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Some fractional Hermite–Hadamard-type integral inequalities with s-(α, m)-convex functions and their applications

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Abstract

Under the new concept of s-(α, m)-convex functions, we obtain some new Hermite–Hadamard inequalities with an s-(α, m)-convex function. We use these inequalities to estimate Riemann–Liouville fractional integrals with second-order differentiable convex functions to enrich the Hermite–Hadamard-type inequalities. We give some applications to special means.

Keywords: Hermite–Hadamard inequality; Convex functions; Riemann–Liouville fractional integral; Power-mean inequality

1 Introduction

Convex functions are a kind of important functions widely used in mathematical programming. They are not only closely related to continuity and differentiability, but also play important roles in inequalities. Therefore convex functions has been widely used in many research fields such as life and management science, optimization [1, 2], and so on. In optimization inequalities, generalized classical convexity is often used together with convexity theory and inequality theory, in which Hermite–Hadamard integral inequalities containing convex functions are valued by many mathematicians because of their pertinence and ease of use. The classical Hermite–Hadamard-type integral inequality is the following [3]:

Let $g : I \subseteq R \to R$ be a convex function on the interval I of real numbers, and let $c, d \in I$ with $c < d$. Then

$$g\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d g(t)\,dt \leq \frac{g(c)+f(d)}{2}. \tag{1}$$

In recent years, with the development of convex function inequalities, the Hermite–Hadamard integral inequality has attracted interest of many researchers. Dragomir and Agarwal [4] and Hwang et al. [5] provided the Hermite–Hadamard inequalities of integer orders of general concave and convex functions, applied them to the error terms in special mean values, and estimated the trapezoid formulas: Let $g : I^0 \subseteq R \to R$ be a differentiable function on I^0, and let $a, b \in I^0$ with $a < b$. If $|g'|$ is convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{g(c)+g(d)}{2} - \frac{1}{d-c}\int_c^d g(t)\,dt\right| \leq \frac{(d-c)(|g'(c)|+|g'(d)|)}{8}. \tag{2}$$

Let $g : I^0 \subseteq R \to R$ be a differentiable function on I^0, let $a, b \in I^0$ with $a < b$, and let $p > 1$. If the function $|g'|^{p/p-1}$ is convex on $[a, b]$, then we have the following inequality:

$$\begin{aligned}&\left|\frac{g(c)+g(d)}{2} - \frac{1}{d-c}\int_c^d g(t)\,dt\right| \\ &\quad \leq \frac{(d-c)}{2(p+1)^{1/p}}\left[\frac{|g'(c)|^{p/p-1}+|g'(d)|^{p/p-1}}{2}\right]^{(p-1)/p}.\end{aligned} \tag{3}$$

If $q \geq 1$ and the function $|g'|^q$ is convex on $[c, d]$, then

$$\begin{aligned}&\left|(A-a)g(a) + (b-B)g(b) + (B-A)g(c) - \int_a^b g(x)\,dx\right| \\ &\quad \leq \begin{cases} M(A,B,c;p,q)\cdot N(A,B,c;p,q), & q>1, 0\leq p\leq q, \\ N(A,B,c;p,q), & p=q=1.\end{cases}\end{aligned} \tag{4}$$

Khaled and Agarwal [6] extended the interval $[a, b]$ and made new estimates of the Hermite–Hadamard inequality on the interval $[\frac{3a-b}{2}, \frac{3b-a}{2}]$:

Let $g : I \subseteq R \to R$ be a differentiable function on I, let $a, b \in I$ with $a < b$, and let its derivative $g' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \to R$ be a continuous function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$. If $|g'|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then we have the following inequality:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(x)\,dx - g\left(\frac{a+b}{2}\right)\right| \\ &\quad \leq \frac{b-a}{8}\left(\left|g'\left(\frac{3a-b}{2}\right)\right|^q + \left|g'\left(\frac{3b-a}{2}\right)\right|^q\right)^{1/q}.\end{aligned} \tag{5}$$

Özcan and Íscan [7] generalized the Hermite–Hadamard inequality for s-convex functions. Let $g : I \subseteq R \to R$ be a differentiable function on I, and let $a, b \in I$ with $a < b$. If $g' \in L[a, b]$, then we have the following inequality:

$$\left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|g'(a)|^q+|g'(b)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{6}$$

All these different estimates of integral inequalities of integer order hold under the convexity of $|g'|$.

With the in-depth study of integer-order Hermite–Hadamard inequality, more and more scholars have also done a lot of research and extensions of fractional Hermite–Hadamard integral inequality, among which there are many papers related to the Riemann–Liouville fractional integral. Sarikaya et al. [8] studied the Hermite–Hadamard integral inequality to estimate arithmetic means and Riemann–Liouville fractional integrals using a convex function $|g'|$:

Let $g:[a,b]\to R$ be a positive function with $0\le a<b$ such that $g'\in L_1[a,b]$. If g is a convex function on $[a,b]$, then we have the following inequalities for fractional integrals:

$$g\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha g(b)+J_{b^-}^\alpha g(a)\right]\le\frac{g(a)+g(b)}{2}. \tag{7}$$

Let $g:[a,b]\to R$ be a differentiable function on (a,b) with $a<b$. If $|g'|$ is convex on $[a,b]$, then we have the following inequalities for fractional integrals:

$$\begin{aligned}&\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha g(b)+J_{b^-}^\alpha g(a)\right]\right|\\&\quad\le\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\left|g'(a)\right|+\left|g'(b)\right|\right].\end{aligned} \tag{8}$$

Chun et al. [9] studied the Hermite–Hadamard integral inequality to estimate geometric means and Riemann–Liouville fractional integrals using a convex function $|g'|$:

Let $g:[a,b]\to R$ be a differentiable function on (a,b) with $a<b$. If $|g'|$ is convex on $[a,b]$, then we have the following inequalities for fractional integrals:

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha g(b)+J_{b^-}^\alpha g(a)\right]-g\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{b-a}{4(\alpha+1)}\left(\alpha+3-\frac{1}{2^{\alpha-1}}\right)\left[\left|g'(a)\right|+\left|g'(b)\right|\right].\end{aligned} \tag{9}$$

Li Xiaoling and Shahid [10] studied the Hermite–Hadamard inequality of s-(α,m)-convex functions with parameter Riemann–Liouville fractional integral:

Let $g:[c,d]\to R$ be a differentiable function on $[c,d]$ with $c<d$ such that g' is s-(α,m)-convex on $[a,b]$. Then we have the following inequality for Riemann–Liouville fractional integrals with $0<\alpha\le 1$:

$$\begin{aligned}&\left|\left(1-\frac{2}{2^\alpha\lambda}\right)g'\left(\frac{a+b}{2}\right)+\lambda\frac{g(a)+g(b)}{2^\alpha}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha g(b)+J_{b^-}^\alpha g(a)\right]\right|\\&\quad\le\frac{(b-a)}{2^{\alpha+1}}\left\{\left[M_1\left|g'(a)\right|+2m(M_2-M_1)\left|g'\left(\frac{a+b}{2}\right)\right|+M_1\left|g'(b)\right|\right]\right.\\&\qquad\left.+\left[M_3\left|g'(a)\right|+m(M_4-M_3)\left|f'\left(\frac{a+b}{2}\right)\right|+M_3\left|g'(b)\right|\right]\right\}.\end{aligned} \tag{10}$$

There are many other Hermite–Hadamard integral inequalities for convex functions; we refer the interested readers to [11–22].

In [10] the author studies the inequalities of first-order differentiable convex functions on the right side of the Hermite–Hadamard inequality. In this paper, using s-(α,m)-convex functions and Riemann–Liouville fractional integrals, we study some Hermite–Hadamard inequalities of second-order differentiable convex functions on the right side of the inequality and apply them to special means.

The arrangement of this paper is as follows. In Sect. 2, we introduce the classes of convex functions to prepare the work; In Sect. 3, we prove new Hermite–Hadamard integral inequalities using new concepts and the Riemann–Liouville fractional integral; In Sect. 4, we apply the results to special mean values.

2 Preliminaries

In this section, we recall some important definitions and results.

The general classical convexity is defined as follows.

Definition 2.1 Let R be the set of real numbers. A function $g : I \subseteq R \to R$ is said to be convex on an interval I if

$$g\big(tc + (1-t)d\big) \leq tg(c) + (1-t)g(d) \tag{11}$$

for all $c, d \in I$ and $t \in [0,1]$.

Muddassar [23] presented the class of s-(α, m)-convex functions as follows.

Definition 2.2 A function $g : [0,+\infty) \to [0,+\infty)$ is said to be s-(α, m)-convex in the first sense or to belong to the class $K_{m,1}^{\alpha,s}$ if for all $c, d \in [0,+\infty)$ and $t \in [0,1]$, we have the following inequality:

$$g\big(tc + m(1-t)d\big) \leq t^{\alpha s} g(c) + m\big(1 - t^{\alpha s}\big) g(d), \tag{12}$$

where $(\alpha, m) \in [0,1]^2$ and $s \in (0,1]$.

Definition 2.3 A function $g : [0,+\infty) \to [0,+\infty)$ is said to be s-(α, m)-convex in the second sense or to belong to the class $K_{m,2}^{\alpha,s}$ if for all $c, d \in [0,+\infty)$ and $t \in [0,1]$, we have the following inequality:

$$g\big(tc + m(1-t)d\big) \leq \big(t^{\alpha}\big)^s g(c) + m\big(1 - t^{\alpha}\big)^s g(d), \tag{13}$$

where $(\alpha, m) \in [0,1]^2$ and $s \in (0,1]$.

Definition 2.4 ([24]) Let $g \in L_1[a,b]$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$, with $a \geq 0$, are defined by

$$J_{a^+}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\,dt \quad (x > a) \tag{14}$$

and

$$J_{b^-}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t)\,dt \quad (x < b), \tag{15}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1}\,du$. In the case $\alpha = 1$, the fractional integral reduces to the classical integral. Properties relating to this operator can be found in [25].

Lemma 2.1 ([3], Lemma 4) *Let $g : I \subseteq R \to R$ be a twice differentiable function on I^0 such that g'' is integrable on $[a,b] \subseteq I^0$ with $a < b$. Then we have the identity*

$$\frac{g(a)+g(b)}{2} - \frac{1}{b-a} \int_a^b g(x)\,dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) g''\big(ta + (1-t)b\big)\,dt. \tag{16}$$

3 Main result and proof

In [10], all the Hermite–Hadamard integral inequalities were based on the s-(α,m)-convexity of $|g'|$. If we do not know the convexity of $|g'|$, but $|g''|$ is convex, then we will get new Hermite–Hadamard inequalities. Next, we will study fractional Hermite–Hadamard integral inequalities based on the convexity of $|g''|$, where the s-(α,m)-convexity is in the first sense.

First, we give a lemma, which will be used in later important conclusions.

Lemma 3.1 *Let $g,g':[c,d]\to R$ be differentiable functions on $[c,d]$, and suppose g'' is integrable. Then we have the following equation for the Riemann–Liouville fractional integral with $0<\alpha\le 1, 0\le\lambda\le 1$:*

$$\begin{aligned}&\frac{\lambda}{2^\alpha}(d-c)\big[g'(c)+g'(d)\big]+\left(2-\frac{2}{2^\alpha}\lambda\right)(d-c)g'\left(\frac{c+d}{2}\right)\\&\quad+(\alpha+1)\big[g(c)-g(d)\big]-\frac{\Gamma(\alpha+2)}{(d-c)^\alpha}\big[J^\alpha_{c^+}g(d)+J^\alpha_{d^-}g(c)\big]\\&\quad=\frac{(d-c)^2}{2^{\alpha+2}}(M_1+M_2+M_3+M_4),\end{aligned}\tag{17}$$

where

$$\begin{aligned}M_1&=\int_0^1\big[(1-t)^{\alpha+1}-\lambda\big]g''\left(tc+(1-t)\frac{c+d}{2}\right)dt,\\M_2&=\int_0^1\big[\lambda-(1-t)^{\alpha+1}\big]g''\left(td+(1-t)\frac{c+d}{2}\right)dt,\\M_3&=\int_0^1\big[2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\big]g''\left(t\frac{c+d}{2}+(1-t)c\right)dt,\\M_4&=\int_0^1\big[\lambda-2^{\alpha+1}+(2-t)^{\alpha+1}\big]g''\left(t\frac{c+d}{2}+(1-t)d\right)dt.\end{aligned}$$

Proof The proof is obtained by integration by parts based on equation (16). We have

$$\begin{aligned}M_1&=\int_0^1\big[(1-t)^{\alpha+1}-\lambda\big]g''\left(tc+(1-t)\frac{c+d}{2}\right)dt\\&=\frac{2}{c-d}\big[(1-t)^{\alpha+1}-\lambda\big]g'\left(tc+(1-t)\frac{c+d}{2}\right)\bigg|_0^1\\&\quad+\frac{2}{c-d}\int_0^1(\alpha+1)(1-t)^\alpha g'\left(tc+(1-t)\frac{c+d}{2}\right)dt\\&=\frac{-2\lambda}{c-d}g'(c)-\frac{2(1-\lambda)}{c-d}g'\left(\frac{c+d}{2}\right)+\frac{4(\alpha+1)}{(c-d)^2}(1-t)^\alpha g\left(tc+(1-t)\frac{c+d}{2}\right)\bigg|_0^1\\&\quad+\frac{4}{(c-d)^2}\int_0^1(\alpha+1)\alpha(1-t)^{\alpha-1}g\left(tc+(1-t)\frac{c+d}{2}\right).\end{aligned}$$

Let $u=tc+(1-t)\frac{c+d}{2}$. Then

$$M_1=\frac{-2\lambda}{c-d}g'(c)-\frac{2(1-\lambda)}{c-d}g'\left(\frac{c+d}{c}\right)-\frac{4(\alpha+1)}{(c-d)^2}g\left(\frac{c+d}{2}\right)$$

$$
\begin{aligned}
&+\frac{4(\alpha+1)\alpha}{(c-d)^2}\int_{\frac{c+d}{2}}^{c}\left(\frac{2}{c-d}\right)^{\alpha}(c-u)^{\alpha-1}g(u)\,du\\
&=\frac{2\lambda}{d-c}g'(c)+\frac{2(1-\lambda)}{d-c}g'\left(\frac{c+d}{c}\right)-\frac{4(\alpha+1)}{(d-c)^2}g\left(\frac{c+d}{2}\right)\\
&-\frac{2^{\alpha+2}(\alpha+1)\alpha}{(d-c)^{\alpha+2}}\int_{c}^{\frac{c+d}{2}}(u-c)^{\alpha-1}g(u)\,du.
\end{aligned}
$$

Using the same algorithm, we get:

$$
\begin{aligned}
M_2&=\int_0^1\left[\lambda-(1-t)^{\alpha+1}\right]g''\left(td+(1-t)\frac{c+d}{2}\right)dt\\
&=\frac{2\lambda}{d-c}g'(d)+\frac{2(1-\lambda)}{d-c}g'\left(\frac{c+d}{c}\right)+\frac{4(\alpha+1)}{(d-c)^2}g\left(\frac{c+d}{2}\right)\\
&-\frac{2^{\alpha+2}(\alpha+1)\alpha}{(d-c)^{\alpha+2}}\int_{\frac{c+d}{2}}^{d}(d-u)^{\alpha-1}g(u)\,du,\\
M_3&=\int_0^1\left[2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right]g''\left(t\frac{c+d}{2}+(1-t)c\right)dt\\
&=\frac{2\lambda}{d-c}g'(c)+\frac{2^{\alpha+2}+2(1-\lambda)}{d-c}g'\left(\frac{c+d}{c}\right)-\frac{4(\alpha+1)}{(d-c)^2}g\left(\frac{c+d}{2}\right)+\frac{2^{\alpha+2}(\alpha+1)}{(d-c)^2}g(c)\\
&-\frac{2^{\alpha+2}(\alpha+1)\alpha}{(d-c)^{\alpha+2}}\int_{\frac{c+d}{2}}^{c}(d-u)^{\alpha-1}g(u)\,du,\\
M_4&=\int_0^1\left[2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right]g''\left(t\frac{c+d}{2}+(1-t)d\right)dt\\
&=\frac{2\lambda}{d-c}g'(d)+\frac{2^{\alpha+2}+2(1-\lambda)}{d-c}g'\left(\frac{c+d}{c}\right)+\frac{4(\alpha+1)}{(d-c)^2}g\left(\frac{c+d}{2}\right)-\frac{2^{\alpha+2}(\alpha+1)}{(d-c)^2}g(d)\\
&-\frac{2^{\alpha+2}(\alpha+1)\alpha}{(d-c)^{\alpha+2}}\int_{d}^{\frac{c+d}{2}}(u-c)^{\alpha-1}g(u)\,du,\\
&M_1+M_2+M_3+M_4\\
&=\frac{4\lambda}{d-c}\left[g'(c)+g'(d)\right]+\frac{2^{\alpha+3}-8\lambda}{d-c}g'\left(\frac{c+d}{2}\right)+\frac{2^{\alpha+2}(\alpha+1)}{(d-c)^2}\left[g(c)-g(d)\right]\\
&-\frac{2^{\alpha+2}\Gamma(\alpha+2)}{(d-c)^{\alpha+2}}\left[J_{c^+}^{\alpha}g(d)+J_{d^-}^{\alpha}g(c)\right].
\end{aligned}
$$

Multiplying both sides by $\frac{(d-c)^2}{2^{\alpha+2}}$, we get (17). This completes the proof. □

Theorem 3.1 *Let $g, g' : [c,d] \to R$ be differentiable functions on $[c,d]$, and suppose g'' is integrable. If $|g''|$ is s-(α,m)-convex on $[c,d]$, then we have the following inequality for Riemann–Liouville fractional integrals with $0 < \alpha \le 1, 0 \le \lambda \le 1$:*

$$
\begin{aligned}
&\left|\frac{\lambda}{2^{\alpha}}(d-c)\left[g'(c)+g'(d)\right]+\left(2-\frac{2}{2^{\alpha}\lambda}\right)(d-c)g'\left(\frac{c+d}{2}\right)\right.\\
&\left.\quad+(\alpha+1)\left[g(c)-g(d)\right]-\frac{\Gamma(\alpha+2)}{(d-c)^{\alpha}}\left[J_{c^+}^{\alpha}g(d)+J_{d^-}^{\alpha}g(c)\right]\right|
\end{aligned}
$$

$$\le \frac{(d-c)^2}{2^{\alpha+2}}\left\{\left[N_1\left|g''(c)\right| + 2m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right| + N_1\left|g''(d)\right|\right]\right.$$
$$\left. + \left[2N_3\left|g''\left(\frac{c+d}{2}\right)\right| + m(N_4-N_3)\left(\left|g''(c)\right| + \left|g''(d)\right|\right)\right]\right\}, \tag{18}$$

where

$$N_1 = \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right| t^{\alpha s}\,dt, \qquad N_2 = \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right| dt,$$
$$N_3 = \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right| t^{\alpha s}\,dt, \qquad N_4 = \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right| dt.$$

Proof If $|g''|$ is s-(α,m)-convex on [c,d], then for all $t \in [0,1]$, by Lemma 3.1 we obtain:

$$\begin{aligned}
|M_1| &\le \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right|\left|g''\left(tc + (1-t)\frac{c+d}{2}\right)\right| dt \\
&\le \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right|\left|t^{\alpha s}g''(c) + m\left(1-t^{\alpha s}\right)g''\left(\frac{c+d}{2}\right)\right| dt \\
&\le \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right|\left[t^{\alpha s}\left|g''(c)\right| + m\left(1-t^{\alpha s}\right)\left|g''\left(\frac{c+d}{2}\right)\right|\right] dt \\
&= N_1\left|g''(c)\right| + m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|, \\
|M_2| &\le \int_0^1 \left|\lambda - (1-t)^{\alpha+1}\right|\left|g''\left(td + (1-t)\frac{c+d}{2}\right)\right| dt \\
&\le \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right|\left|t^{\alpha s}g''(d) + m\left(1-t^{\alpha s}\right)g''\left(\frac{c+d}{2}\right)\right| dt \\
&\le \int_0^1 \left|(1-t)^{\alpha+1} - \lambda\right|\left[t^{\alpha s}\left|g''(c)\right| + m\left(1-t^{\alpha s}\right)\left|g''\left(\frac{c+d}{2}\right)\right|\right] dt \\
&= N_1\left|g''(d)\right| + m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|, \\
|M_3| &\le \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right|\left|g''\left(t\frac{c+d}{2} + (1-t)c\right)\right| dt \\
&\le \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right|\left|t^{\alpha s}g''\left(\frac{c+d}{2}\right) + m\left(1-t^{\alpha s}\right)g''(c)\right| dt \\
&\le \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right|\left[t^{\alpha s}\left|g''\left(\frac{c+d}{2}\right)\right| + m\left(1-t^{\alpha s}\right)\left|g''(c)\right|\right] dt \\
&= N_3\left|g''\left(\frac{c+d}{2}\right)\right| + m(N_4-N_3)\left|g''(c)\right|, \\
|M_4| &\le \int_0^1 \left|\lambda - 2^{\alpha+1} + (2-t)^{\alpha+1}\right|\left|g''\left(t\frac{c+d}{2} + (1-t)d\right)\right| dt \\
&\le \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right|\left|t^{\alpha s}g''\left(\frac{c+d}{2}\right) + m\left(1-t^{\alpha s}\right)g''(d)\right| dt \\
&\le \int_0^1 \left|2^{\alpha+1} - (2-t)^{\alpha+1} - \lambda\right|\left[t^{\alpha s}\left|g''\left(\frac{c+d}{2}\right)\right| + m\left(1-t^{\alpha s}\right)\left|g''(d)\right|\right] dt
\end{aligned}$$

$$= N_3\left|g''\left(\frac{c+d}{2}\right)\right| + m(N_4 - N_3)\left|g''(d)\right|.$$

Summing the four terms on the right-hand side of the inequality, we get (18). This completes the proof. □

Let $\alpha = s = m = 1$ in Theorem 3.1. Then (18) reduces to an integer-order inequality of general convexity.

Corollary 3.1 *Let g, g' be defined as in Theorem 3.1. If $|g''|$ is convex on $[c,d]$, then*

$$\begin{aligned}
&\left|\frac{\lambda}{2}(d-c)\left[g'(c)+g'(d)\right] + (2-\lambda)(d-c)g'\left(\frac{c+d}{2}\right) + 2\left[g(c)-g(d)\right] - \frac{4}{d-c}\int_c^d g(x)\,dx\right| \\
&\quad\le \frac{(d-c)^2}{8}\left[\left(\frac{4}{3}\lambda(4-\lambda)^{\frac{1}{2}} + \frac{4}{3}\lambda^{\frac{3}{2}} - 5\lambda - \frac{16}{3}(4-\lambda)^{\frac{1}{2}} + \frac{34}{3}\right)\left(\left|g''(c)\right| + \left|g''(d)\right|\right)\right. \\
&\qquad\left. + \left(-\frac{16}{3}\lambda(4-\lambda)^{\frac{1}{2}} + 14\lambda + \frac{64}{3}(4-\lambda)^{\frac{1}{2}} - 40\right)\left|g''\left(\frac{c+d}{2}\right)\right|\right]. \qquad (19)
\end{aligned}$$

In [8] the author used the convexity of $|g'|$ to estimate the error. We can do a similar work by using the convexity of $|g''|$.

Remark 3.1 Taking $\lambda = 0$ and $\lambda = 1$ In Corollary 3.1, we get the following two inequalities:

$$\begin{aligned}
&\left|2(d-c)g'\left(\frac{c+d}{2}\right) + 2\left[g(c)-g(d)\right] - \frac{4}{d-c}\int_c^d g(x)\,dx\right| \\
&\quad\le \frac{(d-c)^2}{8}\left[\frac{2}{3}\left|g''(c)\right| + \frac{2}{3}\left|g''(d)\right| + \frac{8}{3}\left|g''\left(\frac{c+d}{2}\right)\right|\right], \\
&\left|\frac{d-c}{2}\left[g'(c)+g'(d)\right] + (d-c)g'\left(\frac{c+d}{2}\right) + 2\left[g(c)-g(d)\right] - \frac{4}{d-c}\int_c^d g(x)\,dx\right| \\
&\quad\le \frac{(d-c)^2}{8}\left[\frac{23-13\sqrt{3}}{3}\left|g''(c)\right| + \frac{23-13\sqrt{3}}{3}\left|g''(d)\right| + (16\sqrt{3}-26)\left|g''\left(\frac{c+d}{2}\right)\right|\right].
\end{aligned}$$

Theorem 3.2 *Let $g, g' : [c,d] \to R$ be differentiable functions on $[c,d]$, and suppose g'' is integrable. If $|g''|^q$ is s-(α, m)-convex on $[c,d]$ with $q \ge 1$, then we have the following inequality with $0 < \alpha \le 1, 0 \le \lambda \le 1$:*

$$\begin{aligned}
&\left|\frac{\lambda}{2^\alpha}(d-c)\left[g'(c)+g'(d)\right] + \left(2 - \frac{2}{2^\alpha \lambda}\right)(d-c)g'\left(\frac{c+d}{2}\right)\right. \\
&\qquad\left. + (\alpha+1)\left[g(c)-g(d)\right] - \frac{\Gamma(\alpha+2)}{(d-c)^\alpha}\left[J_{c^+}^\alpha g(d) + J_{d^-}^\alpha g(c)\right]\right| \\
&\quad\le \frac{(d-c)^2}{2^{\alpha+2}}\left\{(N_2)^{1-\frac{1}{q}}\left[\left(N_1\left|g''(c)\right|^q + m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\right. \\
&\qquad\left. + \left(N_1\left|g'(d)\right|^q + m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\
&\qquad + (N_4)^{1-\frac{1}{q}}\left[\left(N_3\left|g''(c)\right|^q + m(N_4-N_3)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(N_3\left|g''(c)\right|^q+m(N_4-N_3)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\Bigg]\Bigg\}. \tag{20}$$

Proof Using the well-known power-mean integral inequality

$$\int_a^b \left|f(x)g(x)\right|dx \le \left(\int_a^b \left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b \left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}$$

for $q>1$ and the convexity of $|g''|^q$, we have:

$$\begin{aligned}
|M_1| &= \left|\int_0^1 \left[(1-t)^{\alpha+1}-\lambda\right]g''\left(tc+(1-t)\frac{c+d}{2}\right)dt\right| \\
&\le \left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|\left|g''\left(tc+(1-t)\frac{c+d}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|\left(t^{\alpha s}\left|g''(c)\right|^q+m\left(1-t^{\alpha s}\right)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)dt\right]^{\frac{1}{q}} \\
&= (N_2)^{1-\frac{1}{q}}\left(N_1\left|g''(c)\right|^q+m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}, \\
|M_2| &= \left|\int_0^1 \left[\lambda-(1-t)^{\alpha+1}\right]g''\left(td+(1-t)\frac{c+d}{2}\right)dt\right| \\
&\le \left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|\left|g''\left(td+(1-t)\frac{c+d}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \left(\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right|\left(t^{\alpha s}\left|g''(d)\right|^q+m\left(1-t^{\alpha s}\right)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)dt\right]^{\frac{1}{q}} \\
&= (N_2)^{1-\frac{1}{q}}\left(N_1\left|g''(d)\right|^q+m(N_2-N_1)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}, \\
|M_3| &= \left|\int_0^1 \left[2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right]g''\left(\frac{c+d}{2}t+(1-t)c\right)dt\right| \\
&\le \left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|\left|g''\left(\frac{c+d}{2}t+(1-t)c\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|\left(t^{\alpha s}\left|g''\left(\frac{c+d}{2}\right)\right|^q+m\left(1-t^{\alpha s}\right)\left|g''(c)\right|^q\right)dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$= (N_4)^{1-\frac{1}{q}}\left(N_3\left|g''\left(\frac{c+d}{2}\right)\right|^q + m(N_4-N_3)\left|g''(d)\right|^q\right)^{\frac{1}{q}},$$

$$\begin{aligned}|M_4| &= \left|\int_0^1 \left[\lambda - 2^{\alpha+1} + (2-t)^{\alpha+1}\right]g''\left(\frac{c+d}{2}t+(1-t)d\right)dt\right| \\ &\le \left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\ &\quad\times\left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|\left|g''\left(\frac{c+d}{2}t+(1-t)d\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\le \left(\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|dt\right)^{1-\frac{1}{q}} \\ &\quad\times\left[\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|\left(t^{\alpha s}\left|g''\left(\frac{c+d}{2}\right)\right|^q + m\left(1-t^{\alpha s}\right)\left|g''(d)\right|^q\right)dt\right]^{\frac{1}{q}} \\ &= (N_4)^{1-\frac{1}{q}}\left(N_3\left|g''\left(\frac{c+d}{2}\right)\right|^q + m(N_4-N_3)\left|g''(d)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Summing $|M_1|, |M_2|, |M_3|$, and $|M_4|$, we get formula (20). This completes the proof. □

Taking $\alpha = s = m = 1$ in Theorem 3.2, we get the following integer-order inequalities of general convexity. First, taking $\lambda = 0$, we get the following:

Corollary 3.2 *Let g, g' be defined as in Theorem 3.2. If $|g''|^q$ is convex on $[c,d]$ with $q > 1$, then*

$$\begin{aligned}&\left|2(d-c)g'\left(\frac{c+d}{2}\right) + 2\left[g(c)-g(d)\right] - \frac{4}{d-c}\int_c^d g(x)\,dx\right| \\ &\le \frac{(d-c)^2}{12}\left[\left(\frac{1+13\cdot 5^{q-1}}{8}\left|g''(c)\right|^q + \frac{3+7\cdot 5^{q-1}}{8}\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad\left.+\left(\frac{1+13\cdot 5^{q-1}}{8}\left|g''(d)\right|^q + \frac{3+7\cdot 5^{q-1}}{8}\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Second, taking $\lambda = 1$, we get the following:

Corollary 3.3 *Let g, g' be defined as in Theorem 3.2. If $|g''|^q$ is convex on $[c,d]$ with $q > 1$, then*

$$\begin{aligned}&\left|\frac{d-c}{2}\left[g'(c)+g'(d)\right] + (d-c)g'\left(\frac{c+d}{2}\right) + 2\left[g(c)-g(d)\right] - \frac{4}{d-c}\int_c^d g(x)\,dx\right| \\ &\le \frac{(d-c)^2}{8}\left\{\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{5}{12}\left|g''(c)\right|^q + \frac{1}{4}\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\right. \\ &\quad\left.+\left(\frac{5}{12}\left|g''(d)\right|^q + \frac{1}{4}\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\ &\quad+(4\sqrt{3}-6)^{1-\frac{1}{q}}\left[\left(\left(8\sqrt{3}-\frac{53}{4}\right)\left|g''(c)\right|^q + \left(\frac{29}{4}-4\sqrt{3}\right)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\left(8\sqrt{3}-\frac{53}{4}\right)\left|g''(d)\right|^q+\left(\frac{29}{4}-4\sqrt{3}\right)\left|g''\left(\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Theorem 3.3 *Let $g, g' : [c,d] \to R$ be differentiable functions on $[c,d]$, and suppose g'' is integrable. If $|g''|$ is s-(α,m)-concave on $[c,d]$, then we have the following inequality for Riemann–Liouville fractional integrals with $0 < \alpha \le 1, 0 \le \lambda \le 1$:*

$$\begin{aligned}
&\frac{\lambda}{2^\alpha}(d-c)\big[g'(c)+g'(d)\big]+\left(2-\frac{2}{2^\alpha}\lambda\right)(d-c)g'\left(\frac{c+d}{2}\right)\\
&\quad+(\alpha+1)\big[g(c)-g(d)\big]-\frac{\Gamma(\alpha+2)}{(d-c)^\alpha}\big[J^\alpha_{c^+}g(d)+J^\alpha_{d^-}g(c)\big]\\
&\quad\le\frac{(d-c)^2}{2^{\alpha+2}}\left\{N_2\left[\left|g''\left(\frac{N_5c+(N_2-N_5)\frac{c+d}{2}}{N_2}\right)\right|+\left|g''\left(\frac{N_5d+(N_2-N_5)\frac{c+d}{2}}{N_2}\right)\right|\right]\right.\\
&\quad\left.+N_4\left[\left|g''\left(\frac{N_6\frac{c+d}{2}+(N_4-N_6)c}{N_4}\right)\right|+\left|g''\left(\frac{N_6\frac{c+d}{2}+(N_4-N_6)d}{N_2}\right)\right|\right]\right\},
\end{aligned}\tag{21}$$

where

$$N_5=\int_0^1\left|(1-t)^{\alpha+1}-\lambda\right|t\,dt,\qquad N_6=\int_0^1\left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right|t\,dt.$$

Proof Using the concavity of $|g''|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}
\left|g''\big(tc+(1-t)d\big)\right|^q&\ge t\left|g''(c)\right|^q+(1-t)\left|g''(d)\right|^q\\
&\ge\big(t\left|g''(c)\right|+(1-t)\left|g''(d)\right|\big)^q.
\end{aligned}$$

Then

$$\left|g''\big(tc+(1-t)d\big)\right|\ge t\left|g''(c)\right|+(1-t)\left|g''(d)\right|,$$

so that $|g''|$ is also concave. By the Jensen integral inequality for concave functions

$$\frac{\int_c^d\lambda(x)g(u(x))\,dx}{\int_c^d\lambda(x)\,dx}\le g\left(\frac{\int_c^d\lambda(x)u(x)\,dx}{\int_c^d\lambda(x)\,dx}\right)$$

we have

$$\begin{aligned}
&\left|\frac{\lambda}{2^\alpha}(d-c)\big[g'(c)+g'(d)\big]+\left(2-\frac{2}{2^\alpha\lambda}\right)(d-c)g'\left(\frac{c+d}{2}\right)\right.\\
&\quad\left.+(\alpha+1)\big[g(c)-g(d)\big]-\frac{\Gamma(\alpha+2)}{(d-c)^\alpha}\big[J^\alpha_{c^+}g(d)+J^\alpha_{d^-}g(c)\big]\right|\\
&\quad\le\frac{(d-c)^2}{2^{\alpha+2}}\left\{\int_0^1\left|(1-t)^{\alpha+1}-\lambda\right|dt\frac{\int_0^1|(1-t)^{\alpha+1}-\lambda||g''(tc+(1-t)\frac{c+d}{2})|\,dt}{\int_0^1|(1-t)^{\alpha+1}-\lambda|\,dt}\right.\\
&\quad+\int_0^1\left|(1-t)^{\alpha+1}-\lambda\right|dt\frac{\int_0^1|(1-t)^{\alpha+1}-\lambda||g''(td+(1-t)\frac{c+d}{2})|\,dt}{\int_0^1|(1-t)^{\alpha+1}-\lambda|\,dt}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right| dt \frac{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda||g''(t\frac{c+d}{2}+(1-t)c)|\,dt}{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|\,dt} \\
&+\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right| dt \frac{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda||g''(t\frac{c+d}{2}+(1-t)d)|\,dt}{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|\,dt}\Bigg\} \\
\le{}& \frac{(d-c)^2}{2^{\alpha+2}}\Bigg\{\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right| dt\left|g''\left(\frac{\int_0^1 |(1-t)^{\alpha+1}-\lambda|(tc+(1-t)\frac{c+d}{2})\,dt}{\int_0^1 |(1-t)^{\alpha+1}-\lambda|\,dt}\right)\right| \\
&+\int_0^1 \left|(1-t)^{\alpha+1}-\lambda\right| dt\left|g''\left(\frac{\int_0^1 |(1-t)^{\alpha+1}-\lambda|(td+(1-t)\frac{c+d}{2})\,dt}{\int_0^1 |(1-t)^{\alpha+1}-\lambda|\,dt}\right)\right| \\
&+\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right| dt\left|g''\left(\frac{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|(t\frac{c+d}{2}+(1-t)c)\,dt}{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|\,dt}\right)\right| \\
&+\int_0^1 \left|2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda\right| dt\left|g''\left(\frac{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|(t\frac{c+d}{2}+(1-t)d)\,dt}{\int_0^1 |2^{\alpha+1}-(2-t)^{\alpha+1}-\lambda|\,dt}\right)\right|\Bigg\} \\
={}& \frac{(d-c)^2}{2^{\alpha+2}}\Bigg\{N_2\left[\left|g''\left(\frac{N_5 c+(N_2-N_5)\frac{c+d}{2}}{N_2}\right)\right|+\left|g''\left(\frac{N_5 d+(N_2-N_5)\frac{c+d}{2}}{N_2}\right)\right|\right] \\
&+N_4\left[\left|g''\left(\frac{N_6\frac{c+d}{2}+(N_4-N_6)c}{N_4}\right)\right|+\left|g''\left(\frac{N_6\frac{c+d}{2}+(N_4-N_6)d}{N_2}\right)\right|\right]\Bigg\}.
\end{aligned}
$$

This completes the proof. □

Taking $\alpha = 1$ in Theorem 3.3, we get the following integer-order inequalities. First, taking $\lambda = 0$, we get the following:

Corollary 3.4 *Let g, g' be defined as in Theorem 3.2. If $|g''|$ is convex on $[c,d]$, then*

$$
\begin{aligned}
&\left|2(d-c)g'\left(\frac{c+d}{2}\right)+2\big[g(c)-g(d)\big]-\frac{4}{d-c}\int_c^d g(x)\,dx\right| \\
&\quad\le \frac{(d-c)^2}{24}\left[\left(\left|g''\left(\frac{5c+3d}{8}\right)\right|+\left|g''\left(\frac{3c+5d}{8}\right)\right|\right)\right. \\
&\qquad\left.+5\left(\left|g''\left(\frac{27c+13d}{40}\right)\right|+\left|g''\left(\frac{13c+27d}{40}\right)\right|\right)\right].
\end{aligned}
$$

Second, taking $\lambda = 1$, we get the following:

Corollary 3.5 *Let g, g' be defined as in Theorem 3.3. If $|g''|$ is convex on $[c,d]$, then*

$$
\begin{aligned}
&\left|\frac{d-c}{2}\big[g'(c)+g'(d)\big]+(d-c)g'\left(\frac{c+d}{2}\right)+2\big[g(c)-g(d)\big]-\frac{4}{d-c}\int_c^d g(x)\,dx\right| \\
&\quad\le \frac{(d-c)^2}{8}\left[\frac{2}{3}\left(\left|g''\left(\frac{13c+3d}{16}\right)\right|+\left|g''\left(\frac{3c+13d}{16}\right)\right|\right)\right. \\
&\qquad\left.+(4\sqrt{3}-6)\left(\left|g''\left(\frac{5c+(32\sqrt{3}-53)d}{32\sqrt{3}-48}\right)\right|+\left|g''\left(\frac{(32\sqrt{3}-53)c+5d}{32\sqrt{3}-48}\right)\right|\right)\right].
\end{aligned}
$$

4 Applications of the result

Using the results obtained, we can get new estimates for the following special means.

1. The arithmetic mean: $A(c,d) = \frac{c+d}{2}$ for $c, d \in R$.
2. The geometric mean: $G(c,d) = \sqrt{ab}$ for $c, d > 0$.
3. The harmonic mean: $H(c,d) = \frac{2cd}{c+d}$ for $c, d \in R\backslash\{0\}$.
4. The index mean:

$$I(c,d) = \begin{cases} c, & c = d, \\ \frac{1}{e}(\frac{d^d}{c^c})^{\frac{1}{d-c}}, & c \neq d, c, d > 0. \end{cases}$$

5. The logarithmic mean:

$$L(c,d) = \begin{cases} c, & c = d, \\ \frac{d-c}{\ln d - \ln c}, & c \neq d, c, d > 0. \end{cases}$$

6. Generalized logarithmic mean:

$$L_n(c,d) = \begin{cases} c, & c = d, \\ [\frac{d^{n+1}-c^{n+1}}{(n+1)(d-c)}]^{\frac{1}{n}}, & c \neq d, n \in Z\backslash\{-1,0\}, c, d > 0. \end{cases}$$

Proposition 4.1 *Let $n \in Z\backslash\{-1,0\}$ and $c, d > 0$. Then we have the following inequality*:

$$\begin{aligned} &\left|n\lambda(d-c)A\left(c^{n-1},d^{n-1}\right) + n(2-\lambda)(d-c)A^{n-1}(c,d) + 2\left(c^n - d^n\right) - 4L_n^n(c,d)\right| \\ &\quad\leq \frac{n(n-1)(d-c)^2}{4}\left[\left(\frac{4\lambda-16}{3}(4-\lambda)^{\frac{1}{2}} + \frac{4}{3}\lambda^{\frac{3}{2}} + 3\lambda + \frac{34}{3}\right)A\left(\left|c^{n-2}\right|,\left|d^{n-2}\right|\right)\right. \\ &\quad\quad\left. + \left(\frac{32-8\lambda}{3}(4-\lambda)^{\frac{1}{2}} + 7\lambda - 20\right)A^{n-2}\left(|c|,|d|\right)\right]. \end{aligned} \tag{22}$$

Proof The statement follows from Corollary 3.1 for $g(x) = x^n$, $x \in [c,d]$:

$$\begin{aligned} &\frac{4}{d-c}\int_c^d g(x)\,dx = \frac{4(d^{n+1}-c^{n+1})}{(d-c)(n+1)}, \\ &(2-\lambda)(d-c)g'\left(\frac{c+d}{2}\right) = n(2-\lambda)(d-c)\left(\frac{c+d}{2}\right)^{n-1}, \\ &\frac{\lambda(d-c)}{2}\left[g'(c) + g'(d)\right] = n\lambda(d-c)\left(\frac{c^{n-1}+d^{n-1}}{2}\right). \end{aligned}$$

Substituting these formulas into Corollary 3.1, we obtain (22). □

Remark 4.1 Taking $\lambda = 0$ in Proposition 1, we have

$$\begin{aligned} &\left|2n(d-c)A^{n-1}(c,d) + 2\left(c^n - d^n\right) - 4L_n^n(c,d)\right| \\ &\quad\leq \frac{n(n-1)(d-c)^2}{4}\left[\frac{2}{3}A\left(\left|c^{n-2}\right|,\left|d^{n-2}\right|\right) + \frac{4}{3}A^{n-2}\left(|c|,|d|\right)\right]. \end{aligned}$$

Remark 4.2 Taking $\lambda = 1$ in Proposition 4.1, we have

$$\left|n(d-c)A\left(c^{n-1},d^{n-1}\right) + n(d-c)A^{n-1}(c,d) + 2\left(c^n - d^n\right) - 4L_n^n(c,d)\right|$$

$$\leq \frac{n(n-1)(d-c)^2}{4}\left[\frac{23-13\sqrt{3}}{3}A\left(\left|c^{n-2}\right|,\left|d^{n-2}\right|\right)+(8\sqrt{3}-13)A^{n-2}\left(|c|,|d|\right)\right].$$

Proposition 4.2 *Suppose $c,d \in R$ with $c,d>0$. Then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{2(c-d)}{A(1+c,1+d)}-4\ln\left[G\left(1+d,\frac{1}{1+c}\right)I(1+d,1+c)\right]\right| \\
&\quad\leq \frac{(d-c)^2}{12}\left[\left(\frac{1+13\cdot 5^{q-1}}{8(1+c)^{2q}}+\frac{3+7\cdot 5^{q-1}}{8A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(\frac{1+13\cdot 5^{q-1}}{8(1+d)^{2q}}+\frac{3+7\cdot 5^{q-1}}{8A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{23}$$

Proof The statement follows from Corollary 3.2 for $g(x)=-\ln(1+x)$, $x\in[c,d]$. Since $g'(x)=\frac{-1}{1+x}$ and $g''(x)=\frac{1}{(1+x)^2}$, we get

$$\begin{aligned}
&2(d-c)g'\left(\frac{c+d}{2}\right)=\frac{2(c-d)}{\frac{2+c+d}{2}},\quad 2\left[g(c)-g(d)\right]=4\ln\left(\frac{1+d}{1+c}\right)^{\frac{1}{2}}, \\
&\frac{4}{d-c}\int_c^d g(x)\,dx=\frac{4}{d-c}\left[\ln\frac{(1+c)^c}{(1+d)^d}+(d-c)+\ln\frac{1+c}{1+d}\right] \\
&\quad\quad=4\ln\frac{e}{\left(\frac{(1+c)^{1+c}}{(1+d)^{1+d}}\right)^{\frac{1}{c-d}}}.
\end{aligned} \tag{24}$$

Substituting formula (24) into Corollary 3.2, we obtain (23). □

Proposition 4.3 *Suppose $c,d \in R$ with $c,d>0$. Then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{(c-d)A(1+c,1+d)}{G^2(1+c,1+d)}+\frac{c-d}{A(1+c,1+d)}-4\ln\left[G\left(1+d,\frac{1}{1+c}\right)I(1+d,1+c)\right]\right| \\
&\quad\leq \frac{(d-c)^2}{8}\left\{\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{5}{12(1+c)^{2q}}+\frac{1}{4A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right.\right. \\
&\quad\quad\left.+\left(\frac{5}{12(1+d)^{2q}}+\frac{1}{4A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right] \\
&\quad\quad+(4\sqrt{3}-6)^{1-\frac{1}{q}}\left[\left(\frac{32\sqrt{3}-53}{4(1+c)^{2q}}+\frac{29-16\sqrt{3}}{4A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.\left.+\left(\frac{32\sqrt{3}-53}{4(1+d)^{2q}}+\frac{29-16\sqrt{3}}{4A^{2q}(1+c,1+d)}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned} \tag{25}$$

Proof The statement follows from Corollary 3.3 for $g(x)=-\ln(1+x)$, $x\in[c,d]$. Using $g'(x)=\frac{-1}{1+x}$ and $g''(x)=\frac{1}{(1+x)^2}$, we get

$$\frac{d-c}{2}\left[g'(c)+g'(d)\right]=\frac{(c-d)(1+c+1+d)}{2(1+c)(1+d)}. \tag{26}$$

Substituting formulas (24) and (26) into Corollary 3.3, we obtain (25). □

Proposition 4.4 *Suppose $c,d \in R$ with $c,d > 0$. Then we have the following inequality*:

$$\begin{aligned}&\left|\frac{2(c-d)}{A^2(c,d)}+\frac{2(c-d)}{G^2(c,d)}-4L^{-1}(c,d)\right|\\&\quad\le\frac{(d-c)^2}{24}\left[\left(\frac{128}{A^3(5c,3d)}+\frac{128}{A^3(3c,5d)}\right)+\left(\frac{10\times 20^3}{A^3(27c,13d)}+\frac{10\times 20^3}{A^3(13c,27d)}\right)\right].\end{aligned}\tag{27}$$

Proof The statement follows from Corollary 3.4 for $g(x)=\frac{1}{x}$, $x\in[c,d]$. Using $g'(x)=-\frac{1}{x^2}$ and $g''(x)=\frac{2}{x^3}$, we get

$$\begin{aligned}&2(d-c)g'\left(\frac{c+d}{2}\right)=\frac{2(c-d)}{(\frac{c+d}{2})^2},\qquad 2\bigl[g(c)-g(d)\bigr]=\frac{2(d-c)}{cd},\\&\frac{4}{d-c}\int_c^d g(x)\,dx=\frac{4(\ln d-\ln c)}{d-c}.\end{aligned}\tag{28}$$

Substituting formula (28) into Corollary 3.4, we obtain (27). □

Proposition 4.5 *Suppose $c,d \in R$ with $c,d > 0$. Then we have the following inequality*:

$$\begin{aligned}&\left|(c-d)H^{-1}\bigl(c^2,d^2\bigr)+\frac{c-d}{A^2(c,d)}+\frac{2(d-c)}{G^2(c,d)}-4L^{-1}(c,d)\right|\\&\quad\le\frac{(d-c)^2}{8}\left[\left(\frac{4\times 8^3}{3A^3(13c,3d)}+\frac{4\times 8^3}{3A^3(3c,13d)}\right)\right.\\&\qquad\left.+\left(\frac{(16\sqrt{3}-24)^4}{2A^3(5c,(32\sqrt{3}-53)d)}\right)+\left(\frac{(16\sqrt{3}-24)^4}{2A^3((32\sqrt{3}-53)c,5d)}\right)\right].\end{aligned}\tag{29}$$

Proof The statement follows from Corollary 3.5 for $g(x)=\frac{1}{x}$, $x\in[c,d]$. Using $g'(x)=-\frac{1}{x^2}$ and $g''(x)=\frac{2}{x^3}$, we get

$$\frac{d-c}{2}\bigl[g'(c)+g'(d)\bigr]=\frac{(c-d)(c^2+d^2)}{2c^2d^2}.\tag{30}$$

Substituting formulas (28) and (30) into Corollary 3.5, we obtain (29). □

5 Conclusions

We first introduced the new function class of s-(α,m)-convex functions. Then we presented a new differentiability condition to establish the important equation (17) for the Riemann–Liouville fractional integral. In Theorems 3.1–3.3, we gave new Hermite–Hadamard integral inequalities depending on (17) by using the associated power-mean inequality and Jensen's integral inequality. Finally, we applied these inequalities to special mean values. These results can be applied to the qualitative theory research of calculus equations in the future.

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Availability of data and materials
We declare that the data and material in the paper can be used publicly.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RNL carried out the main results and completed the corresponding proof. RX participated in the proof and helped to complete Sect. 4. Both authors read and approved the final manuscript.

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References

1. Kanniappan, P., Pandian, P.: On generalized convex functions in optimization theory – a survey. Opsearch **33**(3), 174–185 (1996)
2. Majeed, S.N., Abd Al-Majeed, M.I.: On convex functions, e-convex functions and their generalizations: applications to non-linear optimization problems. Int. J. Pure Appl. Math. **116**(3), 655–673 (2017)
3. Dragomir, S.S., Pearce, C.E.M.: Selected topics on Hermite–Hadamard inequalities and applications. RGMIA Monographs, Victoria University (2000)
4. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
5. Hwang, D.Y., Tseng, K.L.: Some inequalities for differentiable convex and concave mapping. Comput. Math. Appl. **47**(2–3), 207–216 (2004)
6. Khaled, M., Agarwal, P.: New Hermite–Hadamard type integral inequalities for convex functions and their applications. J. Comput. Appl. Math. **350**, 274–285 (2019)
7. Özcan, S., İşcan, İ.: Some new Hermite–Hadamard type inequalities for *s*-convex functions and their applications. J. Inequal. Appl. **2019**, 201 (2019)
8. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403–2407 (2013)
9. Chun, Z., Feckan, M., Rong, W.J.: Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. J. Appl. Math. Stat. Inform. **8**, 2 (2012)
10. Ling, L.X., Qaisar, S., Nasir, J.: Some results on integral inequalities via Riemann–Liouville fractional integrals. J. Inequal. Appl. **2019**, 214 (2019)
11. Hwang, S.R., Tseng, K.L., Hsu, K.C.: New inequalities for fractional integrals and their applications. Turk. J. Math. **40**, 471–486 (2016)
12. Kirmaci, U.-S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**, 137–146 (2004)
13. Qaisar, S., He, C., Hussain, S.: A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications. J. Inequal. Appl. **2013**, Article ID 158 (2013)
14. Kui, L., Rong, W.J., O'Regan, D.: On the Hermite–Hadamard type inequality for ψ-Riemann–Liouville fraction integrals via convex functions. J. Inequal. Appl. **2019**, 27 (2019)
15. Nisar, K.S., Asifa, T., Gauhar, R.: Some inequalities via fractional conformable integral operators. J. Inequal. Appl. **2019**, 217 (2019)
16. Iqbal, M., Qaisar, S., Muddassar, M.: A short note on integral inequality of Hermite–Hadamard type through convexity. J. Comput. Anal. Appl. **21**(5), 946–953 (2016)
17. Qaisar, S., Iqbal, M., Muddassar, M.: New Hermite–Hadamard's inequalities for preinvex function via fractional integrals. J. Comput. Anal. Appl. **20**(7), 1318–1328 (2016)
18. Agarwal, P., Jleli, M., Tomar, M.: Certain Hermite–Hadamard type inequalities via generalized *k*-fraction integrals. J. Inequal. Appl. **2017**, Article ID 55 (2017)
19. Qaisar, S., Iqbal, M., Hussain, S., Butt, S.I., Meraj, M.A.: New inequalities on Hermite–Hadamard utilizing fractional integrals. Kragujev. J. Math. **42**(1), 15–27 (2018)
20. Zhao, D.F., An, T.Q., Ye, G.J., Delfim, F.M.: On Hermite–Hadamard type inequalities for harmonical *h*-convex interval-valued functions. Math. Inequal. Appl. **23**, 95–105 (2020)
21. An, Y.R., Ye, G.J., Zhao, D.F., Liu, W.: Hermite–Hadamard type inequalities for interval (h_1, h_2)-convex functions. Mathematics **7**(5), 436 (2019). https://doi.org/10.3390/math7050436
22. Shi, F.F., Ye, G.J., Zhao, D.F., Liu, W.: Some fractional Hermite–Hadamard type inequalities for interval-valued functions. Mathematics **8**, 534 (2020)
23. Muddassar, M., Bhatti, M.I., Irshad, W.: Generalisation of integral inequalities of Hermite–Hadamard type through convexity. Bull. Aust. Math. Soc. **88**(2), 320–330 (2014)
24. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Reading (1993)
25. Gorenflo, R., Mainard, F.: Fractional calculus: integral and differential equations of fractional order. In: Fractals and Fractional Calculus in Continuum Mechanics, pp. 223–276. Springer, Wien (1997)