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New trapezium type inequalities of coordinated distance-disturbed convex type functions of higher orders via extended Katugampola fractional integrals

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Abstract

In this paper we establish some new results on trapezium type inequalities of coordinated distance-disturbed (ℓ_1, h_1)–(ℓ_2, h_2)-convex functions of higher orders (σ_1, σ_2) by using the Katugampola (k_1, k_2)-fractional integrals. As special cases of our general results, we recapture some earlier proved results.

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1 Introduction

During the most recent couple of decades, the theory of convex functions has been widely considered because of its applications in the theory of optimization and biological systems [15, 31]. In modern days many generalizations of different convexities and combinations of such concepts appears in the literature. These notions adapted and generalized those inequalities which belong to the classical convexity. Hermite–Hadamard type inequalities are significant and very important on the basis of geometric interpretation. Dragomir in [7], presented the definition of convex functions on $\mathbb{R}^2$, with coordinates in a rectangle. He considered a bi-dimensional interval $\Lambda = [\alpha, \beta] \times [\phi, \varphi]$ with $0 \le \alpha < \beta < \infty$, $0 \le \phi < \varphi < \infty$. Indeed, a function $\rho : \Lambda \to \mathbb{R}$, will be called convex on the coordinates on Λ, if the partial mappings $\rho_y : [\phi, \varphi] \to \mathbb{R}$, $\rho_y(u) = \rho(u, y)$ and $\rho_x : [\alpha, \beta] \to \mathbb{R}$, $\rho_x(v) = \rho(x, v)$ are convex for all $y, v \in [\alpha, \beta]$ and for all $x, u \in [\phi, \varphi]$, respectively. A function on Λ is said to be convex if it satisfies the following inequality:

$$\rho\big(\hat{a}x + (1 - \hat{a})u, \hat{a}y + (1 - \hat{a})v\big) \le \hat{a}\rho(x, y) + (1 - \hat{a})\rho(u, v) \tag{1.1}$$

for all $(x, y), (u, v) \in \Lambda$ and $\hat{a} \in [0, 1]$. Every convex function is coordinated convex but the converse is not true [7]. Dragomir in [7], presented Hadamard type inequalities related to one dimensional case. Further the researchers present many generalizations of the coordinated convex functions and introduced new inequalities, we refer the reader to [2–31, 33–35, 37]. The contributions of Noor [27], Yang [38], Sarikaya [32], Chen [6] and Set *et al.* [36] in this regards are remarkable.

The aim of this paper is to develop new trapezium type inequalities of coordinated distance-disturbed (ℓ_1, h_1)–(ℓ_2, h_2)-convex functions of higher orders (σ_1, σ_2) by using the Katugampola (k_1, k_2)-fractional integrals. We establish our results for many special cases like coordinated distance-disturbed (ℓ_1, s_1)–(ℓ_2, s_2)-convex functions, coordinated distance-disturbed $\ell_1\ell_2$-convex functions, coordinated distance-disturbed (h_1, h_2)-convex functions, coordinated distance-disturbed (s_1, s_2)-convex functions. Here results are proved for coordinated distance-disturbed h-convex functions, coordinated distance-disturbed s-convex functions and coordinated distance-disturbed convex functions. At the end, a brief conclusion is given.

2 Preliminaries

Definition 2.1 ([26]) Let $\psi : [\alpha, \beta] \to \mathbb{R}$ be termed convex if the inequality holds on an interval $[\alpha, \beta] \subseteq \mathbb{R}$ as

$$\psi\big(tl + (1-t)r\big) \leq t\psi(l) + (1-t)\psi(r),$$

where $l, r \in [\alpha, \beta]$ and $t \in [0, 1]$.

This inequality provides bounds of the mean value of a continuous convex function is given by the following theorem.

Theorem 2.2 *If $\Psi : U \to \mathbb{R}$ is a convex function on the interval U of real numbers, such that $\alpha, \beta \in U$ with $\alpha < \beta$, then*

$$\Psi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta \Psi(\xi)\,d\xi \leq \frac{\Psi(\alpha)+\Psi(\beta)}{2}.$$

Theorem 2.3 ([7]) *Suppose that $\rho : \Lambda \to \mathbb{R}$ is convex on the coordinates on Λ. Then the following inequalities hold*:

$$\begin{aligned}
&\rho\left(\frac{\alpha+\beta}{2}, \frac{\phi+\varphi}{2}\right) \\
&\quad \leq \frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \rho\left(x, \frac{\phi+\varphi}{2}\right)dx + \frac{1}{\varphi-\phi}\int_\phi^\varphi \rho\left(\frac{\alpha+\beta}{2}, y\right)dy\right] \\
&\quad \leq \frac{1}{(\beta-\alpha)(\varphi-\phi)}\int_\alpha^\beta\int_\phi^\varphi \rho(x,y)\,dy\,dx \\
&\quad \leq \frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \big[\rho(x,\phi)+\rho(x,\varphi)\big]dx + \frac{1}{\varphi-\phi}\int_\phi^\varphi \big[\rho(\alpha,y)+\rho(\beta,y)\big]dy\right] \\
&\quad \leq \frac{\rho(\alpha,\phi)+\rho(\alpha,\varphi)+\rho(\beta,\phi)+\rho(\beta,\varphi)}{4}.
\end{aligned}$$

A formal definition of coordinated convex function may be stated as follows.

Definition 2.4 Let $\rho : \Lambda \to \mathbb{R}$ be coordinated convex function on Λ, then the inequality

$$\rho\big(tx + (1-t)u, sy + (1-s)v\big) \le ts\rho(x,y) + t(1-s)\rho(x,v) + (1-t)s\rho(u,y) + (1-t)(1-s)\rho(u,v)$$

holds for all $(x,y),(x,v),(u,y),(u,v) \in \Lambda$ and $s,t \in [0,1]$.

Let us recall some fundamental definitions and results which are helpful in developing main results. Further details can be found in [21–24, 26–35].

Definition 2.5 The left- and right-sided Riemann Liouville fractional integrals $I^{\mu}_{\alpha^+}\psi$ and $I^{\mu}_{\beta^-}\psi$ of order with $\mu > 0$, on a finite interval $[\alpha,\beta]$, are defined as

$$I^{\mu}_{\alpha^+}\psi(x) = \frac{1}{\Gamma(\mu)}\int_{\alpha}^{x}(x-t)^{\mu-1}\psi(t)\,dt, \quad x > \alpha,$$

and

$$I^{\mu}_{\beta^-}\psi(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{\beta}(t-x)^{\mu-1}\psi(t)\,dt, \quad x < \beta,$$

respectively. Here Γ represents the usual Gamma function defined by

$$\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\,dx, \quad \mathbb{R}(t) > 0.$$

Definition 2.6 Let $\rho \in L_1[\alpha,\beta]$ and $k > 0$. The left and right k-Riemann–Liouville integrals of order $\mu > 0$ with $\alpha \ge 0$ are denoted by

$$I^{\mu,k}_{\alpha+}\rho(x) = \frac{1}{k\Gamma_k(\mu)}\int_{\alpha}^{x}(x-\tau)^{\frac{\mu}{k}-1}\rho(\tau)\,d\tau, \quad x > \alpha,$$

and

$$I^{\mu,k}_{\beta-}\rho(x) = \frac{1}{k\Gamma_k(\mu)}\int_{x}^{\beta}(\tau-x)^{\frac{\mu}{k}-1}\rho(\tau)\,d\tau, \quad x < \beta,$$

respectively. Note that when $k \to 1$, then it reduces to the classical Riemann–Liouville fractional integral.

Definition 2.7 Let $\rho \in L_1(\Lambda)$. The Riemann–Liouville integrals $J^{\mu,\nu}_{\hat{a}+,\hat{c}+}$, $J^{\mu,\nu}_{\hat{a}+,\hat{d}-}$, $J^{\mu,\nu}_{\hat{b}-,\hat{c}+}$ and $J^{\mu,\nu}_{\hat{b}-,\hat{d}-}$ of order $\mu,\nu > 0$ with $\hat{a},\hat{c} \ge 0$ are defined by

$$J^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(x,y) = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_{\hat{a}}^{x}\int_{\hat{c}}^{y}(x-\jmath_1)^{\mu-1}(y-\jmath_2)^{\nu-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x > \hat{a}, y > \hat{c},$$

$$J^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(x,y) = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_{\hat{a}}^{x}\int_{y}^{\hat{d}}(x-\jmath_1)^{\mu-1}(\jmath_2-y)^{\nu-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x > \hat{a}, y < \hat{d},$$

$$J^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(x,y) = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_{x}^{\hat{b}}\int_{\hat{c}}^{y}(\jmath_1-x)^{\mu-1}(y-\jmath_2)^{\nu-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x < \hat{b}, y > \hat{c},$$

and

$$J^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(x,y)=\frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_x^{\hat{b}}\int_y^{\hat{d}}(\jmath_1-x)^{\mu-1}(\jmath_2-y)^{\nu-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1,\quad x<\hat{b},y<\hat{d},$$

respectively. Furthermore,

$$J^{0,0}_{\hat{a}+,\hat{c}+}\rho(x,y)=J^{0,0}_{\hat{a}+,\hat{d}-}\rho(x,y)=J^{0,0}_{\hat{b}-,\hat{c}+}\rho(x,y)=J^{0,0}_{\hat{b}-,\hat{d}-}\rho(x,y)=\rho(x,y)$$

and

$$J^{1,1}_{\hat{a}+,\hat{d}-}\rho(x,y)=\frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_{\hat{a}}^{x}\int_y^{\hat{d}}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1.$$

Similar to Definition 2.5, Sarikaya [32] introduced the following fractional integrals:

$$\begin{aligned}
J^{\mu}_{\hat{a}+}\rho\left(x,\frac{\hat{c}+\hat{d}}{2}\right)&=\frac{1}{\Gamma(\mu)}\int_{\hat{a}}^{x}(x-\jmath_1)^{\mu-1}\rho\left(\jmath_1,\frac{\hat{c}+\hat{d}}{2}\right)d\jmath_1,\quad x>\hat{a},\\
J^{\mu}_{\hat{b}-}\rho\left(x,\frac{\hat{c}+\hat{d}}{2}\right)&=\frac{1}{\Gamma(\mu)}\int_{x}^{\hat{b}}(\jmath_1-x)^{\mu-1}\rho\left(\jmath_1,\frac{\hat{c}+\hat{d}}{2}\right)d\jmath_1,\quad x<\hat{b},\\
J^{\nu}_{\hat{c}+}\rho\left(\frac{\hat{a}+\hat{b}}{2},y\right)&=\frac{1}{\Gamma(\nu)}\int_{\hat{c}}^{y}(y-\jmath_2)^{\nu-1}\rho\left(\frac{\hat{a}+\hat{b}}{2},\jmath_2\right)d\jmath_2,\quad y>\hat{c},\\
J^{\nu}_{\hat{d}-}\rho\left(\frac{\hat{a}+\hat{b}}{2},y\right)&=\frac{1}{\Gamma(\nu)}\int_{y}^{\hat{d}}(\jmath_2-y)^{\nu-1}\rho\left(\frac{\hat{a}+\hat{b}}{2},\jmath_2\right)d\jmath_2,\quad y<\hat{d}.
\end{aligned}$$

Sarikaya gave the following remarkable results in [32].

Theorem 2.8 *Let* $\rho:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ *be a coordinated convex function on* $\Omega:=[\hat{a},\hat{b}]\times[\hat{c},\hat{d}]\in\mathbb{R}^2$ *with* $0\le\hat{a}<\hat{b}$, $0\le\hat{c}<\hat{d}$ *and* $\rho\in L_1(\Omega)$. *Then one has the inequalities*:

$$\begin{aligned}
\rho\left(\frac{\hat{a}+\hat{b}}{2},\frac{\hat{c}+\hat{d}}{2}\right)&\le\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{4(\hat{b}-\hat{a})^{\mu}(\hat{d}-\hat{c})^{\nu}}\\
&\quad\times\left[J^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+J^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+J^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+J^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\le\frac{\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})}{4}.
\end{aligned}$$

Now, we are in a position to introduce the following extended Riemann–Liouville integrals.

Definition 2.9 Let $\rho\in L_1(\Omega)$ and $k_1,k_2>0$. The (k_1,k_2)-Riemann–Liouville integrals $I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}$, $I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}$, $I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}$ and $I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}}$ of order $\mu,\nu>0$ with $\hat{a},\hat{c}\ge0$ are defined by

$$\begin{aligned}
&I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(x,y)\\
&\quad=\frac{1}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{\hat{a}}^{x}\int_{\hat{c}}^{y}(x-\jmath_1)^{\frac{\mu}{k_1}-1}(y-\jmath_2)^{\frac{\nu}{k_2}-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1,\quad x>\hat{a},y>\hat{c},
\end{aligned}$$

$$I_{\hat{a}+,\hat{d}-}^{\mu,\nu,k_1,k_2}\rho(x,y)$$
$$= \frac{1}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{\hat{a}}^{x}\int_{y}^{\hat{d}}(x-\jmath_1)^{\frac{\mu}{k_1}-1}(\jmath_2-y)^{\frac{\nu}{k_2}-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x>\hat{a}, y<\hat{d},$$
$$I_{\hat{b}-,\hat{c}+}^{\mu,\nu,k_1,k_2}\rho(x,y)$$
$$= \frac{1}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{x}^{\hat{b}}\int_{\hat{c}}^{y}(\jmath_1-x)^{\frac{\mu}{k_1}-1}(y-\jmath_2)^{\frac{\nu}{k_2}-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x<\hat{b}, y>\hat{c}$$

and

$$I_{\hat{b}-,\hat{d}-}^{\mu,\nu,k_1,k_2}\rho(x,y)$$
$$= \frac{1}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{x}^{\hat{b}}\int_{y}^{\hat{d}}(\jmath_1-x)^{\frac{\mu}{k_1}-1}(\jmath_2-y)^{\frac{\nu}{k_2}-1}\rho(\jmath_1,\jmath_2)\,d\jmath_2\,d\jmath_1, \quad x<\hat{b}, y<\hat{d},$$

respectively. Note that when $k_1, k_2 \to 1$, then it reduces to Definition 2.7.

Noor *et al.* in [27], introduced the notion of coordinated $\ell_1\ell_2$-convex functions to generalize the ℓ_1-convex functions as follows.

Definition 2.10 Let $\Lambda \subset \mathbb{R}^2$ be a rectangle. A function $\rho : \Lambda \to \mathbb{R}$ is said to be two dimensional (coordinated) $\ell_1\ell_2$-convex function, if

$$\rho\left(\left[tx^{\ell_1} + (1-t)u^{\ell_1}\right]^{\frac{1}{\ell_1}}, \left[ry^{\ell_2} + (1-r)v^{\ell_2}\right]^{\frac{1}{\ell_2}}\right)$$
$$\leq tr\rho(x,y) + t(1-r)\rho(x,v) + (1-t)r\rho(u,y) + (1-t)(1-r)\rho(u,v)$$

for all $(x,y),(x,v),(u,y),(u,v) \in \Lambda$ and $r,t \in [0,1]$.

Theorem 2.11 ([27]) *Let $\rho : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a $\ell_1\ell_2$-convex function on the coordinates on Ω, then the following inequalities hold*:

$$\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, \left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)$$
$$\leq \frac{\ell_1\ell_2}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})}\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}} x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)\,dy\,dx$$
$$\leq \frac{\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})}{4}.$$

Yang in [38], generalized this concept by defining a larger class of coordinated convex functions termed coordinated (ℓ_1,h_1)–(ℓ_2,h_2)-convex function as follows:

Definition 2.12 Let $\hat{h}_1, \hat{h}_2 : J \to \mathbb{R}$ be two non-negative and non-zero mappings. A mapping $\rho : \Lambda \to \mathbb{R}$ is said to be $(\ell_1,\hat{h}_1)$–$(\ell_2,\hat{h}_2)$-convex function on the coordinates on Λ, if the mappings $\rho_y : [\hat{a},\hat{b}] \to \mathbb{R}$, $\rho_y(u) = \rho(u,y)$ and $\rho_x : [\hat{c},\hat{d}] \to \mathbb{R}$, $\rho_x(v) = \rho(x,v)$ are $(\ell_1,\hat{h}_1)$-convex with respect to u on $[\hat{a},\hat{b}]$ and $(\ell_2,\hat{h}_2)$-convex with respect to v on $[\hat{c},\hat{d}]$, respectively, for all $y \in [\hat{c},\hat{d}]$ and $x \in [\hat{a},\hat{b}]$.

From the above definition, we can say that, if ρ is a coordinated $(\ell_1,\hat{h}_1)$–$(\ell_2,\hat{h}_2)$-convex function, then the following inequality holds:

$$\begin{aligned}
&\rho\Big(\big[tx^{\ell_1}+(1-t)u^{\ell_1}\big]^{\frac{1}{\ell_1}},\big[ry^{\ell_2}+(1-r)v^{\ell_2}\big]^{\frac{1}{\ell_2}}\Big)\\
&\quad\le \hat{h}_1(t)\hat{h}_2(r)\rho(x,y)+\hat{h}_1(t)\hat{h}_2(1-r)\rho(x,v)\\
&\qquad+\hat{h}_1(1-t)\hat{h}_2(r)\rho(u,y)\\
&\qquad+\hat{h}_1(1-t)\hat{h}_2(1-r)\rho(u,v).
\end{aligned}$$

Remark 2.13 If $\ell_1=\ell_2=1$, then the function ρ will be reduced to coordinated $(\hat{h}_1,\hat{h}_2)$-convex function.

Remark 2.14 If $\hat{h}_1(t)=t^{s_1}$ and $\hat{h}_2(t)=t^{s_2}$, then the function ρ will be called a coordinated (ℓ_1,s_1)–(ℓ_2,s_2)-convex function.

Remark 2.15 If $\hat{h}_1(t)=t^{s_1}$, $\hat{h}_2(t)=t^{s_2}$ and $\ell_1=\ell_2=1$, then the function ρ will be called a coordinated (s_1,s_2)-convex function.

Yang in [38], gave the following two interesting results.

Theorem 2.16 *Let $\rho:\Omega\to\mathbb{R}$ be a $(\ell_1,\hat{h}_1)$–$(\ell_2,\hat{h}_2)$-convex function on the coordinates on Ω. Then one has the inequalities*

$$\begin{aligned}
&\frac{1}{4\hat{h}_1(\frac{1}{2})\hat{h}_2(\frac{1}{2})}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},\bigg[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\bigg]^{\frac{1}{\ell_2}}\bigg)\\
&\quad\le\frac{\ell_1\ell_2}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})}\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)\,dy\,dx\\
&\quad\le\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big]\int_0^1\hat{h}_1(t)\,dt\int_0^1\hat{h}_2(t)\,dt.
\end{aligned}$$

Theorem 2.17 *Let $\rho:\Omega\to\mathbb{R}$ be a $(\ell_1,\hat{h}_1)$–$(\ell_2,\hat{h}_2)$-convex function on the coordinates on Ω. Then one has the inequalities*

$$\begin{aligned}
&\frac{1}{4\hat{h}_1(\frac{1}{2})\hat{h}_2(\frac{1}{2})}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},\bigg[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\bigg]^{\frac{1}{\ell_2}}\bigg)\\
&\quad\le\frac{\ell_1}{4\hat{h}_1(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})}\int_{\hat{a}}^{\hat{b}}x^{\ell_1-1}\rho\bigg(x,\bigg[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\bigg]^{\frac{1}{\ell_2}}\bigg)\,dx\\
&\qquad+\frac{\ell_2}{4\hat{h}_2(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})}\int_{\hat{c}}^{\hat{d}}y^{\ell_2-1}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},y\bigg)\,dy\\
&\quad\le\frac{\ell_1\ell_2}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})}\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)\,dy\,dx\\
&\quad\le\frac{\ell_1}{2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})}\bigg[\int_{\hat{a}}^{\hat{b}}x^{\ell_1-1}\rho(x,\hat{c})\,dx+\int_{\hat{a}}^{\hat{b}}x^{\ell_1-1}\rho(x,\hat{d})\,dx\bigg]\int_0^1\hat{h}_2(t)\,dt
\end{aligned}$$

$$+\frac{\ell_2}{2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})}\left[\int_{\hat{c}}^{\hat{d}} y^{\ell_2-1}\rho(\hat{a},y)\,dy+\int_{\hat{c}}^{\hat{d}} y^{\ell_2-1}\rho(\hat{b},y)\,dy\right]\int_0^1 \hat{h}_1(t)\,dt$$
$$\le\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\int_0^1 \hat{h}_1(t)\,dt\int_0^1 \hat{h}_2(t)\,dt.$$

Definition 2.18 ([21]) $X_{\hat{c}}^p(\hat{a},\hat{b})$ ($\hat{c}\in\mathbb{R}$, $1\le p\le\infty$) is the set of those complex valued Lebesgue measurable functions ρ of $[\hat{a},\hat{b}]$ for which $\|\rho\|_{X_{\hat{c}}^p}<\infty$, where the norm is defined by

$$\|\rho\|_{X_{\hat{c}}^p}=\left(\int_{\hat{a}}^{\hat{b}}\left|t^{\hat{c}}\rho(t)\right|^p\frac{dt}{t}\right)^{\frac{1}{p}}<\infty\quad\text{for } 1\le p<\infty, \hat{c}\in\mathbb{R},$$

and for the case $p=\infty$,

$$\|\rho\|_{X_{\hat{c}}^p}=ess\sup_{\hat{a}\le t\le\hat{b}}\left[t^{\hat{c}}\left|\rho(t)\right|\right],\quad \hat{c}\in\mathbb{R}.$$

Katugampola introduced a new fractional integral which generalizes the Riemann–Liouville and Hadamard fractional integrals in a single form as follows; see [18–24, 26–30].

Definition 2.19 Let $[\hat{a},\hat{b}]\subseteq\mathbb{R}$ be a finite interval. Then the left- and right-sided Katugampola fractional integrals of order $\mu>0$ of $\in X_{\hat{c}}^p(\hat{a},\hat{b})$ with $\hat{a}\ge 0$ are defined by

$${}^rI_{\hat{a}+}^{\mu}\rho(x)=\frac{r^{1-\mu}}{\Gamma(\mu)}\int_{\hat{a}}^{x}\frac{t^{r-1}}{(x^r-t^r)^{1-\mu}}\rho(t)\,dt$$

and

$${}^rI_{\hat{b}-}^{\mu}\rho(x)=\frac{r^{1-\mu}}{\Gamma(\mu)}\int_{x}^{\hat{b}}\frac{t^{r-1}}{(t^r-x^r)^{1-\mu}}\rho(t)\,dt$$

with $\hat{a}<x<\hat{b}$ and $r>0$, provided the integrals exist.

Definition 2.20 Let $[\hat{a},\hat{b}]\subseteq\mathbb{R}$ be a finite interval and $k>0$. Then the left- and right-sided Katugampola k-fractional integrals of order $\mu>0$ of $\rho\in X_{\hat{c}}^p(\hat{a},\hat{b})$ with $\hat{a}\ge 0$ are defined by

$${}^rI_{\hat{a}+}^{\mu,k}\rho(x)=\frac{r^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_{\hat{a}}^{x}\frac{t^{r-1}}{(x^r-t^r)^{1-\frac{\mu}{k}}}\rho(t)\,dt$$

and

$${}^rI_{\hat{b}-}^{\mu,k}\rho(x)=\frac{r^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_{x}^{\hat{b}}\frac{t^{r-1}}{(t^r-x^r)^{1-\frac{\mu}{k}}}\rho(t)\,dt$$

with $\hat{a}<x<\hat{b}$ and $\rho>0$, provided the integrals exist. Note that when $k\to 1$, then it reduces to Definition 2.19.

Katugampola fractional integrals into two dimensional case may be given as follows.

Definition 2.21 Let $\rho \in X^p(\Omega)$. The Katugampola fractional integrals ${}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}$, ${}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}$, ${}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}$ and ${}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}$ of order $\mu,\nu > 0$ with $\hat{a},\hat{c} \geq 0$ are defined by

$$\begin{aligned}
&{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\mu}\ell_2^{1-\nu}}{\Gamma(\mu)\Gamma(\nu)}\int_{\hat{a}}^{x}\int_{\hat{c}}^{y}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\mu}(y^{\ell_2}-s^{\ell_2})^{1-\nu}}\rho(t,s)\,ds\,dt, \quad x>\hat{a}, y>\hat{c}, \\
&{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\mu}\ell_2^{1-\nu}}{\Gamma(\mu)\Gamma(\nu)}\int_{\hat{a}}^{x}\int_{y}^{\hat{d}}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\mu}(s^{\ell_2}-y^{\ell_2})^{1-\nu}}\rho(t,s)\,ds\,dt, \quad x>\hat{a}, y<\hat{d}, \\
&{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\mu}\ell_2^{1-\nu}}{\Gamma(\mu)\Gamma(\nu)}\int_{x}^{\hat{b}}\int_{\hat{c}}^{y}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\mu}(y^{\ell_2}-s^{\ell_2})^{1-\nu}}\rho(t,s)\,ds\,dt, \quad x<\hat{b}, y>\hat{c},
\end{aligned}$$

and

$$\begin{aligned}
&{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\mu}\ell_2^{1-\nu}}{\Gamma(\mu)\Gamma(\nu)}\int_{x}^{\hat{b}}\int_{y}^{\hat{d}}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\mu}(s^{\ell_2}-y^{\ell_2})^{1-\nu}}\rho(t,s)\,ds\,dt, \quad x<\hat{b}, y<\hat{d},
\end{aligned}$$

respectively, and $\ell_1,\ell_2 > 0$. Moreover,

$${}^{\ell_1,\ell_2}I^{0,0}_{\hat{a}+,\hat{c}+}\rho(x,y) = {}^{\ell_1,\ell_2}I^{0,0}_{\hat{a}+,\hat{d}-}\rho(x,y) = {}^{\ell_1,\ell_2}I^{0,0}_{\hat{b}-,\hat{c}+}\rho(x,y) = {}^{\ell_1,\ell_2}I^{0,0}_{\hat{b}-,\hat{d}-}\rho(x,y) = \rho(x,y)$$

and

$${}^{\ell_1,\ell_2}I^{1,1}_{\hat{a}+,\hat{d}-}\rho(x,y) = \int_{\hat{a}}^{x}\int_{y}^{\hat{d}} t^{\ell_1-1}s^{\ell_2-1}\rho(t,s)\,ds\,dt.$$

Similar to Definition 2.19, we introduce the following fractional integrals:

$$\begin{aligned}
{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho\left(x,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) &= \frac{\ell_1^{1-\mu}}{\Gamma(\mu)}\int_{\hat{a}}^{x}\frac{t^{\ell_1-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\mu}}\rho\left(t,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dt, \quad x>\hat{a}, \\
{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho\left(x,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) &= \frac{\ell_1^{1-\mu}}{\Gamma(\mu)}\int_{x}^{\hat{b}}\frac{t^{\ell_1-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\mu}}\rho\left(t,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dt, \quad x<\hat{b}, \\
{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},y\right) &= \frac{\ell_2^{1-\nu}}{\Gamma(\nu)}\int_{\hat{c}}^{y}\frac{s^{\ell_2-1}}{(y^{\ell_2}-s^{\ell_2})^{1-\nu}}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},s\right)ds, \quad y>\hat{c}, \\
{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},y\right) &= \frac{\ell_2^{1-\nu}}{\Gamma(\nu)}\int_{y}^{\hat{d}}\frac{s^{\ell_2-1}}{(s^{\ell_2}-y^{\ell_2})^{1-\nu}}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},s\right)ds, \quad y<\hat{d}.
\end{aligned}$$

It is important to notice that, if $\ell_1 = \ell_2 = 1$, then the Katugampola fractional integrals reduce to Riemann–Liouville fractional integrals given in Definition 2.7.

Similarly, we can define the extended Katugampola fractional integrals in the two dimensional case as follows.

Definition 2.22 Let $\rho \in X^p(\Omega)$ and $k_1, k_2 > 0$. The Katugampola (k_1, k_2)-fractional integrals ${}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}$, ${}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}$, ${}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}$ and ${}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}$ of order $\mu, \nu > 0$ with $\hat{a}, \hat{c} \geq 0$ are defined by

$$\begin{aligned}
&{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}\ell_2^{1-\frac{\nu}{k_2}}}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{\hat{a}}^{x}\int_{\hat{c}}^{y}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-s^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(t,s)\,ds\,dt, \quad x > \hat{a}, y > \hat{c}, \\
&{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}\ell_2^{1-\frac{\nu}{k_2}}}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{\hat{a}}^{x}\int_{y}^{\hat{d}}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\frac{\mu}{k_1}}(s^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(t,s)\,ds\,dt, \quad x > \hat{a}, y < \hat{d}, \\
&{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}\ell_2^{1-\frac{\nu}{k_2}}}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{x}^{\hat{b}}\int_{\hat{c}}^{y}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-s^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(t,s)\,ds\,dt, \quad x < \hat{b}, y > \hat{c},
\end{aligned}$$

and

$$\begin{aligned}
&{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(x,y) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}\ell_2^{1-\frac{\nu}{k_2}}}{k_1k_2\Gamma_{k_1}(\mu)\Gamma_{k_2}(\nu)}\int_{x}^{\hat{b}}\int_{y}^{\hat{d}}\frac{t^{\ell_1-1}s^{\ell_2-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(s^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(t,s)\,ds\,dt, \quad x < \hat{b}, y < \hat{d},
\end{aligned}$$

respectively, and $\ell_1, \ell_2 > 0$. Note that when $k_1, k_2 \to 1$, then it reduces to Definition 2.21.

We also introduce the following useful fractional integrals:

$$\begin{aligned}
&{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho\left(x, \left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}}{k_1\Gamma_{k_1}(\mu)}\int_{\hat{a}}^{x}\frac{t^{\ell_1-1}}{(x^{\ell_1}-t^{\ell_1})^{1-\frac{\mu}{k_1}}}\rho\left(t, \left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dt, \quad x > \hat{a}, \\
&{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho\left(x, \left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad = \frac{\ell_1^{1-\frac{\mu}{k_1}}}{k_1\Gamma_{k_1}(\mu)}\int_{x}^{\hat{b}}\frac{t^{\ell_1-1}}{(t^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\rho\left(t, \left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dt, \quad x < \hat{b}, \\
&{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, y\right) \\
&\quad = \frac{\ell_2^{1-\frac{\nu}{k_2}}}{k_2\Gamma_{k_2}(\nu)}\int_{\hat{c}}^{y}\frac{s^{\ell_2-1}}{(y^{\ell_2}-s^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, s\right)ds, \quad y > \hat{c},
\end{aligned}$$

$$\begin{aligned}
&{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},y\right)\\
&=\frac{\ell_2^{1-\frac{\nu}{k_2}}}{k_2\Gamma_{k_2}(\nu)}\int_y^{\hat{d}}\frac{s^{\ell_2-1}}{(s^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},s\right)ds,\quad y<\hat{d}.
\end{aligned}$$

Definition 2.23 Let $h_1, h_2 : J \to \mathbb{R}$ be two non-negative and non-zero functions. Assume that $\sigma_1, \sigma_2 > 0$. A mapping $\rho : \Omega \to \mathbb{R}$ is said to be a distance-disturbed (ℓ_1, h_1)–(ℓ_2, h_2)-convex function on the coordinates on Ω with modulus $\mu_1, \mu_2 > 0$ of higher orders (σ_1, σ_2), if the partial mappings $\rho_y : [\hat{a}, \hat{b}] \to \mathbb{R}$, $\rho_y(u) = \rho(u, y)$ and $\rho_x : [\hat{c}, \hat{d}] \to \mathbb{R}$, $\rho_x(v) = \rho(x, v)$ are, respectively, distance-disturbed (ℓ_1, h_1)-convex with modulus $\mu_1 > 0$ of order $\sigma_1 > 0$ with respect to u on $[\hat{a}, \hat{b}]$ and distance-disturbed (ℓ_2, h_2)-convex with modulus $\mu_2 > 0$ of order $\sigma_2 > 0$ with respect to v on $[\hat{c}, \hat{d}]$, for all $y \in [\hat{c}, \hat{d}]$ and $x \in [\hat{a}, \hat{b}]$.

From the above definition, we can say that, if f is a coordinated distance-disturbed (ℓ_1, h_1)–(ℓ_2, h_2)-convex function with modulus $\mu_1, \mu_2 > 0$ of higher orders (σ_1, σ_2), then the following inequality holds:

$$\begin{aligned}
&\rho\big(\big[tx^{\ell_1}+(1-t)u^{\ell_1}\big]^{\frac{1}{\ell_1}},\big[ry^{\ell_2}+(1-r)v^{\ell_2}\big]^{\frac{1}{\ell_2}}\big)\\
&\qquad+\mu_1 t(1-t)\big(u^{\ell_1}-x^{\ell_1}\big)^{\sigma_1}+\mu_2 r(1-r)\big(v^{\ell_2}-y^{\ell_2}\big)^{\sigma_2}\\
&\quad\le h_1(t)h_2(r)\rho(x,y)+h_1(t)h_2(1-r)\rho(x,v)+h_1(1-t)h_2(r)\rho(u,y)\\
&\qquad+h_1(1-t)h_2(1-r)\rho(u,v).
\end{aligned}$$

Remark 2.24 If $\mu_1, \mu_2 \to 0^+$, then Definition 2.23 will be reduced to Definition 2.12.

Remark 2.25 If $\ell_1 = \ell_2 = \ell$, then the function f will be reduced to coordinated distance-disturbed (ℓ, h_1)–(ℓ, h_2)-convex function of higher orders. If $\ell_1 = \ell_2 = 1$, then the function f will be reduced to coordinated distance-disturbed (h_1, h_2)-convex function of higher orders.

Remark 2.26 If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then the function f will be called a coordinated distance-disturbed (ℓ_1, s_1)–(ℓ_2, s_2)-convex function of higher orders. If $h_1(t) = t^{s_1}$, $h_2(t) = t^{s_2}$ and $\ell_1 = \ell_2 = \ell$, then the function f will be called a coordinated distance-disturbed (ℓ, s_1)–(ℓ, s_2)-convex function of higher orders. If $h_1(t) = t^{s_1}$, $h_2(t) = t^{s_2}$ and $\ell_1 = \ell_2 = 1$, then the function f will be called a coordinated distance-disturbed (s_1, s_2)-convex function of higher orders.

3 Main results

In this section we give the trapezium type inequalities by using distance-disturbed (ℓ_1, h_1)-(ℓ_2, h_2)-convex functions with modulus $\mu_1, \mu_2 > 0$ of higher orders (σ_1, σ_2), where $\sigma_1, \sigma_2 > 0$ on the coordinates on Ω.

Theorem 3.1 *Suppose that $\rho : \Omega \to \mathbb{R}$ is a distance-disturbed (ℓ_1, h_1)-(ℓ_2, h_2)-convex function of higher orders on the coordinates on Ω and $\rho \in L_1(\Omega)$. Then one has the inequali-*

ties

$$\begin{aligned}
&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+A \\
&\quad\leq \frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{4(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} \\
&\qquad\times\bigl[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) \\
&\qquad+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\bigr] \\
&\quad\leq \frac{\mu\nu}{4k_1k_2}\bigl[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\bigr] \\
&\qquad\times\int_0^1\int_0^1 \imath_1^{\frac{\mu}{k_1}-1}\imath_2^{\frac{\nu}{k_2}-1}\bigl[h_1(\imath_1)+h_1(1-\imath_1)\bigr]\bigl[h_2(\imath_2)+h_2(1-\imath_2)\bigr]\,d\imath_1\,d\imath_2 \\
&\qquad-\frac{\mu_1[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}]+\mu_2[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}]}{2},
\end{aligned}\tag{3.1}$$

where

$$A=-\frac{\mu\nu}{16k_1k_2}\left[\frac{\mu_1k_2}{\nu}\left(\frac{1}{2\hat{b}^{\ell_1}}\right)^{\frac{\mu}{k_1}}C_1+\frac{\mu_2k_1}{\mu}\left(\frac{1}{2\hat{d}^{\ell_2}}\right)^{\frac{\nu}{k_2}}C_2\right]$$

and

$$C_1=\int_{\hat{a}^{\ell_1}-\hat{b}^{\ell_1}}^{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}\imath_1^{\sigma_1}\bigl[\imath_1-\hat{a}^{\ell_1}+\hat{b}^{\ell_1}\bigr]^{\frac{\mu}{k_1}-1}\,d\imath_1,\qquad C_2=\int_{\hat{c}^{\ell_2}-\hat{d}^{\ell_2}}^{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}\imath_2^{\sigma_2}\bigl[\imath_2-\hat{c}^{\ell_2}+\hat{d}^{\ell_2}\bigr]^{\frac{\nu}{k_2}-1}\,d\imath_2.$$

Proof Let $x^{\ell_1}=\imath_1\hat{a}^{\ell_1}+(1-\imath_1)\hat{b}^{\ell_1}$, $y^{\ell_1}=(1-\imath_1)\hat{a}^{\ell_1}+\imath_1\hat{b}^{\ell_1}$ and $u^{\ell_2}=\imath_2\hat{c}^{\ell_2}+(1-\imath_2)\hat{d}^{\ell_2}$, $v^{\ell_2}=(1-\imath_2)\hat{c}^{\ell_2}+\imath_2\hat{d}^{\ell_2}$, then, by coordinated distance-disturbed (ℓ_1,h_1)-(ℓ_2,h_2)-convexity of higher orders of ρ, we have

$$\begin{aligned}
&\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad=\rho\left(\left[\frac{x^{\ell_1}+y^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{u^{\ell_2}+v^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\bigl[\rho(x,u)+\rho(x,v)+\rho(y,u)+\rho(y,v)\bigr] \\
&\qquad-\frac{\mu_1}{4}\bigl(y^{\ell_1}-x^{\ell_1}\bigr)^{\sigma_1}-\frac{\mu_2}{4}\bigl(v^{\ell_2}-u^{\ell_2}\bigr)^{\sigma_2}.
\end{aligned}$$

Multiplying by $\frac{\mu\nu}{4k_1k_2}\imath_1^{\frac{\mu}{k_1}-1}\imath_2^{\frac{\nu}{k_2}-1}$ and integrating over $([0,1]\times[0,1])$, one has

$$\begin{aligned}
&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad\leq\frac{\mu\nu}{4k_1k_2}\int_0^1\int_0^1\imath_1^{\frac{\mu}{k_1}-1}\imath_2^{\frac{\nu}{k_2}-1}\bigl[\rho(x,u)+\rho(x,v)+\rho(y,u)+\rho(y,v)\bigr]\,d\imath_1\,d\imath_2
\end{aligned}$$

$$- \frac{\mu\nu}{16k_1k_2h_1(\frac{1}{2})h_2(\frac{1}{2})} \times \int_0^1\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\bigl[\mu_1\bigl(y^{\ell_1}-x^{\ell_1}\bigr)^{\sigma_1}+\mu_2\bigl(v^{\ell_2}-u^{\ell_2}\bigr)^{\sigma_2}\bigr]\,d\iota_1\,d\iota_2. \tag{3.2}$$

Note that by the change of variable, we have for the first integral on the right-hand side of the inequality (3.2)

$$\begin{aligned}
&\int_0^1\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\bigl[\rho(x,u)+\rho(x,v)+\rho(y,u)+\rho(y,v)\bigr]\,d\iota_1\,d\iota_2\\
&= \frac{\ell_1\ell_2}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\biggl[\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}} \frac{x^{\ell_1-1}y^{\ell_2-1}}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(x,y)\,dy\,dx\\
&\quad+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}} \frac{x^{\ell_1-1}y^{\ell_2-1}}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(x,y)\,dy\,dx\\
&\quad+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}} \frac{x^{\ell_1-1}y^{\ell_2-1}}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(x,y)\,dy\,dx\\
&\quad+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}} \frac{x^{\ell_1-1}y^{\ell_2-1}}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho(x,y)\,dy\,dx\biggr].
\end{aligned}$$

Now applying Definition 2.22 of the Katugampola (k_1,k_2)-fractional integral, the first inequality of (3.1) is obtained. For the second inequality on the right-hand side of (3.1), we use the coordinated distance-disturbed (ℓ_1,h_1)-(ℓ_2,h_2)-convexity of higher orders of ρ as follows:

$$\begin{aligned}
\rho(x,u) &= \rho\Bigl(\bigl[\iota_1\hat{a}^{\ell_1}+(1-\iota_1)\hat{b}^{\ell_1}\bigr]^{\frac{1}{\ell_1}},\bigl[\iota_2\hat{c}^{\ell_2}+(1-\iota_2)\hat{d}^{\ell_2}\bigr]^{\frac{1}{\ell_2}}\Bigr)\\
&\le h_1(\iota_1)h_2(\iota_2)\rho(\hat{a},\hat{c})+h_1(\iota_1)h_2(1-\iota_2)\rho(\hat{a},\hat{d})+h_1(1-\iota_1)h_2(\iota_2)\rho(\hat{b},\hat{c})\\
&\quad+h_1(1-\iota_1)h_2(1-\iota_2)\rho(\hat{b},\hat{d})-\mu_1\bigl(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\bigr)^{\sigma_1}-\mu_2\bigl(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\bigr)^{\sigma_2}, &(3.3)\\
\rho(x,v) &= \rho\Bigl(\bigl[\iota_1\hat{a}^{\ell_1}+(1-\iota_1)\hat{b}^{\ell_1}\bigr]^{\frac{1}{\ell_1}},\bigl[(1-\iota_2)\hat{c}^{\ell_2}+\iota_2\hat{d}^{\ell_2}\bigr]^{\frac{1}{\ell_2}}\Bigr)\\
&\le h_1(\iota_1)h_2(1-\iota_2)\rho(\hat{a},\hat{c})+h_1(\iota_1)h_2(\iota_2)\rho(\hat{a},\hat{d})+h_1(1-\iota_1)h_2(1-\iota_2)\rho(\hat{b},\hat{c})\\
&\quad+h_1(1-\iota_1)h_2(\iota_2)\rho(\hat{b},\hat{d})-\mu_1\bigl(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\bigr)^{\sigma_1}-\mu_2\bigl(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\bigr)^{\sigma_2}, &(3.4)\\
\rho(y,u) &= \rho\Bigl(\bigl[(1-\iota_1)\hat{a}^{\ell_1}+\iota_1\hat{b}^{\ell_1}\bigr]^{\frac{1}{\ell_1}},\bigl[\iota_2\hat{c}^{\ell_2}+(1-\iota_2)\hat{d}^{\ell_2}\bigr]^{\frac{1}{\ell_2}}\Bigr)\\
&\le h_1(1-\iota_1)h_2(\iota_2)\rho(\hat{a},\hat{c})+h_1(1-\iota_1)h_2(1-\iota_2)\rho(\hat{a},\hat{d})+h_1(\iota_1)h_2(\iota_2)\rho(\hat{b},\hat{c})\\
&\quad+h_1(\iota_1)h_2(1-\iota_2)\rho(\hat{b},\hat{d})-\mu_1\bigl(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\bigr)^{\sigma_1}-\mu_2\bigl(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\bigr)^{\sigma_2}, &(3.5)
\end{aligned}$$

and

$$\begin{aligned}
\rho(y,v) &= \rho\Bigl(\bigl[(1-\iota_1)\hat{a}^{\ell_1}+\iota_1\hat{b}^{\ell_1}\bigr]^{\frac{1}{\ell_1}},\bigl[(1-\iota_2)\hat{c}^{\ell_2}+\iota_2\hat{d}^{\ell_2}\bigr]^{\frac{1}{\ell_2}}\Bigr)\\
&\le h_1(1-\iota_1)h_2(1-\iota_2)\rho(\hat{a},\hat{c})+h_1(1-\iota_1)h_2(\iota_2)\rho(\hat{a},\hat{d})+h_1(\iota_1)h_2(1-\iota_2)\rho(\hat{b},\hat{c})\\
&\quad+h_1(\iota_1)h_2(\iota_2)\rho(\hat{b},\hat{d})-\mu_1\bigl(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\bigr)^{\sigma_1}-\mu_2\bigl(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\bigr)^{\sigma_2}. &(3.6)
\end{aligned}$$

Adding inequalities (3.3), (3.4), (3.5) and (3.6), we arrive at the result

$$\begin{aligned}
&\rho(x,u)+\rho(x,v)+\rho(y,u)+\rho(y,v)\\
&\quad\le\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big]\\
&\qquad\times\big[h_1(\iota_1)h_2(\iota_2)+h_1(\iota_1)h_2(1-\iota_2)\\
&\qquad+h_1(1-\iota_1)h_2(\iota_2)+h_1(1-\iota_1)h_2(1-\iota_2)\big]\\
&\qquad-2\mu_1\big[\big(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\big)^{\sigma_1}+\big(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\big)^{\sigma_1}\big]\\
&\qquad-2\mu_2\big[\big(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\big)^{\sigma_2}+\big(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\big)^{\sigma_2}\big].
\end{aligned}\tag{3.7}$$

Multiplying (3.7) by $\frac{\mu\nu}{4k_1k_2}\iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}$ and integrating over $([0,1]\times[0,1])$, one has the second inequality of (3.1) by applying Definition 2.22, which then completes the proof. □

Corollary 3.2 *Taking $k_1,k_2\to 1$ in Theorem 3.1, we have*

$$\begin{aligned}
&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+A^{\star}\\
&\quad\le\frac{\ell_1^{\mu}\ell_2^{\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{4(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\\
&\qquad\times\big[{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})\\
&\qquad+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\big]\\
&\quad\le\frac{\mu\nu}{4}\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big]\\
&\qquad\times\int_0^1\int_0^1\iota_1^{\mu-1}\iota_2^{\nu-1}\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\,d\iota_1\,d\iota_2\\
&\qquad-\frac{\mu_1[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}]+\mu_2[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}]}{2},
\end{aligned}\tag{3.8}$$

where

$$A^{\star}=-\frac{\mu\nu}{16}\left[\frac{\mu_1}{\nu}\left(\frac{1}{2\hat{b}^{\ell_1}}\right)^{\mu}C_1^{\star}+\frac{\mu_2}{\mu}\left(\frac{1}{2\hat{d}^{\ell_2}}\right)^{\nu}C_2^{\star}\right]$$

and

$$C_1^{\star}=\int_{\hat{a}^{\ell_1}-\hat{b}^{\ell_1}}^{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}\iota_1^{\sigma_1}\big[\iota_1-\hat{a}^{\ell_1}+\hat{b}^{\ell_1}\big]^{\mu-1}\,d\iota_1,\qquad C_2^{\star}=\int_{\hat{c}^{\ell_2}-\hat{d}^{\ell_2}}^{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}\iota_2^{\sigma_2}\big[\iota_2-\hat{c}^{\ell_2}+\hat{d}^{\ell_2}\big]^{\nu-1}\,d\iota_2.$$

Corollary 3.3 *Taking $\mu_1,\mu_2\to 0^+$ in Theorem 3.1, we get*

$$\begin{aligned}
&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\\
&\quad\le\frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{4(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}
\end{aligned}$$

$$\times \left[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c}) + {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]$$
$$\leq \frac{\mu\nu}{4k_1k_2}\left[\rho(\hat{a},\hat{c}) + \rho(\hat{a},\hat{d}) + \rho(\hat{b},\hat{c}) + \rho(\hat{b},\hat{d})\right]$$
$$\times \int_0^1\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\left[h_1(\iota_1) + h_1(1-\iota_1)\right]\left[h_2(\iota_2) + h_2(1-\iota_2)\right]d\iota_1\,d\iota_2. \tag{3.9}$$

Corollary 3.4 *Taking $k_1, k_2 \to 1$ in Corollary 3.3, we obtain*

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, \left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)$$
$$\leq \frac{\ell_1^{\mu}\ell_2^{\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{4(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\mu}(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\nu}}$$
$$\times \left[{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c}) + {}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]$$
$$\leq \frac{\mu\nu}{4}\left[\rho(\hat{a},\hat{c}) + \rho(\hat{a},\hat{d}) + \rho(\hat{b},\hat{c}) + \rho(\hat{b},\hat{d})\right]$$
$$\times \int_0^1\int_0^1 \iota_1^{\mu-1}\iota_2^{\nu-1}\left[h_1(\iota_1) + h_1(1-\iota_1)\right]\left[h_2(\iota_2) + h_2(1-\iota_2)\right]d\iota_1\,d\iota_2. \tag{3.10}$$

Remark 3.5 If $\mu = 1 = \nu$, then Corollary 3.4 becomes Theorem 2.16 which was proved in [38].

Remark 3.6 If $h_1(t) = t = h_2(t)$, then Remark 3.5 coincides with Theorem 2.11 which was proved in [27].

Corollary 3.7 *Taking $\ell_1 = \ell_2 = 1$ in Corollary 3.4, we have*

$$\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\frac{\hat{a}+b}{2}, \frac{\hat{c}+\hat{d}}{2}\right)$$
$$\leq \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{4(\hat{b}-\hat{a})^{\mu}(\hat{d}-\hat{c})^{\nu}}\left[I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c}) + I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]$$
$$\leq \frac{\mu\nu}{4}\left[\rho(\hat{a},\hat{c}) + \rho(\hat{a},\hat{d}) + \rho(\hat{b},\hat{c}) + \rho(\hat{b},\hat{d})\right]$$
$$\times \int_0^1\int_0^1 \iota_1^{\mu-1}\iota_2^{\nu-1}\left[h_1(\iota_1) + h_1(1-\iota_1)\right]\left[h_2(\iota_2) + h_2(1-\iota_2)\right]d\iota_1\,d\iota_2.$$

This result coincides with Theorem 2.1 of [36], if $h_1(t) = h_2(t) = h(t)$. Furthermore, if $\mu = \nu = 1$, it reduces to Theorem 7 of [23].

Remark 3.8 If $h_1(t) = h_2(t) = t$ then Corollary 3.7 coincides with Theorem 2.8.

Corollary 3.9 *Suppose that $\rho : \Omega \to \mathbb{R}$ is distance-disturbed (ℓ_1, s_1)-(ℓ_2, s_2)-convex function of higher orders on the coordinates on Ω and $\rho \in L_1(\Omega)$. Then one has the inequali-*

ties

$$
\begin{aligned}
&2^{s_1+s_2-2}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+A\\
&\leq \frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{4(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\\
&\quad\times\left[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})\right.\\
&\quad\left.+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\leq \frac{\mu\nu}{4k_1k_2}\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\quad\times\left\{\frac{1}{(\mu+s_1)(\nu+s_2)}+\frac{B(\nu,s_2+1)}{(\mu+s_1)}\right.\\
&\quad\left.+\frac{B(\mu,s_1+1)}{(\nu+s_2)}+B(\nu,s_2+1)B(\mu,s_1+1)\right\}\\
&\quad-\frac{\mu_1[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}]+\mu_2[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}]}{2},
\end{aligned}
$$

where $B(x,y)=\int_0^1 \jmath^{x-1}(1-\jmath)^{y-1}\,d\jmath$, *for all* $x,y>0$, *is the Beta function.*

Corollary 3.10 *Suppose that* $\rho:\Omega\to\mathbb{R}$ *is distance-disturbed* (ℓ,s_1)-(ℓ,s_2)*-convex function of higher orders on the coordinates on* Ω *and* $\rho\in L_1(\Omega)$. *Then one has the inequalities*

$$
\begin{aligned}
&2^{s_1+s_2-2}\rho\left(\left[\frac{\hat{a}^{\ell}+\hat{b}^{\ell}}{2}\right]^{\frac{1}{\ell}},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)+B\\
&\leq \frac{\ell^{\frac{\mu}{k_1}+\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{4(\hat{b}^{\ell}-\hat{a}^{\ell})^{\frac{\mu}{k_1}}(\hat{d}^{\ell}-\hat{c}^{\ell})^{\frac{\nu}{k_2}}}\\
&\quad\times\left[{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})\right.\\
&\quad\left.+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\leq \frac{\mu\nu}{4k_1k_2}\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\quad\times\left\{\frac{1}{(\mu+s_1)(\nu+s_2)}+\frac{B(\nu,s_2+1)}{(\mu+s_1)}+\frac{B(\mu,s_1+1)}{(\nu+s_2)}+B(\nu,s_2+1)B(\mu,s_1+1)\right\}\\
&\quad-\frac{\mu_1[(\hat{b}^{\ell}-\hat{a}^{\ell})^{\sigma_1}+(\hat{a}^{\ell}-\hat{b}^{\ell})^{\sigma_1}]+\mu_2[(\hat{d}^{\ell}-\hat{c}^{\ell})^{\sigma_2}+(\hat{c}^{\ell}-\hat{d}^{\ell})^{\sigma_2}]}{2},
\end{aligned}
$$

where

$$
B=-\frac{\mu\nu}{16k_1k_2}\left[\frac{\mu_1k_2}{\nu}\left(\frac{1}{2\hat{b}^{\ell}}\right)^{\frac{\mu}{k_1}}B_1+\frac{\mu_2k_1}{\mu}\left(\frac{1}{2\hat{d}^{\ell}}\right)^{\frac{\nu}{k_2}}B_2\right]
$$

and

$$
B_1=\int_{\hat{a}^{\ell}-\hat{b}^{\ell}}^{\hat{a}^{\ell}+\hat{b}^{\ell}}\imath_1^{\sigma_1}\left[\imath_1-\hat{a}^{\ell}+\hat{b}^{\ell}\right]^{\frac{\mu}{k_1}-1}d\imath_1,\qquad B_2=\int_{\hat{c}^{\ell}-\hat{d}^{\ell}}^{\hat{c}^{\ell}+\hat{d}^{\ell}}\imath_2^{\sigma_2}\left[\imath_2-\hat{c}^{\ell}+\hat{d}^{\ell}\right]^{\frac{\nu}{k_2}-1}d\imath_2.
$$

Corollary 3.11 *Suppose that $\rho : \Omega \to \mathbb{R}$ is distance-disturbed (s_1, s_2)-convex function of order 2 on the coordinates on Ω and $\rho \in L_1(\Omega)$. Then one has the inequalities*:

$$\begin{aligned}
&2^{s_1+s_2-2}\rho\left(\frac{\hat{a}+\hat{b}}{2}, \frac{\hat{c}+\hat{d}}{2}\right) + D \\
&\quad\le \frac{\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{4(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}} \\
&\qquad\times \bigl[{}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c}) \\
&\qquad + {}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\bigr] \\
&\quad\le \frac{\mu\nu}{4k_1k_2}\bigl[\rho(\hat{a},\hat{c}) + \rho(\hat{a},\hat{d}) + \rho(\hat{b},\hat{c}) + \rho(\hat{b},\hat{d})\bigr] \\
&\qquad\times \left\{\frac{1}{(\mu+s_1)(\nu+s_2)} + \frac{B(\nu,s_2+1)}{(\mu+s_1)} + \frac{B(\mu,s_1+1)}{(\nu+s_2)} + B(\nu,s_2+1)B(\mu,s_1+1)\right\} \\
&\qquad - \bigl[\mu_1(\hat{b}-\hat{a})^2 + \mu_2(\hat{d}-\hat{c})^2\bigr],
\end{aligned}$$

where

$$\begin{aligned}
&D = -\frac{\mu\nu}{16k_1k_2}\left[\frac{\mu_1k_2}{\nu}\left(\frac{1}{2\hat{b}}\right)^{\frac{\mu}{k_1}} D_1 + \frac{\mu_2k_1}{\mu}\left(\frac{1}{2\hat{d}}\right)^{\frac{\nu}{k_2}} D_2\right], \\
&D_1 = \int_{\hat{a}-\hat{b}}^{\hat{a}+\hat{b}} \iota_1^2[\iota_1-\hat{a}+\hat{b}]^{\frac{\mu}{k_1}-1}\,d\iota_1, \qquad D_2 = \int_{\hat{c}-\hat{d}}^{\hat{c}+\hat{d}} \iota_2^2[\iota_2-\hat{c}+\hat{d}]^{\frac{\nu}{k_2}-1}\,d\iota_2.
\end{aligned}$$

To prove our next result, we need Proposition 3.12.

Proposition 3.12 *Let $\rho : I = [\hat{a}, \hat{b}] \subseteq (0,\infty) \to \mathbb{R}$ be a distance-disturbed (ℓ, h)-convex function of higher order $\sigma > 0$ and $\rho \in L_1[\hat{a}, \hat{b}]$. Then, for $\alpha, \mu, k > 0$, the following double inequality holds*:

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}}\right) + \frac{\alpha\mu}{4k}\bigl(\hat{b}^\ell-\hat{a}^\ell\bigr)^\sigma W \\
&\quad\le \frac{\ell^{\frac{\mu}{k}}\Gamma_k(\alpha+k)}{(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k}}}\bigl[{}^{\ell}I^{\alpha,k}_{\hat{a}+}\rho(\hat{b}) + {}^{\ell}I^{\alpha,k}_{\hat{b}-}\rho(\hat{a})\bigr] \\
&\quad\le \alpha\left[\frac{\rho(\hat{a})+\rho(\hat{b})}{k}\right]\int_0^1 t^{\frac{\alpha}{k}-1}\bigl[h(t)+h(1-t)\bigr]\,dt \\
&\qquad - \mu\frac{\alpha k}{(\alpha+k)(\alpha+2k)}\bigl[\bigl(\hat{b}^\ell-\hat{a}^\ell\bigr)^\sigma + \bigl(\hat{a}^\ell-\hat{b}^\ell\bigr)^\sigma\bigr],
\end{aligned} \tag{3.11}$$

where

$$W = \int_0^1 t^{\frac{\alpha}{k}-1}(2t-1)^\sigma\,dt.$$

Proof Since ρ is a distance-disturbed (ℓ,h)-convex function of higher order $\sigma>0$ on $[\hat{a},\hat{b}]$, taking $x^\ell = t\hat{a}^\ell + (1-t)\hat{b}^\ell$ and $y^\ell = (1-t)\hat{a}^\ell + t\hat{b}^\ell$, for all $t\in[0,1]$, we have

$$\frac{1}{h(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}}\right) \le \rho\left(\left[t\hat{a}^\ell+(1-t)\hat{b}^\ell\right]^{\frac{1}{\ell}}\right)+\rho\left(\left[(1-t)\hat{a}^\ell+t\hat{b}^\ell\right]^{\frac{1}{\ell}}\right) - \frac{\mu}{4}(2t-1)^\sigma\left(\hat{b}^\ell-\hat{a}^\ell\right)^\sigma. \tag{3.12}$$

Multiplying both sides of (3.12) by $t^{\frac{\alpha}{k}-1}$ and integrating w.r.t. t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{k}{\alpha h(\frac{1}{2})}f\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}}\right)\\ &\quad\le \int_0^1 t^{\frac{\alpha}{k}-1}\rho\left(\left[t\hat{a}^\ell+(1-t)\hat{b}^\ell\right]^{\frac{1}{\ell}}\right)dt\\ &\qquad+\int_0^1 t^{\frac{\alpha}{k}-1}\rho\left(\left[(1-t)\hat{a}^\ell+t\hat{b}^\ell\right]^{\frac{1}{\ell}}\right)dt-\frac{\mu}{4}\left(\hat{b}^\ell-\hat{a}^\ell\right)^\sigma\int_0^1 t^{\frac{\alpha}{k}-1}(2t-1)^\sigma\,dt.\end{aligned} \tag{3.13}$$

By a change of variable in (3.13), we get

$$\frac{k}{\alpha h(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}}\right)\le\frac{\ell}{(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\alpha}{k}}}\left[\int_{\hat{a}}^{\hat{b}}\frac{x^{\frac{\alpha}{k}-1}}{(\hat{b}^\ell-x^\ell)^{\frac{\alpha}{k}}}\rho(x)\,dx+\int_{\hat{a}}^{\hat{b}}\frac{x^{\frac{\alpha}{k}-1}}{(x^\ell-\hat{a}^\ell)^{\frac{\alpha}{k}}}\rho(x)\,dx\right] - \frac{\mu}{4}\left(\hat{b}^\ell-\hat{a}^\ell\right)^\sigma W.$$

Applying Definition 2.20 of Katugampola k-fractional integrals, one has the first inequality of (3.11). For the second inequality on the right-hand side of (3.11), by using the distance-disturbed (ℓ,h)-convexity of higher order $\sigma>0$ of ρ, we have

$$\rho\left(\left[t\hat{a}^\ell+(1-t)\hat{b}^\ell\right]^{\frac{1}{\ell}}\right)+\rho\left(\left[(1-t)\hat{a}^\ell+t\hat{b}^\ell\right]^{\frac{1}{\ell}}\right)\le\left[\rho(\hat{a})+\rho(\hat{b})\right]\left(h(t)+h(1-t)\right) - \mu t(1-t)\left[\left(\hat{b}^\ell-\hat{a}^\ell\right)^\sigma+\left(\hat{a}^\ell-\hat{b}^\ell\right)^\sigma\right].$$

Multiplying by $t^{\frac{\alpha}{k}-1}$ on both sides and integrating over $[0,1]$, we obtained the second inequality of (3.11). □

Corollary 3.13 *Taking $\mu\to 0^+$ in Proposition 3.12, we have*

$$\begin{aligned}\frac{1}{h(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}}\right)&\le\frac{\ell^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\alpha}{k}}}\left[{}^\ell I^{\alpha,k}_{\hat{a}+}\rho(\hat{b})+{}^\ell I^{\mu,k}_{\hat{b}-}\rho(\hat{a})\right]\\ &\le\alpha\left[\frac{\rho(\hat{a})+\rho(\hat{b})}{k}\right]\int_0^1 t^{\frac{\alpha}{k}-1}\left[h(t)+h(1-t)\right]dt.\end{aligned} \tag{3.14}$$

Remark 3.14 If $\alpha=k=1$, then Corollary 3.13 coincides with Theorem 5 of [8].

Remark 3.15 If $\ell=k=1$ and $h(t)=t$, then Corollary 3.13 coincides with Theorem 2 of [35].

Now, using Proposition 3.12 we can give the following result.

Theorem 3.16 *Suppose that $\rho:\Omega\to\mathbb{R}$ is a distance-disturbed (ℓ_1,h_1)-(ℓ_2,h_2)-convex function of higher orders on the coordinates on Ω and $\rho\in L_1(\Omega)$. Then one has the inequalities*

$$\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\\
&\le \frac{\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\right]\\
&\quad+\frac{\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2h_1(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},d\right)+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},c\right)\right]\\
&\quad+\frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}W_2(F_1+F_2)+\frac{\mu\nu\ell_2\mu_1(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}}{8k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}W_1(F_3+F_4)\\
&\le \frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\\
&\quad\times\left[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})\right.\\
&\quad\left.+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\le \frac{\nu\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\right]\\
&\quad\times\int_0^1\iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2)+h_2(1-\iota_2)\right]d\iota_2\\
&\quad-\mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\right)^{\sigma_2}+\left(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\right)^{\sigma_2}\right](F_1+F_2)\\
&\quad+\frac{\mu\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\right]\\
&\quad\times\int_0^1\iota_1^{\frac{\mu}{k_1}-1}\left[h_1(\iota_1)+h_1(1-\iota_1)\right]d\iota_1\\
&\quad-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[\left(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\right)^{\sigma_1}+\left(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\right)^{\sigma_1}\right](F_3+F_4)\\
&\le \frac{\mu\nu}{k_1k_2}\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\quad\times\int_0^1\int_0^1\iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2)+h_2(1-\iota_2)\right]\left[h_1(\iota_1)+h_1(1-\iota_1)\right]d\iota_2\,d\iota_1\\
&\quad-\mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\right)^{\sigma_2}+\left(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\right)^{\sigma_2}\right](F_1+F_2)\\
&\quad-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\\
&\quad\times\left[\left(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\right)^{\sigma_1}+\left(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\right)^{\sigma_1}\right](F_3+F_4),
\end{aligned}\tag{3.15}$$

where

$$F_1 = \int_{\hat{a}}^{\hat{b}} \frac{x^{\ell_1-1}}{(\hat{b}^{\ell_1} - x^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx, \qquad F_2 = \int_{\hat{a}}^{\hat{b}} \frac{x^{\ell_1-1}}{(x^{\ell_1} - \hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx,$$
$$F_3 = \int_{\hat{c}}^{\hat{d}} \frac{y^{\ell_2-1}}{(\hat{d}^{\ell_2} - y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy, \qquad F_4 = \int_{\hat{c}}^{\hat{d}} \frac{y^{\ell_2-1}}{(y^{\ell_2} - \hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy,$$

and

$$W_1 = \int_0^1 \iota_1^{\frac{\mu}{k_1}-1}(2\iota_1 - 1)^{\sigma_1}\,d\iota_1, \qquad W_2 = \int_0^1 \iota_2^{\frac{\nu}{k_2}-1}(2\iota_2 - 1)^{\sigma_2}\,d\iota_2.$$

Proof Since $\rho : \Omega \to \mathbb{R}$ is a distance-disturbed (ℓ_1, h_1)-(ℓ_2, h_2)-convex function of higher orders (σ_1, σ_2), then partial mapping $\rho_x : [\hat{c}, \hat{d}] \to \mathbb{R}$ defined by $\rho_x(v) = \rho(x, v)$ for all $x \in [\hat{a}, \hat{b}]$ is distance-disturbed (ℓ_2, h_2)-convex of order σ_1 on $[\hat{c}, \hat{d}]$. Similarly, $\rho_y : [\hat{a}, \hat{b}] \to \mathbb{R}$ defined by $\rho_y(u) = \rho(u, y)$ for all $y \in [\hat{c}, \hat{d}]$ is distance-disturbed (ℓ_1, h_1)-convex of order σ_2 on $[\hat{a}, \hat{b}]$. Then, by Proposition 3.12 and applying the distance-disturbed (ℓ_2, h_2)-convexity of ρ_x, we have

$$\begin{aligned}
&\frac{1}{h_2(\frac{1}{2})}\rho_x\left(\left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) + \frac{\nu\mu_2}{4k_2}\left(\hat{d}^{\ell_2} - \hat{c}^{\ell_2}\right)^{\sigma_2} W_2 \\
&\quad\le \frac{\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu + k_2)}{(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho_x(\hat{d}) + {}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho_x(\hat{c})\right] \\
&\quad\le \nu\left[\frac{\rho_x(\hat{c}) + \rho_x(\hat{d})}{k_2}\right]\int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2) + h_2(1 - \iota_2)\right]d\iota_2 \\
&\qquad - \mu_2\frac{\nu k_2}{(\nu + k_2)(\nu + 2k_2)}\left[\left(\hat{d}^{\ell_2} - \hat{c}^{\ell_2}\right)^{\sigma_2} + \left(\hat{c}^{\ell_2} - \hat{d}^{\ell_2}\right)^{\sigma_2}\right], \\
&\frac{1}{h_2(\frac{1}{2})}\rho\left(x, \left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) + \frac{\nu\mu_2}{4k_2}\left(\hat{d}^{\ell_2} - \hat{c}^{\ell_2}\right)^{\sigma_2} W_2 \\
&\quad\le \frac{\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu + k_2)}{(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho(x, \hat{d}) + {}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho(x, \hat{c})\right].
\end{aligned}$$

Or

$$\begin{aligned}
&\le \nu\left[\frac{\rho(x, \hat{c}) + \rho(x, \hat{d})}{k_2}\right]\int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2) + h_2(1 - \iota_2)\right]d\iota_2 \\
&\quad - \mu_2\frac{\nu k_2}{(\nu + k_2)(\nu + 2k_2)}\left[\left(\hat{d}^{\ell_2} - \hat{c}^{\ell_2}\right)^{\sigma_2} + \left(\hat{c}^{\ell_2} - \hat{d}^{\ell_2}\right)^{\sigma_2}\right].
\end{aligned} \tag{3.16}$$

Integrating inequality (3.16) w.r.t. x over $[\hat{a}, \hat{b}]$ after multiplying by

$$\frac{\mu\ell_1 x^{\ell_1-1}}{2k_1(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{b}^{\ell_1} - x^{\ell_1})^{1-\frac{\mu}{k_1}}} \quad \text{and} \quad \frac{\mu\ell_1 x^{\ell_1-1}}{2k_1(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(x^{\ell_1} - \hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}},$$

respectively, we obtain

$$\begin{aligned}
&\frac{\mu\ell_1}{2k_1h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\rho\left(x,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dx \\
&\quad+\frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}W_2\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx \\
&\le\frac{\mu\nu\ell_1\ell_2}{2k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \\
&\quad\times\left[\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\right. \\
&\quad\left.+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\right] \\
&\le\frac{\mu\nu\ell_1}{2k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}\rho(x,\hat{c})}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx+\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}\rho(x,\hat{d})}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx\right] \\
&\quad\times\int_0^1\iota_2^{\frac{\nu}{k_2}-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\,d\iota_2 \\
&\quad-\mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\big[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}\big] \\
&\quad\times\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
&\frac{\mu\ell_1}{2k_1h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}}\rho\left(x,\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)dx \\
&\quad+\frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}W_2\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx \\
&\le\frac{\mu\nu\ell_1\ell_2}{2k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \\
&\quad\times\left[\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\right. \\
&\quad\left.+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\right] \\
&\le\frac{\mu\nu\ell_1}{2k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}\rho(x,\hat{c})}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx+\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell_1-1}\rho(x,\hat{d})}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}}\,dx\right] \\
&\quad\times\int_0^1\iota_2^{\frac{\nu}{k_2}-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\,d\iota_2
\end{aligned}$$

$$- \mu_2 \frac{\mu \nu \ell_1 k_2}{2k_1(\nu + k_2)(\nu + 2k_2)(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \left[\left(\hat{d}^{\ell_2} - \hat{c}^{\ell_2}\right)^{\sigma_2} + \left(\hat{c}^{\ell_2} - \hat{d}^{\ell_2}\right)^{\sigma_2} \right] \\ \times \int_{\hat{a}}^{\hat{b}} \frac{x^{\ell_1 - 1}}{(x^{\ell_1} - \hat{a}^{\ell_1})^{1 - \frac{\mu}{k_1}}} \, dx. \tag{3.18}$$

Now again by Proposition 3.12 and applying the distance-disturbed (ℓ_1, h_1)-convexity of ρ_y, we have

$$\frac{1}{h_1(\frac{1}{2})} \rho_y \left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2} \right]^{\frac{1}{\ell_1}} \right) + \frac{\mu \mu_1}{4k_1} \left(\hat{b}^{\ell_1} - \hat{a}^{\ell_1}\right)^{\sigma_1} W_1 \\ \leq \frac{\ell_1^{\frac{\mu}{k_1}} \Gamma_{k_1}(\mu + k_1)}{(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \left[{}^{\ell_1} I_{\hat{a}+}^{\mu, k_1} \rho_y(\hat{b}) + {}^{\ell_1} I_{\hat{b}-}^{\mu, k_1} \rho_y(\hat{a}) \right] \\ \leq \mu \left[\frac{\rho_y(\hat{a}) + \rho_y(\hat{b})}{k_1} \right] \int_0^1 \iota_1^{\frac{\mu}{k_1} - 1} \left[h_1(\iota_1) + h_1(1 - \iota_1) \right] d\iota_1 \\ - \mu_1 \frac{\mu k_1}{(\mu + k_1)(\mu + 2k_1)} \left[\left(\hat{b}^{\ell_1} - \hat{a}^{\ell_1}\right)^{\sigma_1} + \left(\hat{a}^{\ell_1} - \hat{b}^{\ell_1}\right)^{\sigma_1} \right].$$

Or

$$\frac{1}{h_1(\frac{1}{2})} \rho \left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2}, y \right]^{\frac{1}{\ell_1}} \right) + \frac{\mu \mu_1}{4k_1} \left(\hat{b}^{\ell_1} - \hat{a}^{\ell_1}\right)^{\sigma_1} W_1 \\ \leq \frac{\ell_1^{\frac{\mu}{k_1}} \Gamma_{k_1}(\mu + k_1)}{(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \left[{}^{\ell_1} I_{\hat{a}+}^{\mu, k_1} \rho(\hat{b}, y) + {}^{\ell_1} I_{\hat{b}-}^{\mu, k_1} \rho(\hat{a}, y) \right] \\ \leq \mu \left[\frac{\rho(\hat{a}, y) + \rho(\hat{b}, y)}{k_1} \right] \int_0^1 \iota_1^{\frac{\mu}{k_1} - 1} \left[h_1(\iota_1) + h_1(1 - \iota_1) \right] d\iota_1 \\ - \mu_1 \frac{\mu k_1}{(\mu + k_1)(\mu + 2k_1)} \left[\left(\hat{b}^{\ell_1} - \hat{a}^{\ell_1}\right)^{\sigma_1} + \left(\hat{a}^{\ell_1} - \hat{b}^{\ell_1}\right)^{\sigma_1} \right]. \tag{3.19}$$

Integrating (3.19) w.r.t. y over $[\hat{c}, \hat{d}]$ after multiplying by

$$\frac{\nu \ell_2 y^{\ell_2 - 1}}{2k_2(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}} (\hat{d}^{\ell_2} - y^{\ell_2})^{1 - \frac{\nu}{k_2}}} \quad \text{and} \quad \frac{\nu \ell_2 y^{\ell_2 - 1}}{2k_2(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}} (y^{\ell_2} - \hat{c}^{\ell_2})^{1 - \frac{\nu}{k_2}}},$$

respectively, we have

$$\frac{\nu \ell_2}{2k_2 h_1(\frac{1}{2})(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} \int_c^d \frac{y^{\ell_2 - 1}}{(\hat{d}^{\ell_2} - y^{\ell_2})^{1 - \frac{\nu}{k_2}}} \rho \left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2} \right]^{\frac{1}{\ell_1}}, y \right) dy \\ + \frac{\mu \nu \ell_2 \mu_1 (\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\sigma_1}}{8k_1 k_2 (\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} W_1 \int_{\hat{c}}^{\hat{d}} \frac{y^{\ell_2 - 1}}{(\hat{d}^{\ell_2} - y^{\ell_2})^{1 - \frac{\nu}{k_2}}} \, dy \\ \leq \frac{\mu \nu \ell_1 \ell_2}{2k_1 k_2 (\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}} (\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \\ \times \left[\int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \frac{x^{\ell_1 - 1} y^{\ell_2 - 1} \rho(x, y)}{(\hat{b}^{\ell_1} - x^{\ell_1})^{1 - \frac{\mu}{k_1}} (\hat{d}^{\ell_2} - y^{\ell_2})^{1 - \frac{\nu}{k_2}}} \, dy \, dx \right.$$

$$
\begin{aligned}
&+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\Bigg] \\
&\leq \frac{\mu\nu\ell_2}{2k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigg[\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}\rho(\hat{a},y)}{(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy+\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}\rho(\hat{b},y)}{(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\bigg] \\
&\times\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_1 \\
&-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\big[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}\big] \\
&\times\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}}{(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
&\frac{\nu\ell_2}{2k_2h_1(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}}{(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},y\bigg)\,dy \\
&+\frac{\mu\nu\ell_2\mu_1(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}}{8k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}W_1\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}}{(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy \\
&\leq \frac{\mu\nu\ell_1\ell_2}{2k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} \\
&\times\Bigg[\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(\hat{b}^{\ell_1}-x^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx \\
&+\int_{\hat{a}}^{\hat{b}}\int_{\hat{c}}^{\hat{d}}\frac{x^{\ell_1-1}y^{\ell_2-1}\rho(x,y)}{(x^{\ell_1}-\hat{a}^{\ell_1})^{1-\frac{\mu}{k_1}}(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\,dx\Bigg] \\
&\leq \frac{\mu\nu\ell_2}{2k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigg[\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}\rho(a,y)}{(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy+\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}\rho(\hat{b},y)}{(y^{\ell_2}-\hat{c}^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy\bigg] \\
&\times\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_1 \\
&-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\big[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}\big] \\
&\times\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell_2-1}}{(\hat{d}^{\ell_2}-y^{\ell_2})^{1-\frac{\nu}{k_2}}}\,dy.
\end{aligned}
\tag{3.21}
$$

Adding inequalities (3.17), (3.18), (3.20), (3.21) and applying Definition 2.22, one obtains

$$
\begin{aligned}
&\frac{\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\bigg[{}^{\ell_1}I_{\hat{a}+}^{\mu,k_1}\rho\bigg(\hat{b},\bigg[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\bigg]^{\frac{1}{\ell_2}}\bigg)+{}^{\ell_1}I_{\hat{b}-}^{\mu,k_1}\rho\bigg(\hat{a},\bigg[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\bigg]^{\frac{1}{\ell_2}}\bigg)\bigg] \\
&+\frac{\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2h_1(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigg[{}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},\hat{d}\bigg)+{}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho\bigg(\bigg[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\bigg]^{\frac{1}{\ell_1}},\hat{c}\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}} W_2(F_1+F_2) + \frac{\mu\nu\ell_2\mu_1(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}}{8k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} W_1(F_3+F_4) \\
\le{}& \frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} \\
&\times \bigl[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c}) \\
&+ {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\bigr] \\
\le{}& \frac{\nu\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\bigl[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\bigr] \\
&\times \int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\bigl[h_2(\iota_2)+h_2(1-\iota_2)\bigr]\,d\iota_2 \\
&- \mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\bigl[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}\bigr](F_1+F_2) \\
&+ \frac{\mu\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigl[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\bigr] \\
&\times \int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\bigl[h_1(\iota_1)+h_1(1-\iota_1)\bigr]\,d\iota_1 \\
&- \mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigl[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}\bigr](F_3+F_4),
\end{aligned}
$$

which are the second and third inequalities of (3.15). For the last inequality of (3.15), applying Proposition 3.12 to the last part of the above inequality, we have

$$
\begin{aligned}
&\frac{\nu\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\bigl[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c}) + {}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\bigr] \\
&\quad\times \int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\bigl[h_2(\iota_2)+h_2(1-\iota_2)\bigr]\,d\iota_2 \\
&\quad- \mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\bigl[(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}+(\hat{c}^{\ell_2}-\hat{d}^{\ell_2})^{\sigma_2}\bigr](F_1+F_2) \\
&\quad+ \frac{\mu\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigl[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c}) + {}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\bigr] \\
&\quad\times \int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\bigl[h_1(\iota_1)+h_1(1-\iota_1)\bigr]\,d\iota_1 \\
&\quad- \mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\bigl[(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}+(\hat{a}^{\ell_1}-\hat{b}^{\ell_1})^{\sigma_1}\bigr](F_3+F_4) \\
&\le \frac{\mu\nu}{k_1k_2}\bigl[\rho(\hat{b},\hat{c})+\rho(\hat{a},\hat{c})+\rho(\hat{b},\hat{d})+\rho(\hat{a},\hat{d})\bigr]\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\bigl[h_1(\iota_1)+h_1(1-\iota_1)\bigr]\,d\iota_1 \\
&\quad\times \int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\bigl[h_2(\iota_2)+h_2(1-\iota_2)\bigr]\,d\iota_2
\end{aligned}
$$

$$
\begin{aligned}
&-\mu_2\frac{\mu\nu\ell_1 k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\big[\big(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\big)^{\sigma_2}+\big(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\big)^{\sigma_2}\big](F_1+F_2)\\
&-\mu_1\frac{\mu\nu\ell_2 k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\big[\big(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\big)^{\sigma_1}+\big(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\big)^{\sigma_1}\big](F_3+F_4).
\end{aligned}
$$

For the first inequality of (3.15), we again use Proposition 3.12, which then completes the proof. □

Corollary 3.17 *Taking $k_1, k_2 \to 1$ in Theorem 3.16, we have*

$$
\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\\
&\leq \frac{\ell_1^{\mu}\Gamma(\mu+1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\left[{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\right]\\
&\quad+\frac{\ell_2^{\nu}\Gamma(\nu+1)}{2h_1(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\left[{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{d}\right)+{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{c}\right)\right]\\
&\quad+\frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}U_2(G_1+G_2)+\frac{\mu\nu\ell_2\mu_1(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}}{8(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}U_1(G_3+G_4)\\
&\leq \frac{\ell_1^{\mu}\ell_2^{\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\\
&\quad\times\big[{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\big]\\
&\leq \frac{\nu\ell_1^{\mu}\Gamma(\mu+1)}{2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\big[{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{d})\big]\\
&\quad\times\int_0^1 \iota_2^{\nu-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\,d\iota_2\\
&\quad-\mu_2\frac{\mu\nu\ell_1}{2(\nu+1)(\nu+2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\big[\big(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\big)^{\sigma_2}+\big(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\big)^{\sigma_2}\big](G_1+G_2)\\
&\quad+\frac{\mu\ell_2^{\nu}\Gamma(\nu+1)}{2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\big[{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho(\hat{b},\hat{c})\big]\\
&\quad\times\int_0^1 \iota_1^{\mu-1}\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_1\\
&\quad-\mu_1\frac{\mu\nu\ell_2}{2(\mu+1)(\mu+2)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\big[\big(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\big)^{\sigma_1}+\big(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\big)^{\sigma_1}\big](G_3+G_4)\\
&\leq \mu\nu\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big]\\
&\quad\times\int_0^1\int_0^1 \iota_1^{\mu-1}\iota_2^{\nu-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_2\,d\iota_1\\
&\quad-\mu_2\frac{\mu\nu\ell_1}{2(\nu+1)(\nu+2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\big[\big(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\big)^{\sigma_2}+\big(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\big)^{\sigma_2}\big](G_1+G_2)\\
&\quad-\mu_1\frac{\mu\nu\ell_2}{2(\mu+1)(\mu+2)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\big[\big(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\big)^{\sigma_1}+\big(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\big)^{\sigma_1}\big](G_3+G_4),
\end{aligned}
$$

where

$$G_1 = \int_{\hat{a}}^{\hat{b}} \frac{x^{\ell_1-1}}{(\hat{b}^{\ell_1} - x^{\ell_1})^{1-\mu}}\,dx, \qquad G_2 = \int_{\hat{a}}^{\hat{b}} \frac{x^{\ell_1-1}}{(x^{\ell_1} - \hat{a}^{\ell_1})^{1-\mu}}\,dx,$$
$$G_3 = \int_{\hat{c}}^{\hat{d}} \frac{y^{\ell_2-1}}{(\hat{d}^{\ell_2} - y^{\ell_2})^{1-\nu}}\,dy, \qquad G_4 = \int_{\hat{c}}^{\hat{d}} \frac{y^{\ell_2-1}}{(y^{\ell_2} - \hat{c}^{\ell_2})^{1-\nu}}\,dy,$$

and

$$U_1 = \int_0^1 \iota_1^{\mu-1}(2\iota_1 - 1)^{\sigma_1}\,d\iota_1, \qquad U_2 = \int_0^1 \iota_2^{\nu-1}(2\iota_2 - 1)^{\sigma_2}\,d\iota_2.$$

Corollary 3.18 *Taking* $\mu_1, \mu_2 \to 0^+$ *in Theorem* 3.16, *we get*

$$\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, \left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad\le \frac{\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu + k_1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I_{\hat{a}+}^{\mu,k_1}\rho\left(\hat{b}, \left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) + {}^{\ell_1}I_{\hat{b}-}^{\mu,k_1}\rho\left(\hat{a}, \left[\frac{\hat{c}^{\ell_2} + \hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\right] \\
&\qquad + \frac{\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu + k_2)}{2h_1(\frac{1}{2})(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho\left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, \hat{d}\right) + {}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho\left(\left[\frac{\hat{a}^{\ell_1} + \hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}}, \hat{c}\right)\right] \\
&\quad\le \frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu + k_1)\Gamma_{k_2}(\nu + k_2)}{(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}} \\
&\qquad \times \left[{}^{\ell_1,\ell_2}I_{\hat{a}+,\hat{c}+}^{\mu,\nu,k_1,k_2}\rho(\hat{b}, \hat{d}) + {}^{\ell_1,\ell_2}I_{\hat{a}+,\hat{d}-}^{\mu,\nu,k_1,k_2}\rho(\hat{b}, \hat{c})\right. \\
&\qquad \left. + {}^{\ell_1,\ell_2}I_{\hat{b}-,\hat{c}+}^{\mu,\nu,k_1,k_2}\rho(\hat{a}, \hat{d}) + {}^{\ell_1,\ell_2}I_{\hat{b}-,\hat{d}-}^{\mu,\nu,k_1,k_2}\rho(\hat{a}, \hat{c})\right] \\
&\quad\le \frac{\nu\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu + k_1)}{2k_2(\hat{b}^{\ell_1} - \hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I_{\hat{a}+}^{\mu,k_1}\rho(\hat{b}, \hat{c}) + {}^{\ell_1}I_{\hat{a}+}^{\mu,k_1}\rho(\hat{b}, \hat{d}) + {}^{\ell_1}I_{\hat{b}-}^{\mu,k_1}\rho(\hat{a}, \hat{c}) + {}^{\ell_1}I_{\hat{b}-}^{\mu,k_1}\rho(\hat{a}, \hat{d})\right] \\
&\qquad \times \int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2) + h_2(1 - \iota_2)\right]d\iota_2 \\
&\qquad + \frac{\mu\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu + k_2)}{2k_1(\hat{d}^{\ell_2} - \hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho(\hat{a}, \hat{d}) + {}^{\ell_2}I_{\hat{c}+}^{\nu,k_2}\rho(\hat{b}, \hat{d})\right. \\
&\qquad \left. + {}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho(\hat{a}, \hat{c}) + {}^{\ell_2}I_{\hat{d}-}^{\nu,k_2}\rho(\hat{b}, \hat{c})\right] \\
&\qquad \times \int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\left[h_1(\iota_1) + h_1(1 - \iota_1)\right]d\iota_1 \\
&\quad\le \frac{\mu\nu}{k_1k_2}\left[\rho(\hat{a}, \hat{c}) + \rho(\hat{a}, \hat{d}) + \rho(\hat{b}, \hat{c}) + \rho(\hat{b}, \hat{d})\right] \\
&\qquad \times \int_0^1\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\left[h_2(\iota_2) + h_2(1 - \iota_2)\right]\left[h_1(\iota_1) + h_1(1 - \iota_1)\right]d\iota_2\,d\iota_1.
\end{aligned} \tag{3.22}$$

Corollary 3.19 *Taking $k_1, k_2 \to 1$ in Corollary 3.18, we obtain*

$$\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\\
&\le \frac{\ell_1^{\mu}\Gamma(\mu+1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\left[{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\right]\\
&\quad+\frac{\ell_2^{\nu}\Gamma(\nu+1)}{2h_1(\frac{1}{2})(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\left[{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{d}\right)+{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{c}\right)\right]\\
&\le \frac{\ell_1^{\mu}\ell_2^{\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\\
&\quad\times\left[{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\le \frac{\nu\ell_1^{\mu}\Gamma(\mu+1)}{2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\mu}}\left[{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell_1}I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell_1}I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{d})\right]\\
&\quad\times\int_0^1 \iota_2^{\nu-1}\left[h_2(\iota_2)+h_2(1-\iota_2)\right]d\iota_2\\
&\quad+\frac{\mu\ell_2^{\nu}\Gamma(\nu+1)}{2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\nu}}\left[{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_2}I^{\nu}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_2}I^{\nu}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\right]\\
&\quad\times\int_0^1 \iota_1^{\mu-1}\left[h_1(\iota_1)+h_1(1-\iota_1)\right]d\iota_1\\
&\le \mu\nu\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\quad\times\int_0^1\int_0^1 \iota_1^{\mu-1}\iota_2^{\nu-1}\left[h_2(\iota_2)+h_2(1-\iota_2)\right]\left[h_1(\iota_1)+h_1(1-\iota_1)\right]d\iota_2\,d\iota_1. \qquad (3.23)
\end{aligned}$$

Remark 3.20 If $\mu = 1 = \nu$, then Corollary 3.19 gives Theorem 2.17, which was proved in [38].

Remark 3.21 If $\ell_1 = \ell_2 = 1$ and $h_1(t) = h_2(t) = t$, then Remark 3.20 reduced to Theorem 2.8, which was proved in [32].

Corollary 3.22 *Let ρ be a distance-disturbed (ℓ, h_1)–(ℓ, h_2)-convex function on the coordinates on Ω and $\rho \in L_1(\Omega)$, then one has following inequalities:*

$$\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\rho\left(\left[\frac{\hat{a}^{\ell}+\hat{b}^{\ell}}{2}\right]^{\frac{1}{\ell}},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)\\
&\le \frac{\ell^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2h_2(\frac{1}{2})(\hat{b}^{\ell}-\hat{a}^{\ell})^{\frac{\mu}{k_1}}}\left[{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)\right]\\
&\quad+\frac{\ell^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2h_1(\frac{1}{2})(\hat{d}^{\ell}-\hat{c}^{\ell})^{\frac{\nu}{k_2}}}\left[{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell}+\hat{b}^{\ell}}{2}\right]^{\frac{1}{\ell}},\hat{d}\right)+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell}+\hat{b}^{\ell}}{2}\right]^{\frac{1}{\ell}},\hat{c}\right)\right]\\
&\quad+\frac{\mu\nu\ell\mu_2(\hat{d}^{\ell}-\hat{c}^{\ell})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell}-\hat{a}^{\ell})^{\frac{\mu}{k_1}}}W_2(H_1+H_2)+\frac{\mu\nu\ell\mu_1(\hat{b}^{\ell}-\hat{a}^{\ell})^{\sigma_1}}{8k_1k_2(\hat{d}^{\ell}-\hat{c}^{\ell})^{\frac{\nu}{k_2}}}W_1(H_3+H_4)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\ell^{\frac{\mu}{k_1}+\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}} \\
&\quad\times\Big[{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\Big] \\
&\leq \frac{\nu\ell^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\Big[{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\Big] \\
&\quad\times\int_0^1 \iota_2^{\frac{\nu}{k_2}-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\,d\iota_2 \\
&\quad-\mu_2\frac{\mu\nu\ell k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\big[\big(\hat{d}^\ell-\hat{c}^\ell\big)^{\sigma_2}+\big(\hat{c}^\ell-\hat{d}^\ell\big)^{\sigma_2}\big](H_1+H_2) \\
&\quad+\frac{\mu\ell^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\Big[{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\Big] \\
&\quad\times\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_1 \\
&\quad-\mu_1\frac{\mu\nu\ell k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\big[\big(\hat{b}^\ell-\hat{a}^\ell\big)^{\sigma_1}+\big(\hat{a}^\ell-\hat{b}^\ell\big)^{\sigma_1}\big](H_3+H_4) \\
&\leq \frac{\mu\nu}{k_1k_2}\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big] \\
&\quad\times\int_0^1\int_0^1 \iota_1^{\frac{\mu}{k_1}-1}\iota_2^{\frac{\nu}{k_2}-1}\big[h_2(\iota_2)+h_2(1-\iota_2)\big]\big[h_1(\iota_1)+h_1(1-\iota_1)\big]\,d\iota_2\,d\iota_1 \\
&\quad-\mu_2\frac{\mu\nu\ell k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\big[\big(\hat{d}^\ell-\hat{c}^\ell\big)^{\sigma_2}+\big(\hat{c}^\ell-\hat{d}^\ell\big)^{\sigma_2}\big](H_1+H_2) \\
&\quad-\mu_1\frac{\mu\nu\ell k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\big[\big(\hat{b}^\ell-\hat{a}^\ell\big)^{\sigma_1}+\big(\hat{a}^\ell-\hat{b}^\ell\big)^{\sigma_1}\big](H_3+H_4), \quad (3.24)
\end{aligned}
$$

where

$$
H_1=\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell-1}}{(\hat{b}^\ell-x^\ell)^{1-\frac{\mu}{k_1}}}\,dx,\qquad H_2=\int_{\hat{a}}^{\hat{b}}\frac{x^{\ell-1}}{(x^\ell-\hat{a}^\ell)^{1-\frac{\mu}{k_1}}}\,dx,
$$

and

$$
H_3=\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell-1}}{(\hat{d}^\ell-y^\ell)^{1-\frac{\nu}{k_2}}}\,dy,\qquad H_4=\int_{\hat{c}}^{\hat{d}}\frac{y^{\ell-1}}{(y^\ell-\hat{c}^\ell)^{1-\frac{\nu}{k_2}}}\,dy.
$$

Corollary 3.23 *Let $\rho:\Omega\to\mathbb{R}$ be a distance-disturbed (ℓ_1,s_1)-(ℓ_2,s_2)-convex function on the coordinates on Ω and $\rho\in L_1(\Omega)$. Then one has the inequalities*

$$
\begin{aligned}
&2^{s_1+s_2}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right) \\
&\quad\leq \frac{2^{s_2-1}\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell_2}+\hat{d}^{\ell_2}}{2}\right]^{\frac{1}{\ell_2}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2^{s_1-1}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\\
&\times\left[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{d}\right)+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^{\ell_1}+\hat{b}^{\ell_1}}{2}\right]^{\frac{1}{\ell_1}},\hat{c}\right)\right]\\
&+\frac{\mu\nu\ell_1\mu_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\sigma_2}}{8k_1k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}W_2(F_1+F_2)+\frac{\mu\nu\ell_2\mu_1(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\sigma_1}}{8k_1k_2(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}W_1(F_3+F_4)\\
\le{}&\frac{\ell_1^{\frac{\mu}{k_1}}\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\\
&\times\left[{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})\right.\\
&\left.+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_1,\ell_2}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
\le{}&\frac{\nu\ell_1^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell_1}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell_1}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\right]\\
&\times\left[\frac{k_2}{\nu+k_2s_2}+B\left(\frac{\nu}{k_2},s_2+1\right)\right]\\
&-\mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\right)^{\sigma_2}+\left(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\right)^{\sigma_2}\right](F_1+F_2)\\
&+\frac{\mu\ell_2^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell_2}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell_2}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\right]\\
&\times\left[\frac{k_1}{\mu+k_1s_1}+B\left(\frac{\mu}{k_1},s_1+1\right)\right]\\
&-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[\left(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\right)^{\sigma_1}+\left(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\right)^{\sigma_1}\right](F_3+F_4)\\
\le{}&\frac{\mu\nu}{k_1k_2}\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\times\left\{\frac{k_1k_2}{(\mu+k_1s_1)(\nu+k_2s_2)}+\frac{k_2B(\frac{\mu}{k_1},s_1+1)}{\nu+k_2s_2}+\frac{k_1B(\frac{\nu}{k_2},s_2+1)}{\mu+k_1s_1}\right.\\
&\left.+B\left(\frac{\mu}{k_1},s_1+1\right)B\left(\frac{\nu}{k_2},s_2+1\right)\right\}\\
&-\mu_2\frac{\mu\nu\ell_1k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^{\ell_1}-\hat{a}^{\ell_1})^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^{\ell_2}-\hat{c}^{\ell_2}\right)^{\sigma_2}+\left(\hat{c}^{\ell_2}-\hat{d}^{\ell_2}\right)^{\sigma_2}\right](F_1+F_2)\\
&-\mu_1\frac{\mu\nu\ell_2k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^{\ell_2}-\hat{c}^{\ell_2})^{\frac{\nu}{k_2}}}\left[\left(\hat{b}^{\ell_1}-\hat{a}^{\ell_1}\right)^{\sigma_1}+\left(\hat{a}^{\ell_1}-\hat{b}^{\ell_1}\right)^{\sigma_1}\right](F_3+F_4).
\end{aligned}
$$

Corollary 3.24 *Taking $\ell_1=\ell_2=\ell$ in Corollary 3.23, we have*

$$
\begin{aligned}
&2^{s_1+s_2}\rho\left(\left[\frac{\hat{a}^{\ell}+\hat{b}^{\ell}}{2}\right]^{\frac{1}{\ell}},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)\\
&\le\frac{2^{s_2-1}\ell^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{(\hat{b}^{\ell}-\hat{a}^{\ell})^{\frac{\mu}{k_1}}}\left[{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho\left(\hat{b},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho\left(\hat{a},\left[\frac{\hat{c}^{\ell}+\hat{d}^{\ell}}{2}\right]^{\frac{1}{\ell}}\right)\right]
\end{aligned}
$$

$$+\frac{2^{s_1-1}\ell^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\left[{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}},\hat{d}\right)+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho\left(\left[\frac{\hat{a}^\ell+\hat{b}^\ell}{2}\right]^{\frac{1}{\ell}},\hat{c}\right)\right]$$

$$+\frac{\mu\nu\ell\mu_2(\hat{d}^\ell-\hat{c}^\ell)^{\sigma_2}}{8k_1k_2(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}W_2(H_1+H_2)+\frac{\mu\nu\ell\mu_1(\hat{b}^\ell-\hat{a}^\ell)^{\sigma_1}}{8k_1k_2(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}W_1(H_3+H_4)$$

$$\le\frac{\ell^{\frac{\mu}{k_1}+\frac{\nu}{k_2}}\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}$$

$$\times\left[{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell,\ell}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]$$

$$\le\frac{\nu\ell^{\frac{\mu}{k_1}}\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\left[{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c})+{}^{\ell}I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d})+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c})+{}^{\ell}I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\right]$$

$$\times\left[\frac{k_2}{\nu+k_2s_2}+B\left(\frac{\nu}{k_2},s_2+1\right)\right]$$

$$-\mu_2\frac{\mu\nu\ell k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^\ell-\hat{c}^\ell\right)^{\sigma_2}+\left(\hat{c}^\ell-\hat{d}^\ell\right)^{\sigma_2}\right](H_1+H_2)$$

$$+\frac{\mu\ell^{\frac{\nu}{k_2}}\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\left[{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d})+{}^{\ell}I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d})+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c})+{}^{\ell}I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\right]$$

$$\times\left[\frac{k_1}{\mu+k_1s_1}+B\left(\frac{\mu}{k_1},s_1+1\right)\right]$$

$$-\mu_1\frac{\mu\nu\ell k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\left[\left(\hat{b}^\ell-\hat{a}^\ell\right)^{\sigma_1}+\left(\hat{a}^\ell-\hat{b}^\ell\right)^{\sigma_1}\right](H_3+H_4)$$

$$\le\frac{\mu\nu}{k_1k_2}\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]$$

$$\times\left\{\frac{k_1k_2}{(\mu+k_1s_1)(\nu+k_2s_2)}+\frac{k_2B(\frac{\mu}{k_1},s_1+1)}{\nu+k_2s_2}\right.$$

$$\left.+\frac{k_1B(\frac{\nu}{k_2},s_2+1)}{\mu+k_1s_1}+B\left(\frac{\mu}{k_1},s_1+1\right)B\left(\frac{\nu}{k_2},s_2+1\right)\right\}$$

$$-\mu_2\frac{\mu\nu\ell k_2}{2k_1(\nu+k_2)(\nu+2k_2)(\hat{b}^\ell-\hat{a}^\ell)^{\frac{\mu}{k_1}}}\left[\left(\hat{d}^\ell-\hat{c}^\ell\right)^{\sigma_2}+\left(\hat{c}^\ell-\hat{d}^\ell\right)^{\sigma_2}\right](H_1+H_2)$$

$$-\mu_1\frac{\mu\nu\ell k_1}{2k_2(\mu+k_1)(\mu+2k_1)(\hat{d}^\ell-\hat{c}^\ell)^{\frac{\nu}{k_2}}}\left[\left(\hat{b}^\ell-\hat{a}^\ell\right)^{\sigma_1}+\left(\hat{a}^\ell-\hat{b}^\ell\right)^{\sigma_1}\right](H_3+H_4),$$

where H_1, H_2, H_3 and H_4 are defined in Corollary 3.22.

Corollary 3.25 *Taking $\ell=1$ and $\sigma_1=\sigma_2=2$ in Corollary* 3.24, *we get*

$$2^{s_1+s_2}\rho\left(\frac{\hat{a}+\hat{b}}{2},\frac{\hat{c}+\hat{d}}{2}\right)$$

$$\le\frac{2^{s_2-1}\Gamma_{k_1}(\mu+k_1)}{(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}}\left[I^{\mu,k_1}_{\hat{a}+}\rho\left(\hat{b},\frac{\hat{c}+\hat{d}}{2}\right)+I^{\mu,k_1}_{\hat{b}-}\rho\left(\hat{a},\frac{\hat{c}+\hat{d}}{2}\right)\right]$$

$$+\frac{2^{s_1-1}\Gamma_{k_2}(\nu+k_2)}{(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}\left[I^{\nu,k_2}_{\hat{c}+}\rho\left(\frac{\hat{a}+\hat{b}}{2},\hat{d}\right)+I^{\nu,k_2}_{\hat{d}-}\rho\left(\frac{\hat{a}+\hat{b}}{2},\hat{c}\right)\right]$$

$$
\begin{aligned}
&+\frac{\mu\nu\mu_2(\hat{d}-\hat{c})^2}{8k_1k_2(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}}W_2^{\star}\big(H_1^{\star}+H_2^{\star}\big)+\frac{\mu\nu\mu_1(\hat{b}-\hat{a})^2}{8k_1k_2(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}W_1^{\star}\big(H_3^{\star}+H_4^{\star}\big)\\
&\leq\frac{\Gamma_{k_1}(\mu+k_1)\Gamma_{k_2}(\nu+k_2)}{(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}\\
&\quad\times\big[{}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})\\
&\quad+{}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{1,1}I^{\mu,\nu,k_1,k_2}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\big]\\
&\leq\frac{\nu\Gamma_{k_1}(\mu+k_1)}{2k_2(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}}\big[I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{c})+I^{\mu,k_1}_{\hat{a}+}\rho(\hat{b},\hat{d})+I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{c})+I^{\mu,k_1}_{\hat{b}-}\rho(\hat{a},\hat{d})\big]\\
&\quad\times\left[\frac{k_2}{\nu+k_2s_2}+B\left(\frac{\nu}{k_2},s_2+1\right)\right]\\
&\quad-\mu_2\frac{\mu\nu k_2(\hat{d}-\hat{c})^2}{k_1(\nu+k_2)(\nu+2k_2)(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}}\big(H_1^{\star}+H_2^{\star}\big)\\
&\quad+\frac{\mu\Gamma_{k_2}(\nu+k_2)}{2k_1(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}\big[I^{\nu,k_2}_{\hat{c}+}\rho(\hat{a},\hat{d})+I^{\nu,k_2}_{\hat{c}+}\rho(\hat{b},\hat{d})+I^{\nu,k_2}_{\hat{d}-}\rho(\hat{a},\hat{c})+I^{\nu,k_2}_{\hat{d}-}\rho(\hat{b},\hat{c})\big]\\
&\quad\times\left[\frac{k_1}{\mu+k_1s_1}+B\left(\frac{\mu}{k_1},s_1+1\right)\right]\\
&\quad-\mu_1\frac{\mu\nu k_1(\hat{b}-\hat{a})^2}{k_2(\mu+k_1)(\mu+2k_1)(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}\big(H_3^{\star}+H_4^{\star}\big)\\
&\leq\frac{\mu\nu}{k_1k_2}\big[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\big]\\
&\quad\times\left\{\frac{k_1k_2}{(\mu+k_1s_1)(\nu+k_2s_2)}+\frac{k_2B(\frac{\mu}{k_1},s_1+1)}{\nu+k_2s_2}\right.\\
&\quad\left.+\frac{k_1B(\frac{\nu}{k_2},s_2+1)}{\mu+k_1s_1}+B\left(\frac{\mu}{k_1},s_1+1\right)B\left(\frac{\nu}{k_2},s_2+1\right)\right\}\\
&\quad-\mu_2\frac{\mu\nu k_2(\hat{d}-\hat{c})^2}{k_1(\nu+k_2)(\nu+2k_2)(\hat{b}-\hat{a})^{\frac{\mu}{k_1}}}\big(H_1^{\star}+H_2^{\star}\big)\\
&\quad-\mu_1\frac{\mu\nu k_1(\hat{b}-\hat{a})^2}{k_2(\mu+k_1)(\mu+2k_1)(\hat{d}-\hat{c})^{\frac{\nu}{k_2}}}\big(H_3^{\star}+H_4^{\star}\big),
\end{aligned}
$$

where

$$
\begin{aligned}
H_1^{\star}&=\int_{\hat{a}}^{\hat{b}}\frac{dx}{(\hat{b}-x)^{1-\frac{\mu}{k_1}}},\qquad H_2^{\star}=\int_{\hat{a}}^{\hat{b}}\frac{dx}{(x-\hat{a})^{1-\frac{\mu}{k_1}}},\\
H_3^{\star}&=\int_{\hat{c}}^{\hat{d}}\frac{dy}{(\hat{d}-y)^{1-\frac{\nu}{k_2}}},\qquad H_4^{\star}=\int_{\hat{c}}^{\hat{d}}\frac{dy}{(y-\hat{c})^{1-\frac{\nu}{k_2}}},
\end{aligned}
$$

and

$$
W_1^{\star}=\frac{4k_1}{\mu+2k_1}-\frac{4k_1}{\mu+k_1}+\frac{k_1}{\mu},\qquad W_2^{\star}=\frac{4k_2}{\nu+2k_2}-\frac{4k_2}{\nu+k_2}+\frac{k_2}{\nu}.
$$

Corollary 3.26 *Taking $k_1, k_2 \to 1$ in Corollary 3.25, we have*

$$
\begin{aligned}
&2^{s_1+s_2}\rho\left(\frac{\hat{a}+\hat{b}}{2},\frac{\hat{c}+\hat{d}}{2}\right)\\
&\le \frac{2^{s_2-1}\Gamma(\mu+1)}{(\hat{b}-\hat{a})^{\mu}}\left[I^{\mu}_{\hat{a}+}\rho\left(\hat{b},\frac{\hat{c}+\hat{d}}{2}\right)+I^{\mu}_{\hat{b}-}\rho\left(\hat{a},\frac{\hat{c}+\hat{d}}{2}\right)\right]\\
&\quad+\frac{2^{s_1-1}\Gamma(\nu+1)}{(\hat{d}-\hat{c})^{\nu}}\left[I^{\nu}_{\hat{c}+}\rho\left(\frac{\hat{a}+\hat{b}}{2},\hat{d}\right)+I^{\nu}_{\hat{d}-}\rho\left(\frac{\hat{a}+\hat{b}}{2},\hat{c}\right)\right]\\
&\quad+\frac{\mu\nu\mu_2(\hat{d}-\hat{c})^2}{8(\hat{b}-\hat{a})^{\mu}}W^{\star}_4\left(H^{\star}_5+H^{\star}_6\right)+\frac{\mu\nu\mu_1(\hat{b}-\hat{a})^2}{8(\hat{d}-\hat{c})^{\nu}}W^{\star}_3\left(H^{\star}_7+H^{\star}_8\right)\\
&\le \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{(\hat{b}-\hat{a})^{\mu}(\hat{d}-\hat{c})^{\nu}}\\
&\quad\times\left[{}^{1,1}I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d})+{}^{1,1}I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})+{}^{1,1}I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d})+{}^{1,1}I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\right]\\
&\le \frac{\nu\Gamma(\mu+1)}{2(\hat{b}-\hat{a})^{\mu}}\left[I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{c})+I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{d})+I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{c})+I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{d})\right]\\
&\quad\times\left[\frac{1}{\nu+s_2}+B(\nu,s_2+1)\right]\\
&\quad-\mu_2\frac{\mu\nu(\hat{d}-\hat{c})^2}{(\nu+1)(\nu+2)(\hat{b}-\hat{a})^{\mu}}\left(H^{\star}_5+H^{\star}_6\right)\\
&\quad+\frac{\mu\Gamma(\nu+1)}{2(\hat{d}-\hat{c})^{\nu}}\left[I^{\nu}_{\hat{c}+}\rho(\hat{a},\hat{d})+I^{\nu}_{\hat{c}+}\rho(\hat{b},\hat{d})+I^{\nu}_{\hat{d}-}\rho(\hat{a},\hat{c})+I^{\nu}_{\hat{d}-}\rho(\hat{b},\hat{c})\right]\\
&\quad\times\left[\frac{1}{\mu+s_1}+B(\mu,s_1+1)\right]\\
&\quad-\mu_1\frac{\mu\nu(\hat{b}-\hat{a})^2}{(\mu+1)(\mu+2)(\hat{d}-\hat{c})^{\nu}}\left(H^{\star}_7+H^{\star}_8\right)\\
&\le \mu\nu\left[\rho(\hat{a},\hat{c})+\rho(\hat{a},\hat{d})+\rho(\hat{b},\hat{c})+\rho(\hat{b},\hat{d})\right]\\
&\quad\times\left\{\frac{1}{(\mu+s_1)(\nu+s_2)}+\frac{B(\mu,s_1+1)}{\nu+s_2}+\frac{B(\nu,s_2+1)}{\mu+s_1}+B(\mu,s_1+1)B(\nu,s_2+1)\right\}\\
&\quad-\mu_2\frac{\mu\nu(\hat{d}-c)^2}{(\nu+1)(\nu+2)(\hat{b}-\hat{a})^{\mu}}\left(H^{\star}_5+H^{\star}_6\right)-\mu_1\frac{\mu\nu(\hat{b}-\hat{a})^2}{(\mu+1)(\mu+2)(\hat{d}-\hat{c})^{\nu}}\left(H^{\star}_7+H^{\star}_8\right),
\end{aligned}
$$

where

$$
H^{\star}_5=\int_{\hat{a}}^{\hat{b}}\frac{dx}{(\hat{b}-x)^{1-\mu}},\qquad H^{\star}_6=\int_{\hat{a}}^{\hat{b}}\frac{dx}{(x-\hat{a})^{1-\mu}},
$$

$$
H^{\star}_7=\int_{\hat{c}}^{\hat{d}}\frac{dy}{(\hat{d}-y)^{1-\nu}},\qquad H^{\star}_8=\int_{\hat{c}}^{\hat{d}}\frac{dy}{(y-\hat{c})^{1-\nu}},
$$

and

$$
W^{\star}_3=\frac{4}{\mu+2}-\frac{4}{\mu+1}+\frac{1}{\mu},\qquad W^{\star}_4=\frac{4}{\nu+2}-\frac{4}{\nu+1}+\frac{1}{\nu}.
$$

Corollary 3.27 *Taking* $\mu_1, \mu_2 \to 0^+$ *in Corollary* 3.26, *we obtain*

$$
\begin{aligned}
&2^{s_1+s_2}\rho\left(\frac{\hat{a}+\hat{b}}{2}, \frac{\hat{c}+\hat{d}}{2}\right)\\
&\quad\le \frac{2^{s_2-1}\Gamma(\mu+1)}{(\hat{b}-\hat{a})^{\mu}}\left[I^{\mu}_{\hat{a}+}\rho\left(\hat{b}, \frac{\hat{c}+\hat{d}}{2}\right) + I^{\mu}_{\hat{b}-}\rho\left(\hat{a}, \frac{\hat{c}+\hat{d}}{2}\right)\right]\\
&\qquad + \frac{2^{s_1-1}\Gamma(\nu+1)}{(\hat{d}-\hat{c})^{\nu}}\left[I^{\nu}_{\hat{c}+}\rho\left(\frac{\hat{a}+\hat{b}}{2}, \hat{d}\right) + I^{\nu}_{\hat{d}-}\rho\left(\frac{\hat{a}+\hat{b}}{2}, \hat{c}\right)\right]\\
&\quad\le \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{(\hat{b}-\hat{a})^{\mu}(\hat{d}-\hat{c})^{\nu}}\big[I^{\mu,\nu}_{\hat{a}+,\hat{c}+}\rho(\hat{b},\hat{d}) + I^{\mu,\nu}_{\hat{a}+,\hat{d}-}\rho(\hat{b},\hat{c})\\
&\qquad + I^{\mu,\nu}_{\hat{b}-,\hat{c}+}\rho(\hat{a},\hat{d}) + I^{\mu,\nu}_{\hat{b}-,\hat{d}-}\rho(\hat{a},\hat{c})\big]\\
&\quad\le \frac{\nu\Gamma(\mu+1)}{2(\hat{b}-\hat{a})^{\mu}}\big[I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{c}) + I^{\mu}_{\hat{a}+}\rho(\hat{b},\hat{d}) + I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{c}) + I^{\mu}_{\hat{b}-}\rho(\hat{a},\hat{d})\big]\\
&\qquad \times \left\{\frac{1}{\nu+s_2} + B(\nu, s_2+1)\right\}\\
&\qquad + \frac{\mu\Gamma(\nu+1)}{2(\hat{d}-\hat{c})^{\nu}}\big[I^{\nu}_{\hat{c}+}\rho(\hat{a},\hat{d}) + I^{\nu}_{\hat{c}+}\rho(\hat{b},\hat{d}) + I^{\nu}_{\hat{d}-}\rho(\hat{a},\hat{c}) + I^{\nu}_{\hat{d}-}\rho(\hat{b},\hat{c})\big]\\
&\qquad \times \left\{\frac{1}{\mu+s_1} + B(\mu, s_1+1)\right\}\\
&\quad\le \mu\nu\big[\rho(\hat{a},\hat{c}) + \rho(\hat{a},\hat{d}) + \rho(\hat{b},\hat{c}) + \rho(\hat{b},\hat{d})\big]\\
&\qquad \times \left\{\frac{1}{(\mu+s_1)(\nu+s_2)} + \frac{B(\mu,s_1+1)}{\nu+s_2} + \frac{B(\nu,s_2+1)}{\mu+s_1} + B(\mu,s_1+1)B(\nu,s_2+1)\right\}.
\end{aligned}
$$

Remark 3.28 If $\mu = \nu = 1$ and $s_1 = s_2 = s$, then the inequalities in Corollary 3.27 coincide with Theorem 2.1 of [1].

4 Conclusion

In this paper two inequalities of trapezium type are presented for the Katugampola (k_1, k_2)-fractional integrals taking coordinated distance-disturbed (ℓ_1, h_1)-(ℓ_2, h_2)-convexity of higher orders (σ_1, σ_2) into account. The special cases are discussed to see the compatibility with the previously known results. It is found that the results are highly compatible and they can be extended for other types of convexities. We omit here their proofs and the details are left to the interested reader working in the same domain. We hope that current work will attract the attention of researchers working in mathematical inequalities, fractional calculus, differential equations, difference equations, applied mathematics and other related fields.

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