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An inertially constructed forward–backward splitting algorithm in Hilbert spaces

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Abstract

In this paper, we develop an iterative algorithm whose architecture comprises a modified version of the forward–backward splitting algorithm and the hybrid shrinking projection algorithm. We provide theoretical results concerning weak and strong convergence of the proposed algorithm towards a common solution of the fixed point problem associated to a finite family of demicontractive operators, the split equilibrium problem and the monotone inclusion problem in Hilbert spaces. Moreover, we compute a numerical experiment to show the efficiency of the proposed algorithm. As a consequence, our results improve various existing results in the current literature.

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Keywords: Fixed point problem; Forward–backward splitting algorithm; Monotone inclusion problem; Split equilibrium problem; Demicontractive operator; Hilbert spaces

1 Introduction

The theory of mathematical optimization provides a quantitative optimal solution associated with various real-world problems emerging in the fields of engineering, medicine, economics, management, and industry and other branches of the sciences. One of the main advantages of mathematical optimization is to provide effective iterative algorithms and the corresponding analysis of these iterative algorithms. Moreover, the viability of such iterative algorithms is evaluated in terms of computational performance and complexity. As a consequence, the theory of mathematical optimization has not only emerged as an independent subject to solve real-world problems but also serve as an interdisciplinary bridge between various branches of sciences.

Monotone operator theory is a fascinating field of research in nonlinear functional analysis and found valuable applications in the field of convex optimization, subgradients, partial differential equations, variational inequalities, signal and image processing, evolution equations and inclusions; see, for instance, [4, 12, 14, 30] and the references cited therein. It is noted that the convex optimization problem can be translated into finding a zero of a maximal monotone operator defined on a Hilbert space. On the other hand, the problem of

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finding a zero of the sum of two (maximal-) monotone operators is of fundamental importance in convex optimization and variational analysis [23, 27, 33]. The forward–backward algorithm is prominent among various splitting algorithms to find a zero of the sum of two maximal monotone operators [23]. The class of splitting algorithms has parallel computing architectures and thus reducing the complexity of the problems under consideration. On the other hand, the forward–backward algorithm efficiently tackle the situation for smooth and/or nonsmooth functions. It is worth mentioning that the forward–backward algorithm has been modified by employing the heavy ball method [28] for convex optimization problems.

Fixed point theory has been studied extensively in the current literature owing to its rich abstract structures. These structures and subsequent tools elegantly manipulate various mathematical problems from the areas such as control theory, game theory, mathematical economics, image recovery signal processing and image processing. In 2015, the problem of finding a common solution of the zero point problem and fixed point problem was studied by Takahashi et al. [32]. It is well known that the class of demicontractive operators [15] includes various classes of nonlinear operators and comparatively exhibits powerful applications. Therefore, it is natural to study the fixed point problems associated with the class of demicontractive operators.

The theory of equilibrium problems is a systematic approach to the study of a diverse range of problems arising in the field of physics, optimization, variational inequalities, transportation, economics, network and noncooperative games; see, for example, [5, 11–13] and the references cited therein. The classical equilibrium problem theory has been generalized in several interesting ways to solve real-world problems. In 2012, Censor et al. [9] proposed a theory regarding split variational inequality problem (SVIP) which aims to solve a pair of variational inequality problems in such a way that the solution of a variational inequality problem, under a given bounded linear operator, solves another variational inequality.

In 2011, Moudafi [26] suggested the concept of split monotone variational inclusions (SMVIP) which includes, as a special case, split variational inequality problem, split common fixed point problem, split zeros problem, split equilibrium problem (SEP) and split feasibility problem. These problems have already been studied and successfully employed as a model in intensity-modulated radiation therapy treatment planning; see [6, 8]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [10, 12]. Some methods have been proposed and analyzed to solve SEP and generalized SEP in Hilbert spaces; see, for example, [2, 3, 16–22] and the references cited therein.

Inspired and motivated by the above-mentioned results and the ongoing research in this direction, we aim to employ the modified inertial forward–backward algorithm to find a common solution of fixed point problem associated to a finite family of demicontractive operators, SEP and monotone inclusion problem in Hilbert spaces. The rest of the paper is organized as follows: Section 2 contains preliminary concepts and results regarding fixed point theory, equilibrium problem theory and monotone operator theory. Section 3 comprises weak and strong convergence results of the proposed algorithm. Section 4 deals with the efficiency of the proposed algorithm by a numerical experiment together with theo-

retical applications to the split feasibility problem, the split variational inequality problem and the split minimization problem.

2 Preliminaries

In this section, we recall concepts and results regarding fixed point theory, equilibrium problem theory and monotone operator theory. Throughout this paper, let $\mathcal{H}_1$ be a real Hilbert space with the inner product and the associated norm $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The symbols $\rightharpoonup$ and $\rightarrow$ denotes weak and strong convergence.

An operator P_C is said to be metric projection of $\mathcal{H}_1$ onto nonempty, closed and convex subset C, if for every $x \in \mathcal{H}_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - z\|, \quad \text{for all } z \in C.$$

It is noted that P_C is a firmly nonexpansive operator and $P_C x$ is characterized by the following property:

$$\langle x - P_C x, P_C x - y \rangle \ge 0, \quad \text{for all } x \in \mathcal{H}_1 \text{ and } y \in C.$$

Next, we recall the definitions of nonexpansive and related operators.

Definition 1 ([4]) Let C be a nonempty subset of $\mathcal{H}_1$, for an operator $T : C \to \mathcal{H}_1$, we denote by $Fix(T)$ the set of fixed points of the operator T, that is, $Fix(T) = \{x \in \mathcal{H}_1 | x = Tx\}$. The operator T is considered as:

1. nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C;$$

2. firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \left\| (Id - T)x - (Id - T)y \right\|^2, \quad \forall x, y \in C;$$

3. quasi-nonexpansive if $Fix(T) \ne \emptyset$ such that

$$\|Tx - y\| \le \|x - y\|, \quad \forall x \in C, y \in Fix(T);$$

4. $\Bbbk$-demicontractive if $Fix(T) \ne \emptyset$ and there exists $\Bbbk \in [0, 1)$ such that

$$\|Tx - y\|^2 \le \|x - y\|^2 + \Bbbk \|x - Ty\|^2, \quad \forall x \in C, y \in Fix(T).$$

It follows immediately that a firmly nonexpansive operator is a nonexpansive operator. We now define the concept of SEP. Let $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions, then SEP is to find:

$$x^* \in C \quad \text{such that} \quad F_1\big(x^*, x\big) \ge 0 \quad \text{for all } x \in C \tag{1}$$

and

$$y^* = \hbar x^* \in Q \quad \text{such that} \quad F_2\big(y^*, y\big) \geq 0 \quad \text{for all } y \in Q. \tag{2}$$

The solution set of the SEP (1) and (2) is denoted by

$$\Omega := \big\{ x^* \in C : x^* \in EP(F_1) \text{ and } \hbar x^* \in EP(F_2) \big\}. \tag{3}$$

Now, we recall some important concepts related to monotone operator theory [4].

Let $\mathcal{A} : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a set-valued operator. We denote its domain, range, graph and zeros by $Dom\mathcal{A} = \{x \in \mathcal{H}_1 | \mathcal{A}x \neq 0\}$, $Ran\mathcal{A} = \{u \in \mathcal{H}_1 | (\exists x \in \mathcal{H}_1)u \in \mathcal{A}x\}$, $Gra\mathcal{A} = \{(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1 | u \in \mathcal{A}x\}$ and $Zer\mathcal{A} = \{x \in \mathcal{H}_1 | 0 \in \mathcal{A}x\}$, respectively. Let the set-valued operator $\mathcal{A}$ is said to be monotone, if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in Gra\mathcal{A}.$$

Moreover, $\mathcal{A}$ is said to be maximal monotone if its graph is not strictly contained in the graph of any other monotone operator on $\mathcal{H}_1$. A well-known example of a maximal monotone operator is the subgradient operator of a proper, lower semicontinuous convex function $f : \mathcal{H}_1 \to (-\infty, +\infty]$ defined by

$$\partial f : \mathcal{H}_1 \to 2^{\mathcal{H}_1} : x \mapsto \big\{ u \in \mathcal{H}_1 | f(y) \geq f(x) + \langle u, y - x \rangle, \forall y \in \mathcal{H}_1 \big\}.$$

For a maximal monotone operator, the associated resolvent operator with index $m > 0$ is defined as

$$J_m = (Id + m\mathcal{A})^{-1},$$

where Id denotes the identity operator.

It is well known that the resolvent operator J_m is well-defined everywhere on Hilbert space $\mathcal{H}_1$. Furthermore, J_m is single-valued and satisfies the firmly nonexpansiveness. Furthermore, $x \in \mathcal{A}^{-1}(0)$ if and only if $x = J_m(x)$.

Let $f : \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and let $g : \mathcal{H}_1 \to \mathbb{R}$ be a convex, differentiable and Lipschitz continuous function, then the convex minimization problem for f and g is defined as

$$\min_{x \in \mathcal{H}_1} \big\{ f(x) + g(x) \big\}. \tag{4}$$

Definition 2 ([4]) Let $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonlinear operator. For $\gamma > 0$, the operator $\mathcal{B}$ is said to be γ-inverse strongly monotone (γ-ism) if

$$\langle x - y, \mathcal{B}x - \mathcal{B}y \rangle \geq \gamma \| \mathcal{B}x - \mathcal{B}y \|^2, \quad \forall x, y \in \mathcal{H}_1.$$

The γ-ism is also coined as γ-cocoercive operator. Moreover, γ-ism is $\frac{1}{\gamma}$-Lipschitz continuous. In connection with the problem (4), the monotone inclusion problem with respect to a maximally monotone operator $\mathcal{A}$ and an arbitrary operator $\mathcal{B}$ is to find:

$$x^* \in C \quad \text{such that} \quad 0 \in \mathcal{A}x^* + \mathcal{B}x^*. \tag{5}$$

In the sequel, we list some important results in the form of lemmas for the convergence analysis.

Lemma 2.1 ([4]) *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}_1$. Let $T : C \to C$ be an operator. Then the operator T is said to be demiclosed at zero, if for any sequence (x_k) in C that converges weakly to x and $(Id - T)x_k$ converges strongly to zero, then $x \in Fix(T)$.*

Lemma 2.2 *Let $x, y \in \mathcal{H}_1$ and $\beta \in \mathbb{R}$, then the following relations hold*:
- $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;
- $\|\beta x + (1 - \beta)y\|^2 = \beta\|x\|^2 + (1 - \beta)\|x\|^2 - \beta(1 - \beta)\|x - y\|^2$.

Lemma 2.3 ([31]) *Let E be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in E. Let $l, m \in E$ be such that $\lim_{n\to\infty} \|x_n - l\|$ and $\lim_{n\to\infty} \|x_n - m\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to l and m, respectively, then $l = m$.*

Lemma 2.4 ([24]) *Let E be a Banach space. Let $\mathcal{A} : E \to 2^E$ be an m-accretive operator and let $\mathcal{B} : E \to E$ be an α-inverse strongly accretive operator. Then we have*
   a) *For $r > 0$, $Fix(T_r^{\mathcal{A},\mathcal{B}}) = (\mathcal{A} + \mathcal{B})^{-1}(0)$,*
   b) *for $0 < s \le r$ and $x \in E$, $\|x - T_s^{\mathcal{A},\mathcal{B}}x\| \le 2\|x - T_r^{\mathcal{A},\mathcal{B}}\|$.*

Lemma 2.5 ([24]) *Let E be a uniformly convex and q-uniformly smooth Banach space for some $q \in (0, 2]$. Let $\mathcal{A} : E \to 2^E$ be an m-accretive operator and let $\mathcal{B} : E \to E$ be an α-inverse strongly accretive operator. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\varphi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi_q(0) = 0$ such that for all $x, y \in B_r$*

$$\left\| T_r^{\mathcal{A},\mathcal{B}}x - T_r^{\mathcal{A},\mathcal{B}}y \right\|^q \le \|x - y\|^q - r\left(\alpha q - r^{q-1}k_q\right)\|\mathcal{A}x - \mathcal{A}y\|^q$$
$$- \varphi_q\left(\left\| \left(Id - J_r^{\mathcal{B}}\right)(Id - r\mathcal{A})x - \left(Id - J_r^{\mathcal{B}}\right)(Id - r\mathcal{A})y\right\|\right),$$

where k_q is the q-uniform smoothness coefficient of E.

Lemma 2.6 ([1]) *Let $\{\xi_n\}$, $\{\eta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ satisfying*

$$\xi_{n+1} \le \xi_n + \alpha_n(\xi_n - \xi_{n-1}) + \eta_n, \quad \text{for all } n \ge 1,$$

provided that $\sum_{n=1}^{\infty} \eta_n < +\infty$ and $0 \le \alpha_n \le \alpha < 1$ for all $n \ge 1$. Then the following two relations hold:

$$\begin{cases} a) & \sum_{n\ge1}[\xi_n - \xi_{n-1}]_+ < +\infty, \quad \text{where } [t]_+ = \max\{t, 0\}; \\ b) & \text{there exists } \xi^* \in [0, +\infty) \text{ such that } \lim_{n\to+\infty} \xi_n = \xi^*. \end{cases}$$

Lemma 2.7 ([25]) *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}_1$. For every $x, y \in \mathcal{H}_1$ and $a \in \mathbb{R}$, the set*

$$D = \left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\}$$

is closed and convex.

Assumption 2.8 Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}_1$. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1): $F_1(x, x) = 0$ for all $x \in C$;

(A2): F_1 is monotone, i.e., $F_1(x, y) + F_1(y, x) \leq 0$ for all $x, y \in C$;

(A3): for each $x, y, z \in C$, $\limsup_{t \to 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$;

(A4): for each $x \in C$, $y \mapsto F_1(x, y)$ is convex and lower semicontinuous.

Lemma 2.9 ([11]) *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.8. For $r > 0$ and $x \in \mathcal{H}_1$, there exists $z \in C$ such that*

$$F_1(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \text{for all } y \in C.$$

Moreover, define an operator $T_r^F : \mathcal{H}_1 \to C$ by

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \text{ for all } y \in C \right\},$$

for all $x \in \mathcal{H}_1$. Then we have the following observations:

(1): *for each $x \in \mathcal{H}_1$, $T_r^{F_1}(x) \neq \emptyset$;*

(2): $T_r^{F_1}$ *is single-valued;*

(3): $T_r^{F_1}$ *is firmly nonexpansive;*

(4): $Fix(T_r^{F_1}) = EP(F_1)$;

(5): $EP(F_1)$ *is closed and convex.*

It is noted that if $F_2 : Q \times Q \to \mathbb{R}$ is a bifunction satisfying Assumption 2.8, where Q is a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}_2$. Then, for each $s > 0$ and $w \in \mathcal{H}_2$, we define the operator

$$T_s^{F_2}(w) = \left\{ d \in C : F_2(d, e) + \frac{1}{s}\langle e - d, d - w \rangle \geq 0, \text{ for all } e \in Q \right\}.$$

Similarly, we have the following relations:

(1): for each $w \in \mathcal{H}_2$, $T_s^{F_2}(w) \neq \emptyset$;

(2): $T_s^{F_2}$ is single-valued;

(3): $T_s^{F_2}$ is firmly nonexpansive;

(4): $Fix(T_s^{F_2}) = EP(F_2)$;

(5): $EP(F_2)$ is closed and convex.

3 Algorithm and convergence analysis

In this section, we present an approach to the convergence analysis of inertial forward–backward splitting method for solving the fixed point problem associated to a finite family of demicontractive operators, SEP and monotone inclusion problem in Hilbert spaces. First, we set the following hypotheses required in the sequel: Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and let $C \subseteq \mathcal{H}_1$, $Q \subseteq \mathcal{H}_2$ be nonempty, closed and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We consider the following hypotheses:

(H1) Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.8 such that F_2 is upper semicontinuous;

Algorithm 1 An inertially constructed forward–backward splitting algorithm

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Initialization: Choose $x_0, x_1 \in C$, set $k \geq 1$ and non-increasing sequences $(\beta_k), (\lambda_k) \subset (0, 1)$, $(\Theta_k) \subset [0, 1)$, $0 < m_k < 2\gamma$ and $\delta \in (0, \frac{1}{L})$ such that L is the spectral radius of $\hbar^* \hbar$.

Iterative Steps: Given $x_k \in \mathcal{H}_1$, calculate b_k, ℓ_k, w_k and x_{k+1} as follows:

    Step 1. Compute

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ \ell_k = T_{u_k}^{F_1}(Id - \delta\hbar^*(Id - T_{u_k}^{F_2})\hbar)b_k; \\ w_k = (1 - \beta_k)\ell_k + \beta_k S_i \ell_k. \end{cases}$$

    If $w_k = \ell_k = b_k = x_k$ then stop and x_k is the solution of problem Γ. Otherwise,
    Step 2. Compute

$$x_{k+1} = \lambda_k w_k + (1 - \lambda_k)J_k w_k,$$

    where $J_k = (Id + m_k\mathcal{A})^{-1}(Id - m_k\mathcal{B})$.
    Set $k =: k + 1$ and go back to *Step 1.*

(H2) let $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator;

(H3) let $\mathcal{A} : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximal monotone operator and let $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_1$ be a γ-ism operator;

(H4) for $i \in \{1, 2, \ldots, N\}$, let $S_i : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of $\Bbbk$-demicontractive operators;

(H5) suppose that $\Gamma := zer(\mathcal{A} + \mathcal{B}) \cap \Omega \cap \bigcap_{N}^{i=1} Fix(S_i)$.

Theorem 3.1 *If $\Gamma \neq \emptyset$ with hypotheses* (H1)–(H5), *then the sequence (x_k) generated by Algorithm* 1 *converges weakly to an element $\bar{x} \in \Gamma$, provided the following conditions hold*:

(C1) $\sum_{k=1}^{\infty} \Theta_k \|x_k - x_{k-1}\| < \infty$;

(C2) $0 < a^* \leq \beta_k, \lambda_k \leq b^* < 1$ *and* $\beta_k \in (0, 1 - \Bbbk)$;

(C3) $0 < \liminf_{k \to \infty} \lambda_k \leq \limsup_{k \to \infty} \lambda_k < 1$;

(C4) $\liminf_{k \to \infty} u_k > 0$;

(C5) $0 < \liminf_{k \to \infty} m_k \leq \limsup_{k \to \infty} m_k < 2\gamma$.

Proof First we show that $\hbar^*(Id - T_{u_k}^{F_2})\hbar$ is an $\frac{1}{L}$-ism operator. For this, we utilize the firmly nonexpansiveness of $T_{u_k}^{F_2}$, which implies that $(Id - T_{u_k}^{F_2})$ is an 1-ism operator. Now, observe that

$$\begin{aligned} \left\| \hbar^*\left(Id - T_{u_k}^{F_2}\right)\hbar x - \hbar^*\left(Id - T_{u_k}^{F_2}\right)\hbar y \right\|^2 &= \left\langle \hbar^*\left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y), \hbar^*\left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y) \right\rangle \\ &= \left\langle \left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y), \hbar^*\hbar\left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y) \right\rangle \\ &\leq L\left\langle \left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y), \left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y) \right\rangle \\ &= L\left\| \left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y) \right\|^2 \\ &\leq L\left\langle x - y, \hbar^*\left(Id - T_{u_k}^{F_2}\right)(\hbar x - \hbar y) \right\rangle, \end{aligned}$$

for all $x, y \in \mathcal{H}_1$. So, we observe that $\hbar^*(Id - T_{u_k}^{F_2})\hbar$ is an $\frac{1}{L}$-ism. Moreover, $Id - \delta\hbar^*(Id - T_{u_k}^{F_2})\hbar$ is nonexpansive provided $\delta \in (0, \frac{1}{L})$. Now, we divided the rest of the proof into the following three steps:

Step 1. Show that $\lim_{k\to\infty} \|x_k - \hat{x}\|$ exists for every $\hat{x} \in \Gamma$.

For any $\hat{x} \in \Gamma$, we get

$$
\begin{aligned}
\|b_k - \hat{x}\|^2 &= \|x_k + \Theta_k(x_k - x_{k-1}) - \hat{x}\|^2 \\
&\leq \|x_k - \hat{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 \\
&\quad + 2\Theta_k \langle x_k - \hat{x}, x_k - x_{k-1}\rangle.
\end{aligned}
\tag{6}
$$

Since $T_{u_k}^{F_1}\hat{x} = \hat{x}$ and using (6), we have

$$
\begin{aligned}
\|\ell_k - \hat{x}\|^2 &= \left\| T_{u_k}^{F_1}\left(Id - \delta\hbar^*(Id - T_{u_k}^{F_2})\hbar\right)b_k - \hat{x}\right\|^2 \\
&\leq \left\| b_k - \delta\hbar^*(Id - T_{u_k}^{F_2})\hbar b_k - \hat{x}\right\|^2 \\
&\leq \|b_k - \hat{x}\|^2 + \delta^2 \left\|\hbar^*(Id - T_{u_k}^{F_2})\hbar b_k\right\|^2 \\
&\quad + 2\delta\langle \hat{x} - b_k, \hbar^*(Id - T_{u_k}^{F_2})\hbar b_k\rangle.
\end{aligned}
\tag{7}
$$

Thus, we have

$$
\begin{aligned}
\|\ell_k - \hat{x}\|^2 &\leq \|b_k - \hat{x}\|^2 + \delta^2\langle\hbar b_k - T_{u_k}^{F_2}\hbar b_k, \hbar^*\hbar(Id - T_{u_k}^{F_2})\hbar b_k\rangle \\
&\quad + 2\delta\langle\hat{x} - b_k, \hbar^*(Id - T_{u_k}^{F_2})\hbar b_k\rangle.
\end{aligned}
\tag{8}
$$

Moreover, we have

$$
\begin{aligned}
\delta^2\langle\hbar b_k - T_{u_k}^{F_2}\hbar b_k, \hbar^*\hbar(Id - T_{u_k}^{F_2})\hbar b_k\rangle &\leq L\delta^2\langle\hbar b_k - T_{u_k}^{F_2}\hbar b_k, \hbar b_k - T_{u_k}^{F_2}\hbar b_k\rangle \\
&= L\delta^2\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2.
\end{aligned}
\tag{9}
$$

Note that

$$
\begin{aligned}
2\delta\langle\hat{x} - b_k, &\hbar^*(Id - T_{u_k}^{F_2})\hbar b_k\rangle \\
&= 2\delta\langle\hbar(\hat{x} - b_k), \hbar b_k - T_{u_k}^{F_2}\hbar b_k\rangle \\
&= 2\delta\left[\langle\hbar\hat{x} - T_{u_k}^{F_2}\hbar b_k, \hbar b_k - T_{u_k}^{F_2}\hbar b_k\rangle - \left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2\right] \\
&\leq 2\delta\left[\frac{1}{2}\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2 - \left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2\right] \\
&= -\delta\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2.
\end{aligned}
\tag{10}
$$

Utilizing (8)–(10), we have

$$
\begin{aligned}
\|\ell_k - \hat{x}\|^2 &\leq \|b_k - \hat{x}\|^2 + L\delta^2\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2 - \delta\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2 \\
&\leq \|b_k - \hat{x}\|^2 + \delta(L\delta - 1)\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2.
\end{aligned}
\tag{11}
$$

Since $\delta \in (0, \frac{1}{L})$, the estimate (11) implies that

$$\|\ell_k - \hat{x}\|^2 \le \|x_k - \hat{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \langle x_k - \hat{x}, x_k - x_{k-1} \rangle. \tag{12}$$

Furthermore, by using (6), (12) and (C2), we have

$$
\begin{aligned}
\|w_k - \hat{x}\|^2 &= \left\| (1 - \beta_k)\ell_k + \beta_k S_i \ell_k - \hat{x} \right\|^2 \\
&\le (1 - \beta_k)\|\ell_k - \hat{x}\|^2 + \beta_k \|S_i \ell_k - \hat{x}\|^2 - \beta_k(1 - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\le (1 - \beta_k)\|\ell_k - \hat{x}\|^2 + \beta_k \|\ell_k - \hat{x}\|^2 + \beta_k \Bbbk \|S_i \ell_k - \ell_k\|^2 \\
&\quad - \beta_k(1 - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\le \|\ell_k - \hat{x}\|^2 - \beta_k(1 - \Bbbk - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\le \|x_k - \hat{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \langle x_k - \hat{x}, x_k - x_{k-1} \rangle. \tag{13}
\end{aligned}
$$

Moreover, it follows from (6), (12), (13) and Lemma 2.5 that

$$
\begin{aligned}
\|x_{k+1} - \hat{x}\|^2 &= \left\| \lambda_k w_k + (1 - \lambda_k)J_k w_k - \hat{x} \right\|^2 \\
&\le \lambda_k \|w_k - \hat{x}\|^2 + (1 - \lambda_k)\|J_k w_k - \hat{x}\|^2 \\
&= \|w_k - \hat{x}\|^2 \\
&\le \|x_k - \hat{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \langle x_k - \hat{x}, x_k - x_{k-1} \rangle. \tag{14}
\end{aligned}
$$

From Lemma 2.6 and (C1), we conclude from the estimate (14) that $\lim_{k \to \infty} \|x_k - \hat{x}\|$ exists.

Step 2. Show that $x_k \rightharpoonup \bar{x} \in (\mathcal{A} + \mathcal{B})^{-1}(0)$.

Since $\hat{x} = J_k \hat{x}$, therefore it follows from Lemma 2.2 and Lemma 2.5 that

$$
\begin{aligned}
\|x_{k+1} - \hat{x}\|^2 &= \left\| \lambda_k w_k + (1 - \lambda_k)J_k w_k - \hat{x} \right\|^2 \\
&\le \lambda_k \|w_k - \hat{x}\|^2 + (1 - \lambda_k)\|J_k w_k - \hat{x}\|^2 \\
&\le \|w_k - \hat{x}\|^2 - (1 - \lambda_k)m_k(2\gamma - m_k)\|\mathcal{A}w_k - \mathcal{A}\hat{x}\|^2 \\
&\quad - (1 - \lambda_k)\|w_k - m_k \mathcal{A}w_k - J_k w_k + m_k \mathcal{A}\hat{x}\| \\
&\le \|\ell_k - \hat{x}\|^2 - \beta_k(1 - \Bbbk - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\quad - (1 - \lambda_k)m_k(2\gamma - m_k)\|\mathcal{A}w_k - \mathcal{A}\hat{x}\|^2 \\
&\quad - (1 - \lambda_k)\|w_k - m_k \mathcal{A}w_k - J_k w_k + m_k \mathcal{A}\hat{x}\| \\
&\le \|b_k - \hat{x}\|^2 - \beta_k(1 - \Bbbk - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\quad - (1 - \lambda_k)m_k(2\gamma - m_k)\|\mathcal{A}w_k - \mathcal{A}\hat{x}\|^2 \\
&\quad - (1 - \lambda_k)\|w_k - m_k \mathcal{A}w_k - J_k w_k + m_k \mathcal{A}\hat{x}\| \\
&\le \|x_k - \hat{x}\|^2 + 2\theta_k \langle x_k - x_{k-1}, b_k - \hat{x} \rangle - \beta_k(1 - \Bbbk - \beta_k) \left\| (Id - S_i)\ell_k \right\|^2 \\
&\quad - (1 - \lambda_k)m_k(2\gamma - m_k)\|\mathcal{A}w_k - \mathcal{A}\hat{x}\|^2 \\
&\quad - (1 - \lambda_k)\|w_k - m_k \mathcal{A}w_k - J_k w_k + m_k \mathcal{A}\hat{x}\|. \tag{15}
\end{aligned}
$$

As $\lim_{k\to\infty}\|x_k - \hat{x}\|$ exists, therefore utilizing, (C1), (C4), (C5) and (15), we get

$$\lim_{k\to\infty} (1 - \lambda_k)m_k(2\gamma - m_k)\|\mathcal{A}w_k - \mathcal{A}\hat{x}\| = 0. \tag{16}$$

Also from (15), we get

$$\lim_{k\to\infty} \|w_k - m_k\mathcal{A}w_k - J_kw_k + m_k\mathcal{A}\hat{x}\| = 0. \tag{17}$$

Using (16), (17) and the triangle inequality

$$\|w_k - m_k\mathcal{A}w_k - J_kw_k + m_k\mathcal{A}\hat{x}\| \le \|w_k - J_kw_k\| + m_k\|\mathcal{A}w_k - \mathcal{A}\hat{x}\|,$$

we get

$$\lim_{n\to\infty} \|J_kw_k - w_k\| = 0. \tag{18}$$

Since $\liminf_{k\to\infty} m_k > 0$ there exists $m > 0$ such that $m_k \ge m$ for all $k \ge 0$. It follows from Lemma 2.4(b) that

$$\left\| T_m^{\mathcal{A},\mathcal{B}} w_k - w_k \right\| \le 2\|J_kw_k - w_k\|.$$

Now utilizing (18), the above estimate implies that

$$\lim_{k\to\infty} \left\| T_m^{\mathcal{A},\mathcal{B}} w_k - w_k \right\| = 0. \tag{19}$$

As a consequence, we have

$$\lim_{k\to\infty} \|x_{k+1} - w_k\| = \lim_{k\to\infty} \left(1 - a^*\right)\|J_kw_k - w_k\| = 0. \tag{20}$$

Again, from (15), we have

$$\|x_{k+1} - \hat{x}\|^2 \le \|x_k - \hat{x}\|^2 + 2\theta_k\langle x_k - x_{k-1}, b_k - \hat{x}\rangle - \beta_k(1 - \Bbbk - \beta_k)\left\|(Id - S_i)\ell_k\right\|^2.$$

Rearranging the above estimate and using (C1), (C2), we get

$$\lim_{k\to\infty} \left\|(Id - S_i)\ell_k\right\| = 0. \tag{21}$$

This implies that

$$\lim_{k\to\infty} \|w_k - \ell_k\| = \lim_{k\to\infty} b^*\left\|(Id - S_i)\ell_k\right\| = 0. \tag{22}$$

Again, by Lemma 2.2, Lemma 2.6 and (11), we have

$$\|x_{k+1} - \hat{x}\|^2 \le \|x_k - \hat{x}\|^2 + 2\Theta_k\langle x_k - x_{k-1}, b_k - \hat{x}\rangle + \delta(L\delta - 1)\left\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\right\|^2.$$

Rearranging the above estimate, we have

$$-\delta(L\delta - 1)\big\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\big\|^2 \le \|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2$$

$$+ 2\Theta_k\langle x_k - \hat{x}, x_k - x_{k-1}\rangle. \tag{23}$$

Since $\delta(L\delta - 1) < 0$, it follows from (C1) and (23) that

$$\lim_{k\to\infty}\big\|\hbar b_k - T_{u_k}^{F_2}\hbar b_k\big\| = 0. \tag{24}$$

Note that $T_{u_k}^{F_1}$ is firmly nonexpansive and $Id - \delta\hbar^*(Id - T_{u_k}^{F_2})\hbar$ is nonexpansive, therefore we have

$$\begin{aligned}
\|\ell_k - \hat{x}\|^2 &= \big\|T_{u_k}^{F_1}\big(b_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big) - T_{u_k}^{F_1}\hat{x}\big\|^2 \\
&\le \big\langle T_{u_k}^{F_1}\big(b_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big) - T_{u_k}^{F_1}\hat{x}, b_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k - \hat{x}\big\rangle \\
&= \big\langle \ell_k - \hat{x}, b_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k - \hat{x}\big\rangle \\
&= \frac{1}{2}\Big\{\|\ell_k - \hat{x}\|^2 + \big\|b_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k - \hat{x}\big\|^2 \\
&\quad - \big\|\ell_k - b_k + \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|\ell_k - \hat{x}\|^2 + \|b_k - \hat{x}\|^2 - \big\|\ell_k - b_k + \delta\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|\ell_k - \hat{x}\|^2 + \|b_k - \hat{x}\|^2 - \big(\|\ell_k - b_k\|^2 + \delta^2\big\|\hbar^*\big(Id - T_{b_k}^{F_2}\big)\hbar b_k\big\|^2 \\
&\quad - 2\delta\big\langle \ell_k - b_k, \hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big\rangle\big)\Big\}.
\end{aligned}$$

So, we have

$$\|\ell_k - \hat{x}\|^2 \le \|b_k - \hat{x}\|^2 - \|\ell_k - b_k\|^2 + 2\delta\big\langle \ell_k - b_k, \hbar^*\big(Id - T_{u_k}^{F*}\big)\hbar b_k\big\rangle. \tag{25}$$

Therefore, we have

$$\|\ell_k - b_k\|^2 \le \|b_k - \hat{x}\|^2 - \|\ell_k - \hat{x}\|^2 + 2\delta\|\ell_k - b_k\|\big\|\hbar^*\big(Id - T_{u_k}^{F_2}\big)\hbar b_k\big\|. \tag{26}$$

Utilizing (24) and (C2), we have

$$\lim_{k\to\infty}\|\ell_k - b_k\| = 0. \tag{27}$$

From the definition of (b_k) and (27), we have

$$\lim_{k\to\infty}\|\ell_k - x_k\| = 0. \tag{28}$$

By the definition of (b_k) and (C1), we have

$$\lim_{k\to\infty}\|b_k - x_k\| = \lim_{k\to\infty}\Theta_k\|x_k - x_{k-1}\| = 0. \tag{29}$$

Since (x_k) is bounded and $\mathcal{H}_1$ is reflexive, $\nu_\omega(x_k) = \{x \in \mathcal{H}_1 : x_{k_n} \rightharpoonup x, (x_{k_n}) \subset (x_k)\}$ is nonempty. Let $\bar{x} \in \nu_\omega(x_k)$ be an arbitrary element. Then there exists a subsequence $(x_{k_n}) \subset (x_k)$ converging weakly to $\bar{x}$. Let $\hat{x} \in \nu_\omega(x_k)$ and $(x_{k_m}) \subset (x_k)$ be such that $x_{k_m} \rightharpoonup \hat{x}$. From (24), we also have $\ell_{k_n} \rightharpoonup \bar{x}$ and $\ell_{k_m} \rightharpoonup \hat{x}$. Since $T_m^{\mathcal{A},\mathcal{B}}$ is nonexpansive, from (19) and Lemma 2.1, we have $\hat{x}, \bar{x} \in (\mathcal{A} + \mathcal{B})^{-1}(0)$. By applying Lemma 2.3, we obtain $\hat{x} = \bar{x}$.

Step 3. Show that $\bar{x} \in \Omega$.

Let $\bar{x} \in EP(F_1)$. For any $y \in \mathcal{H}_1$, we have

$$F_1(\ell_k, y) + \frac{1}{u_k} \langle y - \ell_k, \ell_k - x_k - \delta\hbar^*\big(Id - T_{u_k}^{F_2}\hbar b_k\big)\rangle \geq 0.$$

This implies that

$$F_1(\ell_k, y) + \frac{1}{u_k}\langle y - \ell_k, \ell_k - x_k\rangle - \frac{1}{u_k}\langle y - \ell_k, \delta\hbar^*\big(Id - T_{u_k}^{F_2}\hbar b_k\big)\rangle \geq 0.$$

From Assumption 2.8(A2), we have

$$\frac{1}{u_k}\langle y - \ell_k, \ell_k - x_k\rangle - \frac{1}{u_k}\langle y - \ell_k, \delta\hbar^*\big(Id - T_{u_k}^{F_2}\hbar b_k\big)\rangle \geq -F_1(\ell_k, y) \geq F_1(y, \ell_k).$$

So, we have

$$\frac{1}{u_{k_n}}\langle y - \ell_{k_n}, \ell_{k_n} - x_{k_n}\rangle - \frac{1}{u_{k_n}}\langle y - \ell_{k_n}, \delta\hbar^*\big(Id - T_{u_k}^{F_2}\hbar b_{k_n}\big)\rangle \geq F(y, \ell_{k_n}). \tag{30}$$

Utilizing (28) and (C2), we get $\ell_{k_n} \rightharpoonup \bar{x}$. Moreover, from (24) and Assumption 2.8(A4), we get

$$F(y, \bar{x}) \leq 0, \quad \text{for all } y \in \mathcal{H}_1.$$

Let $y_t = ty + (1 - t)\bar{x}$ for some $1 \geq t > 0$ and $y \in \mathcal{H}_1$. Since $\bar{x} \in \mathcal{H}_1$, consequently, $y_t \in \mathcal{H}_1$ and hence $F_1(y_t, \bar{x}) \leq 0$. Using Assumption 2.8((A1) and (A4)), it follows that

$$\begin{aligned}
0 &= F_1(y_t, y_t)\\
&\leq tF_1(y_t, y) + (1 - t)F_1(y_t, \bar{x})\\
&\leq t\big(F_1(y_t, y)\big).
\end{aligned}$$

This implies that

$$F_1(y_t, y) \geq 0, \quad \text{for all } y \in C.$$

Letting $t \to 0$, we have

$$F_1(\bar{x}, y) \geq 0, \quad \text{for all } y \in C.$$

Thus, $\bar{x} \in EP(F_1)$. Similarly, we can show that $\bar{x} \in EP(F_2)$. Since $\hbar$ is a bounded linear operator, we have $\hbar x_{k_n} \rightharpoonup \hbar\bar{x}$. It follows from (26) that

$$T_{u_{k_n}}^{F_2}\hbar b_{k_n} \rightharpoonup \hbar\bar{x} \quad \text{as } n \to \infty. \tag{31}$$

Now, from Lemma 2.7 we have

$$F_2\big(T^{F_2}_{u_{k_n}}\hbar b_{k_n}, y\big) + \frac{1}{u_{k_n}}\big\langle y - T^{F_2}_{u_{k_n}}\hbar b_{k_n}, T^{F_2}_{u_{k_n}}\hbar b_{k_n} - \hbar b_{k_n}\big\rangle \geq 0,$$

for all $y \in \mathcal{H}_1$. Since F_2 is upper semicontinuous in the first argument and from (31), we have

$$F_2(\hbar\bar{x}, y) \geq 0,$$

for all $y \in \mathcal{H}_1$. This implies that $\hbar\bar{x} \in EP(F_2)$. Therefore, $\bar{x} \in \Omega$.

Step 4. From (21) and by using the demiclosed principle for S_i (it is evident that $x_{k_n} \rightharpoonup \bar{x}$ and $\lim_{k \to \infty} \|(Id - S_i)x_{k_n}\| = 0$), we have $\bar{x} \in \bigcap_{N}^{i=1} Fix(S_i)$ and hence $\bar{x} \in \Gamma$. This completes the proof. □

Now, we establish strong convergence results of Algorithm 2.

Theorem 3.2 *If $\Gamma \neq \emptyset$ with hypotheses* (H1)–(H5), *then the sequence* (x_k) *generated by Algorithm 2 converges strongly to an element $\bar{x} \in P_\Gamma x_1$, provided the conditions* (C1)–(C5) *hold.*

Proof The proof is divided into the following steps:

Step 1. Show that the sequence $\{x_k\}$ defined in Algorithm 2 is well-defined.

We know that $(\mathcal{A} + \mathcal{B})^{-1}(0)$, Ω and $Fix(S_i)$ are closed and convex by Lemma 2.4 and Lemma 2.9. Moreover, from Lemma 2.7 we see that C_{k+1} is closed and convex for each $k \geq$

Algorithm 2 An inertially constructed forward–backward splitting algorithm

Initialization: Choose $x_0, x_1 \in C$, set $k \geq 1$ and non-increasing sequences $(\beta_k), (\lambda_k) \subset (0, 1)$, $(\Theta_k) \subset [0, 1)$, $0 < m_k < 2\gamma$ and $\delta \in (0, \frac{1}{L})$ such that L is the spectral radius of $\hbar^*\hbar$.

Step 1. Compute

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ \ell_k = T^{F_1}_{u_k}(Id - \delta\hbar^*(Id - T^{F_2}_{u_k})\hbar)b_k; \\ w_k = (1 - \beta_k)\ell_k + \beta_k S_i \ell_k; \\ y_k = \lambda_k w_k + (1 - \lambda_k)J_k w_k; \end{cases}$$

where $J_k = (Id + m_k\mathcal{A})^{-1}(Id - m_k\mathcal{B})$.

If $y_k = b_k = \ell_k = w_k = x_k$ then stop and x_k is the solution of problem Γ. Otherwise,

Step 2. Compute

$$C_{k+1} = \big\{z \in C_k :$$
$$\|y_k - z\|^2 \leq \|x_k - \hat{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 + 2\Theta_k\langle x_k - \hat{x}, x_k - x_{k-1}\rangle\big\},$$

$$x_{k+1} = P_{C_{k+1}}x_1, \quad \forall k \geq 1.$$

Set $k =: k + 1$ and go back to *Step 1*.

1. Hence the projection $P_{C_{k+1}}x_1$ is well-defined. For any $\hat{x} \in \Gamma$, it follows from Algorithm 2 and the estimates (6), (12) and (13) that

$$
\begin{aligned}
\|y_k - \hat{x}\|^2 &= \lambda_k \|w_k - \hat{x}\|^2 + (1 - \lambda_k)\|J_k w_k - \hat{x}\|^2 \\
&\leq \|w_k - \hat{x}\|^2 \\
&\leq (1 - \beta_k)\|\ell_k - \hat{x}\|^2 + \beta_k \|S_i \ell_k - \hat{x}\|^2 \\
&\leq \|\ell_k - \hat{x}\|^2 - \beta_k(1 - \Bbbk - \beta_k)\|(Id - S_i)\ell_k\|^2 \\
&\leq \|x_k - \hat{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \langle x_k - \hat{x}, x_k - x_{k-1}\rangle.
\end{aligned}
\tag{32}
$$

It follows from the estimate (32) that $\Gamma \subset C_{k+1}$. Summing up these facts, we conclude that C_{k+1} is nonempty, closed and convex for all $k \geq 1$, and hence the sequence (x_k) is well-defined.

Step 2. Show that $\lim_{k\to\infty}\|x_k - x_1\|$ exists.

Since Γ is nonempty closed and convex subset of $\mathcal{H}_1$, there exists a unique $x^* \in \Gamma$ such that $x^* = P_\Gamma x_1$. From $x_{k+1} = P_{C_{k+1}}x_1$, we have $\|x_{k+1} - x_1\| \leq \|x^* - x_1\|$, for all $\bar{x} \in \Gamma \subset C_{k+1}$. In particular $\|x_{k+1} - x_1\| \leq \|P_\Gamma x_1 - x_1\|$. This proves that the sequence (x_k) is bounded. On the other hand, from $x_k = P_{C_k}x_1$ and $x_{k+1} = P_{C_{k+1}}x_1 \in C_{k+1}$, we get

$$
\|x_k - x_1\| \leq \|x_{k+1} - x_1\|.
$$

This implies that (x_k) is nondecreasing and hence

$$
\lim_{k\to\infty}\|x_k - x_1\| \quad \text{exists.}
\tag{33}
$$

Step 3. Show that $\bar{x} \in (\mathcal{A} + \mathcal{B})^{-1}(0)$.

In order to proceed, we first calculate the following estimates which are required in the sequel:

$$
\begin{aligned}
\|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_1 + x_1 - x_k\|^2 \\
&= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_1\rangle \\
&= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k + x_k - x_1\rangle \\
&= \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k\rangle \\
&\leq \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2.
\end{aligned}
$$

Taking $\limsup$ on both sides of the above estimate and utilizing (33), we have $\limsup_{k\to\infty}\|x_{k+1} - x_k\|^2 = 0$. That is,

$$
\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0.
\tag{34}
$$

Note that $x_{k+1} \in C_{k+1}$, therefore we have

$$
\|y_k - x_{k+1}\| \leq \|x_k - x_{k+1}\| + 2\Theta_k \|x_k - x_{k-1}\| - 2\Theta_k \langle x_k - x_{k+1}, x_{k-1} - x_k\rangle.
$$

Utilizing (34) and (C1), the above estimate implies that

$$\lim_{k \to \infty} \|y_k - x_{k+1}\| = 0. \tag{35}$$

From (34), (35) and the triangular inequality

$$\|y_k - x_k\| \le \|y_k - x_{k+1}\| + \|x_{k+1} - x_k\|,$$

we get

$$\lim_{k \to \infty} \|y_k - x_k\| = 0. \tag{36}$$

Also, from Lemma 2.2 and (21), we have

$$
\begin{aligned}
\|y_k - \hat{x}\|^2 &= \lambda_k \|w_k - \hat{x}\|^2 + (1 - \lambda_k)\|J_k w_k - \hat{x}\|^2 - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2 \\
&\le \|w_k - \hat{x}\|^2 - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2 \\
&\le (1 - \beta_k)\|\ell_k - \hat{k}\|^2 + \beta_k \|S_i \ell_k - \hat{x}\|^2 - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2 \\
&\le \|\ell_k - \hat{x}\|^2 - \beta_k(1 - \Bbbk - \beta_k)\|(Id - S_i)\ell_k\|^2 \\
&\quad - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2 \\
&\le \|b_k - \hat{x}\|^2 - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2 \\
&\le \|x_k - \hat{x}\|^2 + 2\Theta_k \langle x_k - x_{k-1}, b_k - \hat{x} \rangle - \lambda_k(1 - \lambda_k)\|J_k w_k - w_k\|^2.
\end{aligned}
$$

Rearranging the above estimate, we have

$$
\begin{aligned}
a^*(1 - b^*)\|J_k w_k - w_k\|^2 &\le \|x_k - \hat{x}\|^2 - \|y_k - \hat{x}\|^2 + 2\Theta_k \langle x_k - x_{k-1}, b_k - \hat{x} \rangle \\
&\le \big(\|x_k - \hat{x}\| + \|y_k - \hat{x}\|\big)\|x_k - y_k\| + 2\Theta_k \langle x_k - x_{k-1}, b_k - \hat{x} \rangle.
\end{aligned}
$$

The above estimate, by using (C1) and (36), implies that

$$\lim_{k \to \infty} \|J_k w_k - w_k\| = 0. \tag{37}$$

Making use of (37), we have the following estimate:

$$\lim_{k \to \infty} \|y_k - w_k\| = \lim_{k \to \infty} (1 - a^*)\|J_k w_k - w_k\| = 0. \tag{38}$$

Reasoning as above, utilizing the estimate (37), the estimate (38) implies that

$$\lim_{k \to \infty} \|w_k - x_k\| = 0. \tag{39}$$

In a similar fashion, we have

$$\lim_{k \to \infty} \|T_m^{\mathcal{A},\mathcal{B}} w_k - w_k\| = 0. \tag{40}$$

Reasoning as above (Theorem 3.1, Step 2), we have the desired result.

Step 4. Show that $\bar{x} \in \Omega$.

See proof of Step 3 in Theorem 3.1.

Step 5. Show that $\bar{x} \in \bigcap_{N}^{i=1} Fix(S_i)$.

See proof of Step 4 in Theorem 3.1.

Step 6. Show that $\bar{x} = P_\Gamma x_1$.

Let $x = P_\Gamma x_1$ imply that $x = P_\Gamma x_1 \in C_{k+1}$. Since $x_{k+1} = P_{C_{k+1}} x_1 \in C_{k+1}$, we have

$$\|x_{k+1} - x_1\| \leq \|x - x_1\|.$$

On the other hand, we have

$$\|x - x_1\| \leq \|\bar{x} - x_1\|$$
$$\leq \liminf_{k \to \infty} \|x_k - x_1\|$$
$$\leq \limsup_{k \to \infty} \|x_k - x_1\|$$
$$\leq \|x - x_1\|.$$

That is,

$$\|\bar{x} - x_1\| = \lim_{k \to \infty} \|x_k - x_1\| = \|x - x_1\|.$$

Therefore, we conclude that $\lim_{k \to \infty} x_k = \bar{x} = P_\Gamma x_1$. This completes the proof.   □

The following remark gives us a stopping criterion of Algorithm 2.

Remark 3.3 We remark here that the condition (C1) is easily implemented in numerical computation since the value of $\|x_k - x_{k-1}\|$ is known before choosing Θ_k. The parameter Θ_k can be taken as $0 \leq \Theta_k \leq \widehat{\Theta_k}$,

$$\widehat{\Theta_k} = \begin{cases} \min\{\frac{\nu_k}{\|x_k - x_{k-1}\|}, \Theta\} & \text{if } x_k \neq x_{k-1}; \\ \Theta & \text{otherwise,} \end{cases}$$

where $\{\nu_k\}$ is a positive sequence such that $\sum_{k=1}^{\infty} \nu_k < \infty$ and $\Theta_k = \Theta \in [0, 1)$.

4 Numerical experiment and results

This section shows the effectiveness of Algorithm 2 by the following given example.

Example 4.1 Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$ be the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy$, for all $x, y \in \mathbb{R}$ and the usual induced norm $|\cdot|$. Let $F_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction defined as $F_1(x, y) = 2x(y - x)$ and let $F_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction defined as $F_2(p, q) = p(q - p)$. For all $x \in \mathbb{R}$, let the operators $\hbar, \mathcal{A}, \mathcal{B} : \mathbb{R} \to \mathbb{R}$ be defined as $\hbar(x) = 3x$, $\mathcal{A}x = 4x$ and $\mathcal{B}x = 3x$, respectively. Let $S_i : \mathbb{R} \to \mathbb{R}$ be a finite family of demicontractive operators defined by

$$S_i(x) = \begin{cases} -\frac{x}{i}, & x \in [0, \infty); \\ x, & x \in (-\infty, 0). \end{cases}$$

Then the sequence (x_k) generated by Algorithm 2 strongly converges to a point in Γ.

Proof It is easy to prove that the bifunctions F_1 and F_2 satisfy Assumptions 2.8 and F_2 is upper semicontinuous with $\Omega = 0$. Moreover, $\hbar$ is bounded linear operator on $\mathbb{R}$ with the adjoint operator $\hbar^*$ such that $\|\hbar\| = \|\hbar^*\| = 3$, $\mathcal{A}$ is a maximal monotone operator and $\mathcal{B}$ is a monotone and γ-Lipschitz operator for some $\gamma > 0$ with $(\mathcal{A} + \mathcal{B})^{-1}(0) = \{0\}$. Note that S_i is a finite family of $\frac{1-i^2}{(1+i)^2}$-demicontractive operators with $\bigcap_N^{i=1} Fix(S_i) = \{0\}$. Hence $\Gamma = (\mathcal{A} + \mathcal{B})^{-1}(0) \cap \Omega \cap \bigcap_\infty^{i=1} Fix(S_i) = 0$. Choose $\Theta = 0.5$, $u_k = \frac{k}{5k+1}$, $\beta_k = \frac{1}{100k+1}$, $\lambda_k = \frac{1}{100k+1}$, $\delta = 0.04$, $L = 3$ and $m = 0.01$. Since

$$
\begin{cases}
\min\{\frac{1}{k^2\|x_k - x_{k-1}\|}, 0.5\} & \text{if } x_k \neq x_{k-1}; \\
0.5 & \text{otherwise.}
\end{cases}
$$

For the rest of the numerical experiment, we proceed as follows:

Step 1. Find $z \in F_2$ such that $F_2(z, y) + \frac{1}{u}\langle y - z, z - \hbar x \rangle \geq 0$ for all $y \in F_2$. We write

$$
\begin{aligned}
F_2(z, y) + \frac{1}{u}\langle y - z, z - \hbar x \rangle \geq 0 \quad &\Leftrightarrow \quad z(y - z) + \frac{1}{u}\langle y - z, z - \hbar x \rangle \geq 0 \\
&\Leftrightarrow \quad uz(y - z) + (y - z)(z - \hbar x) \geq 0 \\
&\Leftrightarrow \quad (y - z)\big((1 + u)z - \hbar x\big) \geq 0,
\end{aligned}
$$

for all $y \in F_2$. Thus, by Lemma 2.9(2), we know that $T_u^{F_2}\hbar x$ is single-valued for each $x \in F_1$. Hence $z = \frac{\hbar x}{1+u}$.

Step 2. Find $g \in F_1$ such that $g = x - \delta\hbar^*(I - T_r^{F_2})\hbar x$. From Step 1, we get

$$
\begin{aligned}
g = x - \delta\hbar^*\big(I - T_u^{F_2}\big)\hbar x &= x - \delta\hbar^*\big(I - T_u^{F_2}\big)\hbar x \\
&= x - \delta\left(3x - \frac{3(\hbar x)}{1 + u}\right) \\
&= (1 - 3\delta)x + \frac{3\delta}{1 + u}(\hbar x).
\end{aligned}
$$

Step 3. Find $p \in F_1$ such that $F_1(p, q) + \frac{1}{u}\langle p - q, p - g \rangle \geq 0$ for all $q \in F_1$. From Step 2, we have

$$
\begin{aligned}
F(p, q) + \frac{1}{u}\langle q - p, p - g \rangle \geq 0 \quad &\Leftrightarrow \quad (2p)(q - p) + \frac{1}{u}\langle q - p, p - g \rangle \geq 0 \\
&\Leftrightarrow \quad u(2p)(q - p) + (q - p)(p - g) \geq 0 \\
&\Leftrightarrow \quad (q - p)\big((1 + 2u)p - g\big) \geq 0,
\end{aligned}
$$

for all $q \in F_1$. Similarly, by Lemma 2.9(2), we obtain $p = \frac{g}{1+2u} = \frac{(1-3\delta)x}{1+2u} + \frac{3\delta\hbar x}{(1+u)(1+2u)}$.
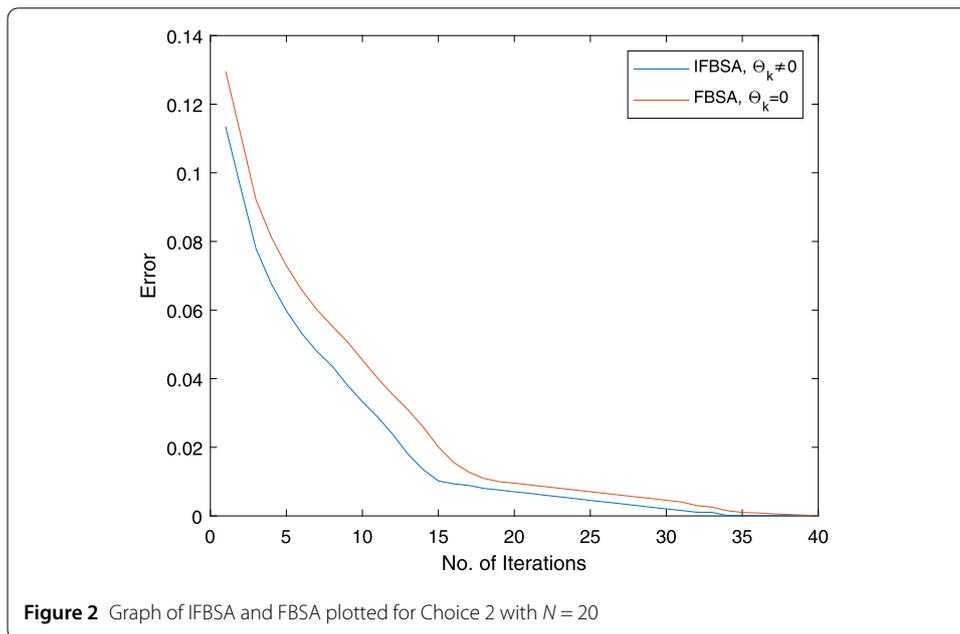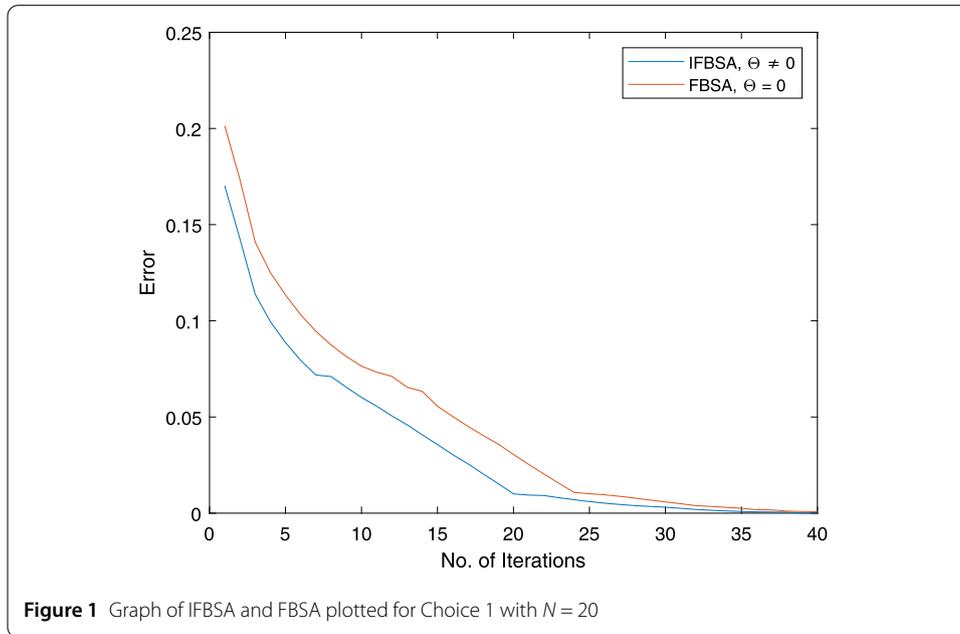
Step 4. Compute the numerical results for x_{k+1}.

We provide a numerical test of a comparison between our Inertial Forward–Backward Splitting Algorithm (IFBSA) defined in Algorithm 2 (i.e., $\Theta_k \neq 0$) and the Forward–Backward Splitting Algorithm (FBSA) (i.e., $\Theta_k = 0$). The stopping criterion is defined as Error $= E_k = \|x_{k+1} - x_k\| < 10^{-6}$. The different choices of x_0 and x_1 are given in tables and figures.

The error plotting E_k and (x_k) against $\Theta_k \neq 0$ and $\Theta_k = 0$ for each choice in Table 1 is shown in Fig. 1. □

Table 1 Numerical results for Example 4.1

	No. of iter.		CPU (sec)	
	$\Theta_k = 0$	$\Theta_k \neq 0$	$\Theta_k = 0$	$\Theta_k \neq 0$
Choice 1. $x_0 = (5)$, $x_1 = (2)$ and $N = 20$	55	46	0.083838	0.064682
Choice 1. $x_0 = (5)$, $x_1 = (2)$ and $N = 4$	53	44	0.094593	0.078660
Choice 2. $x_0 = (-7)$, $x_1 = (-3)$ and $N = 20$	64	54	0.087539	0.064254
Choice 2. $x_0 = (-7)$, $x_1 = (-3)$ and $N = 4$	59	49	0.140981	0.069864

Figure 1 Graph of IFBSA and FBSA plotted for Choice 1 with $N = 20$

Figure 2 Graph of IFBSA and FBSA plotted for Choice 2 with $N = 20$

We can see from Table 1 and Figs. 1 and 2 that IFBSA performs better as compared to FBSA. Elaborating the behavior of this algorithm with respect to Table 1, the error analysis
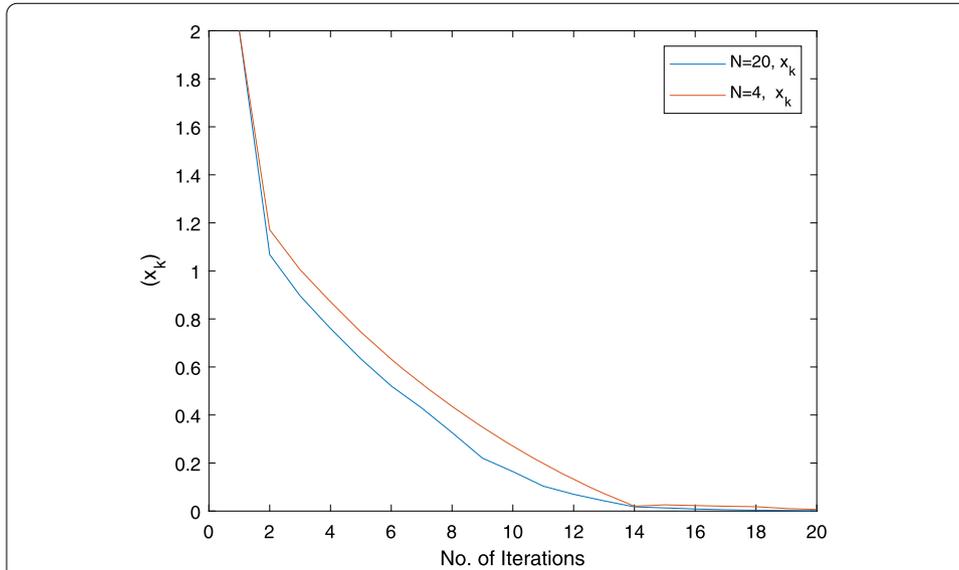
Figure 3 Comparison between trajectories of Algorithm 2 for Choice 1 with $N = 20$ and $N = 4$
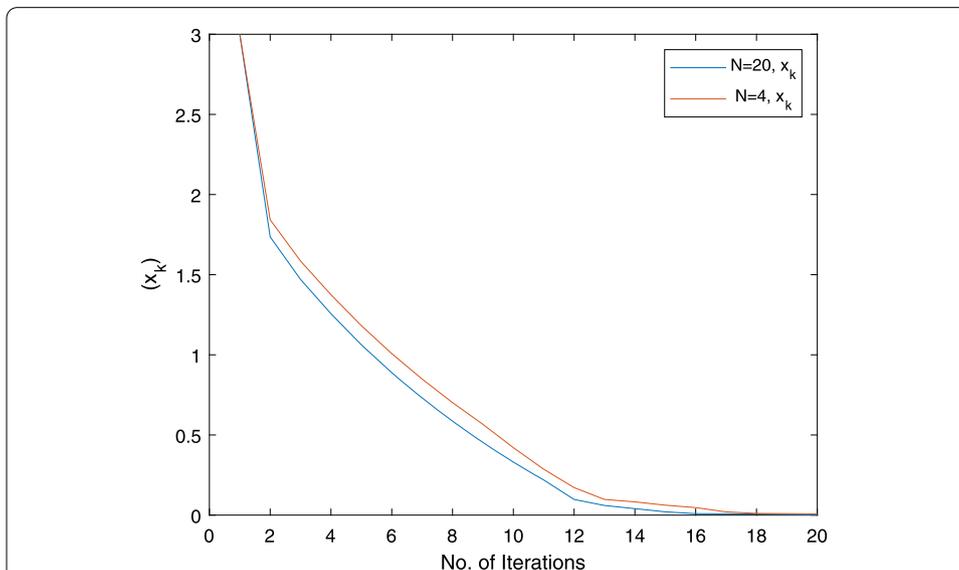

Figure 4 Comparison between trajectories of Algorithm 2 for Choice 2 with $N = 20$ and $N = 4$

is depicted in Figs. 1 and 2 whereas the number of iterations required to converge the sequence (x_k) towards the common solution are expressed in Figs. 3 and 4. Summarizing these facts, we say that the IFBSA exhibits a reduction in the error, time and the number of iterations of the function as compared to the FBSA.

5 Applications

In this section, we illustrate the theoretical results which we have obtained in the previous section.

5.1 Split feasibility problems

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let C and Q be closed, convex and nonempty subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem aims to find $\bar{x} \in C$ such that $S\bar{x} \in Q$. We represent the solution sets by $\omega := C \cap \hbar^{-1}(Q) = \{\bar{y} \in C : \hbar\bar{y} \in Q\}$. Censor and Elfving [7] introduced it to solve inverse problems and their application to medical image reconstruction and radiation therapy in a finite dimensional Hilbert space. For the set C, recall the function

$$b_C(\bar{x}) := \begin{cases} 0, & \bar{x} \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

The proximal operator of b_C is the metric projection on C,

$$prox_{b_C} = \arg\min_{\bar{p} \in C} \|\bar{p} - \bar{x}\|$$
$$= P_C(\bar{x}).$$

Let P_Q be the projection of $\mathcal{H}_2$ onto a nonempty, convex and closed subset Q. Take: $f(\bar{x}) = \frac{1}{2}\|\hbar\bar{x} - P_Q\hbar\bar{x}\|^2$ and $g(\bar{x}) = b_C(\bar{x})$. Then we compute the split feasibility problem from the following result.

Corollary 5.1 *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be nonempty, closed and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\Gamma = \omega \cap \Omega \cap \bigcap_N^{i=1} Fix(S_i) \neq \emptyset$ with hypotheses* (H1)–(H4). *Let Θ_k be a bounded real sequence and $m_k \in (0, \frac{2}{\|\hbar\|^2})$. For given $x_0, x_1 \in \mathcal{H}_1$, let the iterative sequences $(x_k), (b_k), (\ell_k), (w_k), (y_k)$ and (x_{k+1}) be generated by*

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ \ell_k = T_{u_k}^{F_1}(I - \delta_k\hbar^*(I - T_{u_k}^{F_2})\hbar)b_k; \\ w_k = (1 - \beta_k)\ell_k + \beta_k S_k\ell_k; \\ y_k = \lambda_k w_k + (1 - \lambda_k)J_k w_k; \\ C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \le \|x_k - \hat{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 \cdots \\ \qquad\qquad + 2\Theta_k\langle x_k - \hat{x}, x_k - x_{k-1}\rangle\}; \\ x_{k+1} = P_{C_{k+1}}x_1, \quad \forall k \ge 1, \end{cases} \tag{41}$$

where $J_k = P_C(Id - m_k\hbar^*(I - P_Q)\hbar)$. Assume that the conditions (C1)–(C4) *hold, then the sequence (x_k) generated by* (41) *converges strongly to an element $\bar{x} = P_\Gamma x_1$.*

5.2 Monotone variational inequality problems

Let $\mathcal{H}_1$ be a Hilbert space and C be a nonempty, closed and convex subset of $\mathcal{H}_1$. Let $\mathcal{B} : C \to \mathcal{H}_1$ be a nonlinear monotone operator. The variational inequality problem aims to find a point $\bar{x} \in C$ such that

$$\langle \mathcal{B}\bar{x}, \bar{y} - \bar{x}\rangle \ge 0 \quad \forall\bar{y} \in C. \tag{42}$$

The solution set of the above problem is denoted by ω and assume that $\omega \neq \emptyset$. [29], The resolvent operator acts as a projection operator P_C. Then we compute monotone variational inequality problems from the following result.

Corollary 5.2 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces and let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be nonempty, closed and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\Gamma = \omega \cap \Omega \cap \bigcap_N^{i=1} Fix(S_i) \neq \emptyset$ with hypotheses (H1)–(H4). Let Θ_k be a bounded real sequence and $m_k \in (0, \frac{2}{\|\hbar\|^2})$. For given $x_0, x_1 \in \mathcal{H}_1$, let the iterative sequences (x_k), (b_k), (ℓ_k), (w_k), (y_k) and (x_{k+1}) be generated by*

$$
\begin{cases}
b_k = x_k + \Theta_k(x_k - x_{k-1}); \\
\ell_k = T_{u_k}^{F_1}(I - \delta_k \hbar^*(I - T_{u_k}^{F_2})\hbar)b_k; \\
w_k = (1 - \beta_k)\ell_k + \beta_k S_k \ell_k; \\
y_k = \lambda_k w_k + (1 - \lambda_k)J_k w_k; \\
C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \|x_k - \hat{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 \cdots \\
\qquad + 2\Theta_k\langle x_k - \hat{x}, x_k - x_{k-1}\rangle\}; \\
x_{k+1} = P_{C_{k+1}}x_1, \quad \forall k \geq 1,
\end{cases}
\tag{43}
$$

where $J_k = P_C(Id - m_k \mathcal{B})$. Assume that the conditions (C1)–(C4) hold, then the sequence (x_k) generated by (43) converges strongly to an element $\bar{x} = P_\Gamma x_1$.

5.3 Convex minimization problems

Let $f : \mathcal{H}_1 \to \mathbb{R}$ and $g : \mathcal{H}_1 \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions. In Algorithm 2, set that $\mathcal{A} := \partial f$ and $\mathcal{B} := \nabla g$. Assume that ω is the set of solutions of problem (4) and $\omega \neq \emptyset$. Then we compute the convex minimization problem from the following result.

Corollary 5.3 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces and let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be nonempty, closed and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\Gamma = \omega \cap \Omega \cap \bigcap_N^{i=1} Fix(S_i) \neq \emptyset$ with hypotheses (H1)–(H4). Let Θ_k be a bounded real sequence and $m_k \in (0, \frac{2}{\|\hbar\|^2})$. Let $f, g : \mathcal{H}_1 \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions, such that f is sub-differential function and g is differentiable with γ-Lipschitz continuous gradient. For given $x_0, x_1 \in \mathcal{H}_1$, let the iterative sequences (x_k), (b_k), (ℓ_k), (w_k), (y_k) and (x_{k+1}) be generated by*

$$
\begin{cases}
b_k = x_k + \Theta_k(x_k - x_{k-1}); \\
\ell_k = T_{u_k}^{F_1}(I - \delta_k \hbar^*(I - T_{u_k}^{F_2})\hbar)b_k; \\
w_k = (1 - \beta_k)\ell_k + \beta_k S_k \ell_k; \\
y_k = \lambda_k w_k + (1 - \lambda_k)J_k w_k; \\
C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \|x_k - \hat{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 \cdots \\
\qquad + 2\Theta_k\langle x_k - \hat{x}, x_k - x_{k-1}\rangle\}; \\
x_{k+1} = P_{C_{k+1}}x_1, \quad \forall k \geq 1,
\end{cases}
\tag{44}
$$

where $J_k = J_{m_k}^{\partial f}(Id - m_k \nabla g)$. Assume that the conditions (C1)–(C4) hold, then the sequence (x_k) generated by (44) converges strongly to an element $\bar{x} = P_\Gamma x_1$.

6 Conclusions

In this paper, we have devised an inertially constructed forward–backward splitting algorithm for computing a common solution of the finite family of demicontractive operators, SEP and monotone inclusion problem in Hilbert spaces. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. Moreover, this framework has also been implemented to various instances of the monotone inclusion problems. We would like to emphasize that the above mentioned problems occur naturally in many applications, therefore, iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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