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Best approximation of a nonlinear fractional Volterra integro-differential equation in matrix MB-space

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Abstract

In this article, we introduce a class of stochastic matrix control functions to stabilize a nonlinear fractional Volterra integro-differential equation with Ψ-Hilfer fractional derivative. Next, using the fixed-point method, we study the Ulam–Hyers and Ulam–Hyers–Rassias stability of the nonlinear fractional Volterra integro-differential equation in matrix MB-space.

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1 Introduction

Fractional calculus is considered as a branch of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. Therefore, fractional calculus is an extension of the integer-order calculus that considers integrals and derivatives of any real or complex order [1, 2], i.e., unifies and generalizes the notions of integer-order differentiation and n-fold integration.

Different forms of fractional operators have been introduced, like the Riemann–Liouville, Grinwald–Letnikov, Weyl, Caputo, Marchaud, and Hadamard fractional derivatives. The first approach is that Riemann-Liouville, which is based on iterating the classical integral operator n times and then considering the Cauchy's formula where $n!$ is replaced by the Gamma function, and hence the fractional integral of noninteger order is defined.

Fractional calculus has attracted the attention of many mathematicians, but also of some researchers in other areas like physics, chemistry, and engineering. As it is well known, several physical phenomena are often better described by fractional derivatives. This is mainly due to the fact that fractional operators take into consideration the evolution of the system, by taking the global correlation, and not only local characteristics. Moreover, integer-order calculus sometimes contradicts the experimental results, and therefore, derivatives of fractional order may be more suitable [3–5].

Very useful physical applications have given birth to the variable-order fractional calculus, for example, in modeling mechanical behaviors [6]. Nowadays, variable-order fractional calculus is particularly recognized as a useful and promising approach in the modeling of diffusion processes, in order to characterize time- or concentration-dependent anomalous diffusion, or diffusion processes in inhomogeneous porous media [7].

Results on existence and stability of solutions of implicit fractional differential equations can be found in [8–11].

By proposing the study of solution stability via fractional integrals and fractional derivatives, we can generalize the results and obtain the usual ones as particular cases. In this article, we study distribution functions with the ranges in a class of matrix algebras with the generalized triangular norms, to define MB-space and introduce a new class of matrix control functions. Also, we will use two recent fractional operators, that is, of general differentiation and integration [12].

These concepts help us study the Hyers–Ulam (in short HU) and Hyers–Ulam–Rassias (in short HUR) stability of fractional nonlinear Volterra integro-differential equation (in short VIDE),

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\mu(\varsigma) = \boldsymbol{F}(\varsigma,\mu(\varsigma)) + \int_0^{\varsigma} \boldsymbol{H}(\varsigma,\vartheta,\mu(\varsigma))\,d\vartheta, \\ \mathcal{I}_{0+}^{1-\gamma}\mu(0) = \sigma, \end{cases} \tag{1.1}$$

with $\varsigma \in [0,T]$ and a continuous function (in short CF) $\boldsymbol{F}(\varsigma,\mu)$, also $\boldsymbol{H}(\varsigma,\vartheta,\mu)$ is a CF with respect to ς, ϑ and μ on $[0,T]\times\mathbb{R}\times\mathbb{R}$, σ is a fixed number, ${}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\mu(\cdot)$ is defined in (2.1) in which $0<\iota<1$, $0\le\kappa\le 1$, and $\mathcal{I}_{0+}^{1-\gamma}(\cdot)$ is the Ψ-Riemann–Liouville fractional integral in which $0\le\gamma<1$ [12].

2 Preliminaries

Here, we let $\Xi_1 = [0,T]$, with $T>0$, $\Xi_2 = (0,\infty)$, $\Xi_3 = (0,1]$, $\Xi_4 = [0,\infty]$, and $\Xi_5 = [0,1]$ (note that $\Xi_5^{\circ} = (0,1)$ denotes the interior of Ξ_5).

Let

$$\operatorname{diag} M_n(\Xi_5) = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} = \operatorname{diag}[a_1,\dots,a_n], a_1,\dots,a_n \in \Xi_5 \right\},$$

where $\operatorname{diag} M_n(\Xi_5)$ is equipped with the partial order relation:

$$\begin{aligned} &\boldsymbol{a} := \operatorname{diag}[a_1,\dots,a_n], \qquad \boldsymbol{b} := \operatorname{diag}[b_1,\dots,b_n] \in \operatorname{diag} M_n(\Xi_5), \\ &\boldsymbol{a} \preceq \boldsymbol{b} \iff a_j \le b_j \quad \text{for every } j = 1,\dots,n. \end{aligned}$$

Also, $\boldsymbol{a} \prec \boldsymbol{b}$ denotes that $\boldsymbol{a} \preceq \boldsymbol{b}$ and $\boldsymbol{a} \neq \boldsymbol{b}$; $\boldsymbol{a} \ll \boldsymbol{b}$ and $a_j < b_j$ for every $j = 1,\dots,n$. We define $\boldsymbol{e} := \operatorname{diag}[e,\dots,e]$ in $\operatorname{diag} M_n(\Xi_5)$ where $e \in \Xi_5$. For example, $\mathbf{1} = \operatorname{diag}[1,\dots,1]$ and $\mathbf{0} = \operatorname{diag}[0,\dots,0]$.

Now, we extend the concept of triangular norms [13, 14] on $\operatorname{diag} M_n(\Xi_5)$.

Definition 2.1 A generalized triangular norm (in short GTN) on $\operatorname{diag} M_n(\Xi_5)$ is an operation $\circledast : \operatorname{diag} M_n(\Xi_5) \times \operatorname{diag} M_n(\Xi_5) \to \operatorname{diag} M_n(\Xi_5)$ satisfying the following conditions:

(a) $(\forall \boldsymbol{a} \in \operatorname{diag} M_n(\Xi_5))(\boldsymbol{a} \circledast \mathbf{1} = \boldsymbol{a})$ (boundary condition);

(b) $(\forall (\boldsymbol{a},\boldsymbol{b}) \in (\operatorname{diag} M_n(\Xi_5))^2)(\boldsymbol{a} \circledast \boldsymbol{b} = \boldsymbol{b} \circledast \boldsymbol{a})$ (commutativity);

(c) $(\forall(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in (\operatorname{diag} M_n(\Xi_5)^3)(\boldsymbol{a} \circledast (\boldsymbol{b} \circledast \boldsymbol{c}) = (\boldsymbol{a} \circledast \boldsymbol{b}) \circledast \boldsymbol{c})$ (associativity);
(d) $(\forall(\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}') \in (\operatorname{diag} M_n(\Xi_5^4)(\boldsymbol{a} \preceq \boldsymbol{a}'$ and $\boldsymbol{b} \preceq \boldsymbol{b}' \Longrightarrow \boldsymbol{a} \circledast \boldsymbol{b} \preceq \boldsymbol{a}' \circledast \boldsymbol{b}'$ (monotonicity).

For every $\boldsymbol{a},\boldsymbol{b} \in \operatorname{diag} M_n(\Xi_5)$ and all sequences $\{\boldsymbol{a}_k\}$ and $\{\boldsymbol{b}_k\}$ converging to $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, suppose we have

$$\lim_k(\boldsymbol{a}_k \circledast \boldsymbol{b}_k) = \boldsymbol{a} \circledast \boldsymbol{b},$$

then, $\circledast$ on $\operatorname{diag} M_n(\Xi_5)$ is continuous GTN (in short CGTN). Now we present some examples of CGTN:

(1) If $\circledast_M : \operatorname{diag} M_n(\Xi_5) \times \operatorname{diag} M_n(\Xi_5) \to \operatorname{diag} M_n(\Xi_5)$ is defined by

$$\boldsymbol{a} \circledast_M \boldsymbol{b} = \operatorname{diag}[t_1,\dots,t_n] \circledast_M \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}\bigl[\min\{t_1,s_1\},\dots,\min\{t_n,s_n\}\bigr],$$

then $\circledast_M$ is CGTN (minimum CGTN);

(2) If $\circledast_P : \operatorname{diag} M_n(\Xi_5) \times \operatorname{diag} M_n(\Xi_5) \to \operatorname{diag} M_n(\Xi_5)$ is such that

$$\boldsymbol{a} \circledast_P \boldsymbol{b} = \operatorname{diag}[t_1,\dots,t_n] \circledast_P \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}[t_1 \cdot s_1,\dots,t_n \cdot s_n],$$

then $\circledast_P$ is CGTN (product CGTN);

(3) If $\circledast_L : \operatorname{diag} M_n(\Xi_5) \times \operatorname{diag} M_n(\Xi_5) \to \operatorname{diag} M_n(\Xi_5)$ is defined by

$$\begin{aligned}\boldsymbol{a} \circledast_L \boldsymbol{b} &= \operatorname{diag}[t_1,\dots,t_n] \circledast_L \operatorname{diag}[s_1,\dots,s_n] \\ &= \operatorname{diag}\bigl[\max\{t_1+s_1-1,0\},\dots,\max\{t_n+s_n-1,0\}\bigr],\end{aligned}$$

then $\circledast_P$ is CGTN (Lukasiewicz CGTN).

Now, we present some numerical examples:

$$\begin{aligned}\operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right] \circledast_M \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right] &= \begin{bmatrix}\frac{3}{7} & & \\ & 1 & \\ & & \frac{4}{5}\end{bmatrix} \circledast_M \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{2}{3}\end{bmatrix} \\ &= \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right], \\ \operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right] \circledast_P \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right] &= \begin{bmatrix}\frac{3}{7} & & \\ & 1 & \\ & & \frac{4}{5}\end{bmatrix} \circledast_P \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{8}{15}\end{bmatrix} \\ &= \operatorname{diag}\left[0,\frac{1}{5},\frac{8}{15}\right], \\ \operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right] \circledast_L \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right] &= \begin{bmatrix}\frac{3}{7} & & \\ & 1 & \\ & & \frac{4}{5}\end{bmatrix} \circledast_L \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}0 & & \\ & \frac{1}{5} & \\ & & \frac{7}{15}\end{bmatrix} \\ &= \operatorname{diag}\left[0,\frac{1}{5},\frac{7}{15}\right].\end{aligned}$$

Then we get

$$\begin{aligned}&\operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right]\circledast_M \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right]\\&\quad\succeq \operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right]\circledast_P \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right]\\&\quad\succeq \operatorname{diag}\left[\frac{3}{7},1,\frac{4}{5}\right]\circledast_L \operatorname{diag}\left[0,\frac{1}{5},\frac{2}{3}\right].\end{aligned}$$

We consider the set D^+ of matrix-distribution-function-valued (MDF-valued), left continuous and increasing functions $\varphi:\mathbb{R}\cup\{-\infty,\infty\}\to \operatorname{diag} M_n(\Xi_5)$ such that $\varphi_0=\mathbf{0}$ and $\varphi_{+\infty}=\mathbf{1}$. Now $O^+\subseteq D^+$ are all (proper) functions $\varphi\in D^+$ for which $\ell^-\varphi_{+\infty}=\mathbf{1}$ ($\ell^-\varphi_\tau=\lim_{\varsigma\to\tau^-}\varphi_\varsigma$). Note that proper MDF-valued functions are the MDF-valued functions of real random variables (i.e., of those random variables g that a.s. take real values ($P(|g|=\infty)=0$)).

In D^+, we define "$\preceq$" as follows:

$$\varphi\precsim\phi \quad\Longleftrightarrow\quad \varphi_\tau\preceq\phi_\tau,\quad \forall\tau\in\mathbb{R}.$$

Also for each $\varsigma\in\mathbb{R}$,

$$\nabla_\tau^\varsigma=\begin{cases}\mathbf{0}, & \text{if } \tau\le\varsigma,\\ \mathbf{1}, & \text{if } \tau>\varsigma,\end{cases}$$

belongs to D^+ and for every MDF-valued φ we have $\varphi\precsim\nabla^0$ [13, 15]. For example,

$$\varphi_\tau=\begin{cases}\mathbf{0}, & \text{if } \tau\le 0,\\ \operatorname{diag}[1-e^{-\tau},\frac{\tau}{1+\tau},e^{-\frac{1}{\tau}}], & \text{if } \tau>0,\end{cases}$$

is an MDF-valued function in $\operatorname{diag} M_3(\Xi_5)$. Note that $\varphi_\tau=\operatorname{diag}[\varphi_{1,\tau},\dots,\varphi_{n,\tau}]$, with $\varphi_{i,\tau}$ being distribution functions, is MDF-valued.

Definition 2.2 Consider the CGTN $\circledast$, a linear space W, and MDF-valued $\Omega: W\to O^+$. In this case, we define a matrix Menger normed space (MMN-space) $(W,\Omega,\circledast)$ as follows:

(MMN1) $\Omega_\tau^w=\nabla_\tau^0$ for all $\tau>0$ if and only if $w=0$;

(MMN2) $\Omega_\tau^{\alpha w}=\Omega_{\frac{\tau}{|\alpha|}}^w$ for all $w\in W$ and $\alpha\in\mathbb{C}$ with $\alpha\neq 0$;

(MMN3) $\Omega_{\tau+\varsigma}^{w+w'}\succeq\Omega_\tau^w\circledast\Omega_\varsigma^{w'}$ for all $w,w'\in W$ and $\tau,\varsigma\ge 0$.

A complete MMN-space is called MMB-space.

For example, the MDF-valued Ω given by

$$\Omega_\tau^w=\begin{cases}\mathbf{0}, & \text{if } \tau\le 0,\\ \operatorname{diag}[\exp(-\frac{\|w\|}{\tau}),\frac{\tau}{\tau+\|w\|},\exp(-\frac{\|w\|}{\tau})], & \text{if } \tau>0,\end{cases}$$

is a matrix Menger norm and $(W,\Omega,\circledast_M)$ is an MMN-space; here $(W,\|\cdot\|)$ is a normed linear space.

Approximation of functional equations was studied in MN-spaces, fuzzy metric spaces, and random multi-normed space [16, 17]. Also stability results for stochastic fractional differential and integral equations were considered in [18–27].

Theorem 2.3 ([28, 29]) *Let (U,ρ) be a complete Ξ_4-valued metric space and let $\Lambda : U \to U$ be a strictly contractive function with Lipschitz constant $\iota < 1$. Then, for a given element $\xi \in U$, either*

$$\rho\big(\Lambda^n\xi, \Lambda^{n+1}\xi\big) = \infty,$$

for each $n \in \mathbb{N}$ or there is $n_0 \in \mathbb{N}$ such that

(i) $\rho(\Lambda^n\xi, \Lambda^{n+1}\xi) < \infty$, *for every $n \geq n_0$;*

(ii) *the fixed point ζ^* of Λ is the limit point of the sequence $\{\Lambda^n\xi\}$;*

(iii) *in the set $V = \{\zeta \in U \mid \rho(\Lambda^{n_0}\xi, \zeta) < \infty\}$, ζ^* is the unique fixed point of Λ;*

(iv) $(1-\iota)\rho(\zeta,\zeta^*) \leq \rho(\zeta, \Lambda\zeta)$ *for every $\zeta \in V$.*

Definition 2.4 ([30]) The Gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty e^{-\varsigma}\varsigma^{z-1}\, d\varsigma, \quad z \in \mathbb{C}, Re(z) > 0.$$

Let $\iota \in \mathring{\Xi}_5$, let Δ be an integrable function on Ξ_1 and $\Psi \in C^1(\Xi_1)$ an increasing function with $\Psi'(\varsigma) \neq 0$, for each $\varsigma \in \Xi_1$. The Ψ-Hilfer fractional derivative is defined by [15]

$$^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\Delta(\varsigma) = \mathcal{I}_{0+}^{\kappa(1-\iota);\Psi}\left(\frac{1}{\Psi'(\varsigma)}\frac{d}{d\varsigma}\right)\mathcal{I}_{0+}^{(1-\kappa)(1-\iota);\Psi}\Delta(\varsigma). \tag{2.1}$$

Definition 2.5 If for every continuously differentiable function $\Delta(\varsigma)$ and MDF-valued φ satisfying

$$\Omega_\tau^{({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\Delta(\varsigma) - \boldsymbol{F}(\varsigma,\Delta(\varsigma)) - \int_0^\varsigma \boldsymbol{H}(\varsigma,\vartheta,\Delta(\varsigma))\,d\vartheta)} \succeq \varphi_\tau^\varsigma,$$

for every $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$, there exist a solution $\Delta_0(\varsigma)$ of the VIDE Eq. (1.1) and a fixed number $\lambda > 0$ with

$$\Omega_\tau^{(\Delta(\varsigma)-\Delta_0(\varsigma))} \succeq \varphi_{\frac{\tau}{\lambda}}^\varsigma,$$

for every $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$, where λ is independent of $\Delta(\varsigma)$ and $\Delta_0(\varsigma)$, then (1.1) has the HUR stability.

3 Main results

Consider the following hypotheses:

(H0) Assume that M, $L_{\boldsymbol{F}}$, $L_{\boldsymbol{H}}$ are positive real numbers with $2M(\max\{L_{\boldsymbol{F}}, L_{\boldsymbol{H}}\}) \in \mathring{\Xi}_5$ and let $\boldsymbol{F} : \Xi_1 \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{H} : \Xi_1 \times \Xi_1 \times \mathbb{R} \to \mathbb{R}$ be CFs satisfying

$$\Omega_\tau^{(\boldsymbol{F}(\varsigma,\Delta_1)-\boldsymbol{F}(\varsigma,\Delta_2))} \succeq \Omega_{\frac{\tau}{L_{\boldsymbol{F}}}}^{\Delta_1-\Delta_2}, \tag{3.1}$$

for all $\varsigma \in \Xi_1$, $\Delta_1, \Delta_2 \in \mathbb{R}$ and $\tau \in \Xi_2$, and

$$\Omega_\tau^{(\boldsymbol{H}(\varsigma,\vartheta,\Delta_1)-\boldsymbol{H}(\varsigma,\vartheta,\Delta_2))} \succeq \Omega_{\frac{\tau}{L_{\boldsymbol{H}}}}^{\Delta_1-\Delta_2}, \tag{3.2}$$

for all $\varsigma, \vartheta \in \Xi_1$, $\Delta_1, \Delta_2 \in \mathbb{R}$ and $\tau \in \Xi_2$.

Theorem 3.1 *Suppose* ($H0$) *holds and consider a nondecreasing function* $\Psi \in C(\Xi_1)$ *with* $\Psi'(\varsigma) \neq 0$ *and a CDF* $\Delta : \Xi_1 \to \mathbb{R}$ *satisfying*

$$\Omega_\tau^{({}^H\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\Delta(\varsigma)-\boldsymbol{F}(\varsigma,\Delta(\varsigma))-\int_0^\varsigma \boldsymbol{H}(\varsigma,\vartheta,\Delta(\vartheta))\,d\vartheta)} \succeq \varphi_\tau^\varsigma, \tag{3.3}$$

for all $\varsigma, \vartheta \in \Xi_1$, $\Delta \in \mathbb{R}$, *and* $\tau \in \Xi_2$, *where* φ *is MDF-valued with*

$$\mathcal{I}_{0+}^{\iota;\Psi}\Delta(\varsigma) := \frac{1}{\Gamma(\iota)}\int_0^\varsigma \Psi'(\xi)\big(\Psi(\varsigma)-\Psi(\xi)\big)^{\iota-1}\Delta(\xi)\,d\xi, \tag{3.4}$$

satisfying

$$\Omega_\tau^{\Delta(\varsigma)} \succeq \varphi_\tau^\varsigma \quad \Longrightarrow \quad \Omega_\tau^{\mathcal{I}_{0+}^{\iota;\Psi}\Delta(\varsigma)} \succeq \varphi_{\frac{\tau}{M}}^\varsigma, \qquad \inf_{\xi\in\Xi_1}\varphi_{\frac{\tau}{T}}^\xi \succeq \varphi_\tau^\varsigma, \tag{3.5}$$

for each $\varsigma \in \Xi_1$ *and* $\tau \in \Xi_2$. *Then, we can find a unique CF* $\Delta_0 : \Xi_1 \to \mathbb{R}$ *such that*

$$\begin{aligned}\Delta_0(\varsigma) = {} & \frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}\big(\varsigma,\Delta_0(\varsigma)\big) \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\left[\int_0^\xi \boldsymbol{H}\big(\varsigma,\vartheta,\Delta_0(\vartheta)\big)\,d\vartheta\right],\end{aligned} \tag{3.6}$$

with $\mathcal{I}_{0+}^{1-\gamma;\Psi}\Delta(0) = \sigma$, $\iota \in \mathring{\Xi}_5$, $\kappa \in \Xi_5$, *and*

$$\Omega_\tau^{(\Delta(\varsigma)-\Delta_0(\varsigma))} \succeq \varphi^\varsigma_{\frac{M\tau}{1-2M(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})}}, \tag{3.7}$$

for each $\varsigma \in \Xi_1$ *and* $\tau \in \Xi_2$.

Proof For $\alpha, \beta \in U$, we set

$$\rho(\alpha,\beta) = \inf\big\{\lambda \in \Xi_4 : \Omega_\tau^{(\alpha(\varsigma)-\beta(\varsigma))} \succeq \varphi_{\frac{\tau}{\lambda}}^\varsigma\big\}, \tag{3.8}$$

for each $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$, where

$$U = \{\alpha : \Xi_1 \to \mathbb{R} \text{ is a CF}\}.$$

Let $\Lambda : U \to U$ be given by

$$\Lambda\alpha(\varsigma) = \frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \tag{3.9}$$

$$+\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}\big(\varsigma,\alpha(\varsigma)\big)$$
$$+\mathcal{I}_{0+}^{\iota;\Psi}\left[\int_0^{\xi}\boldsymbol{H}\big(\varsigma,\vartheta,\alpha(\vartheta)\big)\,d\vartheta\right],$$

for all $\alpha\in\Xi_1$ and $\varsigma\in\Xi_1$.

First we show that Λ is strictly contractive on U. Let $\lambda_{\alpha\beta}\in\Xi_4$ be a fixed number with $\rho(\alpha,\beta)\le\lambda_{\alpha\beta}$ for any $\alpha,\beta\in U$, so from Eq. (3.8) we have

$$\Omega_{\tau}^{(\alpha(\varsigma)-\beta(\varsigma))}\succeq\varphi^{\varsigma}_{\frac{\tau}{\lambda_{\alpha\beta}}},\tag{3.10}$$

Let $0=\varpi_1<\varpi_2<\cdots<\varpi_k=T$, $\Delta\xi_i=\varpi_i-\varpi_{i-1}=\frac{|T-0|}{k}$, $i=1,2,\dots,k$ and $\|\Delta\xi\|=\max_{1\le i\le k}(\Delta\xi_i)$, for each $\varsigma,\xi\in\Xi_1$ and $\tau\in\Xi_2$. From Eqs. (3.2), (3.5), and (3.10), we have

$$\begin{aligned}
&\Omega_{\tau}^{(\int_0^{\xi}\boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta)}\\
&\quad=\Omega_{\tau}^{(\lim_{\|\Delta\xi\|\to0}\sum_{i=1}^{k}\boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i)}\\
&\quad=\lim_{\|\Delta\xi\|\to0}\Omega_{\tau}^{(\sum_{i=1}^{k}(\boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i))}\\
&\quad\succeq\lim_{\|\Delta\xi\|\to0}\circledast_M\Omega_{\frac{\tau}{k}}^{(\boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i)}\\
&\quad\succeq\inf_{\xi\in\Xi_1}\Omega_{\frac{\tau}{k\Delta\xi_i}}^{(\boldsymbol{H}(\varsigma,\xi,\alpha(\xi))-\boldsymbol{H}(\varsigma,\xi,\beta(\xi)))}\\
&\quad\succeq\inf_{\xi\in\Xi_1}\Omega_{\frac{\tau}{T}}^{(\boldsymbol{H}(\varsigma,\xi,\alpha(\xi))-\boldsymbol{H}(\varsigma,\xi,\beta(\xi)))}\\
&\quad\succeq\inf_{\xi\in\Xi_1}\varphi^{\xi}_{\frac{\tau}{T\lambda_{\alpha\beta}L_{\boldsymbol{H}}}}\\
&\quad\succeq\varphi^{\varsigma}_{\frac{\tau}{\lambda_{\alpha\beta}L_{\boldsymbol{H}}}},
\end{aligned}\tag{3.11}$$

Then, by Eqs. (3.1), (3.4), (3.5), (3.9), (3.10), and (3.11), we have

$$\begin{aligned}
&\Omega_{\tau}^{(\Lambda\alpha(\varsigma)-\Lambda\beta(\varsigma))}\\
&\quad=\Omega_{\tau}^{(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\xi)(\Psi(\varsigma)-\Psi(\xi))^{\iota-1}(\boldsymbol{F}(\xi,\alpha(\xi))-\boldsymbol{F}(\xi,\beta(\xi))+\int_0^{\xi}\boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta)\,d\xi)}\\
&\quad\succeq\Omega_{\frac{\tau}{2}}^{(\mathcal{I}_{0+}^{\iota;\Psi}(\boldsymbol{F}(\xi,\alpha(\xi))-\boldsymbol{F}(\xi,\beta(\xi)))\,d\xi)}\circledast_M\Omega_{\frac{\tau}{2}}^{(\mathcal{I}_{0+}^{\iota;\Psi}(\int_0^{\xi}\boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta))}\\
&\quad\succeq\varphi^{\varsigma}_{\frac{\tau}{2M\lambda_{\alpha\beta}L_{\boldsymbol{F}}}}\circledast_M\varphi^{\varsigma}_{\frac{\tau}{2M\lambda_{\alpha\beta}L_{\boldsymbol{H}}}}\\
&\quad\succeq\varphi^{\varsigma}_{\frac{\tau}{2M\lambda_{\alpha\beta}(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})}},
\end{aligned}\tag{3.12}$$

and we conclude that

$$\rho(\Lambda\alpha,\Lambda\beta)\le2M\lambda_{\alpha\beta}\big(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\}\big),$$

for all $\varsigma\in\Xi_1$ and $\tau\in\Xi_2$. Hence, we deduce that $\rho(\Lambda\alpha,\Lambda\beta)\le[2M(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})]\rho(\alpha,\beta)$ for any $\alpha,\beta\in U$, where $2M(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})\in\mathring{\Xi}_5$.

From Eq. (3.9), we can find a fixed number $\lambda \in \Xi_2$ such that

$$\begin{aligned}&\Omega_\tau^{(\Lambda\beta(\varsigma)-\beta_0(\varsigma))}\\&\quad=\Omega_\tau^{(\frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}(\varsigma,\beta_0(\varsigma))+\mathcal{I}_{0+}^{\iota;\Psi}[\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\beta_0(\vartheta))\,d\vartheta]-\beta_0(\varsigma))}\\&\quad\succeq\varphi_{\frac{\tau}{\lambda}}^{\varsigma},\end{aligned}$$

for arbitrary $\beta_0 \in U$, for all $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$. The boundedness property of

$$\boldsymbol{F}\big(\varsigma,\beta_0(\varsigma)\big),\qquad \boldsymbol{H}\big(\varsigma,\vartheta,\beta_0(\vartheta)\big),\qquad \beta_0(\varsigma),$$

$\min_{\varsigma\in\Xi_1}\varphi_\tau^\varsigma > 0$, and Eq. (3.8) imply that $\rho(\Lambda\beta,\beta_0) < \infty$. From Theorem 2.3, there exists a CF $\Delta_0 : \Xi_1 \to \mathbb{R}$ such that $\Lambda^n\Delta_0 \to \Delta_0$ in (U,ρ) and $\Lambda\Delta_0 = \Delta_0$.

Since β and Δ_0 are bounded on Ξ_1 for each $\beta \in U$ and $\min_{\varsigma\in\Xi_1}\varphi_\tau^\varsigma > 0$, we have a fixed number $\lambda_\beta \in \Xi_4$ with

$$\Omega_\tau^{(\beta_0(\varsigma)-\beta(\varsigma))}\succeq\varphi_{\frac{\tau}{\lambda_\beta}}^{\varsigma},$$

for any $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$. Thus $\rho(\beta_0,\beta) < \infty$ for any $\beta \in U$.

Therefore, $U = \{\beta \in U : \rho(\beta_0,\beta) < \infty\}$. Also Theorem 2.3 and Eq. (3.6) imply the uniqueness of Δ_0.

Using Eqs. (3.3), (3.5), and (3.9), we have

$$\Omega_\tau^{(\Delta(\varsigma)-\frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma-\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}(\varsigma,\Delta(\varsigma))-\mathcal{I}_{0+}^{\iota;\Psi}[\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\Delta(\vartheta))\,d\vartheta])}\succeq\varphi_{\frac{\tau}{M}}^{\varsigma}.$$

Then, we obtain

$$\Omega_\tau^{(\Delta(\varsigma)-\Lambda\Delta(\varsigma))}\succeq\varphi_{\frac{\tau}{M}}^{\varsigma},$$

for any $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$, which implies

$$\rho(\Delta,\Lambda\Delta)\leq M. \tag{3.13}$$

From Theorem 2.3 and Eq. (3.13), we deduce that

$$\rho(\Delta,\Delta_0)\leq\frac{1}{1-2M(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})}\rho(\Lambda\Delta,\Delta)\leq\frac{M}{1-2M(\max\{L_{\boldsymbol{F}},L_{\boldsymbol{H}}\})},$$

which implies Eq. (3.7). □

Theorem 3.2 *Assume that $\iota,\kappa \in \mathring{\Xi}_5$ and consider a nondecreasing function $\Psi \in C^1(\Xi_1)$ with $\Psi'(\varsigma) \neq 0$ for all $\varsigma \in \Xi_1$. Also let $L_{\boldsymbol{F}}, L_{\boldsymbol{H}} \in \Xi_2$ be fixed numbers such that $2M(\max\{L_{\boldsymbol{F}}, TL_{\boldsymbol{H}}\}) \in \mathring{\Xi}_5$. Consider CFs $\boldsymbol{F} : \Xi_1 \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{H} : \Xi_1 \times \Xi_1 \times \mathbb{R} \to \mathbb{R}$ satisfying Eqs. (3.1) and (3.2), respectively. If for $\varepsilon \geq 0$, $\tau \in \Xi_2$, $\boldsymbol{\epsilon}_\tau := \operatorname{diag}[e^{-\frac{\varepsilon}{\tau}},\ldots,e^{-\frac{\varepsilon}{\tau}}]$, a CDF $\Delta : \Xi_1 \to \mathbb{R}$ satisfies*

$$\Omega_\tau^{({}^H\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\Delta(\varsigma)-\boldsymbol{F}(\varsigma,\Delta(\varsigma))-\int_0^\varsigma \boldsymbol{H}(\varsigma,\vartheta,\Delta(\vartheta))\,d\vartheta)}\succeq\boldsymbol{\epsilon}_\tau,$$

for all $\varsigma, \vartheta \in \Xi_1$, $\Delta \in \mathbb{R}$, and $\tau \in \Xi_2$, then we can find a unique CF $\Delta_0 : \Xi_1 \to \mathbb{R}$ satisfying Eq. (3.6) *and*

$$\Omega_\tau^{(\Delta(\varsigma)-\Delta_0(\varsigma))} \succeq \boldsymbol{\epsilon}_{\frac{M\tau}{1-2M(\max\{L_{\boldsymbol{F}}, TL_{\boldsymbol{H}}\})}}, \tag{3.14}$$

for all $\varsigma \in \Xi_1$ and $\Delta \in \mathbb{R}$.

Proof Let $U = \{\alpha : \Xi_1 \to \mathbb{R}$ is a CF$\}$. Consider the Ξ_4-valued metric on U defined by

$$\rho(\alpha, \beta) = \inf\Big\{\lambda \in \Xi_4 : \Omega_\tau^{(\alpha(\varsigma)-\beta(\varsigma))} \succeq \boldsymbol{\epsilon}_{\frac{\tau}{\lambda}}\Big\}, \tag{3.15}$$

for each $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$. In [15] the authors proved the completeness of (U, ρ).

Let $\Lambda : U \to U$ be given by

$$\begin{aligned}\Lambda\alpha(\varsigma) = {} & \frac{(\Psi(\varsigma) - \Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \\ & + \mathcal{I}_{0+}^{\iota;\Psi} \boldsymbol{F}\big(\varsigma, \alpha(\varsigma)\big) \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\left[\int_0^\xi \boldsymbol{H}\big(\varsigma, \vartheta, \alpha(\vartheta)\big)\,d\vartheta\right],\end{aligned} \tag{3.16}$$

for all $\varsigma \in \Xi_1$.

Let $\alpha, \beta \in U$ and consider a fixed number $\lambda_{\alpha\beta} \in \Xi_4$ such that $\rho(\alpha, \beta) \le \lambda_{\alpha\beta}$ and

$$\Omega_\tau^{(\alpha(\varsigma)-\beta(\varsigma))} \succeq \boldsymbol{\epsilon}_{\frac{\tau}{\lambda_{\alpha\beta}}}, \tag{3.17}$$

Let $0 = \varpi_1 < \varpi_2 < \cdots < \varpi_k = T$, $\Delta\xi_i = \varpi_i - \varpi_{i-1} = \frac{|T-0|}{k}$, $i = 1, 2, \ldots, k$, and $\|\Delta\xi\| = \max_{1\le i\le k}(\Delta\xi_i)$, for each $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$.

From Eqs. (3.2) and (3.17), we have

$$\begin{aligned}& \Omega_\tau^{(\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta)} \\ & \quad = \Omega_\tau^{(\lim_{\|\Delta\xi\|\to 0}\sum_{i=1}^k \boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i)} \\ & \quad = \lim_{\|\Delta\xi\|\to 0} \Omega_\tau^{(\sum_{i=1}^k (\boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i))} \\ & \quad \succeq \lim_{\|\Delta\xi\|\to 0} \circledast_M \Omega_{\frac{\tau}{k}}^{(\boldsymbol{H}(\varsigma,\varpi_i,\alpha(\varpi_i))-\boldsymbol{H}(\varsigma,\varpi_i,\beta(\varpi_i))\Delta\xi_i)} \\ & \quad \succeq \inf_{\xi\in\Xi_1} \Omega_{\frac{\tau}{k\Delta\xi_i}}^{(\boldsymbol{H}(\varsigma,\xi,\alpha(\xi))-\boldsymbol{H}(\varsigma,\xi,\beta(\xi)))} \\ & \quad \succeq \inf_{\xi\in\Xi_1} \Omega_{\frac{\tau}{T}}^{(\boldsymbol{H}(\varsigma,\xi,\alpha(\xi))-\boldsymbol{H}(\varsigma,\xi,\beta(\xi)))} \\ & \quad \succeq \inf_{\xi\in\Xi_1} \boldsymbol{\epsilon}_{\frac{\tau}{T\lambda_{\alpha\beta}L_{\boldsymbol{H}}}} \\ & \quad = \boldsymbol{\epsilon}_{\frac{\tau}{T\lambda_{\alpha\beta}L_{\boldsymbol{H}}}},\end{aligned} \tag{3.18}$$

Then, by Eqs. (3.1), (3.16), and (3.17), we have

$$\begin{aligned}
&\Omega_\tau^{(\Lambda\alpha(\varsigma)-\Lambda\beta(\varsigma))} \\
&\quad= \Omega_\tau^{(\frac{1}{\Gamma(\iota)}\int_0^\varsigma \Psi'(\xi)(\Psi(\varsigma)-\Psi(\xi))^{\iota-1}(\boldsymbol{F}(\xi,\alpha(\xi))-\boldsymbol{F}(\xi,\beta(\xi))+\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta)\,d\xi)} \\
&\quad\succeq \Omega_{\frac{\tau}{2}}^{(\mathcal{I}_{0+}^{\iota;\Psi}(\boldsymbol{F}(\xi,\alpha(\xi))-\boldsymbol{F}(\xi,\beta(\xi)))\,d\xi)} \circledast_M \Omega_{\frac{\tau}{2}}^{(\mathcal{I}_{0+}^{\iota;\Psi}(\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\alpha(\vartheta))-\boldsymbol{H}(\varsigma,\vartheta,\beta(\vartheta))\,d\vartheta))} \\
&\quad\succeq \boldsymbol{\epsilon}_{\frac{\tau}{2M\lambda_{\alpha\beta}L_{\boldsymbol{F}}}} \circledast_M \boldsymbol{\epsilon}_{\frac{\tau}{2MT\lambda_{\alpha\beta}L_{\boldsymbol{H}}}} \\
&\quad\succeq \boldsymbol{\epsilon}_{\frac{\tau}{2M\lambda_{\alpha\beta}(\max\{L_{\boldsymbol{F}},TL_{\boldsymbol{H}}\})}},
\end{aligned} \tag{3.19}$$

for each $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$. Therefore $\rho(\Lambda\alpha, \Lambda\beta) \le [2M(\max\{L_{\boldsymbol{F}}, TL_{\boldsymbol{H}}\})]\rho(\alpha,\beta)$ for any $\alpha, \beta \in U$, where $2M(\max\{L_{\boldsymbol{F}}, TL_{\boldsymbol{H}}\}) \in \mathring{\Xi}_5$.

From Eq. (3.16), we can find a fixed number $\lambda \in \Xi_2$ such that

$$\begin{aligned}
&\Omega_\tau^{(\Lambda\beta(\varsigma)-\beta_0(\varsigma))} \\
&\quad= \Omega_\tau^{(\frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}(\varsigma,\beta_0(\varsigma))+\mathcal{I}_{0+}^{\iota;\Psi}[\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\beta_0(\vartheta))\,d\vartheta]-\beta_0(\varsigma))} \\
&\quad\succeq \boldsymbol{\epsilon}_{\frac{\tau}{\lambda}},
\end{aligned}$$

for arbitrary $\beta_0 \in U$, for all $\varsigma \in \Xi_1$ and $\tau \in \Xi_2$. The boundedness property of

$$\boldsymbol{F}(\varsigma, \beta_0(\varsigma)), \boldsymbol{H}(\varsigma, \vartheta, \beta_0(\vartheta)), \beta_0(\varsigma)$$

and Eq. (3.15) imply that $\rho(\Lambda\beta, \beta_0) < \infty$. From Theorem 2.3, there exists a CF $\Delta_0 : \Xi_1 \to \mathbb{R}$ such that $\Lambda^n\Delta_0 \to \Delta_0$ in (U, ρ) and $\Lambda\Delta_0 = \Delta_0$. Using a method similar to that in the proof of Theorem 3.1, we get $\{\beta \in U : \rho(\beta_0, \beta) < \infty\} = U$. Also Theorem 2.3 and Eq. (3.6) imply the uniqueness of Δ_0.

Now, using Eq. (3.3) and [12, Theorem 5], we have

$$\Omega_\tau^{(\Delta(\varsigma)-\frac{(\Psi(\varsigma)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma-\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{F}(\varsigma,\Delta_0(\varsigma))-\mathcal{I}_{0+}^{\iota;\Psi}[\int_0^\xi \boldsymbol{H}(\varsigma,\vartheta,\Delta_0(\vartheta))\,d\vartheta])} \succeq \boldsymbol{\epsilon}_{\frac{\tau}{M}},$$

for all $\varsigma \in \Xi_1$, which implies

$$\rho(\Delta, \Lambda\Delta) \le M.$$

From Theorem 2.3 and Eq. (3.8), we deduce that

$$\Omega_\tau^{(\Delta(\varsigma)-\Delta_0(\varsigma))} \succeq \boldsymbol{\epsilon}_{\frac{M\tau}{1-2M(\max\{L_{\boldsymbol{F}},TL_{\boldsymbol{H}}\})}},$$

which implies Eq. (3.14) for all $\varsigma \in \Xi_1$. □

4 Conclusions

We introduced a new model of stochastic matrix control functions which helped us to stabilize a pseudo-nonlinear fractional Volterra integral equation and get better approximation for it. In fact, two kinds of novel stability concepts, i.e., Hyers–Ulam–Rassias and

Hyers–Ulam stability, of a fractional Volterra integral equation with delay are proved by using an alternative fixed point theorem in generalized complete metric spaces and the concept of stochastic matrix control functions in a matrix MB-space.

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