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Construction and analysis of unified 4-point interpolating nonstationary subdivision surfaces

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Abstract

Subdivision schemes (SSs) have been the heart of computer-aided geometric design almost from its origin, and several unifications of SSs have been established. SSs are commonly used in computer graphics, and several ways were discovered to connect smooth curves/surfaces generated by SSs to applied geometry. To construct the link between nonstationary SSs and applied geometry, in this paper, we unify the interpolating nonstationary subdivision scheme (INSS) with a tension control parameter, which is considered as a generalization of 4-point binary nonstationary SSs. The proposed scheme produces a limit surface having C^1 smoothness. It generates circular images, spirals, or parts of conics, which are important requirements for practical applications in computer graphics and geometric modeling. We also establish the rules for arbitrary topology for extraordinary vertices (valence ≥ 3). The well-known subdivision Kobbelt scheme (Kobbelt in *Comput. Graph. Forum* 15(3):409–420, 1996) is a particular case. We can visualize the performance of the unified scheme by taking different values of the tension parameter. It provides an exact reproduction of parametric surfaces and is used in the processing of free-form surfaces in engineering.

Keywords: Subdivision; Interpolation; Tension control; Regular topology; Arbitrary topology

1 Introduction

The subdivision is a very popular geometric modeling tool. Subdivision algorithms are widely used in computer graphics and computer aided geometric design (CAGD) due to their efficiency, flexibility, and simplicity. There are two common classes of SSs. One is approximating in which the limit surface usually does not go through its control vertices, and in case of interpolating, the limit surface interpolates all subdivision steps of control vertices exactly, which is most appropriate to engineering applications. They seem to have their origin in the geometric problem of smoothing the corners of a given polygon. Further, SSs are classified into stationary and nonstationary schemes. The proposed scheme is in the class of nonstationary interpolating schemes. The idea of nonstationary SSs was given by Dyn and Levin [2]. Nonstationary SSs form a standard structure for

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introducing self-referential sets such as practical applications in computer graphics and provide a potential new technique of researching the texture and shape of images. Yonggang et al. [3] proved that the trigonometric polynomial B-spline curves not only inherit the useful advantage of the polynomial curve but also have reconstructing property of the trigonometric curve. Due to its importance in generating images, several unifications to the nonstationary schemes such as interpolating and approximating are discussed in the literature [4–17]. Being two various subjects that had been improving individually and in parallel, the relation between curves and surfaces of SSs have sought after. Later, it was proved that there is a close relation between curves and surfaces produced by SSs. Kobbelt [1] has extended the technique of [18] and constructed interpolating SS on open quadrilateral meshes with arbitrary topology. Reif [19] has established a generalized technique to the SSs near extraordinary vertices. Fang et al. [20] introduced the unified stationary SS of arbitrary order with image controlling variable, but it does not hold up the surfaces like sphere and hyperboloid. Recently, Ghaffar et al. [21] have introduced tensor products of nine-tic B-spline. Therefore, the natural way to define refinement operators for quadrilateral nets to modify a tensor product scheme such that special rules for the vicinity of nonregular vertices are found. The proposed unified INSS has such potential. The main contribution is as follows:

- Three different schemes (trigonometric, polynomial, and hyperbolic) of surface work under one parameter.
- The proposed schemes reproduce trigonometric functions and hyperbolic functions.
- Results from the hyperbolic scheme are shown in Fig. 7.

The decomposition is extended to regular vertices of quadrilateral surfaces by the tensor product of a unified scheme. Rules for extraordinary and boundary vertices are also established based on repeated local operations. The performance of INSS, which based on quadrilaterals, is comparatively better to triangles for constructing the symmetries of natural and human objects such as legs, arms, and fingers. The major advantage of the proposed scheme is that it has the interpolation property and works on quadrilateral nets, which are most appropriate for engineering applications. Here we present the rule of the unified scheme for arbitrary topology (valence ≥ 3). To achieve this, it may be necessary to use one step of the Catmull Clark method to eliminate extraordinary faces. In quad meshes, there remains only the question of how to compute new edge points and new face points. No new vertex points are computed since the method is interpolating. The nonstationary tension parameter of our scheme is used to control the image of resulting surfaces and the interpolation of the control mesh to limit surface.

Section 2 shows the rules of unified INSS. By topological regularity we mean a tensor product structure with four faces meeting at every vertex. Section 2 is also based on a unified surface subdivision for arbitrary topology. Boundary and crease features are also discussed. Section 3 gives the reconstruction of sine and cosine functions. Also, the analysis of the proposed scheme is presented in this section. Section 4 provides the numerical examples for open (boundary edges occur, which belong to one face) and closed nets (every edge is part of exactly two faces). Section 5 holds the conclusion.

2 Unified four-point binary interpolating nonstationary SS

This section is intended to use a framework for the construction of a unified family of four-point SS for curve and surface designing. The framework has two cases. In the first

case, we construct a univariate scheme. In the second case, we derive the bivariate scheme (regular or irregular surfaces).

2.1 Curve case

Here we introduce the algorithmic technique for the construction of the univariate family of unified four-point binary INSS by applying trigonometric, polynomial, and hyperbolic basis. We can describe the proposed scheme as various stage construction.

- Consider interpolating the limiting curve from the linear space spanned by $\{1, y, \sin(y), \cos(y)\}$, $\{1, y, y^2, y^3\}$, and $\{1, y, \sinh(y), \cosh(y)\}$.
- Describe the control polygon $\{q_m^\ell | i \in \mathbb{Z}\}$ at refinement subdivision step ℓ and the interpolation of initial data q_{i+h}^ℓ , $h = -1, 0, 1, 2$, by a limit curve of the forms $q(y) = \alpha_0 + \alpha_1 y + \alpha_2 \cos(y) + \alpha_3 \sin(y)$, $q(y) = \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3$, and $q(y) = \alpha_0 + \alpha_1 y + \alpha_2 \cosh(y) + \alpha_3 \sinh(y)$.
- Obtain a system of linear equations by the interpolation of initial data q_{i+h}^ℓ corresponding to $y = h\theta/2^\ell$, $0 \leq \theta \leq \frac{\pi}{2}$.
- Now find a solution of the system of equations $f(-2^{-\ell}\theta) = q_{m-1}^\ell$, $q(0) = q_m^\ell$, $q(2^{-\ell}\theta) = q_{m+1}^\ell$ and $q(2^{-\ell+1}\theta) = q_{m+2}^\ell$ for unknowns constants $\alpha_0, \alpha_1, \alpha_2$ and α_3 . We obtained three different schemes (trigonometric, polynomial, and hyperbolic) depending on the spanning set.
- Unifying these SSs, we obtained the following INSS with control polygon $q_m^0 = q_m$, $-2 \leq m \leq N + 2$:

$$\begin{aligned}
 q_{2m}^{\ell+1} &= q_m^\ell, \quad -1 \leq m \leq 2^\ell N + 1, \\
 q_{2m+1}^{\ell+1} &= \alpha^\ell (q_{m-1}^\ell + q_{m+2}^\ell) + \beta^\ell (q_m^\ell + q_{m+1}^\ell), \quad -1 \leq m \leq 2^\ell N,
 \end{aligned} \tag{1}$$

where

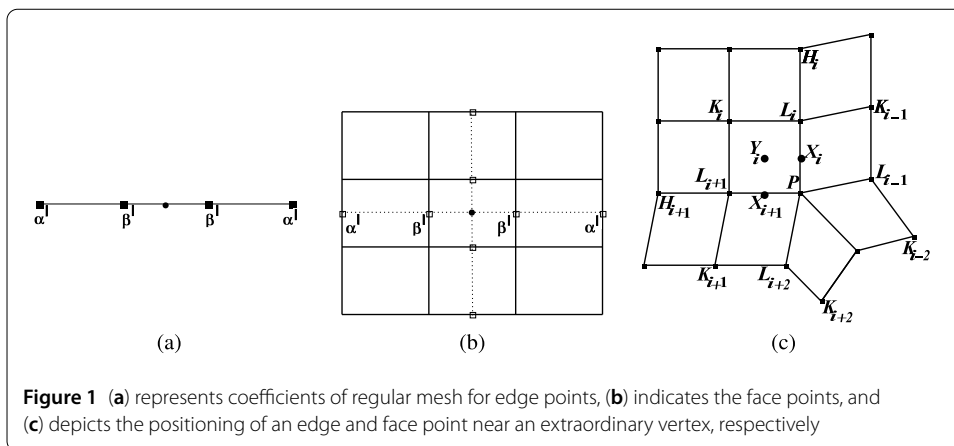
$$\alpha^\ell = -\frac{1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} \quad \text{and} \quad \beta^\ell = \frac{(2\mu_{\ell+1} + 1)^2}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)},$$

where ℓ shows the subdivision step or refinement level of INSS, and $\mu_{\ell+1} = \cos(\theta/2^{\ell+1})$, 1, and $\cosh(\theta/2^{\ell+1})$ for trigonometric, polynomial, and hyperbolic cases, respectively.

2.2 Regular surface case

Let $\{q_{m,n}^0; m = -2, \dots, N + 2, n = -2, \dots, N + 2\}$ be the sequence of control polygon at the initial subdivision step. For the ℓ th subdivision step, the newly generated control points are calculated by using tensor product of univariate INSS (1):

$$\begin{aligned}
 q_{2m,2n}^{\ell+1} &= q_{m,n}^\ell, \quad -1 \leq m \leq 2^\ell N + 1, -1 \leq n \leq 2^\ell N + 1, \\
 q_{2m+1,2n}^{\ell+1} &= \alpha^\ell (q_{m-1,n}^\ell + q_{m+2,n}^\ell) + \beta^\ell (q_{m,n}^\ell + q_{m+1,n}^\ell), \quad -1 \leq m \leq 2^\ell N, -1 \leq n \leq 2^\ell N + 1, \\
 q_{2m,2n+1}^{\ell+1} &= \alpha^\ell (q_{m,n-1}^\ell + q_{m,n+2}^\ell) + \beta^\ell (q_{m,n}^\ell + q_{m,n+1}^\ell), \quad -1 \leq m \leq 2^\ell N + 1, -1 \leq n \leq 2^\ell N, \\
 q_{2m+1,2n+1}^{\ell+1} &= \alpha^{2\ell} (q_{m-1,n-1}^\ell + q_{m+2,n-1}^\ell + q_{m-1,n+2}^\ell + q_{m+2,n+2}^\ell) + \alpha^\ell \beta^\ell (q_{m,n-1}^\ell + q_{m+1,n-1}^\ell \\
 &\quad + q_{m-1,n}^\ell + q_{m+2,n}^\ell + q_{m-1,n+1}^\ell + q_{m+2,n+1}^\ell + q_{m,n+2}^\ell + q_{m+1,n+2}^\ell) \\
 &\quad + \beta^{2\ell} (q_{m,n}^\ell + q_{m+1,n}^\ell + q_{m,n+1}^\ell + q_{m+1,n+1}^\ell), \quad -1 \leq m \leq 2^\ell N, -1 \leq n \leq 2^\ell N.
 \end{aligned}$$



This extended scheme is designed to calculate the algorithm for quadrilateral surfaces. In the considered regular case the valence of vertex is 4, that is, 4 rule for the insertion of points. For insertion of edge points, we use the rules $q_{2m+1,2n}^{\ell+1}$ and $q_{2m,2n+1}^{\ell+1}$ in horizontal and vertical directions, respectively; see Fig. 1(a) as $\alpha^\ell = \alpha^1$. The rules $q_{2m+1,2n+1}^{\ell+1}$ and $q_{2m,2n}^{\ell+1}$ are used to insert new face points as shown in Fig. 1(b) as $\beta^\ell = \beta^1$.

2.3 Irregular surface case

Irregular surfaces are those that cannot be unfolded or unrolled to lie in a flat plane. Solids that have irregular or warped surfaces cannot be created merely by extrusion or revolution. These irregular surfaces are created using surface modeling techniques. In irregular cases, that is, meshes include vertices where other than four faces meet except at boundary, we adopted the subdivision criteria followed by Kobbelt’s SS for arbitrary topology. So the unified INSS requires only one more rule to insert edge points on a nonregular vertex corresponding to the adjacent vertex. All remaining edge and face points are generated by using the unified four point INSS. By applying the proposed scheme it is possible that the points X_m and Y_m are undefined. If we need both possible ways to compute X_m and Y_m by using the proposed scheme to the succeeding edge points, which lead to the same result, then we find a dependence relation for X_{m+1} to X_m , with one edge to the next edge, and we have the notion $K_{m-2}, K_{m-1}, K_m, K_{m+1}, K_{m+2}$ with arbitrary point \mathbf{P} ; see Fig. 1(c). Now we have

$$X_{m+1} - X_m = \beta^\ell(L_{m+1} - L_m) + \alpha^\ell(H_{m+1} - H_m) + \alpha^\ell(L_{m-1} - L_{m+2}) + \frac{\alpha^{2\ell}}{\beta^\ell}(K_{m-2} - K_{m+2}),$$

which implies

$$X_{m+1} - X_m = \alpha^\ell(H_{m+1} - H_m) + \alpha^\ell(L_{m-1} - L_{m+2}) + \zeta^\ell(K_{m-2} - K_{m+2}) + \beta^\ell(L_{m+1} - L_m),$$

where $\zeta^\ell = 1/8\mu_{\ell+1}(1 + \mu_{\ell+1})(1 + 2\mu_{\ell+1})^2$.

The undefined point X_m will be computed by rotation of mask of the SS. Thus the neighborhood points of \mathbf{P} will become symmetric with the refinement process. By using a similar approach of [1] we can define the following equation:

$$\frac{1}{a} \sum_{m=0}^{a-1} X_m = \beta^\ell \mathbf{P} + \frac{1}{2a} \sum_{m=0}^{a-1} L_m + \frac{\alpha^\ell}{a} \sum_{m=0}^{a-1} H_m \quad \text{for } a = 4, \tag{2}$$

where for $n \geq 4$,

$$\frac{1}{a} \sum_{m=0}^{a-1} X_m = \frac{1}{a} \sum_{m=0}^{n-2} (a-1-m) \Delta X_{m+n} + X_n, \quad n = 0, 1, \dots, a-1, \tag{3}$$

with $\Delta X_{m+n} = X_{i+j+1} - X_{m+n}$ denoting the difference, and the face treated as vertex V_n of unified INSS is defined as

$$V_n = \frac{4}{a} \sum_{m=0}^{a-1} L_m - \frac{8\alpha^\ell}{(2\alpha^\ell - 1)n} \sum_{m=0}^{a-1} K_m + \frac{2\alpha^\ell}{(2\alpha^\ell - 1)} (K_{n-2} + K_{n-1} + K_n + K_{n+1}) - (L_{n-1} + L_n + L_{n+1}) \quad \text{for } a \neq 4$$

and

$$V_n = L_{n+2} \quad \text{for } n = 4.$$

Unifying of the common terms of the scheme and putting (2) into (3), we get

$$X_n = \alpha^\ell (H_n + V_n) + \beta^\ell (L_n + \mathbf{P}). \tag{4}$$

By taking $a = 3$ we get $L_{m-1} = L_{m+2} = K_{m-2} = K_{m+2}$. Thus by using our unified four point rule to the neighboring points \mathbf{P}, H_n, L_n , and $V_n, n = 0, \dots, a-1$, the edge points X_n can be computed easily. Similarly, we compute the face points Y_n , and it does not matter whether we compute X_m (horizontally) or X_{m+1} (vertically). In other words, we can compute all vertex points for the face containing an isolated extraordinary vertex from a regular mesh with virtual point V_n .

2.4 Open polygons and boundary curves

It is impossible to insert the first and last edge points of an open polygon by the unified scheme (1). It needs two neighborhood points to compute the edge point $q_1^{\ell+1}$, which refines the first edge point by $q_0^\ell q_1^\ell$. By describing the extrapolated edge point $q_{-1}^\ell = 2q_0^\ell - q_1^\ell$ the initial point $q_1^{\ell+1}$ is computed by using (1) on the subpolygon formed by $q_{-1}^\ell, q_0^\ell, q_1^\ell, q_2^\ell$. The additional rule can be denoted as a linear combination of nonextrapolated initial points $q_0^\ell, q_1^\ell, q_2^\ell$:

$$q_1^{\ell+1} = \frac{(2\mu_{\ell+1} + 1)^2 - 2}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_0^\ell + \frac{(2\mu_{\ell+1} + 1)^2 + 1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_1^\ell - \frac{1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_2^\ell.$$

The rule to insert the point $q_{2n-1}^{\ell+1}$ refining the last edge point $q_{n-1}^\ell q_n^\ell$ is defined as

$$q_{2n+1}^{\ell+1} = \frac{1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_{n-2}^\ell - \frac{(2\mu_{\ell+1} + 1)^2 + 1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_{n-1}^\ell + \frac{(2\mu_{\ell+1} + 1)^2 + 2(2\mu_{\ell+1} + 1)^2 - 2}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} q_n^\ell.$$

If the point on the boundary edge has a corner vertex (valence 2), then we use the boundary rule to insert the edge points on it. Applying the boundary rules $q_{-1}^\ell = 2q_0^\ell - q_1^\ell, q_{n+1}^\ell =$

$2q_n^\ell - q_{n-1}^\ell$, and $q_{n+2}^\ell = 2q_n^\ell - q_{n-2}^\ell$, we can produce limit curves/surfaces of open polygon at the end vertices or boundary vertices (valence > 2).

Lemma 2.1 *The unified IRSS satisfies the affine invariance property.*

Proof 2.1 Since $\alpha^\ell + \beta^\ell + \beta^\ell + \alpha^\ell = 1$, the unified IRSS satisfies the affine invariance property.

Remark 2.1 The unified IRSS is primal because of odd symmetry.

Remark 2.2 The unified scheme is exactly the well-known four-point scheme of [4] and [18] for $\mu_{\ell+1} = 1$. The polynomial case of unified scheme (1) comes from cubic interpolatory polynomial, so the polynomial reproduction will be cubic.

2.5 Analysis of unified scheme

For $\mu_{\ell+1} = \cos(\frac{\theta}{2^{\ell+1}})$, 1, and $\cosh(\frac{\theta}{2^{\ell+1}})$, the masks of the unified schemes coincides with the mask of Kobbelt’s scheme [1] as $\ell \rightarrow \infty$. For $\mu_{\ell+1} = \cos(\frac{\theta}{2^{\ell+1}})$, we have

$$\begin{aligned}
 \text{(i)} \quad & \lim_{\ell \rightarrow \infty} \alpha^\ell = - \lim_{\ell \rightarrow \infty} \frac{1}{8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1)} = -\frac{1}{16}, \\
 \text{(ii)} \quad & \lim_{\ell \rightarrow \infty} \beta^\ell = \lim_{\ell \rightarrow \infty} \frac{(2 \cos(\frac{\theta}{2^{\ell+1}}) + 1)^2}{8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1)} = \frac{9}{16}, \\
 \text{(iii)} \quad & \lim_{\ell \rightarrow \infty} \alpha^{2\ell} = \lim_{\ell \rightarrow \infty} \frac{1}{(8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1))^2} = \frac{1}{256}, \\
 \text{(iv)} \quad & \lim_{\ell \rightarrow \infty} \alpha^\ell \beta^\ell = - \lim_{\ell \rightarrow \infty} \frac{(2 \cos(\frac{\theta}{2^{\ell+1}}) + 1)^2}{(8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1))^2} = -\frac{9}{256}, \\
 \text{(v)} \quad & \lim_{\ell \rightarrow \infty} \beta^{2\ell} = \lim_{\ell \rightarrow \infty} \frac{(2 \cos(\frac{\theta}{2^{\ell+1}}) + 1)^4}{(8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1))^2} = \frac{81}{256}.
 \end{aligned}$$

Similarly, for other values of $\mu_{\ell+1}$, we get the same mask. In [11], it is proved that the scheme of Kobbelt [1] at all control points of closed meshes is C^1 -continuous. Since in the limiting case the Kobbelt scheme is a particular case of the unified scheme, the latter is also C^1 -continuous in this case.

Tables 1–5 indicate the eigenvalues of the proposed SS at $\mu_{\ell+1} = \cos(\frac{\theta}{2^{\ell+1}})$, 1, and $\cosh(\frac{\theta}{2^{\ell+1}})$ for $\theta = \pi/3$ and $\theta = 2\pi/5$. From these tables we observe that the largest eigenvalue is one, and the other eigenvalues are less than one, and thus the unified scheme is convergent. Since the second and third eigenvalues are same, the unified scheme is C^1 -continuous by [19].

Table 1 Eigenvalues of trigonometric (i.e., $\mu_{\ell+1} = \cos(\pi/3 \times 2^{\ell+1})$) subdivision matrix with valence n

n	3	4	5	6	7	8	9	10
e_0	1	1	1	1	1	1	1	1
e_1	0.40973	0.49482	0.56859	0.60675	0.62912	0.64342	0.65314	0.66005
e_2	0.40973	0.49482	0.56859	0.60675	0.62912	0.64342	0.65314	0.66005
e_3	0.22122	0.20877	0.20688	0.19447	0.18626	0.18062	0.17660	0.17364

Table 2 Eigenvalues of trigonometric (i.e., $\mu_{\ell+1} = \cos(2\pi/5 \times 2^{\ell+1})$) subdivision matrix with valence n

n	3	4	5	6	7	8	9	10
e_0	1	1	1	1	1	1	1	1
e_1	0.41486	0.49482	0.50986	0.60675	0.61337	0.65171	0.66085	0.67042
e_2	0.41486	0.49482	0.50986	0.60675	0.61337	0.65171	0.66085	0.67042
e_3	0.22561	0.22280	0.21640	0.20820	0.19910	0.18955	0.17977	0.16988

Table 3 Eigenvalues of polynomial (i.e., $\mu_{\ell+1} = 1$) subdivision matrix with valence n

n	3	4	5	6	7	8	9	10
e_0	1	1	1	1	1	1	1	1
e_1	0.42633	0.50000	0.53794	0.55968	0.57322	0.58213	0.58834	0.59123
e_2	0.42633	0.50000	0.53794	0.55968	0.57322	0.58213	0.58834	0.59123
e_3	0.25000	0.25000	0.42633	0.42633	0.46972	0.50000	0.52180	0.53894

Table 4 Eigen-values of of hyperbolic (i.e., $\mu_{\ell+1} = \cosh(\pi/3 \times 2^{\ell+1})$) subdivision matrix with valence n

n	3	4	5	6	7	8	9	10
e_0	1	1	1	1	1	1	1	1
e_1	0.44057	0.49863	0.53192	0.55174	0.56423	0.57254	0.57833	0.58185
e_2	0.44057	0.49863	0.53192	0.55174	0.56423	0.57254	0.57833	0.58185
e_3	0.18659	0.17300	0.16644	0.16125	0.15776	0.15533	0.15358	0.15209

Table 5 Eigenvalues of of hyperbolic (i.e., $\mu_{\ell+1} = \cosh(2\pi/5 \times 2^{\ell+1})$) subdivision matrix with valence n

n	3	4	5	6	7	8	9	10
e_0	1	1	1	1	1	1	1	1
e_1	0.43198	0.46520	0.49856	0.52515	0.54257	0.55628	0.56815	0.57907
e_2	0.44057	0.49863	0.53192	0.55174	0.56423	0.57254	0.57833	0.58185
e_3	0.21359	0.208900	0.19344	0.18342	0.17776	0.16533	0.15328	0.15009

3 Reproduction of functions

Consider the functions $\sin(\cdot)$, $\cos(\cdot)$, $\sin(\cdot)\sin(\cdot)$, $\cos(\cdot)\cos(\cdot)$, $\cos(\cdot)\sin(\cdot)$, $\cosh(\cdot)$, $\cosh(\cdot)\cosh(\cdot)$, $\sinh(\cdot)$, $\sinh(\cdot)\sinh(\cdot)$, and $\cosh(\cdot)\sinh(\cdot)$ can be reproduced by unified SSs.

Lemma 3.1 Let $q_m^\ell = \cos(\frac{m\theta}{2^\ell})$ (or $q_m^\ell = \sin(\frac{m\theta}{2^\ell})$) for $-1 \leq m \leq 2^\ell N + 1$, $l \geq 0$. Then $q_{2m}^{\ell+1} = \cos(\frac{2m\theta}{2^{\ell+1}})$ (or $q_{2m}^{\ell+1} = \sin(\frac{2m\theta}{2^{\ell+1}})$).

Proof 3.1 Since $q_m^\ell = \cos(\frac{m\theta}{2^\ell})$, it is obvious that $q_{2m}^{\ell+1} = \cos(\frac{2m\theta}{2^{\ell+1}})$. Similarly, for $q_m^\ell = \sin(\frac{m\theta}{2^\ell})$, we have $q_{2m}^{\ell+1} = \sin(\frac{2m\theta}{2^{\ell+1}})$.

Lemma 3.2 Let $q_n^\ell = \cos(\frac{n\theta}{2^\ell})$ for $-1 \leq n \leq 2^\ell N + 1$, $l \geq 0$. Then $q_{2n+1}^{\ell+1} = \cos(\frac{(2n+1)\theta}{2^{\ell+1}})$.

Proof 3.2 From (1) with $m = n$ we have

$$q_{2n+1}^{\ell+1} = \frac{-1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)}(q_{n-1}^\ell + q_{n+2}^\ell) + \frac{(2\mu_{\ell+1} + 1)^2}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)}(q_n^\ell + q_{n+1}^\ell).$$

Putting $\mu_{\ell+1} = \cos(\frac{\theta}{2^{\ell+1}})$, $q_{n-1}^{\ell} = \cos(\frac{(n-1)\theta}{2^{\ell}})$, $q_n^{\ell} = \cos(\frac{n\theta}{2^{\ell}})$, $q_{n+1}^{\ell} = \cos(\frac{(n+1)\theta}{2^{\ell}})$, and $q_{n+2}^{\ell} = \cos(\frac{(n+2)\theta}{2^{\ell}})$, we have

$$q_{2n+1}^{\ell+1} = \frac{-1}{8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1)} \left(\cos\left(\frac{(n-1)\theta}{2^{\ell}}\right) + \cos\left(\frac{(n+2)\theta}{2^{\ell}}\right) \right) + \frac{(2 \cos(\frac{3\theta}{2^{\ell+1}}) + 1)^2}{8 \cos(\frac{\theta}{2^{\ell+1}})(\cos(\frac{\theta}{2^{\ell+1}}) + 1)} \left(\cos\left(\frac{n\theta}{2^{\ell}}\right) + \cos\left(\frac{(n+1)\theta}{2^{\ell}}\right) \right).$$

Applying the identities $1 + \cos(\alpha) = 2 \cos^2(\frac{\alpha}{2})$ and $\cos(\alpha) + \cos(\beta) = 2 \cos(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$, we get

$$q_{2n+1}^{\ell+1} = \frac{-1}{16 \cos(\frac{\theta}{2^{\ell+1}}) \cos^2(\frac{\theta}{2^{\ell+2}})} \left(2 \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cos\left(\frac{3\theta}{2^{\ell+1}}\right) \right) + \frac{(2 \cos(\frac{3\theta}{2^{\ell+1}}) + 1)^2}{16 \cos(\frac{\theta}{2^{\ell+1}}) \cos^2(\frac{\theta}{2^{\ell+2}})} \left(2 \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cos\left(\frac{\theta}{2^{\ell+1}}\right) \right) \Rightarrow \frac{1}{16 \cos(\frac{\theta}{2^{\ell+1}}) \cos^2(\frac{\theta}{2^{\ell+2}})} = \frac{\sin^2(\frac{3\theta}{2^{\ell+2}})}{2 \sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})}$$

and

$$\left(2 \cos\left(\frac{\theta}{2^{\ell+1}}\right) + 1 \right)^2 = \left(2 \cos\left(\frac{2\theta}{2^{\ell+2}}\right) + 1 \right)^2 = \left(3 - 4 \sin^2\left(\frac{\theta}{2^{\ell+2}}\right) \right)^2.$$

Then

$$q_{2n+1}^{\ell+1} = \frac{-\sin^2(\frac{\theta}{2^{\ell+2}})}{2 \sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} \left(2 \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cos\left(\frac{3\theta}{2^{\ell+1}}\right) \right) + \frac{\sin^2(\frac{\theta}{2^{\ell+2}})(3 - 4 \sin^2(\frac{\theta}{2^{\ell+2}}))^2}{2 \sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} \left(2 \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cos\left(\frac{\theta}{2^{\ell+1}}\right) \right) \Rightarrow q_{2n+1}^{\ell+1} = \left(\frac{(3 \sin(\frac{\theta}{2^{\ell+2}}) - 4 \sin^3(\frac{\theta}{2^{\ell+2}}))^2}{\sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} \cos\left(\frac{\theta}{2^{\ell+1}}\right) - \frac{\sin^2(\frac{\theta}{2^{\ell+2}})}{\sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} \cos\left(\frac{3\theta}{2^{\ell+1}}\right) \right) \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Since

$$q_{2n+1}^{\ell+1} = \left(\frac{\sin^2(\frac{3\theta}{2^{\ell+2}}) \cos(\frac{\theta}{2^{\ell+1}})}{\sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} - \frac{\sin^2(\frac{\theta}{2^{\ell+2}}) \cos(\frac{3\theta}{2^{\ell+1}})}{\sin(\frac{\theta}{2^{\ell}}) \sin(\frac{\theta}{2^{\ell+1}})} \right) \times \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right),$$

applying the identity $\sin^2(\frac{\alpha}{2}) = \frac{1-\cos\alpha}{2}$, we have

$$q_{2n+1}^{\ell+1} = \left(\frac{(1 - \cos(\frac{3\theta}{2^{\ell+1}})) \cos(\frac{\theta}{2^{\ell+1}}) - (1 - \cos(\frac{3\theta}{2^{\ell+1}})) \cos(\frac{3\theta}{2^{\ell+1}})}{2 \sin(\frac{\theta}{2^\ell}) \sin(\frac{t}{2^{\ell+1}})} \right) \times \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

By the identity $\cos a - \cos b = -2 \sin(\frac{a+b}{2}) \sin(\frac{a-b}{2})$ we have

$$q_{2n+1}^{\ell+1} = \cos\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Similarly, putting $\mu_{\ell+1} = \cos(\frac{\theta}{2^{\ell+1}})$, and applying the identities $1 + \cos(\alpha) = 2 \cos^2(\frac{\alpha}{2})$, $\sin(\alpha) + \sin(\beta) = 2 \sin(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$, $(3 \sin(\frac{\theta}{2^{\ell+2}}) - 4 \sin^3(\frac{\theta}{2^{\ell+2}}))^2 = \sin^2(\frac{3\theta}{2^{\ell+2}})$, $\sin^2(\frac{3\alpha}{2}) = \frac{1-\cos 3\alpha}{2}$, $\sin^2(\frac{\alpha}{2}) = \frac{1-\cos\alpha}{2}$, and $\cos \alpha - \cos \beta = -2 \sin(\frac{\alpha+\beta}{2}) \sin(\frac{\alpha-\beta}{2})$, we have the following lemma.

Lemma 3.3 *Let $q_n^\ell = \sin(\frac{n\theta}{2^\ell})$ for $-1 \leq n \leq 2^\ell N + 1, l \geq 0$. Then $q_{2n+1}^{\ell+1} = \sin(\frac{(2n+1)\theta}{2^{\ell+1}})$.*

By Lemmas 3.1–3.3, applying $q_{m,n}^\ell = q_m^\ell q_n^\ell$, we have the following theorems.

Theorem 3.3 *Let $q_m^\ell = \cos(\frac{m\theta}{2^\ell})$ and $q_n^\ell = \cos(\frac{n\theta}{2^\ell})$ for $-1 \leq m, n \leq 2^\ell N + 1, l \geq 0$. Then*

$$q_{2m,2n}^{\ell+1} = \cos\left(\frac{2m\theta}{2^{\ell+1}}\right) \cos\left(\frac{2n\theta}{2^{\ell+1}}\right).$$

Theorem 3.4 *Let $q_m^\ell = \cos(\frac{m\theta}{2^\ell})$ and $q_n^\ell = \sin(\frac{n\theta}{2^\ell})$ for $-1 \leq m, n \leq 2^\ell N + 1, l \geq 0$. Then*

$$q_{2m+1,2n+1}^{\ell+1} = \cos\left(\frac{(2m+1)\theta}{2^{\ell+1}}\right) \sin\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Theorem 3.5 *Let $q_m^\ell = \sin(\frac{m\theta}{2^\ell})$ and $q_n^\ell = \sin(\frac{n\theta}{2^\ell})$ for $-1 \leq m, n \leq 2^\ell N + 1, l \geq 0$. Then*

$$q_{2m+1,2n+1}^{\ell+1} = \sin\left(\frac{(2m+1)\theta}{2^{\ell+1}}\right) \sin\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Lemma 3.4 *Let $q_m^\ell = \cosh(\frac{m\theta}{2^\ell})$ (or $q_m^\ell = \sinh(\frac{m\theta}{2^\ell})$) for $-1 \leq m \leq 2^\ell N + 1, l \geq 0$. Then $q_{2m}^{\ell+1} = \cosh(\frac{2m\theta}{2^{\ell+1}})$ (or $q_{2m}^{\ell+1} = \sinh(\frac{2m\theta}{2^{\ell+1}})$).*

Proof 3.6 Since $q_m^\ell = \cosh(\frac{m\theta}{2^\ell})$, it is obvious that $q_{2m}^{\ell+1} = \cosh(\frac{2m\theta}{2^{\ell+1}})$. Similarly, we get $q_{2m}^{\ell+1} = \sinh(\frac{2m\theta}{2^{\ell+1}})$.

Lemma 3.5 *Let $q_n^\ell = \cosh(\frac{n\theta}{2^\ell})$ for $-1 \leq n \leq 2^\ell N + 1, l \geq 0$. Then $q_{2n+1}^{\ell+1} = \cosh(\frac{(2n+1)\theta}{2^{\ell+1}})$.*

Proof 3.7 From (1) with $i = j$ we have

$$q_{2n+1}^{\ell+1} = \frac{-1}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} (q_{n-1}^\ell + q_{n+2}^\ell) + \frac{(2\mu_{\ell+1} + 1)^2}{8\mu_{\ell+1}(\mu_{\ell+1} + 1)} (q_n^\ell + q_{n+1}^\ell).$$

Substituting $\mu_{\ell+1} = \cosh(\frac{\theta}{2^{\ell+1}})$, $q_{n-1}^{\ell} = \cosh(\frac{(n-1)\theta}{2^{\ell}})$, $q_n^{\ell} = \cosh(\frac{n\theta}{2^{\ell}})$, $q_{n+1}^{\ell} = \cosh(\frac{(n+1)\theta}{2^{\ell}})$, and $q_{n+2}^{\ell} = \cosh(\frac{(n+2)\theta}{2^{\ell}})$, we get

$$q_{2n+1}^{\ell+1} = \frac{-1}{8 \cosh(\frac{\theta}{2^{\ell+1}})(\cosh(\frac{\theta}{2^{\ell+1}}) + 1)} \left(\cosh\left(\frac{(n-1)\theta}{2^{\ell}}\right) + \cosh\left(\frac{(n+2)\theta}{2^{\ell}}\right) \right) + \frac{(2 \cosh(\frac{3\theta}{2^{\ell+1}}) + 1)^2}{8 \cosh(\frac{\theta}{2^{\ell+1}})(\cosh(\frac{\theta}{2^{\ell+1}}) + 1)} \left(\cosh\left(\frac{n\theta}{2^{\ell}}\right) + \cosh\left(\frac{(n+1)\theta}{2^{\ell}}\right) \right).$$

Let $\iota = \text{iota}$. Then applying $\sinh(\alpha) = -\iota \sin(\iota\alpha)$, $\cosh(\alpha) = \cos(\iota\alpha)$, $1 + \cos(\alpha) = 2 \cos^2(\frac{\alpha}{2})$, and $\cos(\alpha) + \cos(\beta) = 2 \cos(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$, we get

$$q_{2n+1}^{\ell+1} = \frac{-1}{16 \cos(\frac{\iota\theta}{2^{\ell+1}}) \cos^2(\frac{\iota\theta}{2^{\ell+1}})} \left\{ 2 \cos\left(\frac{(2n+1)\iota t}{2^{\ell+2}}\right) \cos\left(\frac{3\iota t}{2^{\ell+1}}\right) \right\} + \frac{(2 \cos(\frac{\iota\theta}{2^{\ell+1}}) + 1)^2}{16 \cos(\frac{\iota\theta}{2^{\ell+1}}) \cos^2(\frac{\iota\theta}{2^{\ell+2}})} \left\{ 2 \cos\left(\frac{(2n+1)\iota t}{2^{\ell+1}}\right) \cos\left(\frac{\iota t}{2^{\ell+1}}\right) \right\}.$$

Since

$$\frac{1}{16 \cos(\frac{\iota\theta}{2^{\ell+1}}) \cos^2(\frac{\iota\theta}{2^{\ell+2}})} = \frac{\sinh^2(\frac{\theta}{2^{\ell+2}})}{2 \sinh(\frac{\theta}{2^{\ell}}) \sinh(\frac{\theta}{2^{\ell+1}})}$$

and

$$\left(2 \cosh\left(\frac{\theta}{2^{\ell+1}}\right) + 1 \right)^2 = \left(2 \cos\left(\frac{2\iota t}{2^{\ell+2}}\right) + 1 \right)^2 = \left(3 + 4 \sinh^2\left(\frac{\theta}{2^{\ell+2}}\right) \right)^2,$$

we get

$$q_{2n+1}^{\ell+1} = \frac{-\sinh^2(\frac{\theta}{2^{\ell+2}})}{2 \sinh(\frac{\theta}{2^{\ell}}) \sinh(\frac{\theta}{2^{\ell+1}})} \left(2 \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cosh\left(\frac{3\theta}{2^{\ell+1}}\right) \right) + \frac{\sinh^2(\frac{\theta}{2^{\ell+2}})(3 + 4 \sinh^2(\frac{\theta}{2^{\ell+2}}))^2}{2 \sinh(\frac{\theta}{2^{\ell}}) \sinh(\frac{\theta}{2^{\ell+1}})} \left(2 \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right) \cosh\left(\frac{\theta}{2^{\ell+1}}\right) \right).$$

This implies

$$q_{2n+1}^{\ell+1} = \left(\frac{(3 \sinh(\frac{\theta}{2^{\ell+2}}) + 4 \sinh^3(\frac{\theta}{2^{\ell+2}}))^2}{\sinh(\frac{\theta}{2^{\ell}}) \sinh(\frac{\theta}{2^{\ell+1}})} \cosh\left(\frac{\theta}{2^{\ell+1}}\right) - \frac{\sinh^2(\frac{\theta}{2^{\ell+2}})}{\sinh(\frac{\theta}{2^{\ell}}) \sinh(\frac{\theta}{2^{\ell+1}})} \cosh\left(\frac{3\theta}{2^{\ell+1}}\right) \right) \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right). \tag{5}$$

Consider

$$3 \sinh\left(\frac{\theta}{2^{\ell+2}}\right) + 4 \sinh^3\left(\frac{\theta}{2^{\ell+2}}\right) = -3\iota \sin\left(\frac{\iota\theta}{2^{\ell+2}}\right) + 4\iota \sin^3\left(\frac{\iota\theta}{2^{\ell+2}}\right).$$

This implies

$$3 \sinh\left(\frac{\theta}{2^{\ell+2}}\right) + 4 \sinh^3\left(\frac{\theta}{2^{\ell+2}}\right) = -\iota \sin\left(\frac{3\iota t}{2^{\ell+2}}\right) = \sinh\left(\frac{3\theta}{2^{\ell+2}}\right). \tag{6}$$

By substituting (6) into (5) we get

$$q_{2n+1}^{\ell+1} = \left(\frac{\sinh^2(\frac{3\theta}{2^{\ell+2}}) \cosh(\frac{\theta}{2^{\ell+1}})}{\sinh(\frac{\theta}{2^\ell}) \sinh(\frac{\theta}{2^{\ell+1}})} - \frac{\sinh^2(\frac{\theta}{2^{\ell+2}}) \cosh(\frac{3\theta}{2^{\ell+1}})}{\sinh(\frac{\theta}{2^\ell}) \sinh(\frac{\theta}{2^{\ell+1}})} \right) \times \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

This implies

$$q_{2n+1}^{\ell+1} = \left(\frac{\sin^2(\frac{3it}{2^{\ell+1}}) \cos(\frac{i\theta}{2^{\ell+1}})}{\sin(\frac{i\theta}{2^\ell}) \sin(\frac{i\theta}{2^{\ell+1}})} - \frac{\sin^2(\frac{i\theta}{2^{\ell+2}}) \cos(\frac{3it}{2^{\ell+1}})}{\sin(\frac{i\theta}{2^\ell}) \sin(\frac{i\theta}{2^{\ell+1}})} \right) \times \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Using $\sin^2(\frac{a}{2}) = \frac{1-\cos a}{2}$ and $\sin^2(\frac{3a}{2}) = \frac{1-\cos 3a}{2}$, we get

$$q_{2n+1}^{\ell+1} = \left(\frac{(1 - \cos(\frac{3it}{2^{\ell+1}})) \cos(\frac{i\theta}{2^{\ell+1}}) - (1 - \cos(\frac{i\theta}{2^{\ell+1}})) \cos(\frac{3it}{2^{\ell+1}})}{2 \sin(\frac{i\theta}{2^\ell}) \sin(\frac{i\theta}{2^{\ell+1}})} \right) \times \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Simplifying and using $\cos a - \cos b = -2 \sin(\frac{a+b}{2}) \sin(\frac{a-b}{2})$, we get

$$q_{2n+1}^{\ell+1} = \left(\frac{-2 \sin(\frac{t+3it}{2^{\ell+2}}) \sin(\frac{t-3it}{2^{\ell+2}})}{2 \sin(\frac{i\theta}{2^\ell}) \sin(\frac{i\theta}{2^{\ell+1}})} \right) \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

This implies

$$q_{2n+1}^{\ell+1} = \cosh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

In a similar way, putting $\mu_{\ell+1} = \cosh(\frac{\theta}{2^{\ell+1}})$ and applying $\sinh(\alpha) = -i \sin(i\alpha)$, $\cosh(\alpha) = \cos(i\alpha)$, $1 + \cos(\alpha) = 2 \cos^2(\frac{\alpha}{2})$, $\sin(\alpha) + \sin(\alpha) = 2 \sin(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$, $\sin^2(\frac{\alpha}{2}) = \frac{1-\cos \alpha}{2}$, $\sin^2(\frac{3\alpha}{2}) = \frac{1-\cos 3\alpha}{2}$, and $\cos \alpha - \cos \beta = -2 \sin(\frac{\alpha+\beta}{2}) \sin(\frac{\alpha-\beta}{2})$, we have following lemma.

Lemma 3.6 *Let $q_n^\ell = \sinh(\frac{n\theta}{2^\ell})$ for $-1 \leq n \leq 2^\ell N + 1$, $l \geq 0$. Then $q_{2n+1}^{\ell+1} = \sinh(\frac{(2n+1)\theta}{2^{\ell+1}})$.*

Using Lemmas 3.4–3.6, we get following theorems.

Theorem 3.8 *Let $q_m^\ell = \cosh(\frac{m\theta}{2^\ell})$ and $q_n^\ell = \cosh(\frac{n\theta}{2^\ell})$ for $-1 \leq m, n \leq 2^\ell N + 1$, $l \geq 0$. Then*

$$q_{2m,2n}^{\ell+1} = \cosh\left(\frac{2m\theta}{2^{\ell+1}}\right) \cosh\left(\frac{2n\theta}{2^{\ell+1}}\right).$$

Theorem 3.9 *Let $q_m^\ell = \cosh(\frac{m\theta}{2^\ell})$ and $q_n^\ell = \sinh(\frac{n\theta}{2^\ell})$ for $-1 \leq m, n \leq 2^\ell N + 1$, $l \geq 0$. Then*

$$q_{2m+1,2n+1}^{\ell+1} = \cosh\left(\frac{(2m+1)\theta}{2^{\ell+1}}\right) \sinh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

Theorem 3.10 Let $q_m^\ell = \sinh\left(\frac{m\theta}{2^\ell}\right)$ and $q_n^\ell = \sinh\left(\frac{n\theta}{2^\ell}\right)$ for $-1 \leq m, n \leq 2^\ell N + 1, l \geq 0$. Then

$$q_{2m+1,2n+1}^{\ell+1} = \sinh\left(\frac{(2m+1)\theta}{2^{\ell+1}}\right) \sinh\left(\frac{(2n+1)\theta}{2^{\ell+1}}\right).$$

4 Numerical results

In this section, we present several numerical examples to check the performance and show that the proposed SS reconstruct conics.

4.1 Reproduction of fundamental images

In this section, we have implemented our proposed SS on 10 different examples, which support our theoretical analysis. We observe that the fundamental images can be reconstructed by using our method if the control points are taken from these images.

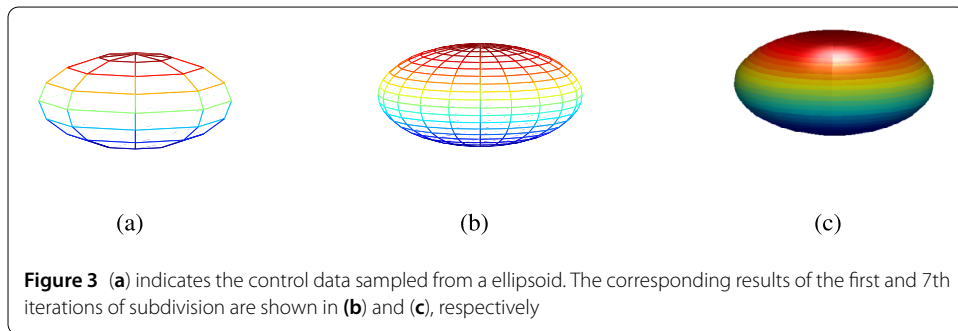
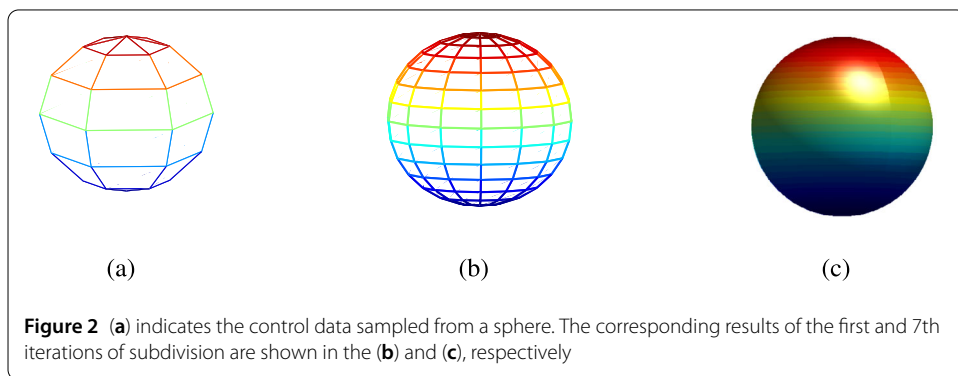
Example 1 By using the control points presented in Fig. 2(a),

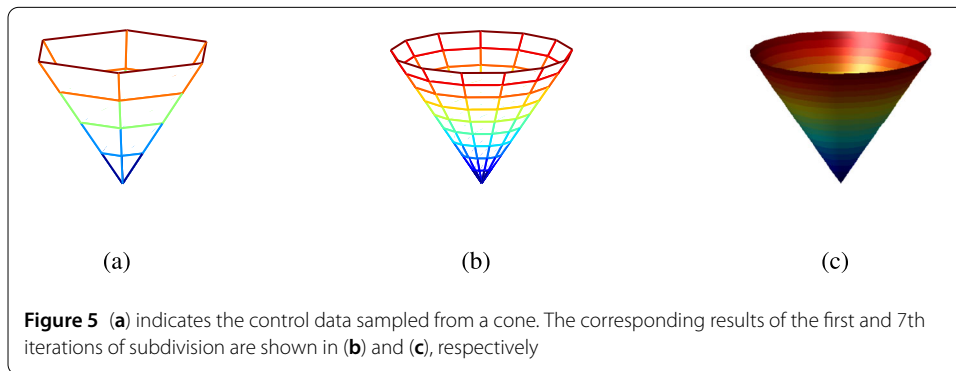
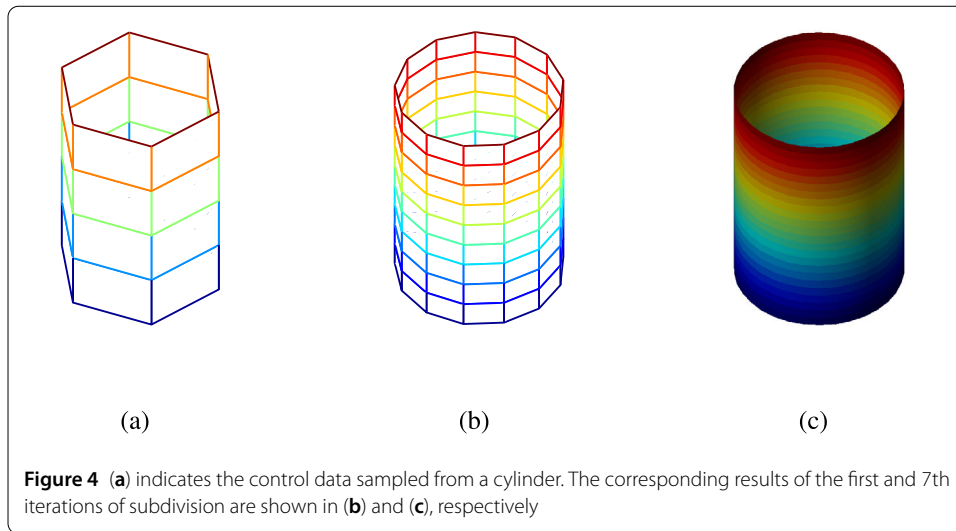
$$q_{m,n}^0 = \left(\sin\left(\frac{2\pi m}{\eta}\right) \cos\left(\frac{2\pi n}{\eta}\right), \sin\left(\frac{2\pi m}{\eta}\right) \sin\left(\frac{2\pi n}{\eta}\right), \cos\left(\frac{2\pi m}{\eta}\right) \right),$$

for $m = 0, 1, 2, \dots, \eta, n = 0, 1, 2, \dots, 2\eta - 1, \eta = 7$, the limiting smooth sphere obtained by the proposed SS is shown in Fig. 2(c), Similarly, the limiting ellipsoid generated in Fig. 3(c), whose control mesh points are

$$q_{m,n}^0 = \left(\sin\left(\frac{2\pi m}{\eta}\right) \cos\left(\frac{2\pi n}{\eta}\right), 2 \sin\left(\frac{2\pi m}{\eta}\right) \sin\left(\frac{2\pi n}{\eta}\right), \cos\left(\frac{2\pi m}{\eta}\right) \right),$$

$m = 0, 1, 2, \dots, \eta, n = 0, 1, 2, \dots, 2\eta - 1, \eta = 7$, is shown in Fig. 3(a).





Example 2 By using the mesh with initial points

$$q_{m,n}^0 = \left(\cos\left(\frac{2\pi m}{\eta}\right), \sin\left(\frac{2\pi m}{\eta}\right), \frac{n}{\eta} \right), \quad m = 0, 1, \dots, \eta, n = 0, 1, \dots, \eta - 2, \eta = 6,$$

and

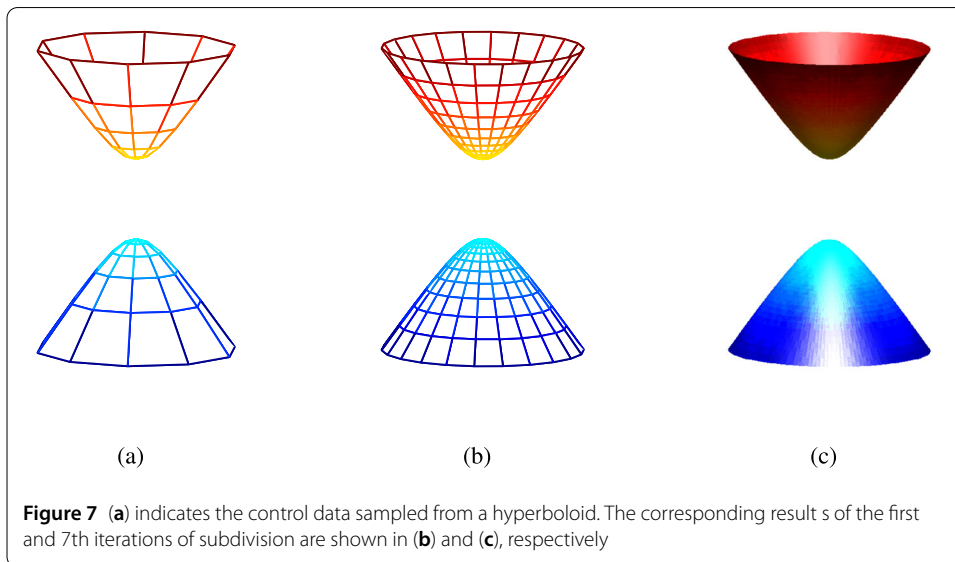
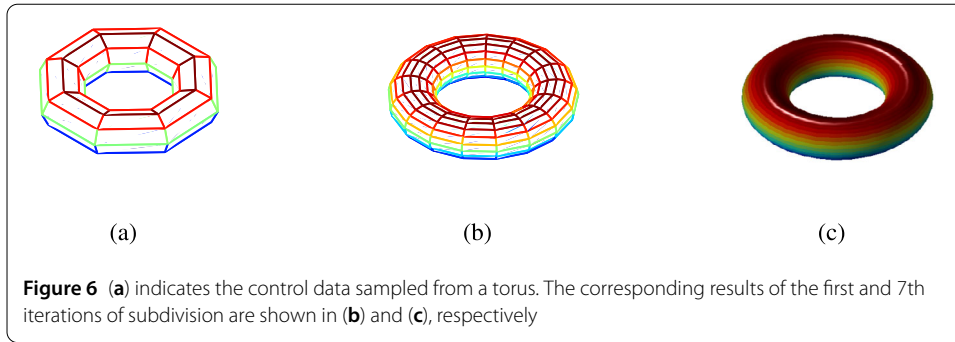
$$q_{m,n}^0 = \left(\frac{n}{\eta} \cos\left(\frac{2\pi m}{\eta}\right), \frac{n}{\eta} \sin\left(\frac{2\pi m}{\eta}\right), \frac{n}{\eta} \right), \quad m, n \in 0, 1, \dots, \eta, \eta = 6,$$

we get the limiting cylinder and cone shown in Figs. 4(c) and 5(c), respectively.

Example 3 The control mesh of torus is presented in Fig. 6(a) with initial points

$$q_{m,n}^0 = \left(3 \cos\left(\frac{2\pi m}{\eta}\right) + \cos\left(\frac{2\pi m}{\eta}\right) \cos\left(\frac{2\pi n}{\eta}\right), \right. \\ \left. 3 \sin\left(\frac{2\pi m}{\eta}\right) + \sin\left(\frac{2\pi m}{\eta}\right) \cos\left(\frac{2\pi n}{\eta}\right), \sin\left(\frac{2\pi n}{\eta}\right) \right),$$

$m, n = 0, 1, 2, \dots, \eta, \eta = 8$, whereas the smooth surface after first subdivision step and the limiting surface of torus are given in Figs. 6(b) and 6(c), respectively.



Example 4 The control meshes of upper and lower hyperboloid surfaces are given in Fig. 7(a) with initial points

$$q_{m,n}^0 = \left(a \sinh\left(\frac{m}{2\eta}\right) \cos\left(\frac{2\pi n}{2\eta}\right), b \sinh\left(\frac{m}{2\eta}\right) \sin\left(\frac{2\pi n}{2\eta}\right), c \cosh\left(\frac{m}{2\eta}\right) \right),$$

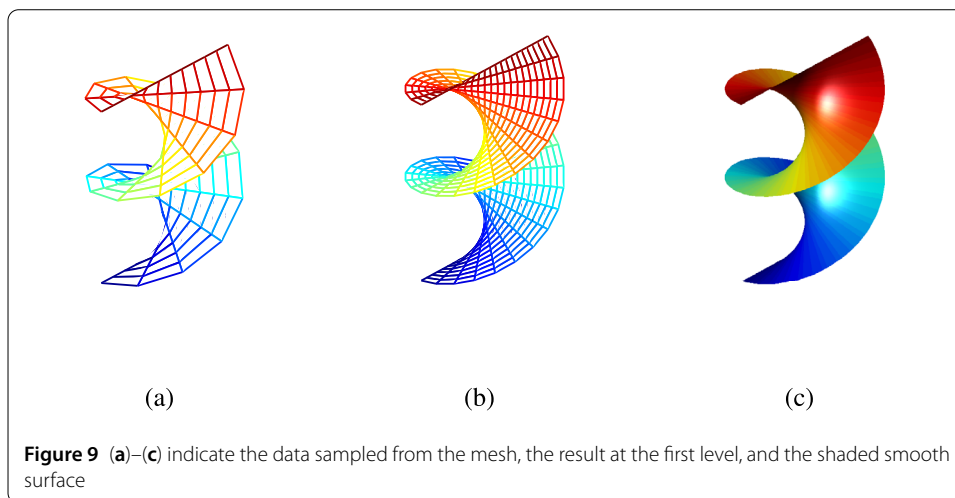
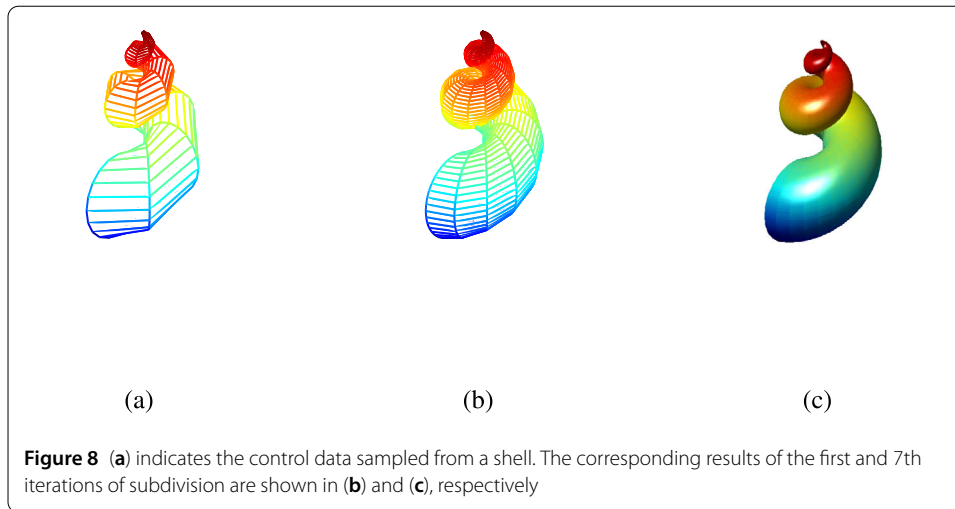
where $m = -\eta, \dots, 0, \dots, \eta$ and $n = 0, 1, 2, \dots, 2\eta$, $\eta = 4$ with $a = b = 1$, $c = 1$ (i.e., for upper hyperbola) and $c = -1$ (i.e., for lower hyperbola). The limiting hyperboloids are shown in Fig. 7(c).

Example 5 The limiting surface of shell and strip constructed by our SS are presented in Figs. 8(c) and 9(c), whereas their control points

$$q_{m,n}^0 = \left(\left(1 - e^{\frac{m}{\eta}}\right) \cos(m\pi) \cos\left(\frac{n\pi}{\eta - 5}\right)^2, -\left(1 - e^{\frac{m}{\eta}}\right) \sin(m\pi) \cos\left(\frac{n\pi}{\eta - 5}\right)^2, \left(1 - e^{\frac{m}{\eta} / (\eta - 4)}\right) - \sin(n\pi) + e^{i\eta} \sin(n\pi) \right),$$

for $m = 0, 1, 2, \dots, \eta$, $n = 0, 1, 2, \dots, \eta - 4$, $\eta = 6$, and

$$q_{m,n}^0 = (n \cos(\pi m), n \sin(\pi m), \pi m), \quad m = 0, 1, 2, \dots, \eta, n = -\eta + 5, \dots, \eta - 5, \eta = 2,$$



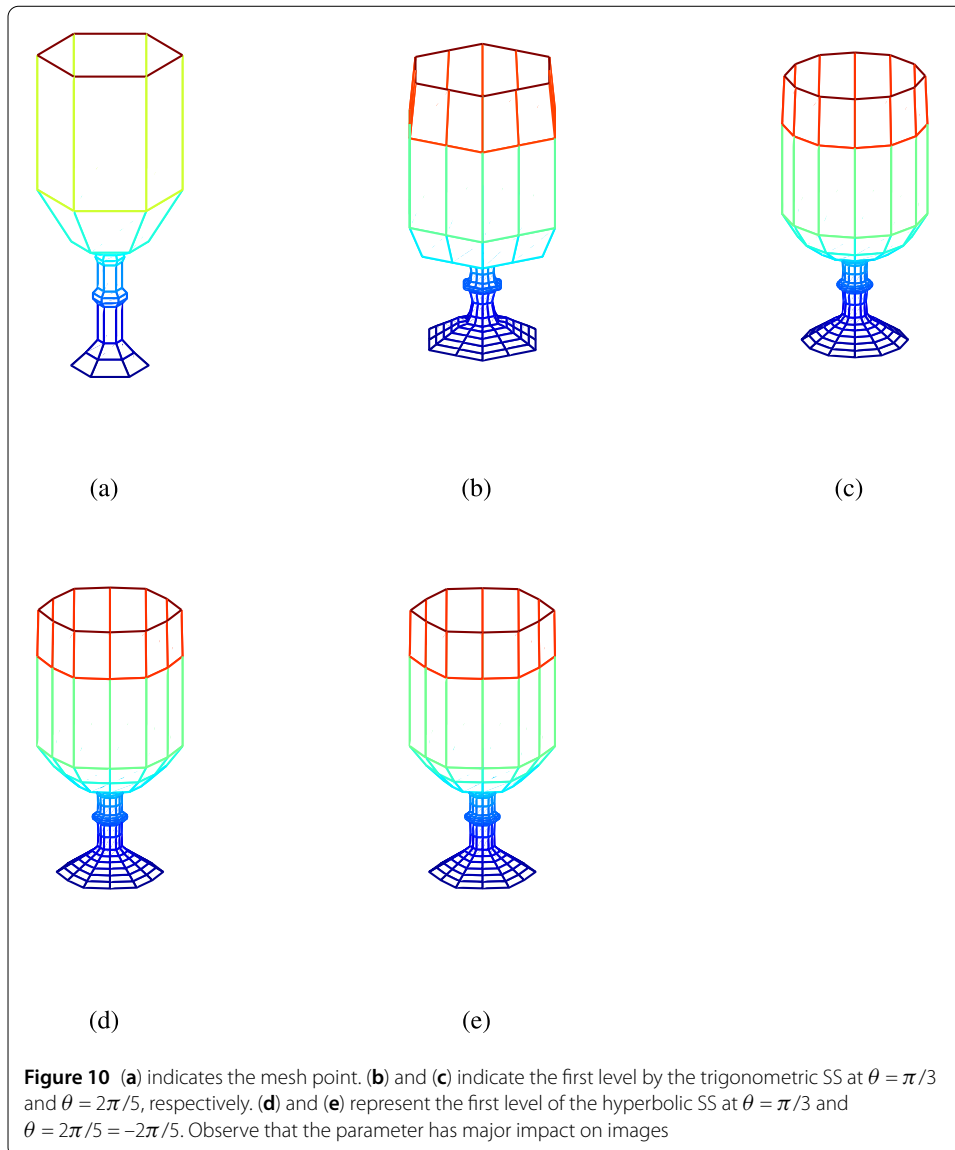
are shown in Figs. 8(a) and 9(a). Figure 10 indicates the impact of parameters on the shapes produced by trigonometric and hyperbolic SS.

5 Conclusion

In this paper, by using the tensor product we extended the four-point univariate nonstationary SS to a unified bivariate nonstationary SS for regular meshes. Recall that the bivariate SS with the tensor product is obtained starting from the unified SS, which can generalize quite several existing SSs, including the Kobbelt [1] scheme as $k \rightarrow \infty$ for $\mu_{\ell+1}$ and $\mu_{\ell+1} = 1$. We observed from the examples that the proposed bivariate SS is suitable to generate different images and texture of the geometric object. Local and adaptive refinement within the domain of image parameter can also be easily applied as the unified univariate SS can generate many classical, analytical, and parametric surfaces. Applications of the univariate SS are not confined for closed meshes; in fact, it works for open meshes as well.

Some properties of the SS are:

- It can reconstruct polynomial, cosine, sine, hyperbolic cosine, and hyperbolic sine functions



- It can be considered as counterpart of the four-point DD SS [18] in one dimension for $\mu_{\ell+1} = 1$ and, in the limiting case, of trigonometric and hyperbolic functions when the data is constant along the other directions.

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Authors' contributions

All the authors contributed equally. All authors read and approved the final manuscript.

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