Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Weighted Hermite–Hadamard type inclusions for products of co-ordinated convex interval-valued functions

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Abstract

In this paper, we establish some Hermite–Hadamard–Fejér type inclusions for the product of two co-ordinated convex interval-valued functions. These inclusions are generalizations of some results given in earlier works.

MSC: 26D07; 26D10; 26D15; 26B15; 26B25

Keywords: Hermite–Hadamard inclusions; Convex interval-valued functions; Co-ordinated convex interval-valued functions

1 Introduction

Over the last century, integral inequalities have attracted the interest of a good many researchers because of the importance in applied and pure mathematics. For example, Hermite–Hadamard inequalities, based on convex functions, have an important place in many areas of mathematics, specifically optimization theory. These inequalities, introduced by C. Hermite and J. Hadamard, express that if $\varpi : I \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $c, d \in I$ with $c < d$, then

$$\varpi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \varpi(x)\,dx \leq \frac{\varpi(c)+\varpi(d)}{2}. \tag{1.1}$$

If ϖ is concave, both inequalities hold in the opposite direction. The best known results associated with these inequalities are midpoint and trapezoid inequalities which are frequently used in special means and estimation errors (see [16, 25]). After that, the authors also gave the fractional version of inequality (1.1) in [49]. For instance, the weighted version of inequality (1.1), which is also named Hermite–Hadamard–Fejér inequality, was established by Fejér in [18] as follows.

Theorem 1 *Suppose that $\varpi : [c, d] \to \mathbb{R}$ is a convex function, and let $\phi : [c, d] \to \mathbb{R}$ be nonnegative, integrable, and symmetric about $x = \frac{c+d}{2}$ (i.e. $\phi(x) = \phi(c + d - x)$). Then we*

have the inequality

$$\varpi\left(\frac{c+d}{2}\right)\int_c^d \phi(x)\,dx \le \int_c^d \varpi(x)\phi(x)\,dx \le \frac{\varpi(c)+\varpi(d)}{2}\int_c^d \phi(x)\,dx. \tag{1.2}$$

Many mathematicians derived some generalizations and new results involving fractional integrals regarding inequality (1.2) to obtain new bounds for the left- and right-hand sides of inequality (1.2) (see [17, 47, 48]). In addition to all these generalizations, a good many of authors have worked on Hermite–Hadamard type inequalities for the product of two convex functions in recent years. Moreover, some of them obtained Hermite–Hadamard type results and included fractional integrals in their works. For instance, Pachpatte provided novel inequalities for the product of two nonnegative and convex mappings in [40]. After that, some authors examined how the results were obtained by multiplying two mappings selected from various convex function classes in the references [3, 10–12, 21, 22, 26, 50–52]. What is more, some inequalities involving the product of two co-ordinated convex mappings were observed by Latif and Alomari in [27]. Thereafter, Ozdemir et al. [38, 39] deduced more general versions of the inequalities presented in [27] by considering the product of two co-ordinated s-convex and the product of two co-ordinated h-convex mappings. In [6], by using the products of two co-ordinated convex mappings, new Hermite–Hadamard type results including fractional integrals were proved by Budak and Sarıkaya.

On the other hand, interval analysis is handled as one of the methods for solving interval uncertainty; it is an important material which is used in mathematical and computer models. Although this theory has a long history which may be dated back to Archimedes' calculation of the circumference of a circle, a considerable study had not been published in this field until 1950s. The first book [33] about interval analysis was published by Ramon E. Moore, known as the pioneer of interval calculus, in 1966. Thereafter, a great many researchers started to investigate the theories and applications of interval analysis. Recently, many authors have focused on integral inclusions obtained by using interval-valued functions. For example, Sadowska [46] established the Hermite–Hadamard inclusion for set-valued functions, that is, a more general version of interval-valued mappings, as follows.

Theorem 2 ([46]) *Suppose that $F:[\varrho,\varsigma]\to\mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued convex function such that $F(\vartheta)=[\underline{F}(\vartheta),\overline{F}(\vartheta)]$. Then we have the inclusions*

$$F\left(\frac{\varrho+\varsigma}{2}\right) \supseteq \frac{1}{\varsigma-\varrho}(IR)\int_\varrho^\varsigma F(x)\,dx \supseteq \frac{F(\varrho)+F(\varsigma)}{2}. \tag{1.3}$$

Furthermore, well-known inclusions such as Ostrowski, Minkowski, and Beckenbach and some of their applications have been provided by considering interval-valued functions in [8, 9, 19, 44]. In addition, some inclusions involving interval-valued Riemann-Liouville fractional integrals have been derived by Budak et al. in [7]. In [28], Liu et al. gave the definition of interval-valued harmonically convex functions, and so they have some Hermite–Hadamard type inclusions including interval fractional integrals. For more details about this topic, you can look over the references [13, 14, 32, 34–36, 45, 55, 56].

The general structure of this paper consists of four main sections including introduction. In this section, we give some necessary inclusions and the concept of interval analysis, and we also mention some related works in the literature. In Sect. 2, some basic information about the interval calculus which forms the basis of this work is presented. In Sect. 3,

we provide Fejér type inclusions for the product of interval-valued convex functions, and we examine the relation between our results and the inclusions presented in the earlier works. Finally, we establish interval-valued Fejér type inclusions including fractional integrals by applying the inclusions given in Sect. 3 to interval-valued fractional integrals in Sect. 4. Briefly, the most important property of this study is that it contains interval-valued Fejér type inclusions for classical and fractional integrals. We note that the opinion and technique of this work may inspire new research in this area.

2 Preliminaries of interval calculus and some inclusions

In this part, we give some necessary notions and notations related to interval analysis, which forms the basis of this paper. The set of all closed intervals of $\mathbb{R}$, the set of all closed positive intervals of $\mathbb{R}$, and the set of all closed negative intervals of $\mathbb{R}$ are denoted by $\mathbb{R}_{\mathcal{I}}$, $\mathbb{R}_{\mathcal{I}}^{+}$, and $\mathbb{R}_{\mathcal{I}}^{-}$, respectively. We also suppose that $\Delta = [\varrho, \varsigma] \times [\zeta, \iota]$. For more details regarding the interval analysis, interested readers can see [1, 15, 30, 41].

Definition 1 ([46]) Let $F : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$. We say that F is a convex interval-valued function if, for all $\xi, \eta \in [\varrho, \varsigma]$ and $\vartheta \in (0, 1)$, we have

$$\vartheta F(\xi) + (1 - \vartheta)F(\eta) \subseteq F\big(\vartheta\xi + (1 - \vartheta)\eta\big).$$

Definition 2 ([54]) Suppose that $h : [\zeta, \iota] \to \mathbb{R}$ is a nonnegative function with $(0, 1) \subseteq [\zeta, \iota]$, $h \neq 0$ and $F : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued function such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$. We say that F is an h-convex interval-valued function if, for all $\xi, \eta \in [\varrho, \varsigma]$ and $\vartheta \in (0, 1)$, we have

$$h(\vartheta)F(\xi) + h(1 - \vartheta)F(\eta) \subseteq F\big(\vartheta\xi + (1 - \vartheta)\eta\big). \tag{2.1}$$

Remark 1 If we set $h(\vartheta) = \vartheta$ in Definition 2, then Definition 2 reduces to Definition 1. Similarly, if we choose $h(\vartheta) = \vartheta^{s}$ in Definition 2, then Definition 2 transforms into s-convex interval-valued functions given in [2].

In [42], Piatek gave the notion of the integral of interval-valued functions and provided its relation with the Riemann integral in the following form.

Theorem 3 *Assume that $F : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued function such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$. A function $F \in \mathcal{IR}_{([\varrho,\varsigma])}$ if and only if $\underline{F}(\vartheta), \overline{F}(\vartheta) \in \mathcal{R}_{([\varrho,\varsigma])}$ and*

$$(IR) \int_{\varrho}^{\varsigma} F(\vartheta)\, d\vartheta = \left[\int_{\varrho}^{\varsigma} \underline{F}(\vartheta)\, d\vartheta, \int_{\varrho}^{\varsigma} \overline{F}(\vartheta)\, d\vartheta\right],$$

where $\mathcal{IR}_{([\varrho,\varsigma])}$ and $\mathcal{R}_{([\varrho,\varsigma])}$ denote all Riemann integrable interval-valued functions and Riemann integrable functions, respectively.

In [54], Zhao et al. gave the following inclusions of Hermite–Hadamard type for h-convex interval-valued functions by using Theorem 3.

Theorem 4 *Let $F : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$ and $F \in \mathcal{IR}_{([\varrho,\varsigma])}$, $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h(\frac{1}{2}) \neq 0$. If F is an h-convex interval-valued function, then the following inclusion holds*:

$$\frac{1}{2h(\frac{1}{2})} F\left(\frac{\varrho+\varsigma}{2}\right) \supseteq \frac{1}{\varsigma-\varrho}(IR)\int_{\varrho}^{\varsigma} F(\xi)\,d\xi \supseteq \big[F(\varrho)+F(\varsigma)\big]\int_{0}^{1} h(\vartheta)\,d\vartheta. \tag{2.2}$$

Remark 2

(i) If $h(\vartheta) = \vartheta$, then inclusion (2.2) reduces to (1.3).

(ii) If $h(\vartheta) = \vartheta^{s}$, then (2.2) reduces to the following result:

$$2^{s-1} F\left(\frac{\varrho+\varsigma}{2}\right) \supseteq \frac{1}{\varsigma-\varrho}(IR)\int_{\varrho}^{\varsigma} F(\xi)\,d\xi \supseteq \frac{F(\varrho)+F(\varsigma)}{s+1},$$

which is obtained by Osuna-Gómez et al. in [37].

Now, let us give the notations $\mathcal{A}_k(\xi; m, n)$ and $\mathcal{B}_k(\xi; m, n)$ used throughout the study:

$$\mathcal{A}_k(\xi; m, n) = \int_{m}^{n} (n-\xi)^2 w_k(\xi)\,d\xi \quad \text{and}$$
$$\mathcal{B}_k(\xi; m, n) = \int_{m}^{n} (n-\xi)(\xi-m) w_k(\xi)\,d\xi, \quad k = 1, 2,$$

where $w_k : [\varrho, \varsigma] \to \mathbb{R}$ is a function.

Budak et al. presented the following results.

Theorem 5 ([5]) *Suppose that $w_1 : [\varrho, \varsigma] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $\xi = \frac{\varrho+\varsigma}{2}$ (i.e. $w_1(\xi) = w_1(\varrho + \varsigma - \xi)$). If $F, \Omega : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ are two interval-valued, nonnegative, and convex interval-valued functions such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$ and $\Omega(\vartheta) = [\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)]$, then we have the following inclusion*:

$$(IR)\int_{\varrho}^{\varsigma} F(\xi)\Omega(\xi) w_1(\xi)\,d\xi \supseteq \frac{\mathbf{M}(\varrho,\varsigma)}{(\varsigma-\varrho)^2}\mathcal{A}_1(\xi;\varrho,\varsigma) + \frac{\mathbf{N}(\varrho,\varsigma)}{(\varsigma-\varrho)^2}\mathcal{B}_1(\xi;\varrho,\varsigma), \tag{2.3}$$

where

$$\mathbf{M}(\varrho,\varsigma) = F(\varrho)\Omega(\varrho) + F(\varsigma)\Omega(\varsigma) \quad \text{and} \quad \mathbf{N}(\varrho,\varsigma) = F(\varrho)\Omega(\varsigma) + F(\varsigma)\Omega(\varrho).$$

Theorem 6 ([5]) *Suppose that the conditions of Theorem 5 hold, then we have the following inclusion*:

$$2F\left(\frac{\varrho+\varsigma}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}\right)\int_{\varrho}^{\varsigma} w_1(\xi)\,d\xi \tag{2.4}$$
$$\supseteq (IR)\int_{\varrho}^{\varsigma} F(\xi)\Omega(\xi) w_1(\xi)\,d\xi + \frac{\mathbf{M}(\varrho,\varsigma)}{(\varsigma-\varrho)^2}\mathcal{B}_1(\xi;\varrho,\varsigma) + \frac{\mathbf{N}(\varrho,\varsigma)}{(\varsigma-\varrho)^2}\mathcal{A}_1(\xi;\varrho,\varsigma),$$

where $\mathbf{M}(\varrho,\varsigma)$ and $\mathbf{N}(\varrho,\varsigma)$ are defined as in Theorem 5.

Motivated by the continuing studies, the authors introduced the fractional integrals for interval-valued functions and proved some associated inclusions of Hermite–Hadamard type.

Definition 3 ([7, 29]) Assume that $F : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an interval-valued function such that $F(\vartheta) = [\underline{F}(\vartheta), \overline{F}(\vartheta)]$. Then interval-valued Riemann–Liouville integrals $\mathcal{J}_{\varrho+}^{\alpha}F$ and $\mathcal{J}_{\varsigma-}^{\alpha}F$ of order $\alpha > 0$ with $\varrho \geq 0$ are defined by

$$\mathcal{J}_{\varrho+}^{\alpha}F(\xi) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\varrho}^{\xi}(\xi - \vartheta)^{\alpha-1}F(\vartheta)\,d\vartheta, \quad \xi > \varrho,$$

and

$$\mathcal{J}_{\varsigma-}^{\alpha}F(\xi) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\xi}^{\varsigma}(\vartheta - \xi)^{\alpha-1}F(\vartheta)\,d\vartheta, \quad \xi < \varsigma,$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $\mathcal{J}_{\varrho+}^{0}F(\xi) = \mathcal{J}_{\varsigma-}^{0}F(\xi) = F(\xi)$.

Theorem 7 ([7]) *If $F = [\underline{F}(\vartheta), \overline{F}(\vartheta)] : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued function, then one has*

$$\mathcal{J}_{\varrho+}^{\alpha}F(\xi) = \left[I_{\varrho+}^{\alpha}\underline{F}(\xi), I_{\varrho+}^{\alpha}\overline{F}(\xi)\right]$$

and

$$\mathcal{J}_{\varsigma-}^{\alpha}F(\xi) = \left[I_{\varsigma-}^{\alpha}\underline{F}(\xi), I_{\varsigma-}^{\alpha}\overline{F}(\xi)\right],$$

where $I_{\varrho+}^{\alpha}$ and $I_{\varsigma-}^{\alpha}$ are left-sided and right-sided Riemann–Liouville fractional integrals of the functions $F : [\varrho, \varsigma] \to \mathbb{R}$, respectively (*see* [20, 24, 31, 43]).

Theorem 8 ([7]) *Let $F = [\underline{F}(\vartheta), \overline{F}(\vartheta)] : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ be a convex interval-valued function and $\alpha > 0$, then we have the following inclusion*:

$$F\left(\frac{\varrho + \varsigma}{2}\right) \supseteq \frac{\Gamma(1+\alpha)}{2(\varsigma - \varrho)^{\alpha}}\left[\mathcal{J}_{\varrho+}^{\alpha}F(\varsigma) + \mathcal{J}_{\varsigma-}^{\alpha}F(\varrho)\right] \supseteq \frac{F(\varrho) + F(\varsigma)}{2}. \tag{2.5}$$

In addition to all these results, Liu et al. refined Hermite–Hadamard type inclusions for interval-valued mappings in [28].

Theorem 9 ([28]) *Suppose that $w : [\varrho, \varsigma] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $\xi = \frac{\varrho+\varsigma}{2}$ (i.e. $w(\xi) = w(\varrho + \varsigma - \xi)$). If $F = [\underline{F}(\vartheta), \overline{F}(\vartheta)] : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ is a convex interval-valued function, then we have the following inclusion*:

$$\begin{aligned} &F\left(\frac{\varrho + \varsigma}{2}\right)\left[I_{\varrho+}^{\alpha}w(\varsigma) + I_{\varsigma-}^{\alpha}w(\varrho)\right] \\ &\quad\supseteq \left[\mathcal{J}_{\varrho+}^{\alpha}F(\varsigma)w(\varsigma) + \mathcal{J}_{\varsigma-}^{\alpha}F(\varrho)w(\varrho)\right] \\ &\quad\supseteq \frac{F(\varrho) + F(\varsigma)}{2}\left[I_{\varrho+}^{\alpha}w(\varsigma) + I_{\varsigma-}^{\alpha}w(\varrho)\right] \end{aligned}$$

for $\alpha > 0$.

Recently, Zhao et al. introduced the co-ordinated convexity of interval-valued functions and proved some new Hermite–Hadamard inclusions like the following.

Definition 4 ([53]) Let $F : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $F(\vartheta, s) = [\underline{F}(\vartheta, s), \overline{F}(\vartheta, s)]$. Then $F = [\underline{F}(\vartheta, s), \overline{F}(\vartheta, s)] : \Delta \to \mathbb{R}_{\mathcal{I}}^+$ is said to be a co-ordinated convex interval-valued function if the following inclusion holds:

$$\begin{aligned} &F\big(\vartheta\xi + (1-\vartheta)\eta, su + (1-s)w\big) \\ &\quad \supseteq \vartheta s F(\xi, u) + \vartheta(1-s)F(\xi, w) + s(1-\vartheta)F(\eta, u) + (1-s)(1-\vartheta)F(\eta, w) \end{aligned}$$

for all $(\xi, \eta), (u, w) \in \Delta$ and $s, \vartheta \in [0, 1]$.

Lemma 1 ([53]) *A function $F = [\underline{F}(\vartheta, s), \overline{F}(\vartheta, s)] : \Delta \to \mathbb{R}_{\mathcal{I}}^+$ is a co-ordinated convex interval-valued function if and only if there exist two functions $F_\xi : [\zeta, \iota] \to \mathbb{R}_{\mathcal{I}}^+$, $F_\xi(w) = F(\xi, w)$, and $F_\eta : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^+$, $F_\eta(u) = F(u, \eta)$ are convex interval-valued functions.*

Theorem 10 ([53]) *Let $F, \Omega : \Delta \to \mathbb{R}_{\mathcal{I}}^+$ be two co-ordinated convex interval-valued functions such that $F(\vartheta, s) = [\underline{F}(\vartheta, s), \overline{F}(\vartheta, s)]$ and $\Omega(\vartheta, s) = [\underline{\Omega}(\vartheta, s), \overline{\Omega}(\vartheta, s)]$, then the following Hermite–Hadamard type inclusions hold*:

$$\begin{aligned} &\frac{1}{(\varsigma-\varrho)(\iota-\zeta)}(IR)\int_\varrho^\varsigma \int_\zeta^\iota F(\xi, \eta)\Omega(\xi, \eta)\, d\eta\, d\xi \\ &\quad \supseteq \frac{1}{9}\mathbf{K}(\varrho, \varsigma, \zeta, \iota) + \frac{1}{18}\big[\mathbf{L}(\varrho, \varsigma, \zeta, \iota) + \mathbf{M}(\varrho, \varsigma, \zeta, \iota)\big] + \frac{1}{36}\mathbf{N}(\varrho, \varsigma, \zeta, \iota) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &4F\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right) \\ &\quad \supseteq \frac{1}{(\varsigma-\varrho)(\iota-\zeta)}(IR)\int_\varrho^\varsigma \int_\zeta^\iota F(\xi, \eta)\Omega(\xi, \eta)\, d\eta\, d\xi + \frac{5}{36}\mathbf{K}(\varrho, \varsigma, \zeta, \iota) \\ &\qquad + \frac{7}{36}\big[\mathbf{L}(\varrho, \varsigma, \zeta, \iota) + \mathbf{M}(\varrho, \varsigma, \zeta, \iota)\big] + \frac{2}{9}\mathbf{N}(\varrho, \varsigma, \zeta, \iota), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathbf{K}(\varrho, \varsigma, \zeta, \iota) &= F(\varrho, \zeta)\Omega(\varrho, \zeta) + F(\varsigma, \zeta)\Omega(\varsigma, \zeta) + F(\varrho, \iota)\Omega(\varrho, \iota) + F(\varsigma, \iota)\Omega(\varsigma, \iota), \\ \mathbf{L}(\varrho, \varsigma, \zeta, \iota) &= F(\varrho, \zeta)\Omega(\varsigma, \zeta) + F(\varsigma, \zeta)\Omega(\varrho, \zeta) + F(\varrho, \iota)\Omega(\varsigma, \iota) + F(\varsigma, \iota)\Omega(\varrho, \iota), \\ \mathbf{M}(\varrho, \varsigma, \zeta, \iota) &= F(\varrho, \zeta)\Omega(\varrho, \iota) + F(\varsigma, \zeta)\Omega(\varsigma, \iota) + F(\varrho, \iota)\Omega(\varrho, \zeta) + F(\varsigma, \iota)\Omega(\varsigma, \zeta), \end{aligned}$$

and

$$\mathbf{N}(\varrho, \varsigma, \zeta, \iota) = F(\varrho, \zeta)\Omega(\varsigma, \iota) + F(\varsigma, \zeta)\Omega(\varrho, \iota) + F(\varrho, \iota)\Omega(\varsigma, \zeta) + F(\varsigma, \iota)\Omega(\varrho, \zeta).$$

For more recent results related to inclusions (2.6) and (2.7), one can read [23].

The definitions of the interval-valued Riemann–Liouville fractional integrals of function $F(\xi,\eta)$ are given as follows.

Definition 5 ([4, 23]) Let $F:\Delta\subset\mathbb{R}^2\to\mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $F(\vartheta,s)=[\underline{F}(\vartheta,s),\overline{F}(\vartheta,s)]$ and $F\in\mathcal{IR}_{([\varrho,\varsigma])}$. The Riemann–Liouville fractional integrals $\mathcal{J}^{\alpha,\beta}_{\varrho+,\zeta+}$, $\mathcal{J}^{\alpha,\beta}_{\varrho+,\iota-}$, $\mathcal{J}^{\alpha,\beta}_{\varsigma-,\zeta+}$, and $\mathcal{J}^{\alpha,\beta}_{\varsigma-,\iota-}$ of order $\alpha,\beta>0$ with $\varrho,\zeta\geq 0$ are defined by

$$\mathcal{J}^{\alpha,\beta}_{\varrho+,\zeta+}F(\xi,\eta)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\varrho}^{\xi}\int_{\zeta}^{\eta}(\xi-\vartheta)^{\alpha-1}(\eta-s)^{\beta-1}F(\vartheta,s)\,ds\,d\vartheta,\quad \xi>\varrho,\eta>\zeta,$$

$$\mathcal{J}^{\alpha,\beta}_{\varrho+,\iota-}F(\xi,\eta)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\varrho}^{\xi}\int_{\eta}^{\iota}(\xi-\vartheta)^{\alpha-1}(s-\eta)^{\beta-1}F(\vartheta,s)\,ds\,d\vartheta,\quad \xi>\varrho,\eta>\iota,$$

$$\mathcal{J}^{\alpha,\beta}_{\varsigma-,\zeta+}F(\xi,\eta)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\xi}^{\varsigma}\int_{\zeta}^{\eta}(\vartheta-\xi)^{\alpha-1}(\eta-s)^{\beta-1}F(\vartheta,s)\,ds\,d\vartheta,\quad \xi<\varsigma,\eta>\zeta,$$

$$\mathcal{J}^{\alpha,\beta}_{\varsigma-,\iota-}F(\xi,\eta)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\xi}^{\varsigma}\int_{\eta}^{\iota}(\vartheta-\xi)^{\alpha-1}(s-\eta)^{\beta-1}F(\vartheta,s)\,ds\,d\vartheta,\quad \xi<\varsigma,\eta<\iota,$$

respectively.

3 Main results

Theorem 11 *Let $F,\Omega:\Delta\subset\mathbb{R}^2\to\mathbb{R}_{\mathcal{I}}^+$ be two co-ordinated convex interval-valued functions on Δ such that $F(\vartheta,s)=[\underline{F}(\vartheta,s),\overline{F}(\vartheta,s)]$ and $\Omega(\vartheta,s)=[\underline{\Omega}(\vartheta,s),\overline{\Omega}(\vartheta,s)]$. In addition, $w_1:[\varrho,\varsigma]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $\xi=\frac{\varrho+\varsigma}{2}$ (i.e. $w_1(\xi)=w_1(\varrho+\varsigma-\xi)$) and $w_2:[\zeta,\iota]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $\eta=\frac{\zeta+\iota}{2}$. Then we have the following Hermite–Hadamard–Fejér type inclusion*:

$$\begin{aligned}&\frac{1}{(\varsigma-\varrho)(\iota-\zeta)}(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\&\quad\supseteq\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\varsigma-\varrho)^3(\iota-\zeta)^3}\big[\mathbf{K}(\varrho,\varsigma,\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)+\mathbf{L}(\varrho,\varsigma,\zeta,\iota)\mathcal{B}_1(\xi;\varrho,\varsigma)\big]\\&\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\varsigma-\varrho)^3(\iota-\zeta)^3}\big[\mathbf{M}(\varrho,\varsigma,\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)+\mathbf{N}(\varrho,\varsigma,\zeta,\iota)\mathcal{B}_1(\xi;\varrho,\varsigma)\big],\end{aligned}$$

where $\mathbf{K}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{L}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{M}(\varrho,\varsigma,\zeta,\iota)$, *and* $\mathbf{N}(\varrho,\varsigma,\zeta,\iota)$ *defined as in Theorem* 10.

Proof Since F and Ω are two co-ordinated convex interval-valued functions on Δ, the functions $F_\xi(\eta)=F(\xi,\eta)$ and $\Omega_\xi(\eta)=\Omega(\xi,\eta)$ are convex interval-valued on $[\zeta,\iota]$. If inclusion (2.3) is applied to the functions F_ξ and Ω_ξ, then we obtain

$$\begin{aligned}&\frac{1}{\iota-\zeta}(IR)\int_{\zeta}^{\iota}F_\xi(\eta)\Omega_\xi(\eta)w_2(\eta)\,d\eta\\&\quad\supseteq\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^3}\big[F_\xi(\zeta)\Omega_\xi(\zeta)+F_\xi(\iota)\Omega_\xi(\iota)\big]\\&\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^3}\big[F_\xi(\zeta)\Omega_\xi(\iota)+F_\xi(\iota)\Omega_\xi(\zeta)\big].\end{aligned}\tag{3.1}$$

This implies that

$$\begin{aligned}&\frac{1}{\iota-\zeta}(IR)\int_{\zeta}^{\iota}F(\xi,\eta)\Omega(\xi,\eta)w_2(\eta)\,d\eta \\ &\quad\supseteq\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^3}\big[F(\xi,\zeta)\Omega(\xi,\zeta)+F(\xi,\iota)\Omega(\xi,\iota)\big] \\ &\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^3}\big[F(\xi,\zeta)\Omega(\xi,\iota)+F(\xi,\iota)\Omega(\xi,\zeta)\big].\end{aligned}\tag{3.2}$$

Multiplying (3.2) by $\frac{w_1(\xi)}{\varsigma-\varrho}$ and integrating the resultant one with respect to ξ over $[\varrho,\varsigma]$, we get

$$\begin{aligned}&\frac{1}{(\varsigma-\varrho)(\iota-\zeta)}(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi \\ &\quad\supseteq\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\varsigma-\varrho)(\iota-\zeta)^3}(IR)\int_{\varrho}^{\varsigma}\big[F(\xi,\zeta)\Omega(\xi,\zeta)+F(\xi,\iota)\Omega(\xi,\iota)\big]w_1(\xi)\,d\xi \\ &\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\varsigma-\varrho)(\iota-\zeta)^3}(IR)\int_{\varrho}^{\varsigma}\big[F(\xi,\zeta)\Omega(\xi,\iota)+F(\xi,\iota)\Omega(\xi,\zeta)\big]w_1(\xi)\,d\xi.\end{aligned}\tag{3.3}$$

Applying (2.3) to each integral in (3.3), we have

$$\begin{aligned}&(IR)\int_{\varrho}^{\varsigma}F(\xi,\zeta)\Omega(\xi,\zeta)w_1(\xi)\,d\xi \\ &\quad\supseteq\frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\zeta)\Omega(\varrho,\zeta)+F(\varsigma,\zeta)\Omega(\varsigma,\zeta)\big] \\ &\qquad+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\zeta)\Omega(\varsigma,\zeta)+F(\varsigma,\zeta)\Omega(\varrho,\zeta)\big],\end{aligned}\tag{3.4}$$

$$\begin{aligned}&(IR)\int_{\varrho}^{\varsigma}F(\xi,\iota)\Omega(\xi,\iota)w_1(\xi)\,d\xi \\ &\quad\supseteq\frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\iota)\Omega(\varrho,\iota)+F(\varsigma,\iota)\Omega(\varsigma,\iota)\big] \\ &\qquad+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\iota)\Omega(\varsigma,\iota)+F(\varsigma,\iota)\Omega(\varrho,\iota)\big],\end{aligned}\tag{3.5}$$

$$\begin{aligned}&(IR)\int_{\varrho}^{\varsigma}F(\xi,\zeta)\Omega(\xi,\iota)w_1(\xi)\,d\xi \\ &\quad\supseteq\frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\zeta)\Omega(\varrho,\iota)+F(\varsigma,\zeta)\Omega(\varsigma,\iota)\big] \\ &\qquad+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\zeta)\Omega(\varsigma,\iota)+F(\varsigma,\zeta)\Omega(\varrho,\iota)\big],\end{aligned}\tag{3.6}$$

and

$$(IR)\int_{\varrho}^{\varsigma}F(\xi,\iota)\Omega(\xi,\zeta)w_1(\xi)\,d\xi\tag{3.7}$$

$$\supseteq \frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\iota)\Omega(\varrho,\zeta)+F(\varsigma,\iota)\Omega(\varsigma,\zeta)\big]$$
$$+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}\big[F(\varrho,\iota)\Omega(\varsigma,\zeta)+F(\varsigma,\iota)\Omega(\varrho,\zeta)\big].$$

Substituting (3.4)–(3.7) in (3.3) and by arranging the resultant one, we get the desired result. On the other hand, the same result can be obtained by using the convexity of interval-valued functions $F_\eta(\xi)=F(\xi,\eta)$ and $\Omega_\eta(\xi)=\Omega(\xi,\eta)$. □

Theorem 12 *Let $F,\Omega:\Delta\subset\mathbb{R}^2\to\mathbb{R}_{\mathcal{I}}^+$ be two co-ordinated convex interval-valued functions on Δ such that $F(\vartheta,s)=[\underline{F}(\vartheta,s),\overline{F}(\vartheta,s)]$ and $\Omega(\vartheta,s)=[\underline{\Omega}(\vartheta,s),\overline{\Omega}(\vartheta,s)]$. In addition, $w_1:[\varrho,\varsigma]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $\xi=\frac{\varrho+\varsigma}{2}$ (i.e. $w_1(\xi)=w_1(\varrho+\varsigma-\xi)$) and $w_2:[\zeta,\iota]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $\eta=\frac{\zeta+\iota}{2}$. Then we have the following Hermite–Hadamard–Fejér type inclusion*:

$$4(IR)\int_\varrho^\varsigma\int_\zeta^\iota F\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi$$
$$\supseteq (IR)\int_\varrho^\varsigma\int_\zeta^\iota F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi$$
$$+\frac{\mathbf{K}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\big[\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota)+\mathcal{B}_2(\eta;\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)$$
$$+\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\big]$$
$$+\frac{\mathbf{L}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\big[\mathcal{B}_2(\eta;\zeta,\iota)\mathcal{B}_1(\xi;\varrho,\varsigma)+\mathcal{A}_2(\eta;\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)$$
$$+\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\big]$$
$$+\frac{\mathbf{M}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\big[\mathcal{B}_2(\eta;\zeta,\iota)\mathcal{B}_1(\xi;\varrho,\varsigma)+\mathcal{A}_2(\eta;\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)$$
$$+\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota)\big]$$
$$+\frac{\mathbf{N}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\big[\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)+\mathcal{A}_2(\eta;\zeta,\iota)\mathcal{B}_1(\xi;\varrho,\varsigma)$$
$$+\mathcal{A}_2(\eta;\zeta,\iota)\mathcal{A}_1(\xi;\varrho,\varsigma)\big].$$

Proof Since F and Ω are co-ordinated convex interval-valued functions on Δ, the functions F_ξ, Ω_ξ, F_η, and Ω_η are convex interval-valued. Applying (2.4) for the functions $F(\xi,\frac{\zeta+\iota}{2})$ and $\Omega(\xi,\frac{\zeta+\iota}{2})$ and multiplying the resultant by $2\int_\zeta^\iota w_2(\eta)\,d\eta$, we get

$$4F\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\int_\varrho^\varsigma\int_\zeta^\iota w_1(\xi)w_2(\eta)\,d\eta\,d\xi \tag{3.8}$$
$$\supseteq 2(IR)\int_\varrho^\varsigma\int_\zeta^\iota F\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi$$
$$+\frac{2}{(\varsigma-\varrho)^2}\left[F\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)+F\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)\right]$$
$$\times\left(\int_\zeta^\iota w_2(\eta)\,d\eta\right)\mathcal{B}_1(\xi;\varrho,\varsigma)$$

$$+\frac{2}{(\varsigma-\varrho)^2}\left[F\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)+F\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)\right]\times\left(\int_\zeta^\iota w_2(\eta)\,d\eta\right)\mathcal{A}_1(\xi;\varrho,\varsigma).$$

Similarly, if we apply (2.4) for the functions $F(\frac{\varrho+\varsigma}{2},\eta)$ and $\Omega(\frac{\varrho+\varsigma}{2},\eta)$ and multiply the resultant inclusion by $2\int_\varrho^\varsigma w_1(\xi)\,d\xi$, we get

$$\begin{aligned}
&4F\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\int_\varrho^\varsigma\int_\zeta^\iota w_1(\xi)w_2(\eta)\,d\eta\,d\xi \qquad (3.9)\\
&\supseteq 2(IR)\int_\varrho^\varsigma\int_\zeta^\iota F\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\quad+\frac{2}{(\iota-\zeta)^2}\left[F\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)+F\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)\right]\\
&\quad\times\left(\int_\varrho^\varsigma w_1(\xi)\,d\xi\right)\mathcal{B}_2(\eta;\zeta,\iota)\\
&\quad+\frac{2}{(\iota-\zeta)^2}\left[F\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)+F\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right]\\
&\quad\times\left(\int_\varrho^\varsigma w_1(\xi)\,d\xi\right)\mathcal{A}_2(\eta;\zeta,\iota).
\end{aligned}$$

Using (2.4) for each term in the right-hand sides of (3.8) and (3.9), we have

$$\begin{aligned}
&2F\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)\int_\zeta^\iota w_2(\eta)\,d\eta \qquad (3.10)\\
&\supseteq(IR)\int_\zeta^\iota F(\varrho,\eta)\Omega(\varrho,\eta)w_2(\eta)\,d\eta\\
&\quad+\left[\frac{F(\varrho,\zeta)\Omega(\varrho,\zeta)+F(\varrho,\iota)\Omega(\varrho,\iota)}{(\iota-\zeta)^2}\right]\mathcal{B}_1(\eta;\zeta,\iota)\\
&\quad+\left[\frac{F(\varrho,\zeta)\Omega(\varrho,\iota)+F(\varrho,\iota)\Omega(\varrho,\zeta)}{(\iota-\zeta)^2}\right]\mathcal{A}_1(\eta;\zeta,\iota),\\
&2F\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)\int_\zeta^\iota w_2(\eta)\,d\eta \qquad (3.11)\\
&\supseteq(IR)\int_\zeta^\iota F(\varsigma,\eta)\Omega(\varsigma,\eta)w_2(\eta)\,d\eta\\
&\quad+\left[\frac{F(\varsigma,\zeta)\Omega(\varsigma,\zeta)+F(\varsigma,\iota)\Omega(\varsigma,\iota)}{(\iota-\zeta)^2}\right]\mathcal{B}_1(\eta;\zeta,\iota)\\
&\quad+\left[\frac{F(\varsigma,\zeta)\Omega(\varsigma,\iota)+F(\varsigma,\iota)\Omega(\varsigma,\zeta)}{(\iota-\zeta)^2}\right]\mathcal{A}_1(\eta;\zeta,\iota),\\
&2F\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)\int_\zeta^\iota w_2(\eta)\,d\eta \qquad (3.12)\\
&\supseteq(IR)\int_\zeta^\iota F(\varrho,\eta)\Omega(\varsigma,\eta)w_2(\eta)\,d\eta\\
&\quad+\left[\frac{F(\varrho,\zeta)\Omega(\varsigma,\zeta)+F(\varrho,\iota)\Omega(\varsigma,\iota)}{(\iota-\zeta)^2}\right]\mathcal{B}_1(\eta;\zeta,\iota)
\end{aligned}$$

$$+\left[\frac{F(\varrho,\zeta)\Omega(\varsigma,\iota)+F(\varrho,\iota)\Omega(\varsigma,\zeta)}{(\iota-\zeta)^2}\right]\mathcal{A}_1(\eta;\zeta,\iota),$$

$$2F\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)\int_\zeta^\iota w_2(\eta)\,d\eta \tag{3.13}$$

$$\supseteq (IR)\int_\zeta^\iota F(\varsigma,\eta)\Omega(\varrho,\eta)w_2(\eta)\,d\eta$$

$$+\left[\frac{F(\varsigma,\zeta)\Omega(\varrho,\zeta)+F(\varsigma,\iota)\Omega(\varrho,\iota)}{(\iota-\zeta)^2}\right]\mathcal{B}_1(\eta;\zeta,\iota)$$

$$+\left[\frac{F(\varsigma,\zeta)\Omega(\varrho,\iota)+F(\varsigma,\iota)\Omega(\varrho,\zeta)}{(\iota-\zeta)^2}\right]\mathcal{A}_1(\eta;\zeta,\iota),$$

$$2F\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)\int_\varrho^\varsigma w_1(\xi)\,d\xi \tag{3.14}$$

$$\supseteq (IR)\int_\varrho^\varsigma F(\xi,\zeta)\Omega(\xi,\zeta)w_1(\xi)\,d\xi$$

$$+\left[\frac{F(\varrho,\zeta)\Omega(\varrho,\zeta)+F(\varsigma,\zeta)\Omega(\varsigma,\zeta)}{(\varsigma-\varrho)^2}\right]\mathcal{B}_1(\xi;\varrho,\varsigma)$$

$$+\left[\frac{F(\varrho,\zeta)\Omega(\varsigma,\zeta)+F(\varsigma,\zeta)\Omega(\varrho,\zeta)}{(\varsigma-\varrho)^2}\right]\mathcal{A}_1(\xi;\varrho,\varsigma),$$

$$2F\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)\int_\varrho^\varsigma w_1(\xi)\,d\xi \tag{3.15}$$

$$\supseteq (IR)\int_\varrho^\varsigma F(\xi,\iota)\Omega(\xi,\iota)w_1(\xi)\,d\xi$$

$$+\left[\frac{F(\varrho,\iota)\Omega(\varrho,\iota)+F(\varsigma,\iota)\Omega(\varsigma,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{B}_1(\xi;\varrho,\varsigma)$$

$$+\left[\frac{F(\varrho,\iota)\Omega(\varsigma,\iota)+F(\varsigma,\iota)\Omega(\varrho,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{A}_1(\xi;\varrho,\varsigma),$$

$$2F\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)\int_\varrho^\varsigma w_1(\xi)\,d\xi \tag{3.16}$$

$$\supseteq (IR)\int_\varrho^\varsigma F(\xi,\iota)\Omega(\xi,\iota)w_1(\xi)\,d\xi$$

$$+\left[\frac{F(\varrho,\iota)\Omega(\varrho,\iota)+F(\varsigma,\iota)\Omega(\varsigma,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{B}_1(\xi;\varrho,\varsigma)$$

$$+\left[\frac{F(\varrho,\iota)\Omega(\varsigma,\iota)+F(\varsigma,\iota)\Omega(\varrho,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{A}_1(\xi;\varrho,\varsigma),$$

$$2F\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)\int_\varrho^\varsigma w_1(\xi)\,d\xi \tag{3.17}$$

$$\supseteq (IR)\int_\varrho^\varsigma F(\xi,\zeta)\Omega(\xi,\iota)w_1(\xi)\,d\xi$$

$$+\left[\frac{F(\varrho,\zeta)\Omega(\varrho,\iota)+F(\varsigma,\zeta)\Omega(\varsigma,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{B}_1(\xi;\varrho,\varsigma)$$

$$+\left[\frac{F(\varrho,\zeta)\Omega(\varsigma,\iota)+F(\varsigma,\zeta)\Omega(\varrho,\iota)}{(\varsigma-\varrho)^2}\right]\mathcal{A}_1(\xi;\varrho,\varsigma),$$

$$
\begin{aligned}
&2F\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)\int_{\varrho}^{\varsigma} w_1(\xi)\,d\xi \qquad (3.18)\\
&\quad\supseteq (IR)\int_{\varrho}^{\varsigma} F(\xi,\iota)\Omega(\xi,\zeta)w_1(\xi)\,d\xi\\
&\qquad+\left[\frac{F(\varrho,\iota)\Omega(\varrho,\zeta)+F(\varsigma,\iota)\Omega(\varsigma,\zeta)}{(\varsigma-\varrho)^2}\right]\mathcal{B}_1(\xi;\varrho,\varsigma)\\
&\qquad+\left[\frac{F(\varrho,\iota)\Omega(\varsigma,\zeta)+F(\varsigma,\iota)\Omega(\varrho,\zeta)}{(\varsigma-\varrho)^2}\right]\mathcal{A}_1(\xi;\varrho,\varsigma).
\end{aligned}
$$

Putting (3.10)–(3.18) in (3.8) and (3.9) and adding the resultant one, we obtain

$$
\begin{aligned}
&8F\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota} w_1(\xi)w_2(\eta)\,d\eta\,d\xi \qquad (3.19)\\
&\quad\supseteq 2(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota} F\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\qquad+2(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota} F\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\qquad+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\left[F(\varrho,\eta)\Omega(\varrho,\eta)+F(\varsigma,\eta)\Omega(\varsigma,\eta)\right]w_2(\eta)\,d\eta\\
&\qquad+\frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\left[F(\varrho,\eta)\Omega(\varsigma,\eta)+F(\varsigma,\eta)\Omega(\varrho,\eta)\right]w_2(\eta)\,d\eta\\
&\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\left[F(\xi,\zeta)\Omega(\xi,\zeta)+F(\xi,\iota)\Omega(\xi,\iota)\right]w_1(\xi)\,d\xi\\
&\qquad+\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\left[F(\xi,\zeta)\Omega(\xi,\iota)+F(\xi,\iota)\Omega(\xi,\zeta)\right]w_1(\xi)\,d\xi\\
&\qquad+\frac{2\mathbf{K}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{L}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{M}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{N}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota).
\end{aligned}
$$

By applying (2.4) to the functions $F(\xi,\frac{\zeta+\iota}{2})$ and $\Omega(\xi,\frac{\zeta+\iota}{2})$, we get

$$
\begin{aligned}
&2(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota} F\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi \qquad (3.20)\\
&\quad\supseteq (IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota} F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\qquad+\frac{\mathcal{B}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\left[F(\xi,\zeta)\Omega(\xi,\zeta)+F(\xi,\iota)\Omega(\xi,\iota)\right]w_1(\xi)\,d\xi\\
&\qquad+\frac{\mathcal{A}_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\left[F(\xi,\zeta)\Omega(\xi,\iota)+F(\xi,\iota)\Omega(\xi,\zeta)\right]w_1(\xi)\,d\xi.
\end{aligned}
$$

Similarly, if we apply (2.4) to the functions $F(\frac{\varrho+\varsigma}{2},\eta)$ and $\Omega(\frac{\varrho+\varsigma}{2},\eta)$, then we have

$$
\begin{aligned}
&2(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}F\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)w_1(\xi)w_2(\eta)\,d\eta\,d\xi \qquad (3.21)\\
&\quad\supseteq (IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\qquad+\frac{\mathcal{B}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\big[F(\varrho,\eta)\Omega(\varrho,\eta)+F(\varsigma,\eta)\Omega(\varsigma,\eta)\big]w_2(\eta)\,d\eta\\
&\qquad+\frac{\mathcal{A}_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\big[F(\varrho,\eta)\Omega(\varsigma,\eta)+F(\varsigma,\eta)\Omega(\varrho,\eta)\big]w_2(\eta)\,d\eta.
\end{aligned}
$$

Substituting (3.20) and (3.21) in (3.19), we have

$$
\begin{aligned}
&8F\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}w_1(\xi)w_2(\eta)\,d\eta\,d\xi \qquad (3.22)\\
&\quad\supseteq 2(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}F(\xi,\eta)\Omega(\xi,\eta)w_1(\xi)w_2(\eta)\,d\eta\,d\xi\\
&\qquad+\frac{2B_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\big[F(\varrho,\eta)\Omega(\varrho,\eta)+F(\varsigma,\eta)\Omega(\varsigma,\eta)\big]w_2(\eta)\,d\eta\\
&\qquad+\frac{2A_1(\xi;\varrho,\varsigma)}{(\varsigma-\varrho)^2}(IR)\int_{\zeta}^{\iota}\big[F(\varrho,\eta)\Omega(\varsigma,\eta)+F(\varsigma,\eta)\Omega(\varrho,\eta)\big]w_2(\eta)\,d\eta\\
&\qquad+\frac{2B_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\big[F(\xi,\zeta)\Omega(\xi,\zeta)+F(\xi,\iota)\Omega(\xi,\iota)\big]w_1(\xi)\,d\xi\\
&\qquad+\frac{2A_2(\eta;\zeta,\iota)}{(\iota-\zeta)^2}(IR)\int_{\varrho}^{\varsigma}\big[F(\xi,\zeta)\Omega(\xi,\iota)+F(\xi,\iota)\Omega(\xi,\zeta)\big]w_1(\xi)\,d\xi\\
&\qquad+\frac{2\mathbf{K}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{L}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{B}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{M}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{B}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota)\\
&\qquad+\frac{2\mathbf{N}(\varrho,\varsigma,\zeta,\iota)}{(\varsigma-\varrho)^2(\iota-\zeta)^2}\mathcal{A}_1(\xi;\varrho,\varsigma)\mathcal{A}_2(\eta;\zeta,\iota).
\end{aligned}
$$

By applying (2.3) to each integral in (3.22), we obtain the desired result. □

Remark 3 If we choose $w_1(\xi)=1$ and $w_2(\eta)=1$ in Theorem 11 and Theorem 12, we get (2.6) and (2.7), respectively.

Remark 4 If we choose $w_1(\xi)=\frac{\alpha}{(\varsigma-\varrho)^{\alpha-1}}[(\varsigma-\xi)^{\alpha-1}+(\xi-\varrho)^{\alpha-1}]$ with $\alpha>0$ and $w_2(\eta)=\frac{\beta}{(\iota-\zeta)^{\beta-1}}[(\iota-\eta)^{\beta-1}+(\eta-\zeta)^{\beta-1}]$ with $\beta>0$ in Theorem 11 and Theorem 12, we get

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\varrho)^{\alpha}(\iota-\zeta)^{\beta}}\\
&\qquad\times\big[\mathcal{J}^{\alpha,\beta}_{\varrho+,\zeta+}F(\varsigma,\iota)\Omega(\varsigma,\iota)+\mathcal{J}^{\alpha,\beta}_{\varrho+,\iota-}F(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\mathcal{J}^{\alpha,\beta}_{\varsigma-,\zeta+}F(\varrho,\iota)\Omega(\varrho,\iota)
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{J}^{\alpha,\beta}_{\varsigma-,\iota-}\digamma(\varrho,\zeta)\Omega(\varrho,\zeta)\Big] \\
&\supseteq \left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\mathbf{K}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\mathbf{L}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\mathbf{N}(\varrho,\varsigma,\zeta,\iota)
\end{aligned}
$$

and

$$
\begin{aligned}
&4\digamma\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\varrho)^{\alpha}(\iota-\zeta)^{\beta}} \\
&\quad\times\Big[\mathcal{J}^{\alpha,\beta}_{\varrho+,\zeta+}\digamma(\varsigma,\iota)\Omega(\varsigma,\iota)+\mathcal{J}^{\alpha,\beta}_{\varrho+,\iota-}\digamma(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\mathcal{J}^{\alpha,\beta}_{\varsigma-,\zeta+}\digamma(\varrho,\iota)\Omega(\varrho,\iota) \\
&\quad+\mathcal{J}^{\alpha,\beta}_{\varsigma-,\iota-}\digamma(\varrho,\zeta)\Omega(\varrho,\zeta)\Big] \\
&\quad+\left\{\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\right\}\mathbf{K}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}\mathbf{L}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}\mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\
&\quad+\left\{\frac{1}{4}-\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}\mathbf{N}(\varrho,\varsigma,\zeta,\iota)
\end{aligned}
$$

which are proved by Kara et al. in [4].

4 Concluding remarks

In this research, some new Hermite–Hadamard–Fejér type inclusions for the product of two co-ordinated convex interval-valued functions are offered. We also proved that the results given in this work generalize the results given in [4, 53]. It is an interesting and new problem that the upcoming researchers can obtain similar inclusions for different kinds of convexities and integrals in their future investigation.

Acknowledgements
We are thankful to the editor and referees for their careful reading and valuable suggestions to make the article reader friendly.

Funding
The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485, 11971241).

Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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Publisher's Note

Received: 24 October 2020 Accepted: 28 January 2021 Published online: 10 February 2021

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