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The generalized U–H and U–H stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving φ-Caputo fractional operators

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Abstract

We investigate the existence and uniqueness of solutions to a coupled system of the hybrid fractional integro-differential equations involving φ-Caputo fractional operators. To achieve this goal, we make use of a hybrid fixed point theorem for a sum of three operators due to Dhage and also the uniqueness result is obtained by making use of the Banach contraction principle. Moreover, we explore the Ulam–Hyers stability and its generalized version for the given coupled hybrid system. An example is presented to guarantee the validity of our existence results.

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1 Introduction

Probably, the fractional differential equation (FrDiffEq) has been preferred instead of the integer order one because of its opportunities for the description of dynamical behaviors of numerous processes in the scientific and engineering fields. To see some improvements in relation to the applicability of FrDiffEqs, we point out the monographs of Hilfer [1], Kilbas et al. [2], Miller and Ross [3], Oldham [4], Pudlubny [5] and the references therein. In view of some strong properties of fractional operators, a number of researchers have studied various abstract fractional applied models in recent years. For example, Abdo et al. [6] derived some existence results of positive solutions for a weighted fractional BVP and Baghani et al. [7] investigated the existence results for a fractional model of Basset–Boussinesq–Oseen equations. In 2020, Baleanu et al. [8] designed a new fractional hybrid model of a thermostat via the hybrid conditions and proved some theorems by means of the Dhage method. By using a prior estimate method, Nazir et al. [9] turned to studying a sequential hybrid fractional equation and Vivek et al. [10] analyzed dynamical behaviors of Hilfer–Hadamard type fractional pantograph equations by utilizing successive approximations. The latest achievements in this field can be found in references such as [11–14].

The notion of the Ulam–Hyers (U–H) stability has been taken into consideration in several publications. The announced stability analysis is a simple manner in this regard. Such a species of stability was developed by Ulam [15]. Later, it was developed by Hyers [16, 17]. Recently, Ben Chikh et al. [18] considered a multi-order BVP via integral conditions and studied the U–H stability for this system. Samina et al. [19] reviewed the qualitative properties of a coupled system of fractional hybrid differential equations by terms of the U–H stability. At the same time, Ahmad et al. [20] derived similar results on the U–H stability for a coupled system of fractional hybrid BVPs with finite delays.

On the other side, φ-fractional operators were introduced by Kilbas [2] as a generalization of Riemann–Liouville (Riem–Lio) operators. These fractional operators are not quite the same as the other classical fractional operators; this is so because their kernel appears with respect to another increasing function φ. Several generalized FODs and their applications were introduced by Agarwal [21].

In 2017, Almeida [22] proposed a kind of Caputo FOD with some applied specifications and after that, he studied the existence results for two distinct φ-fractional models by these new derivatives [23, 24]. Also, In 2020, Derbazi et al. [25] investigated a φ-fractional initial value problem by using a monotone iterative technique and then Wahash et al. studied a singular structure of fractional differential equations based on the newly-defined φ-derivatives and presented a modified Picard iterative method [26]. Abdo et al. [27] obtained some results in two directions of the existence and the U–H stability for a mixed structure of φ-Hilfer fractional intgro-differential equations.

In 2015, Sitho, Ntouyas and Tariboon [28] proved an existence result for an initial value problem of fractional hybrid sequential integro-differential equations given by

$$\begin{cases} {}^{c}\mathbb{D}^{\alpha}_{0^+}\left[\frac{{}^{c}\mathbb{D}^{\omega}_{0^+}\upsilon(z)-\sum_{i=1}^{m}\mathbb{I}^{\beta_i}_{0^+}f_i(z,\upsilon(z))}{g(z,\upsilon(z))}\right] = h(z,\upsilon(z),\mathbb{I}^{\gamma}_{0^+}\upsilon(z)), \quad z\in[0,T], \\ \upsilon(0)=0, \qquad {}^{c}\mathbb{D}^{\omega}_{0^+}\upsilon(0)=0, \end{cases} \tag{1}$$

where $\alpha \in (0,1]$, $\omega \in (0,1]$, $\beta_i > 0$. ${}^{c}\mathbb{D}^{\lambda}_{0^+}$ denotes the Caputo fractional derivative of order $\lambda \in \{\alpha,\omega\}$ and $\mathbb{I}^{\mu}_{0^+}$ denotes the Riemann–Liouville fractional integral of order $\mu \in \{\beta_1,\beta_2,\dots,\beta_m,\gamma\}$ and $g \in C([0,1]\times\mathbb{R},\mathbb{R}\{0\})$ and $h \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and $f_i \in C([0,1]\times\mathbb{R},\mathbb{R})$ for $i=1,2,\dots,m$.

Next in 2019, Jamil, Khan and Shah [29] studied the existence result for a boundary value problem of hybrid fractional sequential integro-differential equations involving Caputo derivatives given by

$$\begin{cases} {}^{c}\mathbb{D}^{\alpha}_{0^+}\left[\frac{{}^{c}\mathbb{D}^{\omega}_{0^+}\upsilon(z)-\sum_{i=1}^{m}\mathbb{I}^{\beta_i}_{0^+}f_i(z,\upsilon(z))}{g(z,\upsilon(z))}\right] = h(z,\upsilon(z),\mathbb{I}^{\gamma}_{0^+}\upsilon(z)), \quad z\in[0,1], \\ \upsilon(0)=0, \qquad {}^{c}\mathbb{D}^{\omega}_{0^+}\upsilon(0)=0, \qquad \upsilon(1)=\delta\upsilon(\eta), \end{cases} \tag{2}$$

where $\alpha \in (0,1]$, $\omega \in (1,2]$, $\beta_i,\gamma > 0$, $\delta \in (0,1)$, $\eta \in (0,1)$. ${}^{c}\mathbb{D}^{\lambda}_{0^+}$ denotes the Caputo fractional derivative of order $\lambda \in \{\alpha,\omega\}$ and $\mathbb{I}^{\mu}_{0^+}$ denotes the Riemann–Liouville fractional integral of order $\mu \in \{\beta_1,\beta_2,\dots,\beta_m,\gamma\}$ and $g \in C([0,1]\times\mathbb{R},\mathbb{R}\{0\})$ and $h \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and $f_i \in C([0,1]\times\mathbb{R},\mathbb{R})$ with $f_i(0,0)=0$ for $i=1,2,\dots,m$. In that work, Jamil et al. proved the existence results with the help of Dhage's criterion [29].

Motivated by the novel advancements in φ-fractional calculus and by the above work, in the current study, we implement the generalized U–H and U–H stability and existence analysis on the coupled system of the fractional hybrid nonlinear integro-differential equations

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\varphi}\left[\frac{\upsilon(z)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k(z,\upsilon(z),\varpi(z))}{\mathbb{K}_1(z,\upsilon(z),\varpi(z))}\right] = \mathbb{H}_1(z,\upsilon(z),\varpi(z)), \\ {}^{c}\mathbb{D}_{a^+}^{\mu;\varphi}\left[\frac{\varpi(z)-\sum_{k=1}^{n}\mathbb{I}_{a^+}^{\xi_k;\varphi}\mathbb{G}_k(z,\upsilon(z),\varpi(z))}{\mathbb{K}_2(z,\upsilon(z),\varpi(z))}\right] = \mathbb{H}_2(z,\upsilon(z),\varpi(z)), \end{cases} \quad z \in \mathfrak{J} := [a,b], \tag{3}$$

with the initial conditions

$$\upsilon(a) = 0, \qquad \varpi(a) = 0, \tag{4}$$

where ${}^{c}\mathbb{D}_{a^+}^{\beta;\varphi}$ is the φ-Caputo FOD of order $\beta \in \{\nu,\mu\} \subseteq (0,1)$, $\mathbb{I}_{a^+}^{\theta;\varphi}$ is the φ-RL-integral of order $\theta > 0$, $\theta \in \{\sigma_1,\sigma_2,\ldots,\sigma_m,\xi_1,\xi_2,\ldots,\xi_n\}$, $\sigma_k (k = 1,2,3,4,\ldots,m)$, $\xi_j > 0$ $(j = 1,2,3,4,\ldots,n)$, the nonlinear functions $\mathbb{K}_1,\mathbb{K}_2 : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and the functions $\mathbb{F}_k,\mathbb{G}_j,\mathbb{H}_1$, $\mathbb{H}_2$: $\mathfrak{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

By a coupled solution of the coupled IVPs of FrDiffEqs (3)–(4), we mean a pair $(\upsilon,\varpi) \in C^2(\mathfrak{J},\mathbb{R}) \times C^2(\mathfrak{J},\mathbb{R})$ that satisfies IVP (3)–(4), where $C^2(\mathfrak{J},\mathbb{R})$ is a space that consists of a collection of the twice continuously differentiable real mappings.

The novelty of our suggested problem in comparison to problems (1) and (2) is that, in this paper, we consider a kind of general case of initial value problem in a configuration of a hybrid coupled system illustrated by (3). Indeed, fractional operators used in our problem are considered as generalized ones with respect to an increasing function φ, which implies that we can cover a wide range of fractional operators in our IVP (3) subject to the generalized kernels. This feature of φ-operators shows the importance and usefulness of these kinds of fractional operators compared to other ones. Also, unlike the two papers mentioned above, we here extend our problem to a hybrid coupled system of fractional integro-differential initial value problems with different sequential orders based on the generalized φ-RL-integral operators and in addition to the establishment of the existence and uniqueness results, we investigate the stability of the suggested coupled system in terms of the Ulam–Hyers stability and the generalized Ulam–Hyers one. In future work, we can implement these techniques on different boundary value problems equipped with complicated integral multi-point boundary conditions.

The structure of this research work is as follows: Sect. 2 provides the auxiliary definitions along with desired lemmas. Section 3 is devoted to generalized U–H and U–H stability and the existence analysis for the given system of integro-differential IVPs (3)–(4). Moreover, we present a concrete example to emphasize the validity of the obtained outcomes.

2 Basic preliminaries

To achieve the desired fundamental purposes, we first review several basic auxiliary notions that are required throughout the manuscript.

The collection $\mathfrak{C} = C(\mathfrak{J},\mathbb{R})$ is designated as a collection consisting of continuous real-valued functions $\upsilon : \mathfrak{J} \to \mathbb{R}$. Apparently, $\mathfrak{C}$ is a Banach space along with the supremum norm

$$\|\upsilon\| = \sup\{|\upsilon(z)| : z \in \mathfrak{J}\}$$

and is a Banach algebra under the action "·" defined by

$$(\upsilon \cdot \varpi)(z) = \upsilon(z) \cdot \varpi(z)$$

for $\upsilon, \varpi \in \mathfrak{C}$ and $z \in \mathfrak{J} = [a,b]$. Given the Banach algebra $\mathfrak{C}$, consider the product space $\mathbb{E} = \mathfrak{C} \times \mathfrak{C}$ which is a vector space equipped with the coordinate-wise addition and scalar multiplication. Define a norm $\|\cdot\|$ in the product linear space $\mathbb{E}$ by

$$\big\|(\upsilon, \varpi)\big\| = \|\upsilon\| + \|\varpi\|.$$

Then the normed linear space $(\mathbb{E}, \|(\cdot,\cdot)\|)$ is a Banach space which further becomes a Banach algebra. The multiplication action between the members of $\mathbb{E}$ is illustrated by

$$\big((\upsilon, \varpi) \cdot (u, v)\big)(z) = (\upsilon, \varpi)(z) \cdot (u, v)(z) = \big(\upsilon(z)u(z), \varpi(z)v(z)\big) \tag{5}$$

for all $z \in \mathfrak{J}$, where $(\upsilon, \varpi), (u, v) \in \mathbb{E}$.

We start by characterizing φ-Riem–Lio fractional integrals and derivatives.

Definition 2.1 ([2]) Let $\alpha > 0$ and an increasing function $\varphi: \mathfrak{J} \longrightarrow \mathbb{R}$ satisfy $\varphi'(z) \neq 0$ for all $z \in \mathfrak{J}$. We define the left-sided φ-Riem–Lio integral of an integrable function υ on $\mathfrak{J}$ in the fractional framework w.r.t. another differentiable function φ as

$$\mathbb{I}_{a^+}^{\alpha;\varphi} \upsilon(z) = \frac{1}{\Gamma(\alpha)} \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{\alpha-1} \upsilon(s)\, ds, \tag{6}$$

where Γ denotes the standard Euler–Gamma function.

Equation (6) turns to the Riem–Lio and Hadamard fractional integrals by taking $\varphi(z) = z$ and $\varphi(z) = \ln z$, respectively. Moreover, the Cauchy formula for m-fold integrals can be obtained by considering $\varphi(z) = z$ and $\alpha = 1$:

$$\int_a^z dz_1 \int_a^{z_1} dz_2 \int_a^{z_2} dz_3 \cdots \int_a^{z_{m-1}} u(z_m)\, dz_m = \frac{1}{(m-1)!} \int_a^z (z-s)^{m-1} u(s)\, ds.$$

Definition 2.2 ([2]) Let $m \in \mathbb{N}$ with $m = [\alpha] + 1$. The left-sided φ-Riem–Lio fractional derivative of an existing function $\upsilon \in C^m(\mathfrak{J}, \mathbb{R})$ w.r.t. a non-decreasing function φ such that $\varphi'(z) \neq 0$, for all $z \in \mathfrak{J}$ in the fractional framework is represented as follows:

$$\begin{aligned}
\mathbb{D}_{a^+}^{\alpha;\varphi} \upsilon(z) &= \left(\frac{1}{\varphi'(z)} \frac{d}{dz}\right)^m \mathbb{I}_{a^+}^{m-\alpha;\varphi} \upsilon(z) \\
&= \frac{1}{\Gamma(m-\alpha)} \left(\frac{1}{\varphi'(z)} \frac{d}{dz}\right)^m \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{m-\alpha-1} \upsilon(s)\, ds.
\end{aligned}$$

Definition 2.3 ([22]) Let $m \in \mathbb{N}$ with $m = [\alpha] + 1$. The left-sided φ-Caputo fractional derivative of an existing function $\upsilon \in C^m(\mathfrak{J}, \mathbb{R})$ w.r.t. a non-decreasing function φ such that $\varphi'(z) \neq 0$, for all $z \in \mathfrak{J}$ in the fractional framework is represented as follows:

$${}^c\mathbb{D}_{a^+}^{\alpha;\varphi} \upsilon(z) = \mathbb{I}_{a^+}^{m-\alpha;\varphi} \left(\frac{1}{\varphi'(z)} \frac{d}{dz}\right)^m \upsilon(z).$$

For simplicity, we have

$$\upsilon_{\varphi}^{[m]}(z) = \left(\frac{1}{\varphi'(z)}\frac{d}{dz}\right)^{m}\upsilon(z).$$

From the definition, it is clear that

$$^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}\upsilon(z) = \begin{cases} \int_{a}^{z}\frac{\varphi'(s)(\varphi(z)-\varphi(s))^{m-\alpha-1}}{\Gamma(m-\alpha)}\upsilon_{\varphi}^{[m]}(s)\,ds & \text{if } \alpha \notin \mathbb{N}, \\ \upsilon_{\varphi}^{[m]}(z) & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{7}$$

Notice that, if $\upsilon \in C^{m}(\mathfrak{J},\mathbb{R})$, then the αth φ-Caputo fractional derivative of υ is determined by

$$^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}\upsilon(z) = \mathbb{D}_{a^{+}}^{\alpha;\varphi}\left[\upsilon(z) - \sum_{k=0}^{n-1}\frac{\upsilon_{\varphi}^{[k]}(a)}{k!}\big(\varphi(z)-\varphi(a)\big)^{k}\right]$$

(see [22, Theorem 3]).

Lemma 2.4 ([2]) *Assuming* $\alpha, \beta > 0$ *and* $\upsilon \in L^{1}(\mathfrak{J},\mathbb{R})$, *we get*

$$\mathbb{I}_{a^{+}}^{\alpha;\varphi}\mathbb{I}_{a^{+}}^{\beta;\varphi}\upsilon(z) = \mathbb{I}_{a^{+}}^{\alpha+\beta;\varphi}\upsilon(z) \quad (z \in \mathfrak{J}).$$

Lemma 2.5 ([23]) *Assuming* $\alpha > 0$, *following assertions hold*:
If $\upsilon \in C(\mathfrak{J},\mathbb{R})$, *then*

$$^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}\mathbb{I}_{a^{+}}^{\alpha;\varphi}\upsilon(z) = \upsilon(z), \quad z \in \mathfrak{J}.$$

If $\upsilon \in C^{m}(\mathfrak{J},\mathbb{R})$ *and* $m-1 < \alpha < m$, *then*

$$\mathbb{I}_{a^{+}}^{\alpha;\varphi}\,{}^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}\upsilon(z) = \upsilon(z) - \sum_{k=0}^{m-1}\frac{\upsilon_{\varphi}^{[k]}(a)}{k!}\big[\varphi(z)-\varphi(a)\big]^{k}, \quad z \in \mathfrak{J}.$$

Lemma 2.6 ([2, 23]) *Let* $z > a$, $\alpha \geq 0$ *and* $\beta > 0$. *Then*

- $\mathbb{I}_{a^{+}}^{\alpha;\varphi}(\varphi(z)-\varphi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\varphi(z)-\varphi(a))^{\alpha+\beta-1}$,
- ${}^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}(\varphi(z)-\varphi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(z)-\varphi(a))^{\beta-\alpha-1}$,
- ${}^{c}\mathbb{D}_{a^{+}}^{\alpha;\varphi}(\varphi(z)-\varphi(a))^{k} = 0$, *for any* $k = 0,\dots,m-1; m \in \mathbb{N}$.

The following definition will be used in the sequel.

Definition 2.7 ([30, 31]) A self-operator $\hbar$ on a Banach space $\mathfrak{C}$ is called Lipschitz if there exists a constant $L_{\hbar} > 0$ satisfying

$$\left\|\hbar(\upsilon) - \hbar(\varpi)\right\| \leq L_{\hbar}\|\upsilon - \varpi\|$$

for all elements $\upsilon, \varpi \in \mathfrak{C}$.

We shall make use of the hybrid fixed point result due to Dhage [30, 32] and the contraction principle due to Banach as a fundamental apparatus for demonstrating the existence–uniqueness result of the coupled solutions of the proposed system given in this paper.

Theorem 2.8 ([30, 32]) *Let X be a convex, bounded and closed set contained in the Banach algebra $\mathfrak{C}$ and the operators $\mathcal{P},\mathcal{S}:\mathfrak{C}\to\mathfrak{C}$ and $\mathcal{Q}:X\to\mathfrak{C}$ be such that:*

($s1$) $\mathcal{P}$ and $\mathcal{S}$ are Lipschitz maps with Lipschitz constants $L_{\mathcal{P}}$ and $L_{\mathcal{S}}$, respectively;

($s2$) $\mathcal{Q}$ is continuous and compact;

($s3$) $\upsilon=\mathcal{P}\upsilon\mathcal{Q}\varpi+\mathcal{S}\upsilon\forall\varpi\in X\Rightarrow\upsilon\in X$; and

($s4$) $L_{\mathcal{P}}M_{\mathcal{Q}}+L_{\mathcal{S}}<1$, where $M_{\mathcal{Q}}=\|\mathcal{Q}(X)\|=\sup\{\|\mathcal{Q}\upsilon\|:\upsilon\in X\}$.

Then the operator equation $\upsilon=\mathcal{P}\upsilon\mathcal{Q}\upsilon+\mathcal{S}\upsilon$ possesses a solution in X.

Theorem 2.9 ([33]) *A contraction mapping $T:\Omega\to\Omega$ possesses a unique fixed point where Ω be a nonempty closed set contained in a Banach space $\mathfrak{C}$.*

3 Main result

To start for verifying the main results, the following assumptions are required for us in the sequel:

($\mathbb{HYP}0$) The real functions $\mathbb{H}_1$ and $\mathbb{H}_2$ are bounded on $\mathfrak{J}\times\mathbb{R}\times\mathbb{R}$ subject to bounds $M_{\mathbb{H}_1}$ and $M_{\mathbb{H}_2}$, respectively. And there exist $M_{\mathbb{K}_i}>0$ for $i=1,2$ such that

$$\bigl|\mathbb{K}_1(z,\upsilon,\varpi)\bigr|\le M_{\mathbb{K}_1},\qquad \bigl|\mathbb{K}_2(z,\upsilon,\varpi)\bigr|\le M_{\mathbb{K}_2}.$$

($\mathbb{HYP}1$) There exist $L_{\mathbb{H}_1}>0$ and $L_{\mathbb{H}_2}>0$ such that

$$\bigl|\mathbb{H}_1(z,\upsilon,\varpi)-\mathbb{H}_1(z,\bar{\upsilon},\bar{\varpi})\bigr|\le L_{\mathbb{H}_1}\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr)$$

and

$$\bigl|\mathbb{H}_2(z,\upsilon,\varpi)-\mathbb{H}_2(z,\bar{\upsilon},\bar{\varpi})\bigr|\le L_{\mathbb{H}_2}\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr)$$

for all $z\in\mathfrak{J}$ and $\upsilon,\bar{\upsilon},\varpi,\bar{\varpi}\in\mathbb{R}$.

($\mathbb{HYP}2$) There exist $L_{\mathbb{K}_2}>0$ and $L_{\mathbb{K}_1}>0$ such that

$$\bigl|\mathbb{K}_1(z,\upsilon,\varpi)-\mathbb{K}_1(z,\bar{\upsilon},\bar{\varpi})\bigr|\le L_{\mathbb{K}_1}\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr)$$

and

$$\bigl|\mathbb{K}_2(z,\upsilon,\varpi)-\mathbb{K}_2(z,\bar{\upsilon},\bar{\varpi})\bigr|\le L_{\mathbb{K}_2}\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr)$$

for all $z\in\mathfrak{J}$ and $\upsilon,\bar{\upsilon},\varpi,\bar{\varpi}\in\mathbb{R}$.

($\mathbb{HYP}3$) There exist bounded functions $L_{\mathbb{F}_k},L_{\mathbb{G}_k}:\mathfrak{J}\to\mathbb{R}_+$ with bounds $\|L_{\mathbb{F}_k}\|$ and $\|L_{\mathbb{G}_j}\|$ such that

$$\bigl|\mathbb{F}_k(z,\upsilon,\varpi)-\mathbb{F}_k(z,\bar{\upsilon},\bar{\varpi})\bigr|\le L_{\mathbb{F}_k}(z)\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr),\quad k=1,\dots,m,$$

and

$$\left|\mathbb{G}_j(z,\upsilon,\varpi)-\mathbb{G}_j(z,\bar{\upsilon},\bar{\varpi})\right|\le L_{\mathbb{G}_j}(z)\bigl(|\upsilon-\bar{\upsilon}|+|\varpi-\bar{\varpi}|\bigr),\quad j=1,\dots,n,$$

for $z\in\mathfrak{J}$ and $\upsilon,\bar{\upsilon},\varpi,\bar{\varpi}\in\mathbb{R}$.

($\mathbb{HYP}$4) There exist $\mathbb{K}_{j,0}>0(j=1,2)$ such that

$$\mathbb{K}_{1,0}=\sup_{z\in\mathfrak{J}}\left|\mathbb{K}_1(z,0,0)\right|,\qquad \mathbb{K}_{2,0}=\sup_{z\in\mathfrak{J}}\left|\mathbb{K}_2(z,0,0)\right|,$$

along with

$$\mathbb{F}_0=\sup_{z\in\mathfrak{J}}\left|\mathbb{F}_k(z,0,0)\right|\quad \forall k\in\{1,2,\dots,m\},$$

and

$$\mathbb{G}_0=\sup_{z\in\mathfrak{J}}\left|\mathbb{G}_j(z,0,0)\right|\quad \forall j\in\{1,2,\dots,n\}.$$

($\mathbb{HYP}$5) The constants in the hypotheses ($\mathbb{HYP}$1)–($\mathbb{HYP}$4) obey the following assertion:

$$[L_{\mathbb{K}_1}\mathcal{A}_1M_{\mathbb{H}_1}]+\bigl[\mathcal{B}_1\|L_{\mathbb{F}_k}\|\bigr]+[L_{\mathbb{K}_2}\mathcal{A}_2M_{\mathbb{H}_2}]+\bigl[\mathcal{B}_2\|L_{\mathbb{G}_j}\|\bigr]<1 \tag{8}$$

To show the existence of solutions of the proposed system of integro-differential IVPs (3)–(4), we require the lemma given below.

Lemma 3.1 *If a function $\upsilon\in C^m(\mathfrak{J},\mathbb{R})$ is taken as a solution for the hybrid fractional integro-differential equation*

$${}^{c}\mathbb{D}_{a^+}^{\nu;\varphi}\left[\frac{\upsilon(z)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k(z,\upsilon(z),\varpi(z))}{\mathbb{K}_1(z,\upsilon(z),\varpi(z))}\right]=\mathbb{H}_1\bigl(z,\upsilon(z),\varpi(z)\bigr),\quad z\in\mathfrak{J}:=[a,b], \tag{9}$$

with the initial condition

$$\upsilon(a)=0, \tag{10}$$

then it satisfies the following hybrid fractional integral equation:

$$\upsilon(z)=\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k\bigl(z,\upsilon(z),\varpi(z)\bigr)+\Bigl(\bigl[\mathbb{K}_1\bigl(z,\upsilon(z),\varpi(z)\bigr)\bigr]\times\bigl[\mathbb{I}^{\nu;\varphi}\mathbb{H}_1\bigl(z,\upsilon(z),\varpi(z)\bigr)\bigr]\Bigr). \tag{11}$$

Proof Applying the νth ψ-Riem–Lio fractional integral on both sides of (9) and using Lemma 2.5, we obtain

$$\left[\frac{\upsilon(z)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k(z,\upsilon(z),\varpi(z))}{\mathbb{K}_1(z,\upsilon(z),\varpi(z))}\right]=\mathbb{I}_{a^+}^{\nu;\varphi}\mathbb{H}_1\bigl(z,\upsilon(z),\varpi(z)\bigr)+c_1, \tag{12}$$

which implies

$$\upsilon(z)=\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k\bigl(z,\upsilon(z),\varpi(z)\bigr)+\bigl[\mathbb{K}_1\bigl(z,\upsilon(z),\varpi(z)\bigr)\bigr]\bigl(\mathbb{I}_{a^+}^{\nu;\varphi}\mathbb{H}_1\bigl(z,\upsilon(z),\varpi(z)\bigr)+c_1\bigr). \tag{13}$$

Using the initial condition $\upsilon(a) = 0$, we have $c_1 = \frac{\upsilon(a)}{\mathbb{K}_1(a,\upsilon(a),\varpi(a))} = 0$. Now, substituting the value of c_1 in (13), we get

$$\upsilon(z) = \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k\bigl(z, \upsilon(z), \varpi(z)\bigr) + \bigl(\bigl[\mathbb{K}_1\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr] \times \bigl[\mathbb{I}^{\nu;\varphi}\mathbb{H}_1\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr]\bigr). \quad (14)$$

The proof is finished. □

Lemma 3.2 *If a function $\varpi \in C^m(\mathfrak{J}, \mathbb{R})$ is taken as a solution for the hybrid fractional integro-differential equation*

$$ {}^c\mathbb{D}_{a^+}^{\mu;\varphi}\left[\frac{\varpi(z) - \sum_{k=1}^{n} \mathbb{I}_{a^+}^{\xi_k;\varphi} \mathbb{G}_k(z, \upsilon(z), \varpi(z))}{\mathbb{K}_2(z, \upsilon(z), \varpi(z))}\right] = \mathbb{H}_2\bigl(z, \upsilon(z), \varpi(z)\bigr), \quad z \in \mathfrak{J} := [a, b], \quad (15)$$

with the initial condition

$$\varpi(a) = 0, \quad (16)$$

then it satisfies the following hybrid fractional integral equation:

$$\begin{aligned}\varpi(z) = &\sum_{k=1}^{n} \mathbb{I}_{a^+}^{\xi_k;\varphi} \mathbb{G}_k\bigl(z, \upsilon(z), \varpi(z)\bigr) \\ &+ \bigl(\bigl[\mathbb{K}_2\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr] \times \bigl[\mathbb{I}^{\mu;\varphi}\mathbb{H}_2\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr]\bigr). \quad (17)\end{aligned}$$

Proof The proof is similar to above. □

Notation 3.3 For simplicity, take

$$\begin{aligned}
&\mathcal{A}_1 = \frac{(\varphi(b) - \varphi(a))^\nu}{\Gamma(\nu + 1)}, \qquad \mathcal{A}_2 = \frac{(\varphi(b) - \varphi(a))^\mu}{\Gamma(\mu + 1)}, \\
&\mathcal{B}_1 = \sum_{k=1}^{m} \frac{(\varphi(b) - \varphi(a))^{\sigma_k}}{\Gamma(\sigma_k + 1)}, \qquad \mathcal{B}_2 = \sum_{j=1}^{n} \frac{(\varphi(b) - \varphi(a))^{\xi_j}}{\Gamma(\xi_j + 1)}\|, \\
&L_{\mathcal{P}} = L_{\mathbb{K}_1} + L_{\mathbb{K}_2}, \qquad L_{\mathcal{S}} = \mathcal{B}_1\|L_{\mathbb{F}_k}\| + \mathcal{B}_2\|L_{\mathbb{G}_j}\|, \\
&L_{\mathcal{Q}} = (\mathcal{A}_1 M_{\mathbb{H}_1} + \mathcal{A}_2 M_{\mathbb{H}_2}). \quad (18)
\end{aligned}$$

Theorem 3.4 *Suppose that the hypotheses* ($\mathbb{HYP}$0)–($\mathbb{HYP}$5) *are obeyed. Furthermore, if*

$$L_{\mathcal{P}} L_{\mathcal{Q}} + L_{\mathcal{S}} < 1, \quad (19)$$

then the coupled system (3)–(4) *possesses a mild coupled solution formulated on* $\mathfrak{J}$.

Proof According to Lemmas 3.1 and 3.2, the mild coupled solutions of the coupled system of fractional integro-differential IVPs in (3)–(4) are the solutions of the coupled fractional integral equations

$$\upsilon(z) = \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k\bigl(z, \upsilon(z), \varpi(z)\bigr) + \bigl[\mathbb{K}_1\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr] \times \bigl[\mathbb{I}_{a^+}^{\nu;\varphi}\mathbb{H}_1\bigl(z, \upsilon(z), \varpi(z)\bigr)\bigr] \quad (20)$$

and

$$\varpi(z) = \sum_{j=1}^{n} \mathbb{I}_{a^+}^{\xi_j;\varphi} \mathbb{G}_k\big(z, \upsilon(z), \varpi(z)\big) + \big[\mathbb{K}_2\big(z, \upsilon(z), \varpi(z)\big)\big] \times \big[\mathbb{I}_{a^+}^{\mu;\varphi} \mathbb{H}_2\big(z, \upsilon(z), \varpi(z)\big)\big]. \quad (21)$$

Choose $\rho > 0$ so that

$$\rho \geq \frac{\mathbb{K}_{1,0}(\mathcal{A}_1 M_{\mathbb{H}_1}) + \mathcal{B}_1 \mathbb{F}_0 + \mathbb{K}_{1,0}(\mathcal{A}_2 M_{\mathbb{H}_2}) + \mathcal{B}_2 \mathbb{G}_0}{1 - [L_{\mathbb{K}_1} \mathcal{A}_1 M_{\mathbb{H}_1}] - [\mathcal{B}_1 \|L_{\mathbb{F}_k}\|] - [L_{\mathbb{K}_2} \mathcal{A}_2 M_{\mathbb{H}_2}] - [\mathcal{B}_2 \|L_{\mathbb{G}_j}\|]} \quad (22)$$

and specify a subset X of the Banach space $\mathfrak{C} \times \mathfrak{C}$ by

$$X = \big\{(\upsilon, \varpi) \in \mathfrak{C} \times \mathfrak{C} : \big\|(\upsilon, \varpi)\big\| \leq \rho\big\}.$$

Evidently, X is a convex, bounded and closed set contained in the Banach space $\mathfrak{C} \times \mathfrak{C} = \mathbb{E}$. Characterize the operators $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2) : \mathbb{E} \to \mathbb{E}, \mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) : \mathbb{E} \to \mathbb{E}$ and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2) : X \to \mathbb{E}$ by

$$\begin{cases} \mathcal{P}_1(\upsilon, \varpi) = \mathbb{K}_1(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}, \\ \mathcal{P}_2(\upsilon, \varpi) = \mathbb{K}_2(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}, \end{cases}$$

and

$$\begin{cases} \mathcal{Q}_1(\upsilon, \varpi) = \mathbb{I}_{a^+}^{\nu;\varphi} \mathbb{H}_1(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}, \\ \mathcal{Q}_2(\upsilon, \varpi) = \mathbb{I}_{a^+}^{\mu;\varphi} \mathbb{H}_2(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}, \end{cases}$$

and

$$\begin{cases} \mathcal{S}_1(\upsilon, \varpi) = \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}, \\ \mathcal{S}_2(\upsilon, \varpi) = \sum_{j=1}^{n} \mathbb{I}_{a^+}^{\xi_j;\varphi}; \mathbb{G}_j(z, \upsilon(z), \varpi(z)), & z \in \mathfrak{J}. \end{cases}$$

In this case, the coupled system of the given hybrid integral equations (20)–(21) can be represented in the framework of a system of operator equations as

$$\mathcal{P}(\upsilon, \varpi)(z)\mathcal{Q}(\upsilon, \varpi)(z) + \mathcal{S}(\upsilon, \varpi)(z) = (\upsilon, \varpi)(z), \quad z \in \mathfrak{J},$$

which further taking into account the multiplication given in (5) reduces to

$$\begin{aligned} &\big(\mathcal{P}_1(\upsilon, \varpi)(z)\mathcal{Q}_1(\upsilon, \varpi)(z) + \mathcal{S}_1(\upsilon, \varpi)(z), \mathcal{P}_2(\upsilon, \varpi)(z)\mathcal{Q}_2(\upsilon, \varpi)(z) + \mathcal{S}_2(\upsilon, \varpi)(z)\big) \\ &\quad = (\upsilon, \varpi)(z) \end{aligned}$$

for $z \in \mathfrak{J}$. This further implies that

$$\begin{cases} \mathcal{P}_1(\upsilon, \varpi)(z)\mathcal{Q}_1(\upsilon, \varpi)(z) + \mathcal{S}_1(\upsilon, \varpi)(z) = \upsilon(z), & z \in \mathfrak{J}, \\ \mathcal{P}_2(\upsilon, \varpi)(z)\mathcal{Q}_2(\upsilon, \varpi)(z) + \mathcal{S}_2(\upsilon, \varpi)(z) = \varpi(z), & z \in \mathfrak{J}. \end{cases}$$

Presently, we demonstrate in the following steps that all three operators $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{S}$ follow the assertions of Theorem 2.8.

Step I: We first show that $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ and $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ are Lipschitzian on $\mathbb{E}$ with Lipschitz constants $L_{\mathcal{P}} = (L_{\mathbb{K}_1} + L_{\mathbb{K}_2})$ and $L_{\mathcal{S}} = (\mathcal{B}_1\|L_{\mathbb{F}_k}\| + \mathcal{B}_2\|L_{\mathbb{G}_j}\|)$, respectively. Let $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$ be arbitrary. Then, using ($\mathbb{HYP}$2), we have

$$\begin{aligned} \left|\mathcal{P}_1(\upsilon,\varpi)(z) - \mathcal{P}_1(\bar{\upsilon},\bar{\varpi})(z)\right| &= \left|\mathbb{K}_1\big(z,\upsilon(z),\varpi(z)\big) - \mathbb{K}_1\big(z,\bar{\upsilon}(z),\bar{\varpi}(z)\big)\right| \\ &\leq L_{\mathbb{K}_1}\big(\left|\upsilon(z)-\bar{\upsilon}(z)\right| + \left|\varpi(z)-\bar{\varpi}(z)\right|\big) \\ &\leq L_{\mathbb{K}_1}\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big) \end{aligned}$$

for all $z \in \mathfrak{J}$. Operating the supremum norm over z, we get

$$\left\|\mathcal{P}_1(\upsilon,\varpi) - \mathcal{P}_1(\bar{\upsilon},\bar{\varpi})\right\| \leq L_{\mathbb{K}_1}\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big)$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$. Along the same lines, we get

$$\left\|\mathcal{P}_2(\upsilon,\varpi) - \mathcal{P}_2(\bar{\upsilon},\bar{\varpi})\right\| \leq L_{\mathbb{K}_2}\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big)$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$. Accordingly, employing the definition of operator $\mathcal{P}$, we get

$$\begin{aligned} \left\|\mathcal{P}(\upsilon,\varpi) - \mathcal{P}(\bar{\upsilon},\bar{\varpi})\right\| &= \left\|\big(\mathcal{P}_1(\upsilon,\varpi),\mathcal{P}_2(\upsilon,\varpi)\big) - \big(\mathcal{P}_1(\bar{\upsilon},\bar{\varpi}),\mathcal{P}_2(\bar{\upsilon},\bar{\varpi})\big)\right\| \\ &= \left\|\big(\mathcal{P}_1(\upsilon,\varpi) - \mathcal{P}_1(\bar{\upsilon},\bar{\varpi}),\mathcal{P}_2(\upsilon,\varpi) - \mathcal{P}_2(\bar{\upsilon},\bar{\varpi})\big)\right\| \\ &\leq \left\|\mathcal{P}_1(\upsilon,\varpi) - \mathcal{P}_1(\bar{\upsilon},\bar{\varpi})\right\| + \left\|\mathcal{P}_2(\upsilon,\varpi) - \mathcal{P}_2(\bar{\upsilon},\bar{\varpi})\right\| \\ &\leq L_{\mathbb{K}_1}\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big) + L_{\mathbb{K}_2}\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big) \\ &= (L_{\mathbb{K}_1} + L_{\mathbb{K}_2})\big(\|\upsilon-\bar{\upsilon}\| + \|\varpi-\bar{\varpi}\|\big) \\ &= L_{\mathcal{P}}\left\|(\upsilon,\varpi) - (\bar{\upsilon},\bar{\varpi})\right\| \end{aligned}$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$, where $L_{\mathcal{P}} = (L_{\mathbb{K}_1} + L_{\mathbb{K}_2})$. Similarly, due to the definition of $\mathcal{S}$ and using ($\mathbb{HYP}$3), we get

$$\begin{aligned} &\left|\mathcal{S}_1(\upsilon,\varpi)(z) - \mathcal{S}_1(\bar{\upsilon},\bar{\varpi})(z)\right| \\ &\quad= \left|\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k\big(z,\upsilon(z),\varpi(z)\big) - \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k\big(z,\bar{\upsilon}(z),\bar{\varpi}(z)\big)\right| \\ &\quad\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\sigma_k)}\int_a^z \varphi\prime(z)\big(\varphi(z)-\varphi(s)\big)^{\sigma_k-1}L_{\mathbb{F}_k}(s)\big(\left|\upsilon(s)-\bar{\upsilon}(s)\right| + \left|\varpi(s)-\bar{\varpi}(s)\right|\big)\,ds \\ &\quad\leq \sum_{k=1}^{m}\frac{(\varphi(b)-\varphi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|L_{\mathbb{F}_k}\|\big(\left\|(\upsilon-\bar{\upsilon})\right\| + \left\|\varpi-\bar{\varpi}\right\|\big) \end{aligned}$$

for all $z \in \mathfrak{J}$. Operating the supremum over z, we get

$$\left\|\mathcal{S}_1(\upsilon,\varpi) - \mathcal{S}_1(\bar{\upsilon},\bar{\varpi})\right\| \leq \sum_{k=1}^{m}\frac{(\varphi(b)-\varphi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|L_{\mathbb{F}_k}\|\left\|(\upsilon,\varpi) - (\bar{\upsilon},\bar{\varpi})\right\|$$

$$= \mathcal{B}_1 \|L_{\mathbb{F}_k}\| \big\|(\upsilon, \varpi) - (\bar{\upsilon}, \bar{\varpi})\big\| \tag{23}$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$. Similarly using ($\mathbb{HYP}3$), we can confirm that $\mathcal{S}_2$ is also Lipschitzian with Lipschitz constant $\mathcal{B}_2\|L_{\mathbb{G}_j}\|$; that is,

$$\begin{aligned}\big\|\mathcal{S}_2(\upsilon, \varpi) - \mathcal{S}_2(\bar{\upsilon}, \bar{\varpi})\big\| &\leq \sum_{j=1}^{n} \frac{(\varphi(b) - \varphi(a))^{\xi_j}}{\Gamma(\xi_j + 1)} \|L_{\mathbb{G}_j}\| \big\|(\upsilon, \varpi) - (\bar{\upsilon}, \bar{\varpi})\big\| \\ &= \mathcal{B}_2 \|L_{\mathbb{G}_j}\| \big\|(\upsilon, \varpi) - (\bar{\upsilon}, \bar{\varpi})\big\| \end{aligned} \tag{24}$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$. Hence, from (23)–(24) it follows that

$$\big\|\mathcal{S}(\upsilon, \varpi) - \mathcal{S}(\bar{\upsilon}, \bar{\varpi})\big\| \leq \big(\mathcal{B}_1\|L_{\mathbb{F}_k}\| + \mathcal{B}_2\|L_{\mathbb{G}_j}\|\big) \big\|(\upsilon, \varpi) - (\bar{\upsilon}, \bar{\varpi})\big\|$$

for all $(\upsilon, \varpi), (\bar{\upsilon}, \bar{\varpi}) \in \mathbb{E}$. In consequence, $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ is a Lipschitz map subject to the constant

$$L_{\mathcal{S}} = \big(\mathcal{B}_1\|L_{\mathbb{F}_k}\| + \mathcal{B}_2\|L_{\mathbb{G}_j}\|\big) > 0.$$

Step II: Now we show that $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ is a continuous and compact operator from X into $\mathbb{E}$. To deduce the continuity of $\mathcal{Q}$, we regard $\{(\upsilon_n, \varpi_n)\}_{n\in\mathbb{N}}$ as a sequence of points contained in X going to $(\upsilon, \varpi) \in X$. Then the dominated convergence result propounded by Lebesgue yields

$$\begin{aligned}\lim_{n\to\infty} \mathcal{Q}_1(\upsilon_n, \varpi_n)(z) &= \frac{1}{\Gamma(\nu)} \lim_{n\to\infty} \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{\nu-1} \mathbb{H}_1\big(s, \upsilon_n(s), \varpi_n(s)\big)\, ds \\ &= \frac{1}{\Gamma(\nu)} \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{\nu-1} \lim_{n\to\infty} \mathbb{H}_1\big(s, \upsilon_n(s), \varpi_n(s)\big)\, ds \\ &= \frac{1}{\Gamma(\nu)} \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{\nu-1} \mathbb{H}_1\big(s, \upsilon(s), \varpi(s)\big)\, ds \\ &= \mathcal{Q}_1(\upsilon, \varpi)(z)\end{aligned}$$

for al $z \in \mathfrak{J}$. Similarly, we prove

$$\lim_{n\to\infty} \mathcal{Q}_2(\upsilon_n, \varpi_n)(z) = \mathcal{Q}_2(\upsilon, \varpi)(z)$$

for all $z \in \mathfrak{J}$. Hence $\mathcal{Q}(\upsilon_n, \varpi_n) = (\mathcal{Q}_1(\upsilon_n, \varpi_n), \mathcal{Q}_2(\upsilon_n, \varpi_n))$ converges to $\mathcal{Q}(\upsilon, \varpi)$ pointwise on $\mathfrak{J}$. In the next, the compactness of $\mathcal{Q}$ is explored on X. Firstly, to ensure the uniform boundedness, by assuming $(\upsilon, \varpi) \in X$ and applying ($\mathbb{HYP}0$), we get

$$\begin{aligned}\big|\mathcal{Q}_1(\upsilon, \varpi)(z)\big| &= \frac{1}{\Gamma(\nu)} \int_a^z \varphi'(s)\big(\varphi(z) - \varphi(s)\big)^{\nu-1} \big|\mathbb{H}_1\big(s, \upsilon(s), \varpi(s)\big)\big|\, ds \\ &\leq \frac{(\varphi(b) - \varphi(a))^{\nu}}{\Gamma(\nu + 1)} M_{\mathbb{H}_1}.\end{aligned}$$

Operating the supremum in terms of z in the above, we arrive at

$$\big\|\mathcal{Q}_1(\upsilon, \varpi)(z)\big\| \leq \frac{(\varphi(b) - \varphi(a))^{\nu}}{\Gamma(\nu + 1)} M_{\mathbb{H}_1} = \mathcal{A}_1 M_{\mathbb{H}_1} < \infty$$

for all $(\upsilon,\varpi)\in X$. Hence $\mathcal{Q}_1$ is a uniformly bounded operator on X. In a similar phase, we can guarantee that $\mathcal{Q}_2$ involves the uniform boundedness specification on X subject to bound $\mathcal{A}_2 M_{\mathbb{H}_2}$. Accordingly, $\mathcal{Q}$ will be a uniformly bounded operator on X, because we have

$$\begin{aligned}\left\|\mathcal{Q}(\upsilon,\varpi)(z)\right\| &= \left\|\mathcal{Q}_1(\upsilon,\varpi)(z)\right\| + \left\|\mathcal{Q}_2(\upsilon,\varpi)(z)\right\| \\ &\le \frac{(\varphi(b)-\varphi(a))^{\nu}}{\Gamma(\nu+1)}M_{\mathbb{H}_1} + \frac{(\varphi(b)-\varphi(a))^{\mu}}{\Gamma(\mu+1)}M_{\mathbb{H}_2} \\ &= \mathcal{A}_1 M_{\mathbb{H}_1} + \mathcal{A}_2 M_{\mathbb{H}_2} = L_{\mathcal{Q}} < \infty.\end{aligned}$$

Next, to confirm the equicontinuity of $\mathcal{Q}$, let $(\upsilon,\varpi)\in X$ be an arbitrary point and let $r,q\in\mathfrak{J}$ subject to $r<q$. Then we have

$$\begin{aligned}&\left|\mathcal{Q}_1(\upsilon,\varpi)(q)-\mathcal{Q}_1(\upsilon,\varpi)(r)\right| \\ &\quad\le \left|\mathbb{I}_{a^+}^{\nu,\varphi}\mathbb{H}_1\big(s,\upsilon(z),\varpi(z)\big)|_{z=q} - \mathbb{I}_{a^+}^{\nu,\varphi}\mathbb{H}_1\big(s,\upsilon(z),\varpi(z)\big)|_{z=r}\right| \\ &\quad\le \frac{1}{\Gamma(\nu)}\int_a^r \varphi'(s)\Big[\big(\varphi(q)-\varphi(s)\big)^{\nu-1}-\big(\varphi(r)-\varphi(s)\big)^{\nu-1}\Big]\left|\mathbb{H}_1\big(s,\upsilon(s),\varpi(s)\big)\right|ds \\ &\qquad + \frac{1}{\Gamma(\nu)}\int_r^q \varphi'(s)\big(\varphi(q)-\varphi(s)\big)^{\nu-1}\left|\mathbb{H}_1\big(s,\upsilon(s),\varpi(s)\big)\right|ds \\ &\quad\to 0 \quad \text{as } r\to q.\end{aligned}$$

This implies

$$\left\|\mathcal{Q}_1(\upsilon,\varpi)(q)-\mathcal{Q}_1(\upsilon,\varpi)(r)\right\|\to 0 \quad \text{as } r\to q$$

uniformly for all $(\upsilon,\varpi)\in X$. Similarly,

$$\left\|\mathcal{Q}_2(\upsilon,\varpi)(q)-\mathcal{Q}_2(\upsilon,\varpi)(r)\right\|\to 0 \quad \text{as } r\to q$$

uniformly for all $(\upsilon,\varpi)\in X$. Hence, it follows that

$$\left\|\mathcal{Q}(\upsilon,\varpi)(q)-\mathcal{Q}(\upsilon,\varpi)(r)\right\|\to 0 \quad \text{as } r\to q$$

uniformly for all $(\upsilon,\varpi)\in X$. Now, it is understood that $\mathcal{Q}$ has the equicontinuity feature on the Banach space $\mathbb{E}$. In consequence, $\mathcal{Q}$ will be relatively compact and thus the conclusion of a result due to Arzelá–Ascoli shows that $\mathcal{Q}$ is completely continuous and in the final step, $\mathcal{Q}$ is compact on X.

Step III: We now proceed to demonstrate the third condition (s3) of Theorem 2.8. Let (u,v) be an element in X such that

$$(\upsilon,\varpi)=\big(\mathcal{P}_1(\upsilon,\varpi)\mathcal{Q}_1(u,v)+\mathcal{S}_1(\upsilon,\varpi),\mathcal{P}_2(\upsilon,\varpi)\mathcal{Q}_2(u,v)+\mathcal{S}_2(\upsilon,\varpi)\big).$$

Then we have

$$\left|\upsilon(z)\right| = \left|\mathcal{P}_1(\upsilon,\varpi)(z)\mathcal{Q}_1(u,v)(z)+\mathcal{S}_1(\upsilon,\varpi)\right|$$

$$\begin{aligned}
&\le \left|\mathcal{P}_1(\upsilon,\varpi)(z)\right|\left|\mathcal{Q}_1(u,v)(z)\right| + \left|\mathcal{S}_1(\upsilon,\varpi)\right| \\
&\le \left[\left|\mathbb{K}_1(z,\upsilon,\varpi) - \mathbb{K}_1(z,0,0)\right| + \left|\mathbb{K}_1(z,0,0)\right|\right] \\
&\quad \times \left(\frac{1}{\Gamma(\nu)}\int_a^z \varphi'(s)\bigl(\varphi(z)-\varphi(s)\bigr)^{\nu-1}\left|\mathbb{H}_1\bigl(s,u(s),v(s)\bigr)\right| ds\right) \\
&\quad + \sum_{K=1}^{m} I^{\sigma_k;\varphi}\left[\left|\mathbb{F}_k(z,\upsilon,\varpi) - \mathbb{F}_k(z,0,0)\right| + \left|\mathbb{F}_k(z,0,0)\right|\right] \\
&\le \left[L_{\mathbb{K}_1}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{K}_{1,0}\right] \times (\mathcal{A}_1 M_{\mathbb{H}_1}) \\
&\quad + \mathcal{B}_1\left[L_{\mathbb{F}_k}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{F}_0\right].
\end{aligned} \tag{25}$$

Taking the supremum in the above inequality (25), we obtain

$$\|\upsilon\| \le \left[L_{\mathbb{K}_1}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{K}_{1,0}\right] \times (\mathcal{A}_1 M_{\mathbb{H}_1}) + \mathcal{B}_1\left[\|L_{\mathbb{F}_k}\|\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{F}_0\right]. \tag{26}$$

Similarly, proceeding with the analogous arguments, we obtain

$$\|\varpi\| \le \left[L_{\mathbb{K}_2}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{K}_{2,0}\right] \times (\mathcal{A}_2 M_{\mathbb{H}_2}) + \mathcal{B}_2\left[\|L_{\mathbb{G}_j}\|\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{G}_0\right]. \tag{27}$$

Adding the inequalities (26) and (27), we obtain

$$\begin{aligned}
&\|\upsilon\| + \|\varpi\| \\
&\quad \le \left[L_{\mathbb{K}_1}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{K}_{1,0}\right] \times (\mathcal{A}_1 M_{\mathbb{H}_1}) + \mathcal{B}_1\left[\|L_{\mathbb{F}_k}\|\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{F}_0\right] \\
&\qquad + \left[L_{\mathbb{K}_2}\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{K}_{2,0}\right] \times (\mathcal{A}_2 M_{\mathbb{H}_2}) + \mathcal{B}_2\left[\|L_{\mathbb{G}_j}\|\bigl(\|\upsilon\| + \|\varpi\|\bigr) + \mathbb{G}_0\right].
\end{aligned}$$

Thus

$$\|\upsilon\| + \|\varpi\| \le \frac{\mathbb{K}_{1,0}(\mathcal{A}_1 M_{\mathbb{H}_1}) + \mathcal{B}_1\mathbb{F}_0 + \mathbb{K}_{1,0}(\mathcal{A}_2 M_{\mathbb{H}_2}) + \mathcal{B}_2\mathbb{G}_0}{1 - [L_{\mathbb{K}_1}\mathcal{A}_1 M_{\mathbb{H}_1}] - [\mathcal{B}_1\|L_{\mathbb{F}_k}\|] - [L_{\mathbb{K}_2}\mathcal{A}_2 M_{\mathbb{H}_2}] - [\mathcal{B}_2\|L_{\mathbb{G}_j}\|]} \le \rho. \tag{28}$$

As $\|(\upsilon,\varpi)\| = \|\upsilon\| + \|\varpi\|$, we have $\|(\upsilon,\varpi)\| \le \rho$. Thus $(\upsilon,\varpi) \in X$ and so the assertion (s3) of Theorem 2.8 follows.

Step IV: At last, we have

$$\begin{aligned}
M_{\mathcal{Q}} = \left\|\mathcal{Q}(X)\right\| &= \sup\left\{\left\|\mathcal{Q}(\upsilon,\varpi)\right\| : (\upsilon,\varpi) \in X\right\} \\
&= \sup\left\{\left\|\mathcal{Q}_1(\upsilon,\varpi)\right\| + \left\|\mathcal{Q}_2(\upsilon,\varpi)\right\| : (\upsilon,\varpi) \in X\right\} \\
&\le \mathcal{A}_1 M_{\mathbb{H}_1} + \mathcal{A}_2 M_{\mathbb{H}_2}.
\end{aligned}$$

From the above estimate and by (18), we obtain

$$L_{\mathcal{P}} M_{\mathcal{Q}} + L_{\mathcal{S}} \le (L_{\mathbb{K}_1} + L_{\mathbb{K}_2})(\mathcal{A}_1 M_{\mathbb{H}_1} + \mathcal{A}_2 M_{\mathbb{H}_2}) + \mathcal{B}_1\|L_{\mathbb{F}_k}\| + \mathcal{B}_2\|L_{\mathbb{G}_j}\| < 1$$

and so the hypothesis (s4) of Theorem 2.8 is obeyed. Accordingly, the operators $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{S}$ obey all four assertions of Theorem 2.8 and thus the equation $\mathcal{P}(\upsilon,\varpi)\mathcal{Q}(\upsilon,\varpi) + \mathcal{S}(\upsilon,\varpi) = (\upsilon,\varpi)$ possesses a solution in X. Consequently, the generalized coupled hybrid system of integro-differential IVPs (3)–(4) involves a mild coupled solution formulated on $\mathfrak{J}$. This finishes the argument. □

3.1 Uniqueness via the Banach contraction principle

This section is devoted to demonstrating the uniqueness subject for the proposed coupled system of φ-Caputo integro-differential IVPs (3)–(4) by making use of Theorem 2.9.

Lemma 3.5 *If the functions $\mathbb{F}_k, \mathbb{G}_j : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$, $\mathbb{K}_1, \mathbb{K}_2 : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}\backslash\{0\}$ and $\mathbb{H}_1, \mathbb{H}_2 : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, then the coupled system of φ-Caputo integro-differential IVPs (3)–(4) is equivalent to the nonlinear fractional integral equations which take the form*

$$\upsilon(z) = \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k\big(z, \upsilon(z), \varpi(z)\big) + \big[\mathbb{K}_1\big(z, \upsilon(z), \varpi(z)\big)\big] \times \big[\mathbb{I}_{a^+}^{\nu;\varphi} \mathbb{H}_1\big(z, \upsilon(z), \varpi(z)\big)\big] \tag{29}$$

and

$$\varpi(z) = \sum_{j=1}^{n} \mathbb{I}_{a^+}^{\xi_j;\varphi} \mathbb{G}_k\big(z, \upsilon(z), \varpi(z)\big) + \big[\mathbb{K}_2\big(z, \upsilon(z), \varpi(z)\big)\big] \times \big[\mathbb{I}_{a^+}^{\mu;\varphi} \mathbb{H}_2\big(z, \upsilon(z), \varpi(z)\big)\big] \tag{30}$$

for all $z \in \mathfrak{J}$.

Theorem 3.6 *Assume that the continuous functions $\mathbb{F}_k, \mathbb{G}_j : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$, $\mathbb{K}_1, \mathbb{K}_2 : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}\backslash\{0\}$ and $\mathbb{H}_1, \mathbb{H}_2 : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the assumptions* ($\mathbb{HYP}$0)–($\mathbb{HYP}$3). *Then the system of the coupled integro-differential IVPs* (3)–(4) *possesses one and only one solution if*

$$\sum_{i=1}^{2} \Lambda_i < 1 \tag{31}$$

subject to the conditions

$$\begin{aligned}
\Lambda_1 &= \big[\mathcal{A}_1(L_{\mathbb{K}_1} M_{\mathbb{K}_1} + L_{\mathbb{H}_1} M_{\mathbb{H}_1}) + \mathcal{B}_1 \|L_{\mathbb{F}_k}\|\big], \\
\Lambda_2 &= \big[\mathcal{A}_2(L_{\mathbb{K}_2} M_{\mathbb{K}_2} + L_{\mathbb{H}_2} M_{\mathbb{H}_2}) + \mathcal{B}_2 \|L_{\mathbb{G}_j}\|\big].
\end{aligned} \tag{32}$$

Proof According to Lemma 3.5, we consider the operators $\mathcal{G}_1 : \mathbb{E} \to \mathbb{E}$ and $\mathcal{G}_2 : \mathbb{E} \to \mathbb{E}$ defined by

$$\begin{aligned}
\mathcal{G}_1\big(\upsilon(z), \varpi(z)\big) = {}& \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k\big(z, \upsilon(z), \varpi(z)\big) \\
& + \big[\mathbb{K}_1\big(z, \upsilon(z), \varpi(z)\big)\big] \times \big[\mathbb{I}_{a^+}^{\nu;\varphi} \mathbb{H}_1\big(z, \upsilon(z), \varpi(z)\big)\big]
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\mathcal{G}_2\big(\upsilon(z), \varpi(z)\big) = {}& \sum_{j=1}^{n} \mathbb{I}_{a^+}^{\xi_j;\varphi} \mathbb{G}_k\big(z, \upsilon(z), \varpi(z)\big) \\
& + \big[\mathbb{K}_2\big(z, \upsilon(z), \varpi(z)\big)\big] \times \big[\mathbb{I}_{a^+}^{\mu;\varphi} \mathbb{H}_2\big(z, \upsilon(z), \varpi(z)\big)\big].
\end{aligned} \tag{34}$$

Therefore, we construct $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ as

$$\mathcal{G}(\upsilon, \varpi)(z) = \mathcal{G}_1(\upsilon, \varpi)(z) + \mathcal{G}_2(\upsilon, \varpi)(z).$$

Let $(\upsilon,\varpi),(\overline{\upsilon},\overline{\varpi})\in\mathbb{E}$. Applying ($\mathbb{HYP}$0)–($\mathbb{HYP}$3), we have

$$\begin{aligned}&\left|\mathcal{G}_1(\upsilon,\varpi)(z)-\mathcal{G}_1(\overline{\upsilon},\overline{\varpi})(z)\right|\\&\quad\le\sum_{k=1}^{m}\frac{(\varphi(b)-\varphi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|L_{\mathbb{F}_k}\|\big(|\upsilon-\overline{\upsilon}|+|\varpi-\overline{\varpi}|\big)\\&\qquad+\frac{(\varphi(b)-\varphi(a))^{\nu}}{\Gamma(\nu+1)}(L_{\mathbb{K}_1}M_{\mathbb{K}_1}+L_{\mathbb{H}_1}M_{\mathbb{H}_1})\big(|\upsilon-\overline{\upsilon}|+|\varpi-\overline{\varpi}|\big)\\&\quad\le\big[\mathcal{A}_1(L_{\mathbb{K}_1}M_{\mathbb{K}_1}+L_{\mathbb{H}_1}M_{\mathbb{H}_1})+\mathcal{B}_1\|L_{\mathbb{F}_k}\|\big]\big(|\upsilon-\overline{\upsilon}|+|\varpi-\overline{\varpi}|\big),\end{aligned}$$

which implies

$$\left\|\mathcal{G}_1(\upsilon,\varpi)(z)-\mathcal{G}_1(\overline{\upsilon},\overline{\varpi})(z)\right\|\le\Lambda_1\big(\|\upsilon-\overline{\upsilon}\|+\|\varpi-\overline{\varpi}\|\big)=\Lambda_1\left\|(\upsilon,\varpi)-(\overline{\upsilon},\overline{\varpi})\right\| \tag{35}$$

subject to Λ_1 given in (32). By the same technique, we can also get

$$\left\|\mathcal{G}_2(\upsilon,\varpi)(z)-\mathcal{G}_2(\overline{\upsilon},\overline{\varpi})(z)\right\|\le\Lambda_2\big(\|\upsilon-\overline{\upsilon}\|+\|\varpi-\overline{\varpi}\|\big)=\Lambda_2\left\|(\upsilon,\varpi)-(\overline{\upsilon},\overline{\varpi})\right\| \tag{36}$$

subject to Λ_2 given in (32). In view of the condition $\sum_{i=1}^{2}\Lambda_i<1$ and

$$\left\|\mathcal{G}(\upsilon,\varpi)(z)-\mathcal{G}(\overline{\upsilon},\overline{\varpi})(z)\right\|\le(\Lambda_1+\Lambda_2)\left\|(\upsilon,\varpi)-(\overline{\upsilon},\overline{\varpi})\right\|, \tag{37}$$

we see that $\mathcal{G}$ is a contraction. In the light of Theorem 2.9, $\mathcal{G}$ possesses a fixed point uniquely which guarantees that the system of the coupled integro-differential IVPs (3)–(4) involves a solution uniquely. □

3.2 U–H stability and its generalized U–H version

In the current subsection, we are interested in studying U–H and the generalized U–H stability types of the proposed system of the coupled integro-differential IVPs (3)–(4).

Definition 3.7 The system of the coupled integro-differential IVPs (3)–(4) is stable with U–H criterion if a real number $c=\max(c_1,c_2)>0$ exists so that, for any $\epsilon=\max(\epsilon_1,\epsilon_2)>0$ and for any $(\overline{\upsilon},\overline{\varpi})\in\mathbb{E}$ satisfying

$$\begin{cases}|{}^{c}\mathbb{D}_{a^+}^{\nu;\varphi}[\frac{\overline{\upsilon}(z)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\varphi}\mathbb{F}_k(z,\overline{\upsilon}(z),\overline{\varpi}(z))}{\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))}]-\mathbb{H}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))|\le\epsilon_1, & z\in\mathfrak{J},\\ |{}^{c}\mathbb{D}_{a^+}^{\mu;\varphi}[\frac{\overline{\varpi}(z)-\sum_{j=1}^{n}\mathbb{I}_{a^+}^{\xi_j;\varphi}\mathbb{G}_j(z,\overline{\upsilon}(z),\overline{\varpi}(z))}{\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))}]-\mathbb{H}_2(z,\overline{\upsilon}(z),\overline{\varpi}_z)|\le\epsilon_2, & z\in\mathfrak{J},\end{cases} \tag{38}$$

there exists a unique solution $(\upsilon,\varpi)\in\mathfrak{C}\times\mathfrak{C}$ of (3)–(4) with

$$\left\|(\overline{\upsilon},\overline{\varpi})-(\upsilon,\varpi)\right\|\le c\epsilon.$$

Definition 3.8 The system of the coupled integro-differential IVPs (3)–(4) is named the generalized stable with U–H criterion if there exists $\sigma=\max(\sigma_1,\sigma_2)\in C(\mathbb{R}^{>0},\mathbb{R}^{>0})$ along with $\sigma(0)=0$ subject to for any $\epsilon=\max(\epsilon_1,\epsilon_2)>0$ and for any $(\overline{\upsilon},\overline{\varpi})\in\mathbb{E}$ satisfying (38), a solution $(\upsilon,\varpi)\in\mathbb{E}$ of (3)–(4) exists uniquely for which

$$\left\|(\overline{\upsilon},\overline{\varpi})-(\upsilon,\varpi)\right\|\le\sigma(\epsilon).$$

Remark 3.9 $(\overline{\upsilon},\overline{\varpi})\in\mathbb{E}$ satisfies the system (38) if there exists a function $(\mathfrak{g}_1,\mathfrak{g}_2)\in\mathbb{E}$ (which depends on $(\overline{\upsilon},\overline{\varpi})$) such that

(i) $|\mathfrak{g}_1(z)|\leq\epsilon_1$ and $|\mathfrak{g}_2(z)|\leq\epsilon_2$ for $z\in\mathfrak{J}$;

(ii) for $z\in\mathfrak{J}$,

$$\begin{cases} {}^{c}\mathbb{D}^{\nu;\varphi}_{a^+}[\frac{\overline{\upsilon}(z)-\sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\varphi}_{a^+}\mathbb{F}_k(z,\overline{\upsilon}(z),\overline{\varpi}(z))}{\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))}]=\mathbb{H}_1(z,\overline{\upsilon}(z),\overline{\varpi}_z)+\mathfrak{g}_1(z),\\ {}^{c}\mathbb{D}^{\mu;\varphi}_{a^+}[\frac{\overline{\varpi}(z)-\sum_{j=1}^{n}\mathbb{I}^{\xi_j;\varphi}_{a^+}\mathbb{G}_j(z,\overline{\upsilon}(z),\overline{\varpi}(z))}{\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))}]=\mathbb{H}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))+\mathfrak{g}_1(z).\end{cases}$$

Theorem 3.10 *Suppose that* $(\mathbb{HYP}2)$ *and* (38) *are fulfilled. Then the system of the coupled integro-differential IVPs* (3)–(4) *is U–H and generalized U–H stable provided that* $(1-\Lambda_1)(1-\Lambda_2)-\Lambda_2\Lambda_1\neq 0$ *where* Λ_1 *and* Λ_2 *are illustrated in* (32).

Proof For $\epsilon_1,\epsilon_2>0$, let $(\overline{\upsilon},\overline{\varpi})\in\mathbb{E}$ be any solution of (38). By Remark 3.9 and Lemma 3.1, we have

$$\begin{cases} \overline{\upsilon}(z)=\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\nu;\varphi}_{a^+}\mathbb{H}_1(z,\overline{\upsilon}(z),\overline{\varpi}_z)+\sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\varphi}_{a^+}\mathbb{F}_k(z,\overline{\upsilon}(z),\overline{\varpi}(z))\\ \qquad +\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\nu;\varphi}_{a^+}\mathfrak{g}_1(z),\\ \overline{\varpi}(z)=\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\mu;\varphi}_{a^+}\mathbb{H}_2(z,\overline{\upsilon}(z),\overline{\varpi}_z)+\sum_{j=1}^{n}\mathbb{I}^{\xi_j;\varphi}_{a^+}\mathbb{G}_j(z,\overline{\upsilon}(z),\overline{\varpi}(z))\\ \qquad +\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\mu;\varphi}_{a^+}\mathfrak{g}_2(z),\end{cases}\tag{39}$$

for $z\in\mathfrak{J}$ and

$$\overline{\upsilon}(a)=0,\qquad \overline{\varpi}(a)=0.$$

From (39) and for $z\in\mathfrak{J}$, we obtain

$$\begin{cases} |\overline{\upsilon}(z)-\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\nu;\varphi}_{a^+}\mathbb{H}_1(z,\overline{\upsilon}(z),\overline{\varpi}_z)-\sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\varphi}_{a^+}\mathbb{F}_k(z,\overline{\upsilon}(z),\overline{\varpi}(z))|\\ \quad\leq|\mathbb{K}_1(z,\overline{\upsilon}(z),\overline{\varpi}(z))|\mathbb{I}^{\nu;\varphi}_{a^+}|\mathfrak{g}_1(z)|\leq\frac{(\varphi(z)-\varphi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{K}_1}\epsilon_1,\\ |\overline{\varpi}(z)-\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))\mathbb{I}^{\nu;\varphi}_{a^+}\mathbb{H}_2(z,\overline{\upsilon}(z),\overline{\varpi}_z)-\sum_{j=1}^{n}\mathbb{I}^{\xi_j;\varphi}_{a^+}\mathbb{G}_j(z,\overline{\upsilon}(z),\overline{\varpi}(z))|\\ \quad\leq|\mathbb{K}_2(z,\overline{\upsilon}(z),\overline{\varpi}(z))|\mathbb{I}^{\nu;\varphi}_{a^+}|\mathfrak{g}_2(z)|\leq\frac{(\varphi(z)-\varphi(a))^{\mu}}{\Gamma(\mu+1)}L_{\mathbb{K}_2}\epsilon_2,\end{cases}\tag{40}$$

and

$$\left|\overline{\upsilon}(a)\right|\leq 0,\qquad \left|\overline{\varpi}(a)\right|\leq 0.$$

Let $(\upsilon,\varpi)\in\mathbb{E}$ be the solution of the system

$$\begin{cases} {}^{c}\mathbb{D}^{\nu;\varphi}_{a^+}[\frac{\overline{\upsilon}(z)-\sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\varphi}_{a^+}\mathbb{F}_k(z,\upsilon(z),\varpi(z))}{\mathbb{K}_1(z,\upsilon(z),\varpi(z))}]=\mathbb{H}_1(z,\upsilon(z),\varpi(z)),\\ {}^{c}\mathbb{D}^{\mu;\varphi}_{a^+}[\frac{\overline{\varpi}(z)-\sum_{j=1}^{n}\mathbb{I}^{\xi_j;\varphi}_{a^+}\mathbb{G}_j(z,\upsilon(z),\varpi(z))}{\mathbb{K}_2(z,\upsilon(z),\varpi(z))}]=\mathbb{H}_2(z,\upsilon(z),\varpi(z)),\end{cases}\quad z\in\mathfrak{J},\tag{41}$$

with

$$\begin{cases} \upsilon(a)=\overline{\upsilon}(a)=0,\\ \varpi(a)=\overline{\varpi}(a)=0.\end{cases}\tag{42}$$

Thanks to Lemma 3.1, the equivalent fractional integral system to problem (41)–(42) is

$$\upsilon(z) = \mathbb{K}_1\big(z, \upsilon_z, \varpi(z)\big)\mathbb{I}_{a^+}^{\nu,\varphi}\mathbb{H}_1\big(z, \upsilon_z, \varpi(z)\big) + \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k,\varphi}\mathbb{F}_k\big(z, \upsilon(z), \varpi(z)\big), \quad \text{if } z \in \mathfrak{J}, \tag{43}$$

$$\varpi(z) = \mathbb{K}_2\big(z, \upsilon_z, \varpi(z)\big)\mathbb{I}_{a^+}^{\mu,\varphi}\mathbb{H}_2\big(z, \upsilon(z), \varpi(z)\big) + \sum_{j=1}^{n}\mathbb{I}_{a^+}^{\xi_j,\varphi}\mathbb{G}_j\big(z, \upsilon(z), \varpi(z)\big) \quad \text{if } z \in \mathfrak{J}. \tag{44}$$

Since $\upsilon(a) = \overline{\upsilon}(a)$ and $\varpi(a) = \overline{\varpi}(a)$ we have $|\overline{\upsilon}(a) - \upsilon(a)| = 0$ and $|\overline{\varpi}(a) - \varpi(a)| = 0$. On the other hand, for any $z \in \mathfrak{J}$ and by the same arguments in Theorem 3.6 with (40) and (43), we get

$$\begin{aligned}
&\big|\overline{\upsilon}(z) - \upsilon(z)\big| \\
&\quad = \left|\overline{\upsilon}(z) - \mathbb{K}_1\big(z, \upsilon_z, \varpi(z)\big) \times \mathbb{I}_{a^+}^{\nu,\varphi}\mathbb{H}_1\big(z, \upsilon(z), \varpi(z)\big) - \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k,\varphi}\mathbb{F}_k\big(z, \upsilon(z), \varpi(z)\big)\right| \\
&\quad \le \frac{(\varphi(z) - \varphi(a))^{\nu}}{\Gamma(\nu + 1)}L_{\mathbb{K}_1}\epsilon_1 + \Lambda_1\big(\|\overline{\upsilon} - \upsilon\|_{\mathfrak{C}} + \|\overline{\varpi} - \varpi\|\big),
\end{aligned}$$

which implies

$$(1 - \Lambda_1)\|\overline{\upsilon} - \upsilon\| - \Lambda_1\|\overline{\varpi} - \varpi\| \le \Omega_1\epsilon_1, \tag{45}$$

where $\Omega_1 := \frac{(\varphi(b)-\varphi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{K}_1}$. Similarly, we have

$$(1 - \Lambda_2)\|\overline{\varpi} - \varpi\| - \Lambda_2\|\overline{\upsilon} - \upsilon\| \le \Omega_2\epsilon_2, \tag{46}$$

where $\Omega_2 := \frac{(\varphi(b)-\varphi(a))^{\mu}}{\Gamma(\mu+1)}L_{\mathbb{K}_2}$.

Representing (45) and (46) as matrices, we get

$$\begin{pmatrix} 1 - \Lambda_1 & -\Lambda_1 \\ -\Lambda_2 & 1 - \Lambda_2 \end{pmatrix}\begin{pmatrix} \|\overline{\upsilon} - \upsilon\| \\ \|\overline{\varpi} - \varpi\| \end{pmatrix} \le \begin{pmatrix} \Omega_1\epsilon_1 \\ \Omega_2\epsilon_2 \end{pmatrix}. \tag{47}$$

After straightforward calculations of (47), we find that

$$\|\overline{\upsilon} - \upsilon\| \le \frac{1 - \Lambda_1}{\Delta}\Omega_1\epsilon_1 + \frac{\Lambda_1}{\Delta}\Omega_2\epsilon_2, \tag{48}$$

$$\|\overline{\varpi} - \varpi\| \le \frac{\Lambda_2}{\Delta}\Omega_1\epsilon_1 + \frac{1 - \overline{\Lambda}_2}{\Delta}\Omega_2\epsilon_2, \tag{49}$$

where $\Delta = (1 - \Lambda_1)(1 - \Lambda_2) - \Lambda_2\Lambda_1 \neq 0$. By collecting (48) and (49), we obtain

$$\|\overline{\upsilon} - \upsilon\| + \|\overline{\varpi} - \varpi\| \le \left(\frac{1 - \Lambda_1}{\Delta} + \frac{\Lambda_2}{\Delta}\right)\Omega_1\epsilon_1 + \left(\frac{\Lambda_1}{\Delta} + \frac{1 - \Lambda_2}{\Delta}\right)\Omega_2\epsilon_2.$$

For $\epsilon = \max(\epsilon_1, \epsilon_2)$ and $c = (\frac{(1-\Lambda_1+\Lambda_2)\Omega_1+(\Lambda_1+1-\Lambda_2)\Omega_2}{\Delta})$, we get

$$\big\|(\overline{\upsilon}, \overline{\varpi}) - (\upsilon, \varpi)\big\| = \|\overline{\upsilon} - \upsilon\| + \|\overline{\varpi} - \varpi\| \le c\epsilon.$$

Therefore, by means of Definition 3.7, the solution of the problem (3)–(4) is U–H stable. Similarly, it gives the existence of a function $\sigma \in C(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ so that $\sigma(\epsilon) = c\epsilon$ along with $\sigma(0) = 0$. Accordingly, the solution of the system of the coupled integro-differential IVPs (3)–(4) is generalized U–H stable. □

4 Example

In this section, we present an illustrative coupled system of the given coupled hybrid integro-differential IVPs (3)–(4) to ensure the correctness of results obtained above.

Example 4.1 We formulate the coupled system of hybrid integro-differential IVPs which take the form

$$\begin{cases} \mathbb{D}_{0^+}^{\nu;\varphi}\left[\frac{\lambda(z)-\mathbb{I}_{0^+}^{\sigma_1;\varphi}\left(\frac{z^2}{10}\left(\frac{1}{2}(|\lambda|+|\hat{\lambda}|)+e^{-z}\right)\right)}{\frac{\sqrt{\pi}\cos(\pi z)}{(7\pi+15e^z)^2}\frac{|\lambda|+|\hat{\lambda}|}{1+z}+\frac{z}{10}}\right] = \frac{\sqrt{3}\cos^2(2\pi z)}{3(27-z)}(|\lambda| + |\hat{\lambda}|), \\ \mathbb{D}_{0^+}^{\mu;\varphi}\left[\frac{\hat{\lambda}(z)-\mathbb{I}_{0^+}^{\xi_1;\varphi}\left(\frac{1}{100}e^z+\frac{2+|\lambda|+|\hat{\lambda}|}{8e^{2+z}(1+|\lambda|+|\hat{\lambda}|)}\right)}{\frac{1}{10}\left(\frac{|\lambda|+|\hat{\lambda}|}{1+z^2}+z^2\right)}\right] = \frac{\sqrt{2}\pi}{4(4\pi-z)^2}(|\lambda| + |\hat{\lambda}|), \\ \lambda(a) = 0, \qquad \hat{\lambda}(a) = 0. \end{cases} \tag{50}$$

Let us consider the hybrid system (50) with specific data:

$$\nu = \frac{1}{2}, \qquad \mu = \frac{1}{4}, \qquad \sigma_1 = \frac{2}{3}, \qquad \xi_1 = \frac{3}{5}, \qquad a = 0,$$
$$b = 1, \qquad m = n = 1, \qquad \varphi(z) = z, \qquad \mathfrak{J} = [0, 1].$$

Using the given data, we find that

$$\mathbb{F}_0 = \frac{1}{10}, \qquad \mathbb{G}_0 = \frac{1}{100}, \qquad \mathbb{K}_{1,0} = \frac{1}{10}, \qquad \mathbb{K}_{2,0} = \frac{1}{10},$$

and

$$\begin{aligned} \left|\mathbb{F}_1(z,\lambda,\hat{\lambda}) - \mathbb{F}_1(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{1}{10}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right), \\ \left|\mathbb{G}_1(z,\lambda,\hat{\lambda}) - \mathbb{G}_1(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{1}{8e^2}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right), \\ \left|\mathbb{K}_1(z,\lambda,\hat{\lambda}) - \mathbb{K}_1(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{\sqrt{\pi}}{(7\pi+15)^2}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right), \\ \left|\mathbb{K}_2(z,\lambda,\hat{\lambda}) - \mathbb{K}_2(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{1}{10}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right), \\ \left|\mathbb{H}_1(z,\lambda,\hat{\lambda}) - \mathbb{H}_1(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{\sqrt{3}\cos^2(2\pi z)}{3(27-z)}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right), \\ \left|\mathbb{H}_2(z,\lambda,\hat{\lambda}) - \mathbb{H}_2(z,\bar{\lambda},\bar{\hat{\lambda}})\right| &\le \frac{\sqrt{2}\pi}{4(4\pi-z)^2}\left(\|\lambda-\bar{\lambda}\| + \|\hat{\lambda}-\bar{\hat{\lambda}}\|\right). \end{aligned}$$

Hence, the hypotheses ($\mathbb{HYP}$1)–($\mathbb{HYP}$3) are satisfied with $L_{\mathbb{F}_1} = \frac{1}{10}$, $L_{\mathbb{G}_1} = \frac{1}{8e^2}$, $L_{\mathbb{K}_1} = \frac{\sqrt{\pi}}{(7\pi+15)^2}$, $L_{\mathbb{K}_2} = \frac{1}{10}$, $\|L_{\mathbb{H}_1}\| = \frac{\sqrt{3}}{81}$ and $\|L_{\mathbb{H}_2}\| = \frac{\sqrt{2}}{64\pi}$. We demonstrate that the assertion (31) holds with $z \in \mathfrak{J} = [0,1]$ in which

$$\Lambda_1 = 0.1108, \qquad \Lambda_2 = 0.0562, \qquad \sum_{i=1}^{2} \Lambda_i = 0.167 < 1$$

and also

$$\Omega_1 = 0.0015, \qquad \Omega_2 = 0.11033.$$

Since the assertions of Theorem 3.6 are verified, a solution exists uniquely for the coupled hybrid system of integro-differential IVPs (50) on $[0,1]$. Moreover, Theorem 3.10 ensures the U–H and generalized U–H stability for problem (50). As shown in Theorem 3.10, for every $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$, if $(\overline{\upsilon}, \overline{\varpi}) \in \mathbb{R}$ satisfies

$$\begin{cases} |{}^{c}\mathbb{D}_{a^+}^{\nu;\varphi} [\frac{\overline{\upsilon}(z) - \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\varphi} \mathbb{F}_k(z, \overline{\upsilon}(z), \overline{\varpi}(z))}{\mathbb{K}_1(z, \overline{\upsilon}(z), \overline{\varpi}(z))}] - \mathbb{H}_1(z, \overline{\upsilon}(z), \overline{\varpi}(z))| \leq \epsilon_1, & z \in \mathfrak{J}, \\ |{}^{c}\mathbb{D}_{a^+}^{\mu;\varphi} [\frac{\overline{\varpi}(z) - \sum_{k=1}^{n} \mathbb{I}_{a^+}^{\xi_k;\varphi} \mathbb{G}_k(z, \overline{\upsilon}(z), \overline{\varpi}(z))}{\mathbb{K}_2(z, \overline{\upsilon}(z), \overline{\varpi}(z))}] - \mathbb{H}_2(z, \overline{\upsilon}(z), \overline{\varpi}(z))| \leq \epsilon_1, & z \in \mathfrak{J}, \end{cases} \tag{51}$$

then there exists a unique solution $(\upsilon, \varpi) \in \mathbb{R}$ such that

$$\left\| (\overline{\upsilon}, \overline{\varpi}) - (\upsilon, \varpi) \right\| \leq c\epsilon,$$

where $c = (\frac{(1-\Lambda_1+\Lambda_2)\Omega_1 + (\Lambda_1+1-\Lambda_2)\Omega_2}{\Delta}) \approx 0.1414 > 0$ and $\Delta = (1-\Lambda_1)(1-\Lambda_2) - \Lambda_2\Lambda_1 = 0.8330 \neq 0$. Hence it is confirmed that the coupled hybrid system of integro-differential IVPs (50) is U–H and generalized U–H stable.

5 Conclusion

In this research article, we investigate the existence and uniqueness of solutions to a coupled hybrid system of fractional integro-differential equations involving φ-Caputo fractional operators. To achieve the goals, we make use of a hybrid fixed point theorem for a sum of three operators due to Dhage and at the same time the uniqueness result is obtained by making use of the contraction principle. Moreover, we explore the Ulam–Hyers stability and its generalized version for the given coupled hybrid system. An example is presented to confirm the viability of our obtained results.

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