begin{aligned} \sum_{k=1}^m c_k \zeta(\tau_k) &= \left(\frac{\sum_{k=1}^m c_k}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \right) \\ &\quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta c_k \varpi(\tau_k) \\ &= \left(\frac{\sum_{k=1}^m c_k}{1 - \sum_{k=1}^m c_k} + 1 \right) \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right) \\ &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k \varpi(\tau_k) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta \varpi(\tau_k) \right). \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\zeta(a) = \sum_{k=1}^m c_k \zeta(\tau_k). \tag{3.8}$$

□

As a result of Theorem 3.1, we have the subsequent theorem.

Theorem 3.2 *Let $0 < \vartheta \leq 1$, $\sum_{k=1}^m c_k \neq 1$, and $f : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function along with $f(a, \zeta(a), \zeta(\gamma a)) = 0$, $\zeta \in \mathfrak{D}$. Then the ABC- type pantograph FDEs (1.1) are equivalent to the following FIE:*

$$\begin{aligned} \zeta(\tau) &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k f(\tau_k, \zeta(\tau_k), \zeta(\gamma(\tau_k))) \right. \\ &\quad \left. + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta f(\tau_k, \zeta(\tau_k), \zeta(\gamma(\tau_k))) \right) \\ &\quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} f(\tau, \zeta(\tau), \zeta(\gamma \tau)) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta f(\tau, \zeta(\tau), \zeta(\gamma \tau)), \quad \tau \in \mathfrak{J}. \end{aligned}$$

4 Existence and uniqueness theorems

This section is devoted to proving the existence and uniqueness theorems for the ABC- type pantograph FDEs (1.1). Before proceeding with the main findings, we are obligated to provide the following assumption:

(A₁) The function $f : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and there exists $L_f > 0$ such that

$$|f(\tau, u, \bar{u}) - f(\tau, v, \bar{v})| \leq L_f (|u - v| + |\bar{u} - \bar{v}|), \quad \tau \in \mathfrak{J}, \text{ and } u, v, \bar{u}, \bar{v} \in \mathbb{R}.$$

Theorem 4.1 *Suppose that hypothesis (A₁) holds. Then the ABC-type pantograph FDEs (1.1) have a unique solution, provided that*

$$\mathfrak{R}_1 := \left(\frac{(1 - \vartheta)}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left[\frac{1}{(1 - \sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^{\vartheta} + (T - a)^{\vartheta} \right] \right) \frac{2L_f}{\mathfrak{N}(\vartheta)} < 1. \tag{4.1}$$

Proof Define the operator $\mathbb{T} : \mathfrak{D} \rightarrow \mathfrak{D}$ by $\mathbb{T}\zeta = \zeta, \zeta \in \mathfrak{D}$, i.e.,

$$\begin{aligned} (\mathbb{T}\zeta)(\tau) &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k f(\tau_k, \zeta(\tau_k), \zeta(\gamma(\tau_k))) \right. \\ &\quad \left. + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^{\vartheta} f(\tau_k, \zeta(\tau_k), \zeta(\gamma(\tau_k))) \right) \\ &\quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} f(\tau, \zeta(\tau), \zeta(\gamma\tau)) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^{\vartheta} f(\tau, \zeta(\tau), \zeta(\gamma\tau)). \end{aligned}$$

The operator \mathbb{T} is well defined, that is, $\mathbb{T}(\mathfrak{D}) \subseteq \mathfrak{D}$. Indeed, for any $\zeta \in \mathfrak{D}, f(\cdot, \zeta(\cdot), \zeta(\gamma(\cdot)))$ is continuous. Besides, by Lemma 2.2, $\mathbb{T}\zeta \in \mathfrak{D}$. Also, by Lemma 2.1 with Remark 2.1, we end up at

$$\begin{aligned} {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}(\mathbb{T}\zeta)(\tau) &= {}^{ABC}\mathbb{D}_{a^+}^{\vartheta} \zeta(a) + {}^{ABC}\mathbb{D}_{a^+}^{\vartheta} {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \zeta(\tau), \zeta(\gamma\tau)) \\ &= f(\tau, \zeta(\tau), \zeta(\gamma\tau)). \end{aligned}$$

Since $f(\tau, \cdot, \cdot)$ is continuous on $[a, T]$, then ${}^{ABC}\mathbb{D}_{a^+}^{\vartheta}(\mathbb{T}\zeta)(\tau) \in \mathfrak{D}$.

Now, we need to prove that \mathbb{T} is a condensing map. Let $\zeta, \bar{\zeta} \in \mathfrak{D}$ and $\tau \in \mathfrak{J}$. Then

$$\begin{aligned} &|(\mathbb{T}\zeta)(\tau) - (\mathbb{T}\bar{\zeta})(\tau)| \\ &\leq \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k |f(\tau_k, \zeta(\tau_k), \zeta(\gamma\tau_k)) - f(\tau_k, \bar{\zeta}(\tau_k), \bar{\zeta}(\gamma\tau_k))| \right. \\ &\quad \left. + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^{\vartheta} |f(\tau_k, \zeta(\tau_k), \zeta(\gamma\tau_k)) - f(\tau_k, \bar{\zeta}(\tau_k), \bar{\zeta}(\gamma\tau_k))| \right) \\ &\quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} |f(\tau, \zeta(\tau), \zeta(\gamma\tau)) - f(\tau, \bar{\zeta}(\tau), \bar{\zeta}(\gamma\tau))| \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^{\vartheta} |f(\tau, \zeta(\tau), \zeta(\gamma\tau)) - f(\tau, \bar{\zeta}(\tau), \bar{\zeta}(\gamma\tau))|. \end{aligned}$$

By assumption (A₁), we obtain

$$\begin{aligned} &I_{a^+}^{\vartheta} |f(\tau_k, \zeta(\tau_k), \zeta(\gamma(\tau_k))) - f(\tau_k, \bar{\zeta}(\tau_k), \bar{\zeta}(\gamma(\tau_k)))| \\ &\leq \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_k} (\tau_k - s)^{\vartheta-1} |f(s, \zeta(s), \zeta(\gamma(s))) - f(s, \bar{\zeta}(s), \bar{\zeta}(\gamma(s)))| ds \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_k} (\tau_k - s)^{\vartheta-1} L_f(|\varsigma(s) - \bar{\varsigma}(s)| + |\varsigma(\gamma(s)) - \bar{\varsigma}(\gamma(s))|) ds \\
 &\leq \frac{2L_f(\tau_k - a)^\vartheta}{\Gamma(\vartheta + 1)} \|\varsigma - \bar{\varsigma}\|.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 &I_{a^+}^\vartheta |f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)) - f(\tau, \bar{\varsigma}(\tau), \bar{\varsigma}(\gamma\tau))| \\
 &\leq \frac{2L_f(T - a)^\vartheta}{\Gamma(\vartheta + 1)} \|\varsigma - \bar{\varsigma}\|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\|(\mathbb{T}\varsigma) - (\mathbb{T}\bar{\varsigma})\| \\
 &= \max_{\tau \in \mathfrak{J}} |(\mathbb{T}\varsigma)(\tau) - (\mathbb{T}\bar{\varsigma})(\tau)| \\
 &\leq \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{2L_f(1 - \vartheta)}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k + \frac{2L_f}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta \right) \|\varsigma - \bar{\varsigma}\| \\
 &\quad + \left(\frac{2L_f(1 - \vartheta)}{\mathfrak{N}(\vartheta)} + \frac{2L_f(T - a)^\vartheta}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \right) \|\varsigma - \bar{\varsigma}\| \\
 &= \left(\frac{4(1 - \vartheta) \sum_{k=1}^m c_k}{1 - \sum_{k=1}^m c_k} + \frac{2}{\Gamma(\vartheta)} \left[\frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta + (T - a)^\vartheta \right] \right) \\
 &\quad \times \frac{L_f}{\mathfrak{N}(\vartheta)} \|\varsigma - \bar{\varsigma}\| \\
 &= \mathfrak{P}_1 \|\varsigma - \bar{\varsigma}\|.
 \end{aligned}$$

Condition (4.1) shows that \mathbb{T} is a condensing operator. Hence, by Theorem 2.1, \mathbb{T} has a unique fixed point. □

Theorem 4.2 *Suppose that hypothesis (A₁) holds. Then there exists at least one solution of the ABC-type pantograph FDEs (1.1), provided that condition (4.1) is satisfied.*

Proof Consider the operator $\mathbb{T} : \mathfrak{D} \rightarrow \mathfrak{D}$ defined by

$$(\mathbb{T}\varsigma)(\tau) = (\mathbb{T}_1\varsigma)(\tau) + (\mathbb{T}_2\varsigma)(\tau), \quad \varsigma \in \mathfrak{D}, \tau \in \mathfrak{J},$$

where

$$\begin{aligned}
 (\mathbb{T}_1\varsigma)(\tau) &= \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k f(\tau_k, \varsigma(\tau_k), \varsigma(\gamma\tau_k)) \right. \\
 &\quad \left. + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta f(\tau_k, \varsigma(\tau_k), \varsigma(\gamma\tau_k)) \right) \tag{4.2}
 \end{aligned}$$

and

$$(\mathbb{T}_2\varsigma)(\tau) = \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)) + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)). \tag{4.3}$$

Since $f : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\mu_f := \max\{|f(\tau, 0, 0)| : \tau \in \mathfrak{J}\}$ exists. Let

$$B_\xi = \{\zeta \in \mathfrak{D} : \|\zeta\| \leq \xi\} \tag{4.4}$$

with the radius

$$\xi \geq \frac{\mathfrak{P}_2}{1 - \mathfrak{P}_1}, \tag{4.5}$$

where

$$\mathfrak{P}_2 := \left(\frac{1 - \vartheta}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left[\frac{1}{(1 - \sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta + (T - a)^\vartheta \right] \right) \frac{\mu_f}{\mathfrak{N}(\vartheta)}. \tag{4.6}$$

We will complete the proof in the following several steps.

Step1: We show that $\mathbb{T}_1\zeta + \mathbb{T}_2\nu \in B_\xi$ for all $\zeta, \nu \in B_\xi$.

By (4.2), we have

$$\begin{aligned} |(\mathbb{T}_1\zeta)(\tau)| &\leq \frac{1}{1 - \sum_{k=1}^m c_k} \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k |f(\tau_k, \zeta(\tau_k), \zeta(\gamma\tau_k))| \right. \\ &\quad \left. + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \sum_{k=1}^m c_k I_{a^+}^\vartheta |f(\tau_k, \zeta(\tau_k), \zeta(\gamma\tau_k))| \right). \end{aligned} \tag{4.7}$$

Using hypothesis (A₁), for $\zeta \in B_\xi$ and for any $\tau \in \mathfrak{J}$, we have

$$\begin{aligned} |f(\tau, \zeta(\tau), \zeta(\gamma\tau))| &\leq |f(\tau, \zeta(\tau), \zeta(\gamma\tau)) - f(\tau, 0, 0)| + |f(\tau, 0, 0)| \\ &\leq L_f |\zeta(\tau)| + L_f |\zeta(\gamma\tau)| + \mu_f \\ &\leq 2L_f \xi + \mu_f. \end{aligned} \tag{4.8}$$

Further, by using (4.8), for any $\tau \in \mathfrak{J}$, we have

$$\begin{aligned} I_{a^+}^\vartheta |f(\tau, \zeta(\tau), \zeta(\gamma\tau))| &\leq I_{a^+}^\vartheta (2L_f \xi + \mu_f) \\ &= (2L_f \xi + \mu_f) \frac{(\tau - a)^\vartheta}{\Gamma(\vartheta + 1)}. \end{aligned} \tag{4.9}$$

Using inequalities (4.8) and (4.9) into inequality (4.7), we obtain

$$\begin{aligned} \|(\mathbb{T}_1\zeta)\| &= \max_{\tau \in \mathfrak{J}} |(\mathbb{T}_1\zeta)(\tau)| \\ &\leq \frac{(2L_f \xi + \mu_f)}{\mathfrak{N}(\vartheta)(1 - \sum_{k=1}^m c_k)} \left((1 - \vartheta) \sum_{k=1}^m c_k + \frac{1}{\Gamma(\vartheta)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta \right) \\ &= \left((1 - \vartheta) \frac{\sum_{k=1}^m c_k}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)(1 - \sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta \right) \frac{2L_f}{\mathfrak{N}(\vartheta)} \xi \\ &\quad + \left((1 - \vartheta) \frac{\sum_{k=1}^m c_k}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)(1 - \sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta \right) \frac{\mu_f}{\mathfrak{N}(\vartheta)}. \end{aligned} \tag{4.10}$$

Also, for $v \in B_\xi$,

$$\begin{aligned}
 \|(\mathbb{T}_2 v)\| &= \max_{\tau \in \mathfrak{J}} |(\mathbb{T}_2 v)(\tau)| \\
 &\leq \max_{\tau \in \mathfrak{J}} \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau, v(\tau), v(\gamma\tau))| + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta |f(\tau, v(\tau), v(\gamma\tau))| \right) \\
 &\leq \frac{(1-\vartheta)}{\mathfrak{N}(\vartheta)} (2L_f \xi + \mu_f) + \frac{(2L_f \xi + \mu_f)(T-a)^\vartheta}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \\
 &= \left((1-\vartheta) + \frac{(T-a)^\vartheta}{\Gamma(\vartheta)} \right) \frac{2L_f}{\mathfrak{N}(\vartheta)} \xi \\
 &\quad + \left((1-\vartheta) + \frac{(T-a)^\vartheta}{\Gamma(\vartheta)} \right) \frac{\mu_f}{\mathfrak{N}(\vartheta)}. \tag{4.11}
 \end{aligned}$$

Inequalities (4.10) and (4.11) give

$$\begin{aligned}
 \|(\mathbb{T}_1 \zeta) + (\mathbb{T}_2 v)\| &\leq \|(\mathbb{T}_1 \zeta)\| + \|(\mathbb{T}_2 v)\| \\
 &\leq \left(\frac{1-\vartheta}{1-\sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left[\frac{1}{(1-\sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta + (T-a)^\vartheta \right] \right) \frac{2L_f}{\mathfrak{N}(\vartheta)} \xi \\
 &\quad + \left(\frac{1-\vartheta}{1-\sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left[\frac{1}{(1-\sum_{k=1}^m c_k)} \sum_{k=1}^m c_k (\tau_k - a)^\vartheta + (T-a)^\vartheta \right] \right) \frac{\mu_f}{\mathfrak{N}(\vartheta)} \\
 &= \mathfrak{P}_1 \xi + \mathfrak{P}_2.
 \end{aligned}$$

Using (4.1) and (4.5), we get

$$\| \mathbb{T}_1 \zeta + \mathbb{T}_2 v \| \leq \xi.$$

Thus, $\mathbb{T}_1 \zeta + \mathbb{T}_2 v \in B_\xi$ for all $\zeta, v \in B_\xi$.

Step 2: \mathbb{T}_1 is a condensing map. This is evident due to \mathbb{T} is a contraction map.

Step 3: \mathbb{T}_2 is continuous and compact.

$\mathbb{T}_2 : B_\xi \rightarrow B_\xi$ is continuous due to f is continuous. Indeed, let ζ_n be a sequence such that $\zeta_n \rightarrow \zeta$ in \mathfrak{D} . Then, for all $\tau \in \mathfrak{J}$, one has

$$\begin{aligned}
 |(\mathbb{T}_2 \zeta_n)(\tau) - (\mathbb{T}_2 \zeta)(\tau)| &\leq \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau, \zeta_n(\tau), \zeta_n(\gamma\tau)) - f(\tau, \zeta(\tau), \zeta(\gamma\tau))| \\
 &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta |f(\tau, \zeta_n(\tau), \zeta_n(\gamma\tau)) - f(\tau, \zeta(\tau), \zeta(\gamma\tau))| \\
 &\leq \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \|f(\cdot, \zeta_n(\cdot), \zeta_n(\gamma(\cdot))) - f(\cdot, \zeta(\cdot), \zeta(\gamma(\cdot)))\| \\
 &\quad + \frac{(T-a)^\vartheta}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \|f(\cdot, \zeta_n(\cdot), \zeta_n(\gamma(\cdot))) - f(\cdot, \zeta(\cdot), \zeta(\gamma(\cdot)))\|.
 \end{aligned}$$

Since f is continuous, the operator \mathbb{T}_2 is also continuous. Thus, we have

$$\|(\mathbb{T}_2 \zeta_n) - (\mathbb{T}_2 \zeta)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Next, \mathbb{T}_2 is uniformly bounded on B_ξ . For any $\varsigma \in B_\xi$ and $\tau \in \mathfrak{J}$, we have

$$\begin{aligned} |(\mathbb{T}_2\varsigma)| &= \max_{\tau \in \mathfrak{J}} |(\mathbb{T}_2\varsigma)(\tau)| \\ &\leq \max_{\tau \in \mathfrak{J}} \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau, \varsigma(\tau), \varsigma(\gamma\tau))| + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^\vartheta |f(\tau, \varsigma(\tau), \varsigma(\gamma\tau))| \right) \\ &\leq \max_{\tau \in \mathfrak{J}} \left(\frac{(1-\vartheta)}{\mathfrak{N}(\vartheta)} (2L_f\xi + \mu_f) + \frac{(2L_f\xi + \mu_f)(\tau - a)^\vartheta}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \right) \\ &\leq \frac{2L_f\xi + \mu_f}{\mathfrak{N}(\vartheta)} \left((1-\vartheta) + \frac{(T-a)^\vartheta}{\Gamma(\vartheta)} \right). \end{aligned}$$

This leads to \mathbb{T}_2 is uniformly bounded on B_ξ .

Now, we show that $\mathbb{T}_2(B_\xi)$ is equicontinuous. For that, let $\varsigma \in B_\xi$ and $a \leq \tau_1 < \tau_2 \leq T$. Then, by using (4.8), we have

$$\begin{aligned} &|(\mathbb{T}_2\varsigma)(\tau_2) - (\mathbb{T}_2\varsigma)(\tau_1)| \\ &= \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau_2, \varsigma(\tau_2), \varsigma(\gamma(\tau_2))) - f(\tau_1, \varsigma(\tau_1), \varsigma(\gamma(\tau_1)))| \\ &\quad + \left| \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_2} (\tau_2 - s)^{\vartheta-1} f(s, \varsigma(s), \varsigma(\gamma(s))) ds \right. \\ &\quad \left. - \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_1} (\tau_1 - s)^{\vartheta-1} f(s, \varsigma(s), \varsigma(\gamma(s))) ds \right| \\ &\leq \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau_2, \varsigma(\tau_2), \varsigma(\gamma(\tau_2))) - f(\tau_1, \varsigma(\tau_1), \varsigma(\gamma(\tau_1)))| \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\tau_1} |(\tau_1 - s)^{\vartheta-1} - (\tau_2 - s)^{\vartheta-1}| |f(s, \varsigma(s), \varsigma(\gamma(s)))| ds \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\vartheta-1} |f(s, \varsigma(s), \varsigma(\gamma(s)))| ds \\ &\leq \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} |f(\tau_2, \varsigma(\tau_2), \varsigma(\gamma(\tau_2))) - f(\tau_1, \varsigma(\tau_1), \varsigma(\gamma(\tau_1)))| \\ &\quad + \frac{2(2L_f\xi + \mu_f)}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} (\tau_2 - \tau_1)^\vartheta. \end{aligned}$$

Since $f(\cdot, \varsigma(\cdot), \varsigma(\gamma(\cdot)))$ is continuous, $|(\mathbb{T}_2\varsigma)(\tau_2) - (\mathbb{T}_2\varsigma)(\tau_1)| \rightarrow 0$ as $\tau_2 \rightarrow \tau_1$. In light of the former steps with Arzela–Ascoli theorem, we derive that (\mathbb{T}_2B_ξ) is relatively compact, and hence \mathbb{T}_2 is completely continuous. So, Theorem 2.2 shows that (1.1) has at least one solution. □

5 Ulam–Hyers stability

In this section, we discuss two types of stability for (1.1), namely Ulam–Hyers and generalized Ulam–Hyers stabilities. For $\varepsilon > 0$, we consider the following inequations:

$$|{}^{ABC}\mathbb{D}_{a^+}^\vartheta \tilde{\zeta}(\tau) - f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau))| \leq \varepsilon, \quad \tau \in \mathfrak{J}. \tag{5.1}$$

Definition 5.1 The pantograph ABC-FDEs (1.1) are Ulam–Hyers stable if there exists a real number $C_f > 0$ such that, for each $\varepsilon > 0$ and for each solution $\tilde{\zeta} \in \mathfrak{D}$ of inequality (5.1),

there exists a unique solution $\zeta \in \mathfrak{D}$ of (1.1) with

$$|\tilde{\zeta}(\tau) - \zeta(\tau)| \leq C_f \varepsilon, \quad \tau \in \mathfrak{J}.$$

And the pantograph ABC-FDEs (1.1) are generalized Ulam–Hyers stable if we can find $\phi_f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\phi_f(0) = 0$ such that

$$|\tilde{\zeta}(\tau) - \zeta(\tau)| \leq \phi_f(\varepsilon), \quad \tau \in \mathfrak{J}.$$

Remark 5.1 Let $\tilde{\zeta} \in \mathfrak{D}$ be the solution of inequality (5.1) if and only if we have a function $h \in \mathfrak{D}$ which depends on $\tilde{\zeta}$ such that

- i) $|h(\tau)| \leq \varepsilon$ for all $\tau \in \mathfrak{J}$,
- ii) ${}^{ABC}\mathbb{D}_{a^+}^\vartheta \tilde{\zeta}(\tau) = f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) + h(\tau), \tau \in \mathfrak{J}$.

Lemma 5.1 *If $\tilde{\zeta} \in \mathfrak{D}$ is a solution of inequality (5.1), then $\tilde{\zeta}$ is a solution of the following inequality:*

$$|\tilde{\zeta}(\tau) - \mathcal{R}_{\tilde{\zeta}} - {}^{AB}\mathbb{I}_{a^+}^\vartheta f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau))| \leq \Lambda \varepsilon, \tag{5.2}$$

where

$$\mathcal{R}_{\tilde{\zeta}} := \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k {}^{AB}\mathbb{I}_{a^+}^\vartheta f(\tau_k, \tilde{\zeta}(\tau_k), \tilde{\zeta}(\gamma\tau_k))$$

and

$$\Lambda := \left(\frac{1 - \vartheta}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left(\frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k (\tau_k - a) \right)^\vartheta + (T - a)^\vartheta \right) \frac{1}{\mathfrak{N}(\vartheta)}.$$

Proof In view of Remark 5.1, we have

$$\begin{aligned} {}^{ABC}\mathbb{D}_{a^+}^\vartheta \tilde{\zeta}(\tau) &= f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) + h(\tau), \quad \tau \in \mathfrak{J}, \\ \tilde{\zeta}(a) &= \sum_{k=1}^m c_k \tilde{\zeta}(\tau_k), \quad \tau_k \in \mathfrak{J}'. \end{aligned}$$

Then, by Theorem 3.1, we get

$$\begin{aligned} \tilde{\zeta}(\tau) &= \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k {}^{AB}\mathbb{I}_{a^+}^\vartheta (f(\tau_k, \tilde{\zeta}(\tau_k), \tilde{\zeta}(\gamma\tau_k)) + h(\tau_k)) \\ &\quad + {}^{AB}\mathbb{I}_{a^+}^\vartheta (f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) + h(\tau)) \\ &= \mathcal{R}_{\tilde{\zeta}} + \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k {}^{AB}\mathbb{I}_{a^+}^\vartheta h(\tau_k) + {}^{AB}\mathbb{I}_{a^+}^\vartheta (f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) + h(\tau)). \end{aligned}$$

From this it follows that

$$\begin{aligned}
 & |\tilde{\zeta}(\tau) - \mathcal{R}_{\tilde{\zeta}} - {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau))| \\
 & \leq \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k {}^{AB}\mathbb{I}_{a^+}^{\vartheta} |h(\tau_k)| + {}^{AB}\mathbb{I}_{a^+}^{\vartheta} |h(\tau)| \\
 & = \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k \left(\frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} |h(\tau_k)| + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^{\vartheta} |h(\tau_k)| \right) \\
 & \quad + \frac{1 - \vartheta}{\mathfrak{N}(\vartheta)} |h(\tau)| + \frac{\vartheta}{\mathfrak{N}(\vartheta)} I_{a^+}^{\vartheta} |h(\tau)| \\
 & \leq \left(\frac{1 - \vartheta}{1 - \sum_{k=1}^m c_k} + \frac{1}{\Gamma(\vartheta)} \left(\frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k (\tau_k - a) \right)^{\vartheta} + (T - a) \right)^{\vartheta} \frac{\varepsilon}{\mathfrak{N}(\vartheta)} \\
 & = \Lambda \varepsilon. \tag*{\square}
 \end{aligned}$$

Theorem 5.1 *Suppose that hypothesis (A₁) holds. If $\mathfrak{N}(\vartheta) - (1 - \vartheta)2L_f < 1$, then the pantograph ABC-FDEs (1.1) are Ulam–Hyers stable.*

Proof Let $\varepsilon > 0$ and $\tilde{\zeta} \in \mathfrak{D}$ be a function which satisfies inequality (5.1), and let $\varsigma \in \mathfrak{D}$ be the unique solution of the following problem:

$$\begin{cases}
 {}^{ABC}\mathbb{D}_{a^+}^{\vartheta} \varsigma(\tau) = f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)), & \tau \in \mathfrak{J}, \\
 \varsigma(a) = \sum_{k=1}^m c_k \varsigma(\tau_k) = \sum_{k=1}^m c_k \tilde{\zeta}(\tau_k) = \tilde{\zeta}(a), & \tau_k \in \mathfrak{J}'.
 \end{cases} \tag{5.3}$$

Using Theorem 3.1, we obtain

$$\varsigma(\tau) = \mathcal{R}_{\varsigma} + {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)),$$

where

$$\mathcal{R}_{\varsigma} := \frac{1}{1 - \sum_{k=1}^m c_k} \sum_{k=1}^m c_k {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau_k, \varsigma(\tau_k), \varsigma(\gamma\tau_k)).$$

Since $\varsigma(a) = \tilde{\zeta}(a)$ and $\sum_{k=1}^m c_k \varsigma(\tau_k) = \sum_{k=1}^m c_k \tilde{\zeta}(\tau_k)$, then $\mathcal{R}_{\varsigma} = \mathcal{R}_{\tilde{\zeta}}$. Hence

$$\varsigma(\tau) = \mathcal{R}_{\tilde{\zeta}} + {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \varsigma(\tau), \varsigma(\gamma\tau)).$$

It follows from Lemma 5.1 and (A₁) that

$$\begin{aligned}
 & |\tilde{\zeta}(\tau) - \varsigma(\tau)| \\
 & \leq |\tilde{\zeta}(\tau) - \mathcal{R}_{\tilde{\zeta}} - {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau))| \\
 & \quad + |{}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) - {}^{AB}\mathbb{I}_{a^+}^{\vartheta} f(\tau, \varsigma(\tau), \varsigma(\gamma\tau))| \\
 & \leq \Lambda \varepsilon + {}^{AB}\mathbb{I}_{a^+}^{\vartheta} |f(\tau, \tilde{\zeta}(\tau), \tilde{\zeta}(\gamma\tau)) - f(\tau, \varsigma(\tau), \varsigma(\gamma\tau))| \\
 & \leq \Lambda \varepsilon + 2L_f {}^{AB}\mathbb{I}_{a^+}^{\vartheta} |\tilde{\zeta}(\tau) - \varsigma(\tau)|.
 \end{aligned}$$

Using Theorem 2.3 with $\sigma(\tau) = |\tilde{\zeta}(\tau) - \zeta(\tau)|$, $a(\tau) = \Lambda \varepsilon$, and $b(\tau) = 2L_f$, we get

$$\begin{aligned} |\tilde{\zeta}(\tau) - \zeta(\tau)| &\leq \frac{\Lambda \varepsilon \mathfrak{N}(\vartheta)}{\mathfrak{N}(\vartheta) - (1 - \vartheta)2L_f} \mathbb{E}_\vartheta \left(\frac{2\vartheta L_f (\tau - a)^\vartheta}{\mathfrak{N}(\vartheta) - 2(1 - \vartheta)L_f} \right) \\ &\leq C_f \varepsilon, \end{aligned}$$

where

$$C_f := \frac{\Lambda \mathfrak{N}(\vartheta)}{\mathfrak{N}(\vartheta) - (1 - \vartheta)2L_f} \mathbb{E}_\vartheta \left(\frac{2\vartheta L_f (T - a)^\vartheta}{\mathfrak{N}(\vartheta) - 2(1 - \vartheta)L_f} \right). \quad \square$$

Corollary 5.1 *Under the hypotheses of Theorem 5.1, if there exists $\phi_f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\phi_f(0) = 0$, then the pantograph ABC problem (1.1) has generalized Ulam–Hyers stability.*

Proof Choosing $\phi_f(\varepsilon) = C_f \varepsilon$ and $\phi_f(0) = 0$, from Theorem 5.1 we obtain $|\tilde{\zeta}(\tau) - \zeta(\tau)| \leq \phi_f(\varepsilon)$. □

6 Examples

In this section, we justify the validity of Theorems 4.1, 4.2, and 5.1 through an example.

Example 6.1 For $\vartheta \in (0, 1]$, we consider the following ABC fractional problem:

$$\begin{cases} {}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{2}} \zeta(\tau) = \frac{\tau^2}{10} \left(e^{-\tau} + \frac{|\zeta(\tau)|}{10(1+|\zeta(\tau)|)} + \frac{|\zeta(\frac{\tau}{2})|}{10(1+|\zeta(\frac{\tau}{2})|)} \right), & \tau \in [0, 1], \\ \zeta(0) = \frac{1}{4} \zeta\left(\frac{1}{3}\right), \end{cases} \quad (6.1)$$

where $\vartheta = \frac{1}{2}$, $c_1 = \frac{1}{4}$, $\tau_1 = \frac{1}{3}$ ($m = 1$), $f(\tau, \zeta(\tau), \zeta(\gamma\tau)) = \frac{\tau^2}{10} \left(e^{-\tau} + \frac{|\zeta(\tau)|}{10(1+|\zeta(\tau)|)} + \frac{|\zeta(\frac{\tau}{2})|}{10(1+|\zeta(\frac{\tau}{2})|)} \right)$, and $\gamma = \frac{1}{2}$.

Clearly, $f(0, \zeta(0), \zeta(0)) = 0$. Moreover,

$$\mu_f = \max_{\tau \in [0,1]} |f(\tau, 0, 0)| = \max_{\tau \in [0,1]} \frac{\tau^2}{10} e^{-\tau} = \frac{1}{10e}.$$

Let $\tau \in [0, 1]$ and $\zeta, \nu \in \mathbb{R}$. Then

$$\begin{aligned} &\left| f(\tau, \zeta(\tau), \zeta\left(\frac{1}{2}\tau\right)) - f(\tau, \nu(\tau), \nu\left(\frac{1}{2}\tau\right)) \right| \\ &\leq \left| \frac{\tau^2}{10} \left(e^{-\tau} + \frac{|\zeta(\tau)|}{10(1+|\zeta(\tau)|)} + \frac{|\zeta(\frac{\tau}{2})|}{10(1+|\zeta(\frac{\tau}{2})|)} \right) \right. \\ &\quad \left. - \frac{\tau^2}{10} \left(e^{-\tau} + \frac{|\nu(\tau)|}{10(1+|\nu(\tau)|)} + \frac{|\nu(\frac{\tau}{2})|}{10(1+|\nu(\frac{\tau}{2})|)} \right) \right| \\ &\leq \frac{1}{10} \left(\frac{10|\zeta(\tau) - \nu(\tau)|}{100(1+|\zeta(\tau)|)(1+|\nu(\tau)|)} + \frac{10|\zeta(\frac{\tau}{2}) - \nu(\frac{\tau}{2})|}{100(1+|\zeta(\frac{\tau}{2})|)(1+|\nu(\frac{\tau}{2})|)} \right) \\ &\leq \frac{1}{10} \left(|\zeta(\tau) - \nu(\tau)| + \left| \zeta\left(\frac{\tau}{2}\right) - \nu\left(\frac{\tau}{2}\right) \right| \right). \end{aligned}$$

Therefore, hypothesis (A_1) holds with $L_f = \frac{1}{10}$. We shall examine that condition (4.1) is satisfied with $\mathfrak{N}(\vartheta) = 1$ and $m = 1$. Hence, by some simple calculations, we find that

$$\mathfrak{B}_1 = \left(\frac{1}{15} + \frac{9 + \sqrt{3}}{45\sqrt{\pi}} \right) \approx 0.2 < 1.$$

Thus Theorems 4.1 and 4.2 guarantee the existence and uniqueness of solution on $[0, 1]$ problem (6.1).

Finally, since $\mathfrak{N}(\vartheta) - (1 - \vartheta)2L_f = \frac{9}{10} < 1$, problem (6.1) is Ulam–Hyers and generalized Ulam–Hyers stable with

$$C_f = \frac{10}{9} \Lambda E_{\frac{1}{2}} \left(\frac{1}{9} \right) \quad \text{and} \quad \Lambda = \left(\frac{1}{3} + \frac{9 + \sqrt{3}}{9\sqrt{\pi}} \right).$$

Remark 6.1

- (1) If $\gamma = 1$, then our problem (1.1) reduces to problem (1.2). Therefore, all the results mentioned in this work are also valid for problem (1.2).
- (2) If we replace the nonlocal condition $\zeta(a) = \sum_{k=1}^m c_k \zeta(\tau_k)$ with the initial condition $\zeta(0) = \zeta_0$ ($a = 0$) and use ${}^{CF}\mathbb{D}_{a^+}^\vartheta$ (Caputo–Fabrizio FD) instead of ${}^{ABC}\mathbb{D}_{a^+}^\vartheta$ (ABC derivative), then our problem (1.1) reduces to the following problem:

$$\begin{cases} {}^{CF}\mathbb{D}_{0^+}^\vartheta \zeta(\tau) = f(\tau, \zeta(\tau), \zeta(\gamma\tau)), & \tau \in [a, T], 0 < \vartheta \leq 1, \\ \zeta(0) = \zeta_0. \end{cases}$$

7 Conclusions

The theory of fractional operators including nonsingular kernels is novel and of significant recent interest, thus there is a need to study the qualitative properties of FDEs involving such operators. In this paper, we have obtained the existence and uniqueness of solutions for the pantograph FDEs with nonlocal conditions involving ABC fractional derivative. Our approach is based on the reduction of ABC-type pantograph FDEs into FIE and some fixed point theorems of Banach and Krasnoselskii. Further, we have applied Gronwall’s inequality in the frame of AB fractional integral operator to develop adequate results for various types of Ulam–Hyers stability. Pertinent examples are provided to justify the results. The problems scrutinized are also valid for some special cases, in other words, they are reduced to the corresponding problems that contain Caputo–Fabrizio fractional derivative operator. Besides, the analysis of the obtained results was restricted to a minimum of assumptions.

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