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Differential equations of even-order with p-Laplacian like operators: qualitative properties of the solutions

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Abstract
In this paper, we study the oscillation of solutions for an even-order differential equation with middle term, driven by a p-Laplace differential operator of the form

$$\begin{cases} (r(x)\Phi_p[z^{(\kappa-1)}(x)])' + a(x)\Phi_p[f(z^{(\kappa-1)}(x))] + \sum_{i=1}^{j} q_i(x)\Phi_p[h(z(\delta_i(x)))] = 0, \\ j \geq 1, r(x) > 0, r'(x) + a(x) \geq 0, x \geq x_0 > 0. \end{cases}$$

The oscillation criteria for these equations have been obtained. Furthermore, an example is given to illustrate the criteria.

Keywords: Even-order; Differential equation; Oscillation; p-Laplacian equation

1 Introduction

It is worth mentioning in this context that delay differential equations have many real-life applications in all branches of science and engineering; see [1, 2]. On the other hand, the p-Laplace equations have crucial applications in different areas such as in elasticity theory, see, for example, Aronsson–Janfalk [3], and in general nonlinear phenomena, see Vetro [4]. Therefore, the literature reveals results of various studies concerning the oscillatory behavior of equations driven by a p-Laplace differential operator; see, by way of example not exhaustive enumeration, Li–Baculikova–Dzurina–Zhang [5], Liu–Zhang–Yu [6], Zhang–Agarwal–Li [7]. Additionally, the oscillatory properties of differential equations are studied intensively by many scientists; see, for example, [8–22].

The aim of this work is to investigate the oscillatory behavior of the even-order delay differential equation (DDE) with damping of the form

$$\begin{cases} (r(x)\Phi_p[z^{(\kappa-1)}(x)])' + a(x)\Phi_p[f(z^{(\kappa-1)}(x))] + \sum_{i=1}^{j} q_i(x)\Phi_p[h(z(\delta_i(x)))] = 0, \\ j \geq 1, r(x) > 0, r'(x) + a(x) \geq 0, x \geq x_0 > 0, \end{cases} \tag{1}$$

under the following conditions:

(G1) $\Phi_p[s] = |s|^{p-2}s$;

(G2) $r, a, q_i \in C([x_0, \infty), [0, \infty))$, $q_i(x) > 0$, $i = 1, 2, \dots, j$ are such that

$$\int_{x_0}^{\infty} \left[\frac{1}{r(s)} \exp\left(- \int_{x_0}^{s} \frac{a(u)}{r(u)} \, du \right) \right]^{1/(p-1)} ds < \infty; \tag{2}$$

(G3) $\delta_i \in C([x_0, \infty), (0, \infty))$, $\delta_i(x) \leq x$ and $\lim_{x\to\infty} \delta_i(x) = \infty$, $i = 1, 2, \dots, j$;
(G4) $f, h \in C(\mathbb{R}, \mathbb{R})$, $f(x) \geq m|x|^{p-2}x > 0$, $h(x) \geq \ell |x|^{p-2}x > 0$ for $x \neq 0$, $m \geq 1$ and $\ell > 0$,
where the first term of equation (1) means the p-Laplace-type operator with $1 < p < \infty$.

To achieve our target, we implemented several relevant facts and auxiliary results from the existing literature [7, 23–26]. Notice that Liu–Zhang–Yu [6] provided some theoretical information on the oscillation of half-linear functional differential equations with damping, i.e.,

$$\begin{cases} (r(x)\Phi(z^{(n-1)}(x)))' + a(x)\Phi(z^{(n-1)}(x)) + q(x)\Phi(z(g(x))) = 0, \\ \quad \Phi = |s|^{p-2}s, x \geq x_0 > 0, \end{cases}$$

where n is even. The authors used the comparison method with second order equations. In Bazighifan–Poom [23] and Bazighifan–Abdeljawad [24], the comparison method with the first and second order equations is used to establish oscillation criteria for

$$\begin{cases} (r(x)|z^{(n-1)}(x)|^{p-2}z^{(n-1)}(x))' + \sum_{i=1}^{j} q_i(x) g(z(\delta_i(x))) = 0, \\ \quad j \geq 1, x \geq x_0 > 0, \end{cases}$$

where n is even and p is a real number greater than 1, in the case where $\delta_i(x) \geq \upsilon$, $\alpha \leq \beta$ (with $r \in C^1((0, \infty), \mathbb{R})$, $q_i \in C([0, \infty), \mathbb{R})$, $i = 1, 2, \dots, j$).

For the special case when $p = 1$, Elabbasy et al. [16] provided some information on the asymptotic behavior of (1). The authors used the comparison method with second order equations to achieve their targets. We must point out that Li et al. [5] had used the Riccati transformation, together with the integral averaging technique, to discuss the oscillation of the following equation:

$$\begin{cases} (r(x)|z'''(x)|^{p-2}z'''(x))' + q(x)|z(\delta_i(x))|^{p-2}z(\delta(x)) = 0, \\ \quad 1 < p < \infty, x \geq x_0 > 0. \end{cases}$$

In Park et al. [26], the Riccati technique is used to obtain oscillation criteria of

$$\begin{cases} (r(x)|z^{(n-1)}(x)|^{p-2}z^{(n-1)}(x))' + q(x) g(z(\delta(x))) = 0, \\ \quad 1 < p < \infty, x \geq x_0 > 0, \end{cases}$$

where n is even. Zhang et al. in [7] studied the equation

$$\begin{cases} L_z' + p(x)|(z^{(\kappa-1)}(x))|^{p-2}z^{(\kappa-1)}(x) + q(x)|(z(\delta(x)))|^{p-2}z(\delta(x)) = 0, \\ \quad 1 < p < \infty, x \geq x_0 > 0, \end{cases}$$

where

$$L_z = r(x)\left|\left(z^{(\kappa-1)}(x)\right)\right|^{p-2}z^{(\kappa-1)}(x).$$

As a matter of fact, the investigation of the half-linear/p-Laplace equation (1) has become an important area of research due to the fact that such equations arise in a variety of real-world problems such as in the study of non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, etc.; see the following papers for more details [27–33]. In this work, we will partially use the tools and findings of [7, 23–26] to obtain new oscillation conditions for (1). Theoretical results will be illustrated via an example.

2 Oscillation criteria

For further convenience, we denote:

$$\sigma(x_0,x) := \exp\biggl(\int_{x_0}^{x}\frac{a(u)}{r(u)}\,du\biggr),$$
$$\zeta(x) := \int_x^\infty \frac{ds}{(r(s)\sigma(x_0,s))^{1/(p-1)}},$$
$$\varpi(x) := \frac{\delta_i'(x)}{\delta_i(x)} - \frac{ma(x)}{r(x)},$$
$$\psi(x) := \frac{1}{\sigma^{1/(p-1)}(x_0,x)} - \frac{\zeta(x)a(x)r^{(2-p)/(p-1)}(x)}{(p-1)},$$
$$\psi^*(x) := \frac{a(x)}{r(x)} + \frac{(p-1)^p\delta_i(x)\psi^p(x)\sigma(x_0,x)}{\zeta(x)r^{1/(p-1)}(x)}.$$

Next, we recall some technical tools useful throughout the paper:

Lemma 2.1 ([34]) *Let $z \in C^\kappa([x_0,\infty),(0,\infty))$. If $\lim_{x\to\infty} z(x) \neq 0$ and*

$$z^{(\kappa-1)}(x)z^{(\kappa)}(x) \le 0,$$

then

$$z(x) \ge \frac{\lambda}{(\kappa-1)!}x^{\kappa-1}\bigl|z^{(\kappa-1)}(x)\bigr|, \quad \lambda \in (0,1).$$

Lemma 2.2 ([35]) *Let $C > 0$ and D be constants. Then*

$$Dz - Cz^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{D^{\alpha+1}}{C^\alpha}, \quad \alpha \ge 1.$$

Lemma 2.3 ([34]) *Let $z \in C^\kappa([x_0,\infty),(0,\infty))$. If $z^{(\kappa-1)}(x)z^{(\kappa)}(x) \le 0$, then for every $\theta \in (0,1)$ and $\kappa > 0$ one has*

$$z(\theta x) \ge \kappa x^{\kappa-1}z^{(\kappa-1)}(x).$$

Lemma 2.4 ([36]) *Let $z \in C^{n-1}([x_z,\infty),\mathbb{R})$ be an (eventually) positive solution of (1). Then, we distinguish the following situations*:

$$(I_1)\quad z(x) > 0, \qquad z'(x) > 0, \qquad z^{(\kappa-1)}(x) > 0, \qquad z^{(\kappa)}(x) < 0;$$
$$(I_2)\quad z(x) > 0, \qquad z^{(\kappa-2)}(x) > 0, \qquad z^{(\kappa-1)}(x) < 0.$$

Lemma 2.5 *Let (I_1) hold and $z(x) > 0$. If*

$$\varsigma(x) := \delta_i(x)\frac{r(x)(z^{(\kappa-1)})^{p-1}(x)}{z^{p-1}(x/2)}, \quad \varsigma(x) > 0, \tag{3}$$

where $\delta_i \in C^1([x_0,\infty))$, then there exists a constant $\kappa > 0$ such that

$$\varsigma'(x) \le -\ell\delta_i(x)\sum_{i=1}^{j} q_i(x) + \varpi_+(x)\varsigma(x) - \frac{(p-1)\kappa x^{\kappa-2}}{2(r(x)\delta_i(x))^{1/(p-1)}}\varsigma^{\frac{p}{(p-1)}}(x). \tag{4}$$

Proof Let (I_1) hold and $z(x) > 0$. Using Lemma 2.3, we obtain

$$z'(x/2) \ge \kappa x^{\kappa-2} z^{(\kappa-1)}(x). \tag{5}$$

From (3), we get

$$\begin{aligned}\varsigma'(x) &= \delta_i'(x)\frac{r(x)(z^{(\kappa-1)})^{p-1}(x)}{z^{p-1}(x/2)} + \delta_i(x)\frac{(r(z^{(\kappa-1)})^{p-1})'(x)}{z^{p-1}(x/2)} \\ &\quad - (p-1)\delta_i(x)\frac{z'(x/2)r(x)(z^{(\kappa-1)})^{p-1}(x)}{2z^{p}(x/2)}.\end{aligned}$$

From (3) and (5), we find

$$\begin{aligned}\varsigma'(x) &\le \frac{\delta_i'(x)}{\delta_i(x)}\varsigma(x) + \delta_i(x)\frac{(r(z^{(\kappa-1)})^{p-1})'(x)}{z^{p-1}(x/2)} \\ &\quad - (p-1)\kappa x^{\kappa-2}\delta_i(x)\frac{r(x)(z^{(\kappa-1)})^{p}(x)}{2z^{p}(x/2)}.\end{aligned} \tag{6}$$

From (1), we get

$$\begin{aligned}\bigl(r(x)\Phi_p\bigl[z^{(\kappa-1)}(x)\bigr]\bigr)' &= -a(x)\Phi_p\bigl[f\bigl(z^{(\kappa-1)}(x)\bigr)\bigr] - \sum_{i=1}^{j} q_i(x)\Phi_p\bigl[h\bigl(z\bigl(\delta_i(x)\bigr)\bigr)\bigr] \\ &= -ma(x)\bigl|z^{(\kappa-1)}(x)\bigr|^{p-2}z^{(\kappa-1)}(x) \\ &\quad - \ell\sum_{i=1}^{j} q_i(x)\bigl|z^{(\kappa-1)}\bigl(\delta_i(x)\bigr)\bigr|^{p-2}z^{(\kappa-1)}\bigl(\delta_i(x)\bigr) \\ &= -ma(x)\bigl(z^{(\kappa-1)}(x)\bigr)^{p-1} - \ell\sum_{i=1}^{j} q_i(x)\bigl(z^{(\kappa-1)}\bigl(\delta_i(x)\bigr)\bigr)^{p-1}.\end{aligned} \tag{7}$$

From (6) and (7), we find

$$\begin{aligned}\varsigma'(x) &\le \frac{\delta_{i+}'(x)}{\delta_i(x)}\varsigma(x) - ma(x)\frac{\varsigma(x)}{r(x)} \\ &\quad - \ell\delta_i(x)\sum_{i=1}^{j} q_i(x)\frac{z^{p-1}(\delta_i(x))}{z^{p-1}(x/2)} - (p-1)\kappa x^{\kappa-2}\frac{\varsigma^{\frac{p}{(p-1)}}(x)}{2(\delta_i(x)r(x))^{1/(p-1)}}\end{aligned}$$

$$\leq -\ell \delta_i(x) \sum_{i=1}^{j} q_i(x) + \left(\frac{\delta'_{i+}(x)}{\delta_i(x)} - m\frac{a(x)}{r(x)} \right) \varsigma(x)$$
$$- (p-1)\kappa x^{\kappa-2} \frac{\varsigma^{\frac{p}{(p-1)}}(x)}{2(\delta_i(x) r(x))^{1/(p-1)}}.$$

Hence, we find

$$\varsigma'(x) \leq -\ell \delta_i(x) \sum_{i=1}^{j} q_i(x) + \varpi_+(x)\varsigma(x) - (p-1)\kappa x^{\kappa-2} \frac{\varsigma^{\frac{p}{(p-1)}}(x)}{2(\delta_i(x) r(x))^{1/(p-1)}}.$$

The proof is complete. □

Lemma 2.6 *Let* (I_2) *hold and* $z(x) > 0$. *If*

$$\vartheta(x) := -\frac{r(x)(-z^{(\kappa-1)})^{p-1}(x)}{(z^{(\kappa-2)})^{p-1}(x)}, \quad \vartheta(x) < 0, \tag{8}$$

then there exists a constant $\mu \in (0,1)$ *such that*

$$\vartheta'(x) \leq \frac{ma(x)}{r(x)\zeta^{p-1}(x)\sigma(x_0,x)} - \ell \sum_{i=1}^{j} q_i(x) \left(\frac{\mu}{(\kappa-2)!} \delta_i^{\kappa-2}(x) \right)^{p-1} - (p-1) \frac{\vartheta^{\frac{p}{(p-1)}}(x)}{r^{\frac{1}{(p-1)}}(x)}. \tag{9}$$

Proof Assume that (I_2) holds and $z(x) > 0$. Since

$$\begin{aligned}
&\left(-r(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}\sigma(x_0,x)\right)' \\
&\quad = \left(-r(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}\right)'\sigma(x_0,x) \\
&\qquad + \left(-r(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}\right)\sigma(x_0,x)\frac{a(x)}{r(x)} \\
&\quad = (-1)^p \left(-a(x) f\left(z^{(\kappa-1)}(x)\right) - \sum_{i=1}^{j} q_i(x) g\left(z\left(\delta_i(x)\right)\right) \right) \sigma(x_0,x) \\
&\qquad - a(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}\sigma(x_0,x) \\
&\quad \leq (-1)^p \left(-ma(x)\left(z^{(\kappa-1)}(x)\right)^{p-1} - \ell \sum_{i=1}^{j} q_i(x) z^{p-1}\left(\delta_i(x)\right) \right) \sigma(x_0,x) \\
&\qquad - a(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}\sigma(x_0,x) \\
&\quad = \left(-a(x)\left(-z^{(\kappa-1)}(x)\right)^{p-1}(1-m) + \ell \sum_{i=1}^{j} q_i(x)\left(-z^{p-1}\left(\delta_i(x)\right)\right) \right) \sigma(x_0,x) \\
&\quad = (-1)^{p-1} \left(-a(x)\left(z^{(\kappa-1)}(x)\right)^{p-1}(1-m) + \ell \sum_{i=1}^{j} q_i(x)\left(z^{p-1}\left(\delta_i(x)\right)\right) \right) \sigma(x_0,x) \\
&\quad \leq -\ell \sum_{i=1}^{j} q_i(x) z^{p-1}\left(\delta_i(x)\right)\sigma(x_0,x) < 0,
\end{aligned}$$

we deduce that $-r(x)(-z^{(\kappa-1)}(x))^{p-1}\sigma(x_0,x)$ is decreasing. Thus, for $s \geq x \geq x_1$, one has

$$\big(r(s)\sigma(x_0,s)\big)^{1/(p-1)}z^{(\kappa-1)}(s) \leq \big(r(x)\sigma(x_0,x)\big)^{1/(p-1)}z^{(\kappa-1)}(x). \tag{10}$$

Dividing both sides of (10) by $(r(s)\sigma(x_0,s))^{1/(p-1)}$ and integrating the resulting inequality from x to u, we get

$$z^{(\kappa-2)}(u) \leq z^{(\kappa-2)}(x) + \big(r(x)\sigma(x_0,x)\big)^{1/(p-1)}z^{(\kappa-1)}(x)\int_x^u \frac{ds}{(r(s)\sigma(x_0,s))^{1/\alpha}}.$$

Letting $u \to \infty$, we arrive at

$$0 \leq z^{(\kappa-2)}(x) + \big(r(x)\sigma(x_0,x)\big)^{1/(p-1)}z^{(\kappa-1)}(x)\zeta(x),$$

which yields

$$-\frac{z^{(\kappa-1)}(x)}{z^{(\kappa-2)}(x)}\zeta(x)\big(r(x)\sigma(x_0,x)\big)^{1/(p-1)} \leq 1.$$

Hence,

$$\frac{r(x)(z^{(\kappa-1)}(x))^{p-1}}{(z^{(\kappa-2)}(x))^{p-1}} \geq \frac{-1}{\zeta^{p-1}(x)\sigma(x_0,x)}.$$

From (8), we have

$$\vartheta(x) \geq \frac{-1}{\zeta^{p-1}(x)\sigma(x_0,x)} \tag{11}$$

and

$$\vartheta'(x) = \frac{(-r(x)(-z^{(\kappa-1)}(x))^{p-1})'}{(z^{(\kappa-2)}(x))^{p-1}} - (p-1)\frac{-r(x)(-z^{(\kappa-1)}(x))^{p}}{(z^{(\kappa-2)}(x))^{p}}.$$

From (1) and (8), we obtain

$$\begin{aligned}\vartheta'(x) &\leq -m\frac{a(x)}{r(x)}\vartheta(x) - \ell\sum_{i=1}^{j} q_i(x)\frac{z^{p-1}(\delta_i(x))}{(z^{(\kappa-2)}(x))^{p-1}} - (p-1)\frac{\vartheta^{\frac{p}{(p-1)}}(x)}{r^{\frac{1}{(p-1)}}(x)} \\ &= -m\frac{a(x)}{r(x)}\vartheta(x) - \ell\sum_{i=1}^{j} q_i(x)\frac{z^{p-1}(\delta_i(x))}{(z^{(\kappa-2)}(\delta_i(x)))^{p-1}}\frac{(z^{(\kappa-2)}(\delta_i(x)))^{p-1}}{(z^{(\kappa-2)}(x))^{p-1}} - (p-1)\frac{\vartheta^{\frac{p}{(p-1)}}(x)}{r^{\frac{1}{(p-1)}}(x)}.\end{aligned} \tag{12}$$

Using Lemma 2.1, we find

$$z(x) \geq \frac{\mu}{(\kappa-2)!}x^{\kappa-2}z^{(\kappa-2)}(x). \tag{13}$$

Thus, from (11) and (13), we get

$$\vartheta'(x) \leq \frac{ma(x)}{r(x)\zeta^{p-1}(x)\sigma(x_0,x)} - \ell\sum_{i=1}^{j} q_i(x)\left(\frac{\mu}{(\kappa-2)!}\delta_i^{\kappa-2}(x)\right)^{p-1} - (p-1)\frac{\vartheta^{\frac{p}{(p-1)}}(x)}{r^{\frac{1}{(p-1)}}(x)}.$$

The proof is complete. □

Theorem 2.1 *Let functions $\delta_i, \zeta \in C^1([x_0,\infty),(0,\infty))$ and $\kappa > 0$, $\mu \in (0,1)$ be such that*

$$\lim_{x\to\infty}\sup\int_{x_0}^{x}\left(\ell\delta_i(s)\sum_{i=1}^{j}q_i(s)-\left(\frac{2}{\kappa s^{\kappa-2}}\right)^{p-1}\frac{r(s)\delta_i(s)(\varpi_+(s))^p}{p^p}\right)ds=\infty \tag{14}$$

and

$$\lim_{x\to\infty}\sup\int_{x_0}^{x}\left(\ell\sum_{i=1}^{j}q_i(s)\left(\frac{\mu\delta_i^{\kappa-2}(s)}{(\kappa-2)!}\zeta(s)\right)^{p-1}\sigma(x_0,s)-\psi^*(s)\right)ds=\infty. \tag{15}$$

Then all solutions of (1) *are oscillatory.*

Proof Let z be a nonoscillatory solution of equation (1) and $z(x) > 0$. Applying Lemma 2.2 to (4) and setting

$$D=\varpi_+(x),\qquad C=(p-1)\kappa x^{\kappa-2}/\bigl(2\bigl(r(x)\delta_i(x)\bigr)^{1/(p-1)}\bigr),\quad\text{and}\quad z=\varsigma,$$

we have

$$\varsigma'(x)\le-\ell\delta_i(x)\sum_{i=1}^{j}q_i(x)+\left(\frac{2}{\kappa x^{\kappa-2}}\right)^{p-1}\frac{r(x)\delta_i(x)(\varpi_+(x))^p}{p^p}. \tag{16}$$

Integrating from x_1 to x, we find

$$\int_{x_1}^{x}\left(\ell\delta_i(s)\sum_{i=1}^{j}q_i(s)-\left(\frac{2}{\kappa s^{\kappa-2}}\right)^{p-1}\frac{r(s)\delta_i(s)(\varpi_+(s))^p}{p^p}\right)ds\le\varsigma(x_1),$$

which contradicts (14).

Now, multiplying (9) by $\zeta^{p-1}(x)\sigma(x_0,x)$ and integrating the resulting inequality from x_1 to x, we get

$$\begin{aligned}&\zeta^{p-1}(x)\sigma(x_0,x)\vartheta(x)-\zeta^{p-1}(x_1)\sigma(x_0,x_1)\vartheta(x_1)-\int_{x_1}^{x}\frac{a(s)}{r(s)}\,ds\\&\quad+(p-1)\int_{x_1}^{x}r^{\frac{-1}{(p-1)}}(s)\zeta^{p-2}(s)\sigma(x_0,s)\psi(s)\vartheta(s)\,ds\\&\quad+\int_{x_1}^{x}\ell\sum_{i=1}^{j}q_i(s)\left(\frac{\mu}{(\kappa-2)!}\delta_i^{\kappa-2}(s)\right)^{p-1}\zeta^{p-1}(s)\sigma(x_0,s)\,ds\\&\quad+(p-1)\int_{x_1}^{x}\frac{\vartheta^{\frac{p}{(p-1)}}(s)}{r^{\frac{1}{(p-1)}}(s)}\zeta^{p-1}(s)\sigma(x_0,s)\,ds\\&\le0.\end{aligned}$$

In view of Lemma 2.2, we put

$$C=\zeta^{p-1}(s)\sigma(x_0,s)/r^{\frac{1}{(p-1)}}(s),\qquad D=\int_{x_1}^{x}r^{\frac{-1}{(p-1)}}(s)\zeta^{p-2}(s)\sigma(x_0,s)\psi(s),\qquad z=\vartheta(x).$$

Thus, we get

$$\begin{aligned}
&\zeta^{p-1}(x)\sigma(x_0,x)\vartheta(x) - \zeta^{p-1}(x_1)\sigma(x_0,x_1)\vartheta(x_1) - \int_{x_1}^{x} \frac{a(s)}{r(s)}\,ds \\
&\quad + \int_{x_1}^{x} \ell \sum_{i=1}^{j} q_i(s)\left(\frac{\mu}{(\kappa-2)!}\delta_i^{\kappa-2}(s)\right)^{p-1} \zeta^{p-1}(s)\sigma(x_0,s)\,ds \\
&\quad + \int_{x_1}^{x} \frac{(p-1)^p \delta_i(s)\psi^p(s)\sigma(x_0,s)}{\zeta(s) r^{\frac{1}{(p-1)}}(x)}\,ds \\
&\leq 0.
\end{aligned}$$

Hence, by (11), we obtain

$$\int_{x_1}^{x}\left(\ell \sum_{i=1}^{j} q_i(s)\left(\frac{\mu \delta_i^{\kappa-2}(s)}{(\kappa-2)!}\zeta(s)\right)^{p-1}\sigma(x_0,s) - \psi^*(s)\right)ds \leq \zeta^{p-1}(x)\sigma(x_0,x)\vartheta(x_1) + 1,$$

which contradicts (15). The proof is complete. □

Remark 2.1 For interested researchers, there is a good problem of finding new results in the following cases:

- **(S$_1$)** $z(x) > 0$, $z'(x) > 0$, $z^{(\kappa-2)}(x) > 0$, $z^{(\kappa-1)}(x) \leq 0$, $(r(x)(z^{(m-1)}(x))^{p-1})' \leq 0$,
- **(S$_2$)** $z(x) > 0$, $z^{(r)}(x) < 0$, $z^{(r+1)}(x) > 0$ for all odd integer $r \in \{1,3,\ldots,\kappa-3\}$, $z^{(\kappa-1)}(x) < 0$, $(r(x)(w^{(\kappa-1)}(x))^{p-1})' \leq 0$.

Example 2.1 For $x \geq 1$, consider the equation

$$\left(x^2\left(z'(x)\right)\right)' + \frac{x}{2}z'(x) + q_0 z\left(\frac{x}{2}\right) = 0, \quad x \geq 1, \tag{17}$$

where $q_0 > 0$ is a constant. Let $p = 2$, $\kappa = 2$, $x_0 = 1$, $r(x) = x^2$, $a(x) = x/2$, $q(x) = q_0$, $\delta_i(x) = x/2$. We now set $\delta_i(x) = m = \ell = 1$, then

$$\begin{aligned}
\sigma(x_0,x) &:= \exp\left(\int_{x_0}^{x} \frac{a(u)}{r(u)}\,du\right) = x^{1/2}, \\
\zeta(x) &:= \int_{x}^{\infty} \frac{ds}{(r(s)\sigma(x_0,s))^{\frac{1}{(p-1)}}} = \frac{2}{3x^{3/2}}, \\
\varpi(x) &:= \frac{\delta_i'(x)}{\delta_i(x)} - \frac{ma(x)}{r(x)} = \frac{-1}{2x}, \\
\psi(x) &:= \frac{1}{\sigma^{\frac{1}{(p-1)}}(x_0,x)} - \frac{\zeta(x)a(x)r^{(2-p)/(p-1)}(x)}{(p-1)} = \frac{2}{3x^{1/2}}, \\
\psi^*(x) &:= \frac{a(x)}{r(x)} + \frac{(p-1)^p\delta_i(x)\psi^p(x)\sigma(x_0,x)}{\zeta(x)r^{\frac{1}{(p-1)}}(x)} = \frac{7}{6x},
\end{aligned}$$

thus, we get

$$\lim_{x\to\infty}\sup \int_{x_0}^{x}\left(\ell\delta_i(s)\sum_{i=1}^{j} q_i(s) - \left(\frac{2}{\kappa s^{\kappa-2}}\right)^{p-1}\frac{r(s)\delta_i(s)(\varpi_+(s))^p}{p^p}\right)ds = \infty$$

and, for some $\mu \in (0,1)$,

$$\begin{aligned} &\lim_{x\to\infty}\sup \int_{x_0}^{x}\left(\ell\sum_{i=1}^{j} q_i(s)\left(\frac{\mu\delta_i^{\kappa-2}(s)}{(\kappa-2)!}\zeta(s)\right)^{p-1}\sigma(x_0,s)-\psi^*(s)\right)ds \\ &\quad= \lim_{x\to\infty}\sup \int_{x_0}^{x}\left(\frac{q_0\mu}{s}-\frac{7}{6s}\right)ds. \end{aligned}$$

Thus, by Theorem 2.1, every solution of (17) is oscillatory if $q_0 > \frac{7}{6\mu}$.

3 Conclusion

In this article, we studied the oscillatory properties of even-order differential equations. New oscillation criteria were established. We used Riccati technique to prove that every solution of (1) is oscillatory. Further, we shall study equation (1) under the condition $\delta_i(t) \geq t$ in the future work.

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