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On some new double dynamic inequalities associated with Leibniz integral rule on time scales

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Abstract

In 2020, El-Deeb et al. proved several dynamic inequalities. It is our aim in this paper to give the retarded time scales case of these inequalities. We also give a new proof and formula of Leibniz integral rule on time scales. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we study boundedness of some delay initial value problems by applying our results as application.

Keywords: Gronwall-type inequality; Boundedness; Time scales

1 Introduction

In 2020, El-Deeb et al. [1] have proved the following inequalities:

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\leq a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\qquad+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}b(\hat{\xi}_1,\hat{\xi}_2)\bigg[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2\Delta\hat{\xi}_1
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\leq a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\qquad+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\bigg(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

The objective of the theory of time scales, which was introduced by Stefan Hilger in his PhD thesis [2] in 1988, is to unify continuous and discrete calculus. Several foundational definitions and notations of basic calculus of time scales introduced in the excellent recent books [3, 4] by Bohner and Peterson will be employed in the sequel. For some Gronwall–Bellman-type integral, dynamic inequalities and other type of inequalities on time scales, see the papers [5–36].

We use the following notations:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} &\sigma(t) = t, \\ &\mu(t) = 0, \\ &f^{\Delta}(t) = f'(t), \\ &\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt; \end{aligned} \tag{1.1}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(t) = t + 1, \\ &\mu(t) = 1, \\ &f^{\Delta}(t) = \Delta f(t), \\ &\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \end{aligned} \tag{1.2}$$

where Δ is the forward difference operator.

Theorem 1.1 (Chain rule on time scales [3]) *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and Δ-differentiable on $\mathbb{T}^{\kappa}$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then there exists $c \in [t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'\bigl(g(c)\bigr)g^{\Delta}(t). \tag{1.3}$$

Theorem 1.2 (Chain rule on time scales [3]) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 \bigl[f'\bigl(hg^{\sigma}(t) + (1-h)g(t)\bigr)\bigr]\,dh\right\}(g)^{\Delta}(t), \tag{1.4}$$

holds.

Theorem 1.3 ([3]) *Let $t_0 \in \mathbb{T}^{\kappa}$ and $k : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ be continuous at (t, t), where $t > t_0$ and $t \in \mathbb{T}^{\kappa}$. Assume that $k^{\Delta}(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\bigl|\bigl[k\bigl(\sigma(t), \tau\bigr) - k(s, \tau)\bigr] - k^{\Delta}(t, \tau)\bigl[\sigma(t) - s\bigr]\bigr| \le \varepsilon\bigl|\sigma(t) - s\bigr|, \quad \forall s \in U.$$

If k^{Δ} denotes the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^{t} k(t, \tau)\Delta\tau$$

yields

$$f^{\Delta}(t)=\int_{t_0}^{t}k^{\Delta}(t,\tau)\Delta\tau+k\big(\sigma(t),t\big).$$

Other dynamic inequalities on time scales may be found in [37–40]. In this manuscript, we will discuss the retarded time scale case of the inequalities obtained in [1] using new techniques by replacing the upper limit $\hat{\varsigma}$ and $\hat{\varrho}$ of the integral by the delay function $\hat{\alpha}(\hat{\varsigma})\leq\hat{\varsigma}$ and $\hat{\beta}(\hat{\varrho})\leq\hat{\varrho}$. Furthermore, these inequalities that we obtained here extend some known results in the literature, and they also unify the continuous and discrete cases.

2 Main results

Throughout the paper, we suppose that $\mathbb{T}_1$ and $\mathbb{T}_2$ are two time scales.

First, we prove the following result.

Theorem 2.1 (Leibniz integral rule on time scales) *In the following by $f^{\Delta}(t,s)$ we mean the delta derivative of $f(t,s)$ with respect to t. Similarly, $f^{\nabla}(t,s)$ is understood. If f, f^{Δ} and f^{∇} are continuous, and $u,h:\mathbb{T}\to\mathbb{T}$ are delta differentiable functions, then the following formulas hold $\forall t\in\mathbb{T}^{\kappa}$:*

(i) $[\int_{u(t)}^{h(t)}f(t,s)\Delta s]^{\Delta}=\int_{u(t)}^{h(t)}f^{\Delta}(t,s)\Delta s+h^{\Delta}(t)f(\sigma(t),h(t))-u^{\Delta}(t)f(\sigma(t),u(t))$;

(ii) $[\int_{u(t)}^{h(t)}f(t,s)\Delta s]^{\nabla}=\int_{u(t)}^{h(t)}f^{\nabla}(t,s)\Delta s+h^{\nabla}(t)f(\rho(t),h(t))-u^{\nabla}(t)f(\rho(t),u(t))$;

(iii) $[\int_{u(t)}^{h(t)}f(t,s)\nabla s]^{\Delta}=\int_{u(t)}^{h(t)}f^{\Delta}(t,s)\nabla s+h^{\Delta}(t)f(\sigma(t),h(t))-u^{\Delta}(t)f(\sigma(t),u(t))$;

(iv) $[\int_{u(t)}^{h(t)}f(t,s)\nabla s]^{\nabla}=\int_{u(t)}^{h(t)}f^{\nabla}(t,s)\nabla s+h^{\nabla}(t)f(\rho(t),h(t))-u^{\nabla}(t)f(\rho(t),u(t))$.

Proof We will only prove part (i); the others may be proved similarly. Define a function g by

$$g(t)=\int_{u(t)}^{h(t)}f(t,s)\Delta s,\quad\text{for } t\in\mathbb{T}^{\kappa}.\tag{2.1}$$

We notice that g is a continuous function. Indeed, we have two cases for t. In the first case, if t is right-scattered, from (2.1), we get

$$\begin{aligned}
g^{\Delta}(t)&=\frac{g(\sigma(t))-g(t)}{\sigma(t)-t}\\
&=\frac{1}{\sigma(t)-t}\left[\int_{u(\sigma(t))}^{h(\sigma(t))}f\big(\sigma(t),s\big)\Delta s-\int_{u(t)}^{h(t)}f(t,s)\Delta s\right]\\
&=\frac{1}{\sigma(t)-t}\left[-\int_{u(t)}^{u(\sigma(t))}f\big(\sigma(t),s\big)\Delta s+\int_{u(t)}^{h(t)}f\big(\sigma(t),s\big)\Delta s\right.\\
&\quad\left.+\int_{h(t)}^{h(\sigma(t))}f\big(\sigma(t),s\big)\Delta s-\int_{u(t)}^{h(t)}f(t,s)\Delta s\right]\\
&=\int_{u(t)}^{h(t)}\frac{f(\sigma(t),s)-f(t,s)}{\sigma(t)-t}\Delta s+\frac{1}{\sigma(t)-t}\int_{h(t)}^{h(\sigma(t))}f\big(\sigma(t),s\big)\Delta s\\
&\quad-\frac{1}{\sigma(t)-t}\int_{u(t)}^{u(\sigma(t))}f\big(\sigma(t),s\big)\Delta s\\
&=\int_{u(t)}^{h(t)}f^{\Delta}(t,s)\Delta s+\frac{h(\sigma(t))-h(t)}{\sigma(t)-t}f\big(\sigma(t),h(t)\big)
\end{aligned}$$

$$\begin{aligned} &- \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t} f\big(\sigma(t), u(t)\big) \\ &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t) f\big(\sigma(t), h(t)\big) - u^{\Delta}(t) f\big(\sigma(t), u(t)\big). \end{aligned} \tag{2.2}$$

From (2.2), we get the required result.

Now consider the second case when t is right-dense. Since f is continuous, it is rd-continuous, hence it has a delta partial anti-derivative with respect to the second variable s, say $F(t,s)$, that is, $f(t,s) = F^{\Delta_s}(t,s)$, and then we have

$$\begin{aligned} \left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\Delta} &= g^{\Delta}(t) \\ &= \lim_{r\to t} \frac{g(t) - g(r)}{t - r} \\ &= \lim_{r\to t} \frac{1}{t-r}\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s - \int_{u(r)}^{h(r)} f(r,s)\Delta s\right] \\ &= \lim_{r\to t} \frac{1}{t-r}\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s - \int_{u(r)}^{u(t)} f(r,s)\Delta s\right. \\ &\quad \left. - \int_{u(t)}^{h(t)} f(r,s)\Delta s - \int_{h(t)}^{h(r)} f(r,s)\Delta s\right] \\ &= \lim_{r\to t} \int_{u(t)}^{h(t)} \frac{f(t,s) - f(r,s)}{t-r}\Delta s + \lim_{r\to t}\frac{1}{t-r}\int_{h(r)}^{h(t)} F^{\Delta_s}(r,s)\Delta s \\ &\quad - \lim_{r\to t}\frac{1}{t-r}\int_{u(r)}^{u(t)} F^{\Delta_s}(r,s)\Delta s. \end{aligned} \tag{2.3}$$

Thus, from (2.3), we get

$$\begin{aligned} \left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\Delta} &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t}\frac{1}{t-r}\big[F\big(r,h(t)\big) - F\big(r,h(r)\big)\big] \\ &\quad - \lim_{r\to t}\frac{1}{t-r}\big[F\big(r,u(t)\big) - F\big(r,u(r)\big)\big] \\ &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t}\frac{h(t)-h(r)}{t-r}\,\frac{F(r,h(t)) - F(r,h(r))}{h(t)-h(r)} \\ &\quad - \lim_{r\to t}\frac{u(t)-u(r)}{t-r}\,\frac{F(r,u(t)) - F(r,u(r))}{u(t)-u(r)} \\ &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t}\frac{h(t)-h(r)}{t-r}\lim_{r\to t}\frac{F(r,h(t)) - F(r,h(r))}{h(t)-h(r)} \\ &\quad - \lim_{r\to t}\frac{u(t)-u(r)}{t-r}\lim_{r\to t}\frac{F(r,u(t)) - F(r,u(r))}{u(t)-u(r)} \\ &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)F^{\Delta_s}\big(t,h(t)\big) - u^{\Delta}(t)F^{\Delta_s}\big(t,u(t)\big) \\ &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)f\big(t,h(t)\big) - u^{\Delta}(t)f\big(t,u(t)\big). \end{aligned}$$

This completes the proof. □

Remark 2.2 If we take $h(t) = t$ and $u(t) = a$ (where a is constant), then Theorem 2.1 reduces to [4, Theorem 5.37, p. 139].

Now, by using the result of Theorem 2.1, we state and prove the rest of our main results:

Theorem 2.3 *Suppose $a \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ is nondecreasing with respect to $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, and g, u, p, $f \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. Also let $\hat{\alpha} \in C^1_{\mathrm{rd}}(\mathbb{T}_1, \mathbb{T}_1)$ and $\hat{\beta} \in C^1_{\mathrm{rd}}(\mathbb{T}_2, \mathbb{T}_2)$ be nondecreasing functions with $\hat{\alpha}(\hat{\varsigma}) \le \hat{\varsigma}$ on $\mathbb{T}_1$, $\hat{\beta}(\hat{\varrho}) \le \hat{\varrho}$ on $\mathbb{T}_2$. Furthermore, suppose $\tilde{\Phi}$, $\tilde{\Psi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\tilde{\Phi}, \tilde{\Psi}\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \tilde{\Phi}(u) = +\infty$. If $u(\hat{\varsigma}, \hat{\varrho})$ satisfies*

$$\begin{aligned}\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \end{aligned}\tag{2.4}$$

for $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\bigg\{\tilde{\Lambda}^{-1}\bigg[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg]\bigg\} \tag{2.5}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where

$$q(\hat{\varsigma},\hat{\varrho}) = a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.6}$$

$$\tilde{\Lambda}(r) = \int_{r_0}^{r} \frac{\Delta\hat{\xi}_1}{\omega\circ\tilde{\Phi}^{-1}(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Lambda}(+\infty) = \int_{r_0}^{+\infty} \frac{\Delta\hat{\xi}_1}{\omega\circ\tilde{\Phi}^{-1}(\hat{\xi}_1)} = +\infty, \tag{2.7}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ is chosen so that

$$\bigg(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg) \in \mathrm{Dom}\big(G^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma}, \hat{\varrho}) > 0$. Since $q \ge 0$ and it is nondecreasing, fixing an arbitrary point $(\breve{\xi}, \breve{\zeta}) \in \Omega$ and defining $z(\hat{\varsigma}, \hat{\varrho})$ by

$$\begin{aligned} z(\hat{\varsigma},\hat{\varrho}) = q(\breve{\xi},\breve{\zeta}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \end{aligned}$$

which is a positive and nondecreasing function for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varsigma} \le \breve{\zeta} \le \hat{\varrho}_1$, we then get $z(0, \hat{\varrho}) = z(\hat{\varsigma}, 0) = q(\breve{\xi}, \breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big). \tag{2.8}$$

By applying Theorem 2.1, differentiating $z(\hat{\varsigma},\hat{\varrho})$ with respect to $\hat{\varsigma}$, and using (2.8), we get

$$\begin{aligned}
&z_{\hat{\varsigma}}^{\Delta}(\hat{\varsigma},\hat{\varrho})\\
&\quad= \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg[\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2\\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg[\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2.
\end{aligned}$$

Since $\tilde{\Psi}\circ\tilde{\Phi}^{-1}$ is nondecreasing with respect to $(\hat{\varsigma},\hat{\varrho})\in\mathbb{R}_+\times\mathbb{R}_+$, we then have

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})\\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg[\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\\
&\qquad + \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2\\
&\quad\le \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg[1+\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2,
\end{aligned} \tag{2.9}$$

from which $\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})))\le\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, so from (2.9), we get

$$\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))} \le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(1+\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2. \tag{2.10}$$

Now from (2.10), we get

$$\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big) \le \tilde{\Lambda}\big(q(\breve{\xi},\breve{\zeta})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(1+\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.$$

Since $(\breve{\xi},\breve{\zeta})\in\Omega$ is chosen arbitrarily,

$$z(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Lambda}^{-1}\bigg[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(1+\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg]. \tag{2.11}$$

So from (2.11) and (2.8), we get the desired inequality in (2.5). For $a(\hat{\varsigma},\hat{\varrho})=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon\to 0$. This completes the proof. □

Remark 2.4 If we take $\hat{\alpha}(\hat{\varsigma})=\hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho})=\hat{\varrho}$, then Theorem 2.3 reduces to [1, Theorem 2.1].

Corollary 2.5 *The discrete form can be obtained by letting* $\mathbb{T}=\mathbb{Z}$, *with the help of relations* (1.2), *and* $\hat{\alpha}(\hat{\varsigma})=\hat{\varsigma}$, $\hat{\beta}(\hat{\varrho})=\hat{\varrho}$ *in Theorem* 2.3. *If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big] + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right)$$

holds for $(\hat{\varsigma},\hat{\varrho})\in\Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\tilde{\Lambda}^{-1}\left[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1+\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right]\right\}$$

for $0\le\hat{\varsigma}\le\hat{\varsigma}_1$, $0\le\hat{\varrho}\le\hat{\varrho}_1$, *where*

$$q(\hat{\varsigma},\hat{\varrho}) = a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} p(\hat{\xi}_1,\hat{\xi}_2),$$

$$\tilde{\Lambda}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1}\frac{1}{\omega\circ\tilde{\Phi}^{-1}(\hat{\xi}_1)}, \quad r\ge r_0>0, \qquad \tilde{\Lambda}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty}\frac{1}{\omega\circ\tilde{\Phi}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1)\in\Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\breve{st}=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1+\sum_{\hat{\zeta}0}^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\right)\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Theorem 2.6 *Assume that* h, $b\in C_{\mathrm{rd}}(\Omega,\mathbb{R}_+)$. *Let* g, f, p, a, u, $\tilde{\Phi}$, *and* $\tilde{\Psi}$ *be as in Theorem* 2.3. *If* $u(\hat{\varsigma},\hat{\varrho})$ *satisfies*

$$\begin{aligned}\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le{}& a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\bigg[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) \\ &+ \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \end{aligned} \tag{2.12}$$

for $(\hat{\varsigma},\hat{\varrho})\in\Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{G^{-1}\left[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]\right\} \tag{2.13}$$

for $0\le\hat{\varsigma}\le\hat{\varsigma}_1$, $0\le\hat{\varrho}\le\hat{\varrho}_1$, *where* $\tilde{\Lambda}$ *is defined by* (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.14}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right) \in \mathrm{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma},\hat{\varrho}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and non-decreasing function $z(\hat{\varsigma},\hat{\varrho})$ by

$$\begin{aligned} z(\hat{\varsigma},\hat{\varrho}) = q(\breve{\xi},\breve{\zeta}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\Big[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) \\ &+ \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \end{aligned}$$

for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \breve{\zeta} \le y_1$, then $z(0,\hat{\varrho}) = z(\hat{\varsigma},0) = q(\breve{\xi},\breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned} &z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho}) \\ &= \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 + \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\ &\quad \times \left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\ &\le \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\ &\quad \times \left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\ &\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big)\bigg[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\ &\quad + \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\bigg]\Delta\hat{\xi}_2. \end{aligned}$$

Since $\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho})) \le \tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we then get

$$\begin{aligned} &\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))} \\ &\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\bigg[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\ &\quad + \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\bigg]\Delta\hat{\xi}_2. \end{aligned} \tag{2.15}$$

Integrating (2.15), we get

$$\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big) \le \tilde{\Lambda}\big(q(\breve{\xi},\breve{\zeta})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Lambda}^{-1}\left[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]. \tag{2.16}$$

Thus, from (2.16) and $u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we get the required inequality in (2.13). For $a(\hat{\varsigma},\hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.7 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.6 reduces to [1, Theorem 2.4].

Corollary 2.8 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 2.6, then, with the help of relations (1.1), we have the following inequality due to Boudeliou [41]. If*

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\,d\hat{\xi}_2\,d\hat{\xi}_1\\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\,d\hat{\zeta}\right]d\hat{\xi}_2\,d\hat{\xi}_1
\end{aligned}$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{G^{-1}\left[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right]\right\}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where $\tilde{\Lambda}$ is defined by (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right]d\hat{\xi}_2\,d\hat{\xi}_1,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ is chosen so that

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right) \in \operatorname{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Corollary 2.9 *The discrete form can be obtained by letting* $\mathbb{T} = \mathbb{Z}$, *with the help of relations* (1.2) *and* $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ *in Theorem* 2.6. *If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big] + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right]$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{G^{-1}\left[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right]\right\}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where* $\tilde{\Lambda}$ *is defined by* (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2) + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right],$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right) \in \operatorname{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Theorem 2.10 *Assume that* g, a, u, f, p, $\tilde{\Phi}$, *and* $\tilde{\Psi}$ *are as in Theorem* 2.3. *If* $u(\hat{\varsigma},\hat{\varrho})$ *satisfies*

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \\
&\quad\le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,
\end{aligned} \tag{2.17}$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$\begin{aligned}
&u(\hat{\varsigma},\hat{\varrho}) \\
&\quad\le \tilde{\Phi}^{-1}\Bigg\{\tilde{\Lambda}^{-1}\Bigg(\tilde{\Theta}^{-1}\Bigg[\tilde{\Theta}\big(q_1(\hat{\varsigma},\hat{\varrho})\big) \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\Bigg]\Bigg)\Bigg\},
\end{aligned} \tag{2.18}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$q_1(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.19}$$

$$\begin{aligned}
&\tilde{\Theta}(r) = \int_{r_0}^{r} \frac{\Delta\hat{\xi}_1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1})(\hat{\xi}_1)}, \quad r \geq r_0 > 0, \\
&\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{\Delta\hat{\xi}_1}{(\omega\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1}(\hat{\xi}_1)} = +\infty,
\end{aligned} \tag{2.20}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Theta}\big(q_1(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right) \in \operatorname{Dom}\big(\tilde{\Theta}^{-1}\big).$$

Proof Suppose that $a(\breve{\xi},\breve{\zeta}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and nondecreasing function $z(\hat{\varsigma},\hat{\varrho})$ by

$$\begin{aligned}
z(\hat{\varsigma},\hat{\varrho}) = a(\breve{\xi},\breve{\zeta}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,
\end{aligned}$$

for $0 \leq \hat{\varsigma} \leq \breve{\xi} \leq \hat{\varsigma}_1$, $0 \leq \hat{\varrho} \leq \breve{\zeta} \leq \hat{\varrho}_1$, then $z(0,\hat{\varrho}) = z(\hat{\varsigma},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \leq \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho}) \\
&= \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\
&\leq \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) \\
&\quad \times \left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\
&\leq \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma}) \\
&\quad \times \int_0^{\hat{\beta}(\hat{\varrho})} \big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\
&\quad \times \left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))} \\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)+p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\qquad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2. \qquad (2.21)
\end{aligned}$$

Integrating (2.21), we get

$$\begin{aligned}
\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big)&\le\tilde{\Lambda}\big(a(\breve{\xi},\breve{\zeta})\big)+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

If $(\breve{\xi},\breve{\zeta})\in\Omega$ is chosen arbitrarily, then

$$\begin{aligned}
\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big)&\le q_1(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

Since $q_1(\hat{\varsigma},\hat{\varrho})>0$ is a nondecreasing function, fixing an arbitrary point $(\breve{\xi},\breve{\zeta})\in\Omega$ and defining $\nu(\hat{\varsigma},\hat{\varrho})>0$ to be a nondecreasing function given by

$$\begin{aligned}
\nu(\hat{\varsigma},\hat{\varrho})&=q_1(\breve{\xi},\breve{\zeta})+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,
\end{aligned}$$

for $0\le\hat{\varsigma}\le\breve{\xi}\le\hat{\varsigma}_1$, $0\le\hat{\varrho}\le\breve{\zeta}\le y_1$, we obtain $\nu(0,\hat{\varrho})=\nu(\hat{\varsigma},0)=q_1(\breve{\xi},\breve{\zeta})$ and

$$z(\hat{\varsigma},\hat{\varrho})\le\tilde{\Lambda}^{-1}\big(\nu(\hat{\varsigma},\hat{\varrho})\big). \qquad (2.22)$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned}
\nu^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})&=\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 \\
&\quad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\
&\le\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(G^{-1}\big(\nu\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big)\Delta\hat{\xi}_2 \\
&\quad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(G^{-1}\big(\nu(\hat{\zeta},\hat{\xi}_2)\big)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\tilde{\Psi} \circ \tilde{\Phi}^{-1}\right) \circ \tilde{\Lambda}^{-1}\big(\nu\big(\hat{\alpha}(\hat{\varsigma}), \hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma}) \\
&\quad \times \left[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\big)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\big)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\right],
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{\nu^{\Delta\hat{\varsigma}}(\hat{\varsigma}, \hat{\varrho})}{(\tilde{\Psi} \circ \tilde{\Phi}^{-1}) \circ \tilde{\Lambda}^{-1}(\nu(\hat{\varsigma}, \hat{\varrho}))} \\
&\quad \leq \hat{\alpha}^{\Delta}(\hat{\varsigma})\left[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\big)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\big)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\right]. \qquad (2.23)
\end{aligned}
$$

Integrating (2.23), we get

$$
\tilde{\Theta}\big(\nu(\hat{\varsigma}, \hat{\varrho})\big) \leq \tilde{\Theta}\big(q_1(\breve{\xi}, \breve{\zeta})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
$$

Since we chose $(\breve{\xi}, \breve{\zeta}) \in \Omega$ arbitrarily,

$$
\begin{aligned}
&\nu(\hat{\varsigma}, \hat{\varrho}) \\
&\quad \leq \tilde{\Theta}^{-1}\left[\tilde{\Theta}\big(q_1(\hat{\varsigma}, \hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]. \qquad (2.24)
\end{aligned}
$$

From (2.24), (2.22), and $u(\hat{\varsigma}, \hat{\varrho}) \leq \tilde{\Phi}^{-1}(z(\hat{\varsigma}, \hat{\varrho}))$, we get the desired inequality in (2.18). For $a(\hat{\varsigma}, \hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma}, \hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.11 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.10 reduces to [1, Theorem 2.7].

Corollary 2.12 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 2.10, then, with the help of relations (1.1), we get the following inequality due to Boudeliou [41]. If*

$$
\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma}, \hat{\varrho})\big) \\
&\quad \leq a(\hat{\varsigma}, \hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u(\hat{\xi}_1, \hat{\xi}_2)\big)\big[f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1, \hat{\xi}_2)\big) + p(\hat{\xi}_1, \hat{\xi}_2)\big]\,d\hat{\xi}_2\,d\hat{\xi}_1 \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1, \hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta}, \hat{\xi}_2)\big)\Delta\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1
\end{aligned}
$$

holds for $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, then

$$
\begin{aligned}
u(\hat{\varsigma}, \hat{\varrho}) \leq \tilde{\Phi}^{-1}\Bigg\{\tilde{\Lambda}^{-1}\Bigg(\tilde{\Theta}^{-1}\Bigg[&\tilde{\Theta}\big(q_2(\hat{\varsigma}, \hat{\varrho})\big) \\
&+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1\Bigg]\Bigg)\Bigg\},
\end{aligned}
$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$q_2(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1,$$

$$\tilde{\Theta}(r) = \int_{r_0}^{r} \frac{d\hat{\xi}_1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1})(\hat{\xi}_1)}, \quad r \ge r_0 > 0,$$

$$\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{d\hat{\xi}_1}{(\omega\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1\right) \in \operatorname{Dom}\big(\tilde{\Theta}^{-1}\big).$$

Corollary 2.13 *The discrete form, due to El-Deeb et al.* [1], *can be obtained by letting* $\mathbb{T} = \mathbb{Z}$ *in Theorem* 2.10, *with the help of relations* (1.2) *and* $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ *as follows. If*

$$\begin{aligned}\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le{}& a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big] \\ &+ \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right),\end{aligned}$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\bar{G}^{-1}\left(\bar{F}^{-1}\left[\bar{F}\big(\bar{q}_2(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right]\right)\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$\bar{q}_2(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} p(\hat{\xi}_1,\hat{\xi}_2),$$

$$\bar{F}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1} \frac{1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\bar{G}^{-1})(\hat{\xi}_1)}, \quad r \ge r_0 > 0,$$

$$\bar{F}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty} \frac{1}{(\omega\circ\tilde{\Phi}^{-1})\circ\bar{G}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\bar{F}\big(\bar{q}_2(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right) \in \operatorname{Dom}\big(\bar{F}^{-1}\big).$$

Theorem 2.14 *Assume that* $g, a, f, u, \tilde{\Phi}$, *and* $\tilde{\Psi}$ *are as in Theorem 2.3. If* $u(\hat{\varsigma}, \hat{\varrho})$ *satisfies*

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \\
&\quad\le a(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2 \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.25)
\end{aligned}$$

for $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\beta}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2\right]\right\}, \qquad (2.26)$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.27)$$

$$\begin{aligned}
&\breve{H}(r) = \int_{r_0}^{r} \frac{\Delta\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \\
&\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{\Delta\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,
\end{aligned} \qquad (2.28)$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + 2\left(\int_0^{\sigma(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma}, \hat{\varrho}) > 0$. Taking $(\breve{\xi}, \breve{\zeta}) \in \Omega$ as a fixed arbitrary point, we define $z(\hat{\varsigma}, \hat{\varrho}) > 0$ to be a nondecreasing function by

$$\begin{aligned}
&z(\hat{\varsigma},\hat{\varrho}) \\
&\quad= a(\breve{\xi},\breve{\zeta}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2 \qquad (2.29) \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.30)
\end{aligned}$$

for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \breve{\zeta} \le \hat{\varrho}_1$, hence $z(0, \hat{\varrho}) = z(\hat{\varsigma}, 0) = a(\breve{\xi}, \breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

From (2.29), and applying the chain rule on time scales (1.2), we get

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho}) \\
&\quad= 2\left(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2 \\
\le{}& 2\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg) \\
&\times \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) \\
&\times \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2 \\
\le{}& 2\big(\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\big)^2\hat{\alpha}^{\Delta}(\hat{\varsigma})\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg) \\
&\times \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\
&+ \big(\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\big)^2\hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2,
\end{aligned}
$$

thus we have

$$
\begin{aligned}
\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho})))^2} \le{}& 2\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2, \\
={}& \Bigg[\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2\Bigg]^{\Delta_{\hat{\varsigma}}} \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2. \qquad (2.31)
\end{aligned}
$$

Integrating (2.31), we get

$$
\begin{aligned}
\breve{H}\big(z(\hat{\varsigma},\hat{\varrho})\big) \le{}& \breve{H}\big(a(\breve{\xi},\breve{\zeta})\big) + \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2 \\
&+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}
$$

Since $(\breve{\xi},\breve{\zeta})\in\Omega$ is chosen arbitrarily,

$$
z(\hat{\varsigma},\hat{\varrho}) \le \breve{H}^{-1}\Bigg[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2\Bigg]. \qquad (2.32)
$$

From (2.32) and $u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we get the desired inequality (2.26). For $a(\hat{\varsigma},\hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.15 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.14 reduces to [1, Theorem 10].

Theorem 2.16 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 2.14, with the help of relations (1.1), we have the following inequality due to Boudeliou. If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2 + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1,$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\beta}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2\right]\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1,$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{d\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ is chosen so that

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + 2\left(\int_0^{\sigma(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Corollary 2.17 *The discrete form, due to El-Deeb et al. [1], can be obtained by letting $\mathbb{T} = \mathbb{Z}$ and $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ in Theorem 2.14 as follows. If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\right)^2 + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right)$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right)^2\right]\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1, 0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right),$$

$$\breve{H}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1} \frac{1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Theta}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty} \frac{1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

3 Applications

In this section we would like to show the beauty behind our results by applying Theorems 2.10 and 2.3 to study the boundedness of the solutions of some delay initial boundary value problems.

Consider the problem

$$u^{\Delta\hat{\varsigma}\Delta\hat{\varrho}}(\hat{\varsigma},\hat{\varrho}) = \tilde{\Theta}\left(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big),\int_0^{\hat{\alpha}(\hat{\varsigma})} \breve{k}\big(\hat{\xi}_1,\hat{\varrho},u(s,\hat{\varrho})\big)\Delta\hat{\xi}_1\right), \tag{3.1}$$

$$u(\hat{\varsigma},0) = a_1(\hat{\varsigma}), \qquad u(0,\hat{\varrho}) = a_2(\hat{\varrho}), \qquad a_1(0) = a_2(0) = 0, \tag{3.2}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, where $\breve{k} \in C_{\mathrm{rd}}(\Omega\times\mathbb{R},\mathbb{R})$, $\tilde{\Theta} \in C_{\mathrm{rd}}(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $a_1 \in C_{\mathrm{rd}}(\mathbb{T}_1,\mathbb{R})$, and $a_2 \in C_{\mathrm{rd}}(\mathbb{T}_2,\mathbb{R})$.

Theorem 3.1 *Suppose that the functions* $\breve{k}, \tilde{\Theta}, a_2, a_1$ *in* (3.1) *and* (3.2) *satisfy the conditions*

$$\begin{aligned}
&\big|\tilde{\Theta}\big(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}),v\big)\big| \\
&\quad\le \tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big)\big[f(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big) + p(\hat{\varsigma},\hat{\varrho})\big] \\
&\qquad + f(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big)v,
\end{aligned} \tag{3.3}$$

$$\big|\breve{k}\big(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big| \le g(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big), \tag{3.4}$$

$$\big|a_1(\hat{\varsigma}) + a_2(\hat{\varrho})\big| \le a(\hat{\varsigma},\hat{\varrho}), \tag{3.5}$$

where the functions $p, g, a, f, \hat{\alpha}, \hat{\beta}$, *and* $\tilde{\Psi}$ *are defined as in Theorem* 2.10 *with* $a(\hat{\varsigma},\hat{\varrho}) > 0$, *for all* $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. *Then*

$$\begin{aligned}
\big|u(\hat{\varsigma},\hat{\varrho})\big| \le{}& \tilde{\Lambda}^{-1}\left(\tilde{\Theta}^{-1}\left[\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\right.\right. \\
&\left.\left.\times\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]\right),
\end{aligned} \tag{3.6}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1, 0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where F and G are defined as in Theorem* 2.10,

$$q_2\big(\hat{\varsigma},\hat{\varrho} = G\big(a(\hat{\varsigma},\hat{\varrho})\big)\big) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \frac{p(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\Delta t\Delta s, \tag{3.7}$$

and $(\hat{\varsigma},\hat{\varrho}) \in \Omega$ *is chosen so that*

$$\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \in \operatorname{Dom}\big(F^{-1}\big).$$

Proof If the problem (3.1) and (3.2) has a solution $u(\hat{\varsigma},\hat{\varrho})$, it can be written as

$$u(\hat{\varsigma},\hat{\varrho}) = a_1(\hat{\varsigma}) + a_2(\hat{\varrho}) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \tilde{\Theta}\left(\hat{\xi}_1,\hat{\xi}_2,u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big),\int_0^{\hat{\xi}_1}\breve{k}\big(\hat{\zeta},\hat{\xi}_2,u(\hat{\zeta},t)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{3.8}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. Using the conditions (3.3), (3.4), and (3.5) in (3.8), we get

$$\begin{aligned}\big|u(\hat{\varsigma},\hat{\varrho})\big| &\le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big)\\ &\quad\times\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\ &\quad+ \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} f(s,t)\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big)\\ &\quad\times\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(\big|u(\hat{\zeta},\hat{\xi}_2)\big|\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.\end{aligned} \tag{3.9}$$

Now, from (3.9), we get

$$\begin{aligned}\big|u(\hat{\varsigma},\hat{\varrho})\big| &\le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \frac{\varphi(|u(\hat{\xi}_1,\hat{\xi}_2)|)}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\\ &\quad\times\big[f\big(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2)\big)\varphi\big(\big|u(\hat{\xi}_1,\hat{\xi}_2)\big|\big)\\ &\quad+ p\big(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2)\big)\big]\Delta t\Delta s\\ &\quad+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\varphi\big(\big|u(\hat{\xi}_1,\hat{\xi}_2)\big|\big)\\ &\quad\times\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\varphi\big(\big|u(\hat{\zeta},\hat{\xi}_2)\big|\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,\end{aligned} \tag{3.10}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. Now, an application of Theorem 2.10 to (3.10) yields the required inequality in (3.6). □

Consider the initial boundary value problem of the form

$$\big(z^q\big)^{\Delta\hat{\varsigma}\Delta\hat{\varrho}}(\hat{\varsigma},\hat{\varrho}) = \breve{A}\left(\hat{\varsigma},\hat{\varrho},z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big),\int_0^{\hat{\alpha}(\hat{\varsigma})} h\big(\hat{\xi}_1,\hat{\varrho},z(\hat{\xi}_1,\hat{\varrho})\big)\Delta\hat{\xi}_1\right), \tag{3.11}$$

$$z(\hat{\varsigma},0) = a_1(\hat{\varsigma}), \qquad z(0,\hat{\varrho}) = a_2(\hat{\varrho}), \qquad a_1(0) = a_2(0) = 0, \tag{3.12}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$.

Theorem 3.2 *Assume that the functions $h, \breve{A}, a_2, a_1$ in* (3.11) *and* (3.12) *satisfy the conditions*

$$\left|\breve{A}(\hat{\varsigma},\hat{\varrho},z(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}),v)\right| \leq f(\hat{\varsigma},\hat{\varrho})\left|z^r(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}))\right| + f(\hat{\varsigma},\hat{\varrho})v, \tag{3.13}$$

$$\left|h(\hat{\varsigma},\hat{\varrho},z(\hat{\varsigma},\hat{\varrho}))\right| \leq g(\hat{\varsigma},\hat{\varrho})\left|z^r(\hat{\varsigma},\hat{\varrho})\right|, \tag{3.14}$$

$$\left|a_1(\hat{\varsigma}) + a_2(\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}), \tag{3.15}$$

where $r \geq q > 0$. Then

$$\begin{aligned}\left|z(\hat{\varsigma},\hat{\varrho})\right| \leq \Bigg[&\left(a(\hat{\varsigma},\hat{\varrho})\right)^{\frac{q-r}{q}} + \frac{q-r}{q}\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))} \\ &\times \left(1 + \int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\Bigg]^{\frac{1}{q-r}},\end{aligned} \tag{3.16}$$

for $0 \leq \hat{\varsigma} \leq \hat{\varsigma}_1, 0 \leq \hat{\varrho} \leq \hat{\varrho}_1$.

Proof If the problem (3.11) and (3.12) has a solution $z(\hat{\varsigma},\hat{\varrho})$, it can be written as

$$\begin{aligned}z^q(\hat{\varsigma},\hat{\varrho}) = a_1(x) + a_2(y) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \tilde{\Theta}\Bigg(&\hat{\xi}_1,\hat{\xi}_1,u(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)), \\ &\int_0^{\hat{\alpha}(\hat{\xi}_1)} \breve{k}(\hat{\varsigma},\hat{\xi}_2,u(\hat{\varsigma},\hat{\xi}_2))\Delta\hat{\varsigma}\Bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,\end{aligned} \tag{3.17}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. Using the conditions (3.13), (3.14), and (3.15) in (3.17), we get

$$\begin{aligned}\left|z^q(\hat{\varsigma},\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}) &+ \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left|z^r(\hat{\alpha}(s),\hat{\beta}(t))\right|\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\left|z^r(\hat{\varsigma},\hat{\xi}_2)\right|\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.\end{aligned} \tag{3.18}$$

From (3.18), we get

$$\begin{aligned}\left|z^q(\hat{\varsigma},\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\alpha}(\hat{\varrho})} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\left|z^r(\hat{\xi}_1,\hat{\xi}_2)\right|\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))} \\ &\times \left(\int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\left|z^r(\hat{\varsigma},\hat{\xi}_2)\right|\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,\end{aligned} \tag{3.19}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. A suitable application of Theorem 2.3 to (3.19) with $\tilde{\Phi}(u) = u^q$, $\tilde{\Psi}(u) = u^r$ and $p(\hat{\varsigma},\hat{\varrho}) = 0$ gives the required inequality in (3.16). □

4 Conclusion

In this work, by using a new technique, we proved several nonlinear retarded dynamic inequalities in two independent variables of Gronwall type on time scales. We also gave a new proof and formula of Leibniz integral rule on time scales. Further, we also applied our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we studied the boundedness of some delay initial value problems by applying our results.

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