Saadati et al. *Advances in Difference Equations* (2021) 2021:153
https://doi.org/10.1186/s13662-021-03287-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

n-Expansively super-homogeneous and (n,k)-contractively sub-homogeneous fuzzy control functions and stability results with numerical examples

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Abstract

We consider fuzzy sets and generalized triangular norms on positive elements of order commutative C^*-algebras to study the concept of C^*-algebra valued normed algebras with uncertainty. Using n-expansively super-homogeneous and (n,k)-contractively sub-homogeneous control functions, we make stochastic (Θ,Υ,Ξ)-derivations stable and get a better estimated error. We present some numerical examples of control functions and approximations to illustrate the applicability of the main results.

MSC: 37H10; 39B52; 54A20; 39A50; 47H10

Keywords: n-expansively super-homogeneous; (n,k)-contractively sub-homogeneous; Stochastic derivations; Hyers–Ulam stability

1 Introduction

In this paper, we define some new control functions with uncertainty named n-expansively super-homogeneous and (n,k)-contractively sub-homogeneous mappings. These control functions help us to make stochastic derivations stable. Also, we can get a better approximation for these stochastic derivations.

We consider the positive cone of an order commutative C^*-algebra and generalize the concept of triangular norm and fuzzy sets on it; we refer the reader to [1–3] for more details. Also, we define C^*-algebra valued normed algebras using generalized triangular norms and fuzzy sets.

Definition 1 Let $\mathcal{A}$ be an order commutative C^*-algebra and $\mathcal{A}^+$ be the positive cone of $\mathcal{A}$. Let $U\neq\emptyset$. A C^*-algebra valued fuzzy set (in short, C^*-AVF set) $\mathcal{C}$ on U is a function $\mathcal{C}:U\longrightarrow\mathcal{A}^+$. For each u in U, $\mathcal{C}(u)$ represents the degree (in $\mathcal{A}^+$) to which u satisfies $\mathcal{A}^+$.

We put $\mathbf{0}=\inf\mathcal{A}^+$ and $\mathbf{1}=\sup\mathcal{A}^+$. Now, we define a class of generalized t-norms (triangular norm) on $\mathcal{A}^+$.

Definition 2 A t-norm on $\mathcal{A}^+$ is an operation $\odot : \mathcal{A}^+ \times \mathcal{A}^+ \to \mathcal{A}^+$ satisfying the following conditions:

(a) $t \odot \mathbf{1} = t$ for every $t \in \mathcal{A}^+$ (boundary condition);
(b) $t \odot s = s \odot t$ for every $(t,s) \in (\mathcal{A}^+)^2$ (commutativity);
(c) $t \odot (s \odot p) = (t \odot s) \odot p$ for every $(t,s,p) \in (\mathcal{A}^+)^3$ (associativity);
(d) $t \preceq t' \text{and} s \preceq s' \Longrightarrow t \odot s \preceq t' \odot s'$ for every $(t,t',s,s') \in (\mathcal{A}^+)^4$ (monotonicity).

Now suppose that, for $t,s \in \mathcal{A}^+$ and sequences $\{t_n\}$ and $\{s_n\}$ converging to t and s, we have

$$\lim_n (t_n \odot s_n) = t \odot s.$$

Then $\odot$ on $\mathcal{A}^+$ is continuous (in short, CTN).

Definition 3 Assume that a decreasing mapping $\mathcal{F} : \mathcal{A}^+ \to \mathcal{A}^+$ satisfies $\mathcal{F}(\mathbf{0}) = \mathbf{1}$ and $\mathcal{F}(\mathbf{1}) = \mathbf{0}$. Then $\mathcal{F}$ is called a negation on $\mathcal{A}^+$.

Example 1 Let

$$\operatorname{diag} M_n\bigl([0,1]\bigr) = \left\{ \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} = \operatorname{diag}[t_1,\dots,t_n], t_1,\dots,t_n \in [0,1] \right\}.$$

We denote $\operatorname{diag}[t_1,\dots,t_n] \preceq \operatorname{diag}[s_1,\dots,s_n]$ if and only if $t_i \leq s_i$ for all $i = 1,\dots,n$; also, $\mathbf{1} = \operatorname{diag}[1,\dots,1]$ and $\mathbf{0} = \operatorname{diag}[0,\dots,0]$. Now, we know that if $\mathcal{A} = \operatorname{diag} M_n([0,1])$, then $\operatorname{diag} M_n([0,1]) = \mathcal{A}^+$. Define $\odot_P : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ such that

$$\operatorname{diag}[t_1,\dots,t_n] \odot_P \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}[t_1.s_1,\dots,t_n.s_n].$$

Then $\odot_P$ is a t-norm (product t-norm). Also note that $\odot_P$ is a CTN.

Example 2 Let $\operatorname{diag} M_n([0,1]) = \mathcal{A}^+$. Define $\odot_M : \operatorname{diag} M_n([0,1]) \times \operatorname{diag} M_n([0,1]) \to \operatorname{diag} M_n([0,1])$ such that

$$\operatorname{diag}[t_1,\dots,t_n] \odot_M \operatorname{diag}[s_1,\dots,s_n] = \operatorname{diag}\bigl[\min(t_1,s_1),\dots,\min(t_n,s_n)\bigr].$$

Then $\odot_M$ is a t-norm (minimum t-norm). Also note that $\odot_M$ is a CTN.

Definition 4 The triple $(T,\mathcal{N},\odot)$ is called a *C^*-AVF normed space* (in short, C^*AVFN-space) if T is a vector space over $\mathbb{C}$, $\odot$ is a CTN on $\mathcal{A}^+$, and $\mathcal{N}$ is a C^*AVF-set on $T \times [0,+\infty)$ such that, for each $t,s \in T$ and τ, ς in $[0,+\infty)$, we have

(a) $\mathcal{N}(t,0) = \mathbf{0}$;
(b) $\mathcal{N}(t,\tau) = \mathbf{1}$ for all $\tau > 0$ if and only if $t = 0$;
(c) $\mathcal{N}(\alpha t,\tau) = \mathcal{N}(t,\frac{\tau}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{N}(t+s,\tau+\varsigma) \succeq \mathcal{N}(t,\tau) \odot \mathcal{N}(s,\varsigma)$;
(e) $\mathcal{N}(t,\cdot) : [0,\infty) \to \mathcal{A}^+$ is left continuous;

(f) $\lim_{t\to\infty}\mathcal{N}(t,\tau)=\mathbf{1}$.
Also, $\mathcal{N}$ is called a *C^*-AVF norm*.

Let $(T,\mathcal{N},\odot)$ be a C^*-AVFN-space. For $\tau>0$, define the *open ball* $O_{(t,\varrho)}(\tau)$ as

$$O_{(t,\varrho)}(\tau)=\{s\in T:\mathcal{N}(t-s,\tau)\succ\mathcal{F}(\varrho)\},$$

in which $t\in T$ is the center and $\varrho\in\mathcal{A}^+\setminus\{\mathbf{0},\mathbf{1}\}$ is the radius. We say that $A\subseteq T$ is *open* if for each $t\in A$, there exist $\tau>0$ and $\varrho\in\mathcal{A}^+\setminus\{\mathbf{0},\mathbf{1}\}$ such that $O_{(t,\varrho)}(\tau)\subseteq A$. We denote the family of all open subsets of T by $\tau_{\mathcal{N}}$ and so $\tau_{\mathcal{N}}$ is the *C^*-AVF topology induced by the C^*-AVF norm* $\mathcal{N}$.

Example 3 Consider a normed space $(T,\|\cdot\|)$. Let $\odot=\odot_M$ and define the fuzzy set $\mathcal{N}$ on $T\times(0,\infty)$ as

$$\mathcal{N}(t,\tau)=\operatorname{diag}\left[\frac{h\tau}{h\tau+m\|t\|},\exp\left(-\frac{\|t\|}{\tau}\right)\right]$$

for all $\tau,h,m\in\mathbb{R}^+$. Then $(T,\mathcal{N},\odot_M)$ is a C^*-AVFN-space.

Example 4 Let $(T,\|\cdot\|)$ be a normed space,

$$u\odot v=\left(u_1v_1,\min\{u_2,v_2\}\right)$$

for all $u=(u_1,u_2),v=(v_1,v_2)\in\mathcal{A}^+$, and define the fuzzy set $\mathcal{N}$ on $T\times(0,\infty)$ as

$$\mathcal{N}(s,\zeta)=\operatorname{diag}\left[\frac{\zeta}{\zeta+\|s\|},\frac{\zeta}{\zeta+\|s\|}\right],\quad\forall\zeta\in\mathbb{R}^+.$$

Then $(T,\mathcal{N},\odot)$ is a C^*-AVFN-space.

Lemma 1 ([4]) *Let $(T,\mathcal{N},\odot)$ be a C^*-AVFN-space. Then $\mathcal{N}(t,\tau)$ is nondecreasing with respect to τ for all $t\in T$.*

Definition 5 Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence C^*-AVFN-space $(T,\mathcal{N},\odot)$. If

$$\forall\varepsilon\in\mathcal{A}^+\setminus\{\mathbf{0}\}\text{ and }\tau>0,\exists n_0\in\mathbb{N}\text{ such that }\forall m\geq n\geq n_0,\mathcal{N}(t_m-t_n,\tau)\succeq\mathcal{F}(\varepsilon),$$

then $\{t_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Also $\{t_n\}_{n\in\mathbf{N}}$ is convergent to $t\in T$ ($t_n\xrightarrow{\mathcal{N}}t$) if $\mathcal{N}(t_n-t,\tau)\to\mathbf{1}$ whenever $n\to+\infty$ for every $\tau>0$. When all Cauchy sequences are convergent in a C^*AVFN-space, the space is *complete*. A complete C^*AVFN-space is called a C^*AVF Banach space (in short, C^*AVFB-space).

Definition 6 A C^*-AVFN algebra $(T,\mathcal{N},\odot,\odot')$ is a C^*-AVFN-space $(T,\mathcal{N},\odot)$ satisfying
(g) $\mathcal{N}(wz,\tau\zeta)\succeq\mathcal{N}(w,\tau)\odot'\mathcal{N}(z,\zeta)$ for every $w,z\in T$ and $\tau,\zeta>0$ in which $\odot'$ is a CTN.

Consider a normed algebra $(T,\|\cdot\|)$. Define a C^*-AVFN algebra $(T,\mathcal{N},\odot_M,\odot_M)$, in which

$$\mathcal{N}(w,\zeta)=\operatorname{diag}\left[\frac{\zeta}{\zeta+\|w\|},\exp\left(-\frac{\|w\|}{\zeta}\right)\right]$$

for all $\zeta > 0$ if and only if

$$\|wz\| \leq \|w\|\|z\| + \zeta\|w\| + \tau\|z\| \quad (w, z \in T; \tau, \zeta > 0),$$

for which we name the standard C^*-AVFN algebra.

Definition 7 Consider a complete C^*AVF-algebra $(\mathcal{V}, \mathcal{N}, \odot, \odot')$. An involution on $\mathcal{V}$ is a mapping $v \to v^*$ from $\mathcal{V}$ into $\mathcal{V}$ with

(i) $v^{**} = v$ for $v \in \mathcal{V}$;

(ii) $(\Upsilon v + \Theta w)^* = \overline{\Upsilon} v^* + \overline{\Theta} w^*$;

(iii) $(vw)^* = w^* v^*$ for $v, w \in \mathcal{V}$.

If, in addition, $\mathcal{N}(v^* v, \Theta\Upsilon) = \mathcal{N}(v, \Theta) \odot' \mathcal{N}(v, \Upsilon)$ for $v \in \mathcal{V}$ and $\Theta, \Upsilon > 0$, then $\mathcal{V}$ is a C^*AVF C^*-algebra.

Novotný and Hrivnák [5] considered (Θ, Υ, Ξ)-derivations on Lie algebras. Let $\mathcal{B}$ be a Lie C^*-algebra. We say that a $\mathbb{C}$-linear mapping $\mathcal{D} : \mathcal{B} \to \mathcal{B}$ is a Lie derivation on $\mathcal{B}$ if $\mathcal{D} : \mathcal{B} \to \mathcal{B}$ satisfies that

$$\mathcal{D}[t, s] = \left[\mathcal{D}(t), s\right] + \left[t, \mathcal{D}(s)\right] \tag{1.1}$$

for all $t, s \in \mathcal{B}$ [6, 7]. Also the $\mathbb{C}$-linear mapping $\mathfrak{H} : \mathcal{B} \to \mathcal{B}$ is a Lie (Θ, Υ, Ξ)-derivation on $\mathcal{B}$ if there exist $\Theta, \Upsilon, \Xi \in \mathbb{C}$ such that

$$\Theta\mathfrak{H}[t, s] = \Upsilon\left[\mathfrak{H}(t), s\right] + \Xi\left[t, \mathfrak{H}(s)\right] \tag{1.2}$$

for all $t, s \in \mathcal{B}$. A C^*AVF C^*-algebra $\mathcal{B}$ with a Lie product $[t, s] = ts - st$ is said to be a *C^*AVF Lie C^*-algebra*. Assume that $\mathcal{B}$ is a C^*AVF Lie C^*-algebra. A $\mathbb{C}$-linear mapping $H : \mathcal{B} \to \mathcal{B}$ is said to be a *C^*AVF Lie derivation* on $\mathcal{B}$ if $H : \mathcal{B} \to \mathcal{B}$ satisfies (1.1). A $\mathbb{C}$-linear mapping $\mathfrak{H} : \mathcal{B} \to \mathcal{B}$ is said to be a *C^*AVF Lie (Θ, Υ, Ξ)-derivation* on $\mathcal{B}$ if there exist $\Theta, \Upsilon, \Xi \in \mathbb{C}$ satisfying (1.2).

Consider a probability measure space (Γ, Σ, ξ) and Borel measurable spaces $(T, \mathfrak{B}_T)$ and $(S, \mathfrak{B}_S)$, where T and S are C^*AVFB-spaces. If for $F : \Gamma \times T \to S$ we have $\{\gamma : F(\gamma, t) \in R\} \in \Sigma$ for every t in T and $R \in \mathfrak{B}_S$, we say that F is a random operator. If $F(\gamma, \alpha t_1 + \beta t_2) = \alpha F(\gamma, t_1) + \beta F(\gamma, t_2)$ almost everywhere for t_1, t_2 in T and scalers α, β, then F is a linear random operator, also if we can find an $M(\gamma) > 0$ such that

$$\nu\big(F(\gamma, t_1) - F(\gamma, t_2), M(\gamma)\tau\big) \geq \nu(t_1 - t_2, \tau)$$

almost everywhere for t_1, t_2 in T and $\tau > 0$, then F is a bounded random operator.

2 Cauchy–Jensen random operator

In this paper, let $\mathcal{G} = [0, \infty]$ and $\mathcal{G}^\circ = (0, \infty)$.

Theorem 1 ([8, 9]) *Let S be a set with the complete $\mathcal{G}$-valued metric δ, and let a self-mapping Λ on S satisfy*

$$\delta(\Lambda s, \Lambda t) \leq \kappa\delta(t, s), \quad \kappa < 1 \textit{ is a Lipschitz constant.}$$

Let $s \in S$. Then we have two options

(I) $\delta(\Lambda^m s, \Lambda^{m+1} s) = \infty, \forall m \in \mathbb{N}$ *or*

(II) *we can find $m_0 \in \mathbb{N}$ such that*

  (1) $\delta(\Lambda^m s, \Lambda^{m+1} s) < \infty, \forall m \geq m_0$;

  (2) *the fixed point t^* of Λ is the convergent point of the sequence $\{\Lambda^m s\}$;*

  (3) *in the set $V = \{t \in S \mid \delta(\Lambda^{m_0} s, t) < \infty\}$, t^* is the unique fixed point of Λ;*

  (4) $(1-\kappa)\delta(t, t^*) \leq \delta(t, \Lambda t)$ *for every $s \in V$.*

In this paper, assume that $(\mathcal{B}, \mathcal{N}, \odot_M, \odot_M)$ is a C^*-AVF Lie C^*-algebra. Also, we use the random operator $g : \Gamma \times \mathcal{B} \to \mathcal{B}$:

$$\Delta_\nu g(\gamma, t_1, \ldots, t_n) := \sum_{i=1}^{n} g\left(\gamma, \nu t_i + \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \nu t_j\right) - 2\nu \sum_{i=1}^{n} g(\gamma, t_i),$$

$$\Delta_{\Theta,\Upsilon,\Xi} g(\gamma, t, s) := \Theta g[\gamma, t, s] - \Upsilon\big[g(\gamma, t), s\big] - \Xi\big[t, g(\gamma, s)\big]$$

for all $t_1, \ldots, t_n \in \mathcal{B}, \gamma \in \Gamma$, all $\nu \in \Omega$ for some set $\Omega \in D_{\mathbb{C}}$ and $\Theta, \Upsilon, \Xi \in \mathbb{C}$. Denote

$$D_{\mathbb{C}} = \{\Omega \subseteq \mathbb{C} | g : \Omega \longrightarrow \mathcal{B} \text{ is additive, bounded and continuous}\}.$$

For more details, see [10–13]. Also, $\mathbb{T}^1_{1/n_0} := \{e^{i\theta}; 0 \leq \theta \leq 2\pi/n_0\} \in D_{\mathbb{C}}$.

Lemma 2 ([14]) *A random operator $g : \Gamma \times T \to S$ satisfies the equation*

$$g\left(\gamma, t_1 + \frac{1}{2}(t_2 + t_3)\right) + g\left(\gamma, t_2 + \frac{1}{2}(t_1 + t_3)\right) + g\left(\gamma, t_3 + \frac{1}{2}(t_1 + t_2)\right) = 2\big(g(\gamma, t_1) + g(\gamma, t_2) + g(\gamma, t_3)\big) \tag{2.1}$$

for all $t_1, t_2, t_3 \in T, \gamma \in \Gamma$ if and only if g is additive.

If we set $t_3 = 0$ in (2.1), then we get that the Cauchy–Jensen random operator

$$g\left(\gamma, \frac{1}{2}(t_1 + t_2)\right) + g\left(\gamma, t_1 + \frac{t_2}{2}\right) + g\left(\gamma, \frac{t_1}{2} + t_2\right) = 2\big(g(\gamma, t_1) + g(\gamma, t_2)\big)$$

is equivalent to $g(\gamma, t_1 + t_2) = g(\gamma, t_1) + g(\gamma, t_2)$ for all $t_1, t_2 \in T, \gamma \in \Gamma$.

Lemma 3 ([15]) *A random operator $g : \Gamma \times T \to S$ satisfies $\Delta_\nu g = 0$ for all $t_1, \ldots, t_n \in T, \gamma \in \Gamma$ if and only if g is additive.*

Lemma 4 ([10]) *Let $g : \Gamma \times \mathcal{B} \to \mathcal{B}$ be an additive random operator such that $g(\gamma, \nu t) = \nu g(\gamma, t)$ for all $\nu \in \Omega, \gamma \in \Gamma$ where the bounded set Ω is in $D_{\mathbb{C}}$. Then the random operator g is $\mathbb{C}$-linear.*

3 Hyers–Ulam–Rassias stability

In this section, we present some stability results. In real phenomena, the concept of stability also appears in mechanical applications as a consequence of real equilibrium problems. Related stability problems take part in mathematical models from mechanics when equilibrium equations are imposed (see [16, 17]). The stability results have numerous applications in the study of stability of porous medium problems (see [18]). For further applications, we refer to [19–21].

Definition 8 Let $n \in \mathbb{N}$. A C^*AVF mapping $\mathcal{R} : \mathcal{B}^n \times (0,\infty) \to \mathcal{A}^+$ is called a C^*AVF n-expansively super-homogeneous function if there is a fixed number $\ell \in (0,1)$ such that

$$\mathcal{R}\big((\mu^{-1}t_1,\dots,\mu^{-1}t_n),\tau\big) \succeq \mathcal{R}\left((t_1,\dots,t_n),\frac{\mu^n\tau}{\ell^n}\right), \tag{3.1}$$

$$\lim_{\varsigma\to\infty}\mathcal{R}\big((t_1,\dots,t_n),\varsigma\big) = \mathbf{1} \tag{3.2}$$

for all $t_i \in \mathcal{B}(1 \le i \le n)$, $1 < \mu \in \mathbb{N}$, and $\tau \in \mathcal{G}^\circ$.

Example 5 Consider a real function $r : \mathbb{R} \to \mathbb{R}$ defined as $r(t) = |t|^4$. Define

$$\mathcal{R}\big((t_1,t_2,t_3),\tau\big) = \operatorname{diag}\left[\frac{\tau}{\tau + \sum_{j=1}^3 r(t_j)}, \exp\left(-\frac{\sum_{j=1}^3 r(t_j)}{\tau}\right)\right]$$

for all $t_1, t_2, t_3 \in \mathbb{R}$ and $\tau \in \mathcal{G}^\circ$. Put $\ell = \frac{1}{\sqrt[3]{2}}$. Then $\mathcal{R}$ is a 3-expansively super-homogeneous function.

Theorem 2 *Consider a C^*-AVF expansively super-homogeneous function $\varphi : \mathcal{B}^n \times (0,\infty) \to \mathcal{A}^+$ and a C^*VAF 2-expansively super-homogeneous function $\psi : \mathcal{B}^2 \times (0,\infty) \to \mathcal{A}^+$ with a fixed number ℓ such that a random operator $g : \Gamma \times \mathcal{B} \to \mathcal{B}$ satisfies*

$$\mathcal{N}\big(\Delta_\eta g(\gamma,t_1,\dots,t_n),t\big) \succeq \varphi\big((t_1,\dots,t_n),\tau\big), \tag{3.3}$$

$$\mathcal{N}\big(\Delta_{\Theta,\Upsilon,\Xi} g(\gamma,t,s),\tau\big) \succeq \psi\big((t,s),\tau\big) \tag{3.4}$$

for all $t_1,\dots,t_n,t,s \in \mathcal{B}, \gamma \in \Gamma$, $\eta \in \Omega$, $\tau \in \mathcal{G}^\circ$ and some $\Theta, \Upsilon, \Xi \in \mathbb{C}$, where $\Omega \in D_{\mathbb{C}}$ is bounded. Then we can find a unique C^*VAF Lie (Θ,Υ,Ξ)-derivation $\mathfrak{H} : \Gamma \times \mathcal{B} \to \mathcal{B}$ which satisfies $\Delta_\nu g = 0$ and the inequality

$$\mathcal{N}\big(g(\gamma,z) - \mathfrak{H}(\gamma,z),\varsigma\big) \succeq \varphi\left((\overbrace{z,\dots,z}^{n\text{-times}}),\frac{(2^n n - 2n\ell^n)\varsigma}{\ell^n}\right) \tag{3.5}$$

for all $z \in \mathcal{B}, \gamma \in \Gamma$ and $\varsigma \in \mathcal{G}^\circ$.

Proof Consider $M := \{k : \Gamma \times \mathcal{B} \to \mathcal{B}, k(\varpi,0) = 0, \forall \varpi \in \Gamma\}$ and define

$$\delta(k,h) := \inf\Bigg\{P \in \Xi^\circ : \mathcal{N}\big(k(\varpi,w) - h(\varpi,w),\tau\big) \succeq \varphi\left((w,\dots,w),\frac{\tau}{P}\right),$$
$$\forall \varpi \in \Gamma, w \in \mathcal{B}, \tau \in \mathcal{G}^\circ\Bigg\}.$$

In [22], Miheț and Radu showed that (M,δ) is a complete $\mathcal{G}$-valued metric space (see [23]).

Define a linear mapping $\Lambda : M \to M$ as

$$(\Lambda k)(\varpi, w) = 2k\left(\varpi, \frac{w}{2}\right), \quad \forall k \in M \text{ and } w \in \mathcal{B}\varpi \in \Gamma.$$

Let $k, h \in M$ and consider a sequence of positive real numbers P_m with $\lim_{m\to\infty} P_m = \delta(k,h)$ and $\delta(k,h) \leq P_m$. Fix m and, for convenience, let $P_m = P$. Then

$$\mathcal{N}\big(k(\varpi, w) - h(\varpi, w), \varsigma\big) \succeq \varphi\left((w, \dots, w), \frac{\varsigma}{P}\right)$$

for all $w \in \mathcal{B}, \varpi \in \Gamma$ and $\varsigma \in \Xi^\circ$. Now we have

$$\begin{aligned}
\mathcal{N}\big((\Lambda k)(\varpi, w) - (\Lambda h)(\varpi, w), \varsigma\big) &= \mathcal{N}\left(2k\left(\varpi, \frac{w}{2}\right) - 2h\left(\varpi, \frac{w}{2}\right), \varsigma\right) \\
&= \mathcal{N}\left(k\left(\varpi, \frac{w}{2}\right) - h\left(\varpi, \frac{w}{2}\right), \frac{\varsigma}{2}\right) \\
&\succeq \varphi\left(\left(\frac{w}{2}, \dots, \frac{w}{2}\right), \frac{\varsigma}{2P}\right) \\
&\succeq \varphi\left((w, \dots, w), \frac{2^{n-1}\varsigma}{\ell^n P}\right)
\end{aligned}$$

for all $w \in \mathcal{B}$ and $\varsigma \in \mathcal{G}^\circ, \varpi \in \Gamma$, and so $\delta(\Lambda k, \Lambda h) \leq \frac{\ell^n}{2^{n-1}} P = \frac{\ell^n}{2^{n-1}} P_m$ for any $k, h \in M$. Now let $m \to \infty$, and we get $\delta(\Lambda k, \Lambda h) \leq \frac{\ell^n}{2^{n-1}} \delta(k,h)$ for any $k, h \in M$.

Let g be as in the statement of the theorem. Putting $t_1, \dots, t_n = w$ and $\eta = 1$ in (3.3), we obtain

$$\mathcal{N}\big(g(\gamma, 2w) - 2g(\gamma, w), \tau\big) \succeq \phi\big((w, \dots, w), n\tau\big)$$

for all $w \in \mathcal{B}$, $\gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Thus

$$\begin{aligned}
\mathcal{N}\left(2g\left(\gamma, \frac{w}{2}\right) - g(\gamma, w), \tau\right) &\succeq \varphi\left(\left(\frac{w}{2}, \dots, \frac{w}{2}\right), n\tau\right) \\
&\succeq \varphi\left((w, \dots, w), \frac{2^n n\tau}{\ell^n}\right)
\end{aligned}$$

for all $w \in \mathcal{B}, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Hence $\delta(\Lambda g, g) \leq \frac{\ell^n}{2^n n}$. Now Theorem 1 guarantees that $\{\Lambda^n g\}$ converges to a unique fixed point $\mathfrak{H} \in M$ of Λ such that $\mathfrak{H}(\gamma, 2w) = 2\mathfrak{H}(\gamma, w)$, i.e.,

$$\mathfrak{H}(\gamma, w) = \lim_{m\to\infty} 2^m g\left(\gamma, \frac{w}{2^m}\right) \tag{3.6}$$

for all $w \in \mathcal{B}, \gamma \in \Gamma$. Also (see Theorem 1)

$$\delta(g, \mathfrak{H}) \leq \frac{1}{1 - \frac{\ell^n}{2^{n-1}}} \delta(g, \Lambda g) \leq \frac{\ell^n}{2^n n - 2n\ell^n},$$

i.e., (3.5) holds for all $t \in \mathcal{B}$ and $\tau \in \mathcal{G}^\circ$. From the property of $\mathfrak{H}$, we get that

$$\mathcal{N}\big(\Delta_\eta \mathfrak{H}(\gamma, t_1, \dots, t_n), \tau\big) = \lim_{m\to\infty} \mathcal{N}\left(\Delta_\eta g\left(\gamma, \frac{t_1}{2^m}, \dots, \frac{t_n}{2^m}\right), \frac{\tau}{2^m}\right)$$

$$\succeq \lim_{m\to\infty} \varphi\left(\left(\frac{t_1}{2^m},\dots,\frac{t_n}{2^m}\right),\frac{\tau}{2^m}\right) = 1$$

holds for all $t_1,\dots,t_n \in \mathcal{B}, \gamma \in \Gamma$, $\eta \in \Omega$, and $\tau \in \mathcal{G}^\circ$. Thus $\Delta_\eta \mathfrak{H}(\gamma, t_1,\dots,t_n) = 0$ for all $t_1,\dots,t_n \in \mathcal{B}, \gamma \in \Gamma$ and all $\eta \in \Omega$. If we put $\eta = 1$ in the above equality, then Lemma 3 implies that $\mathfrak{H}$ is additive. Putting $t_1 = t$ and $t_2 = \cdots = t_n = 0$ in the above equality, we get $\mathfrak{H}(\gamma, \eta t) = \eta\mathfrak{H}(\gamma, t)$ and Lemma 4 implies that $\mathfrak{H} \in M$ is $\mathbb{C}$-linear. Also (3.1) and (3.4) imply that

$$\begin{aligned}
\mathcal{N}\bigl(\Delta_{\Theta,\Upsilon,\Xi}\mathfrak{H}(\gamma,t,s),\tau\bigr) &= \lim_{m\to\infty} \mathcal{N}\left(\Delta_{\Theta,\Upsilon,\Xi} g\left(\gamma,\frac{t}{2^m},\frac{s}{2^m}\right),\frac{\tau}{2^m}\right) \\
&\succeq \lim_{m\to\infty} \psi\left(\left(\frac{t}{2^m},\frac{s}{2^m}\right),\frac{\tau}{2^m}\right) \\
&\succeq \lim_{m\to\infty} \psi\left((t,s),\frac{2^{2m}\tau}{\ell^2 2^m}\right) \\
&= \lim_{m\to\infty} \psi\left((t,s),\frac{2^m\tau}{\ell^2}\right) \\
&= 1
\end{aligned}$$

for all $t,s \in \mathcal{B}$, some $\Theta, \Upsilon, \Xi \in \mathbb{C}$ and $\tau \in \mathcal{G}^\circ$. Then, for some $\Theta, \Upsilon, \Xi \in \mathbb{C}$,

$$\Theta\mathfrak{H}[\gamma,t,s] = \Upsilon\bigl[\mathfrak{H}(\gamma,t),s\bigr] + \Xi\bigl[t,\mathfrak{H}(\gamma,s)\bigr]$$

for all $t,s \in \mathcal{B}, \gamma \in \Gamma$. So the random operator $\mathfrak{H} \in M$ is a C^*VAF Lie (Θ,Υ,Ξ)-derivation on the C^*VAF Lie C^*-algebra $\mathcal{B}$ and (3.5) holds. □

Example 6 Let a random operator $g : \Gamma \times \mathcal{B} \to \mathcal{B}$ satisfy

$$\mathcal{N}\bigl(\Delta_\eta g(\gamma,t_1,\dots,t_4),t\bigr) \succeq \operatorname{diag}\left[\frac{\tau}{\tau+\sum_{j=1}^4 \|t_j\|^5},\exp\left(-\frac{\sum_{j=1}^4\|t_j\|^5}{\tau}\right)\right], \tag{3.7}$$

$$\mathcal{N}\bigl(\Delta_{\Theta,\Upsilon,\Xi} g(\gamma,t_1,t_2),\tau\bigr) \succeq \operatorname{diag}\left[\frac{\tau}{\tau+\sum_{j=1}^2 \|t_j\|^5},\exp\left(-\frac{\sum_{j=1}^2\|t_j\|^5}{\tau}\right)\right] \tag{3.8}$$

for all $t_1,\dots,t_4 \in \mathcal{B}, \gamma \in \Gamma, \eta \in \Omega, \tau \in \mathcal{G}^\circ$ and some $\Theta, \Upsilon, \Xi \in \mathbb{C}$, where $\Omega \in D_{\mathbb{C}}$ is bounded. Then we can find a unique C^*VAF Lie (Θ,Υ,Ξ)-derivation $\mathfrak{H} : \Gamma \times \mathcal{B} \to \mathcal{B}$ which satisfies $\Delta_\nu g = 0$ and the inequality

$$\mathcal{N}\bigl(g(\gamma,z)-\mathfrak{H}(\gamma,z),\tau\bigr) \succeq \operatorname{diag}\left[\frac{30\tau}{30\tau+\|z\|^5},\exp\left(-\frac{\|z\|^5}{30\tau}\right)\right] \tag{3.9}$$

for all $z \in \mathcal{B}, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$.

Define

$$\varphi\bigl((t_1,t_2,t_3,t_4),\tau\bigr) = \operatorname{diag}\left[\frac{\tau}{\tau+\sum_{j=1}^4 \|t_j\|^5},\exp\left(-\frac{\sum_{j=1}^4\|t_j\|^5}{\tau}\right)\right]$$

and

$$\psi\big((t_1,t_2),\tau\big) = \operatorname{diag}\left[\frac{\tau}{\tau + \sum_{j=1}^{2}\|t_j\|^5}, \exp\left(-\frac{\sum_{j=1}^{2}\|t_j\|^5}{\tau}\right)\right]$$

for all $t_1, t_2, t_3 \in \mathbb{B}$ and $\tau \in \mathcal{G}^\circ$. Put $\ell = \frac{1}{\sqrt[4]{2}}$. Then φ and ψ are 4-expansively super-homogeneous function and 2-expansively super-homogeneous function, respectively. Now, applying Theorem 2, we get (3.9).

Definition 9 Let $n, k \in \mathbb{N}$. A C^*AVF map $\mathcal{O} : \mathcal{B}^n \times (0,\infty) \to \mathcal{A}^+$ is called a C^*AVF (n,k)-contractively sub-homogeneous if there exists a fixed number ℓ with $0 < \ell < 1$ such that

$$\mathcal{O}(\mu t_1, \dots, \mu t_n, \tau) \succeq \mathcal{O}\left((t_1, \dots, t_n), \frac{\tau}{\ell^k \mu^{\frac{1}{k}}}\right),$$

$$\lim_{\varsigma \to \infty} \mathcal{O}(t_1, \dots, t_n, \varsigma) = \mathbf{1}$$

for all $t_1, \dots, t_n \in \mathcal{B}$, $1 < \mu \in \mathbb{N}$ and $\tau \in \mathcal{G}^\circ$.

Example 7 Consider a real function $r : \mathbb{R} \to \mathbb{R}$ defined as $r(t) = |t|^{\frac{1}{4}}$. Define

$$\mathcal{O}\big((t_1,t_2,t_3),\tau\big) = \operatorname{diag}\left[\frac{\tau}{\tau + \sum_{j=1}^{3} r(t_j)}, \exp\left(-\frac{\sum_{j=1}^{3} r(t_j)}{\tau}\right)\right]$$

for all $t_1, t_2, t_3 \in \mathbb{R}$ and $\tau \in \mathcal{G}^\circ$. Put $\ell = \frac{1}{\sqrt[8]{2}}$. Then $\mathcal{O}$ is a $(3,2)$-contractively sub-homogeneous function.

Theorem 3 *Consider a C^*AVF $(n+2,k)$-contractively sub-homogeneous function $\varphi : \mathcal{B}^{n+2} \times (0,\infty) \to \mathcal{A}^+$ with a fixed number ℓ such that a random operator $g : \Gamma \times \mathcal{B} \to \mathcal{B}$ holds*

$$\mathcal{N}\big(\Delta_\eta g(\gamma, t_1, \dots, t_n) + \Delta_{\Theta,\Upsilon,\Xi} g(\gamma, t, s), \tau\big) \succeq \varphi\big((t_1, \dots, t_n, t, s), \tau\big) \tag{3.10}$$

for all $t_1, \dots, t_n, t, s \in \mathcal{B}, \gamma \in \Gamma$, all $\eta \in \Omega$ in which $\Omega \in D_{\mathbb{C}}$ is a bounded set, $\Theta, \Upsilon, \Xi \in \mathbb{C}$ and $\tau \in \mathcal{G}^\circ$. Then there is a unique C^*VAF Lie (Θ, Υ, Ξ)-derivation $\mathfrak{H} : \Gamma \times \mathcal{B} \to \mathcal{B}$ which satisfies $\Delta_\nu g = 0$ and the inequality

$$\mathcal{N}\big(g(\gamma, w) - \mathfrak{H}(\gamma, w), \tau\big) \succeq \varphi\left(\big(\overbrace{w, \dots, w}^{n\text{-times}}, 0, 0\big), \frac{2n(\sqrt[k]{2^{k-1}} - \ell^k)}{\sqrt[k]{2^{k-1}}}\tau\right) \tag{3.11}$$

for all $w \in \mathcal{B}, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$.

Proof Putting $t_1, \dots, t_n = t$ and $\eta = 1$ in (3.10), we get

$$\mathcal{N}\big(ng(\gamma, 2t) - 2ng(\gamma, t), \tau\big) \succeq \varphi\big((t, \dots, t, 0, 0)\tau\big) \tag{3.12}$$

for all $t \in \mathcal{B}, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Let $M := \{f : \Gamma \times \mathcal{B} \to \mathcal{B}, f(\varpi, 0) = 0 \forall \varpi \in \Gamma\}$. We introduce a function on M as

$$\delta(f,h) := \inf\left\{u > 0 : \mathcal{N}\big(f(\gamma,t) - h(\gamma,t), \tau\big)\big) \succeq \varphi\left((t, \dots, t, 0, 0), \frac{\tau}{u}\right),\right.$$

$$\left.\forall t \in \mathcal{B}, \gamma \in \Gamma \text{ and } \tau \in \mathcal{G}^\circ \right\}.$$

In [22], Miheţ and Radu showed that (B,δ) is a complete Ξ-valued metric space (see [23]).

Define $\Lambda : M \to M$ as

$$(\Lambda f)(\gamma,t) = \frac{1}{2} f(\gamma,2t) \quad \text{for all } f \in E \text{ and } t \in \mathcal{B}.$$

Now, we have

$$\begin{aligned}
\mathcal{N}\big((\Lambda f)(\varpi,w) - (\Lambda h)(\varpi,w),\varsigma\big) &= \mathcal{N}\left(\frac{1}{2} f(\gamma,2t) - \frac{1}{2} h(\gamma,2t),\varsigma\right) \\
&= \mathcal{N}\big(f(\gamma,2t) - h(\gamma,2t), 2\varsigma\big) \\
&\succeq \varphi\left((2w,\dots,2w,0,0), \frac{2\varsigma}{u}\right) \\
&\succeq \varphi\left((w,\dots,w,0,0), \frac{2^{1-\frac{1}{k}}\varsigma}{\ell^k u}\right)
\end{aligned}$$

for all $w \in \mathcal{B}$ and $\varsigma \in \mathcal{G}^\circ, \varpi \in \Gamma$, and so $\delta(\Lambda f,\Lambda h) \leq \frac{\ell^k}{2^{1-\frac{1}{k}}}\delta(f,h)$ for any $f,h \in E$. Let g be as in the statement of the theorem. Using (3.12) we get

$$\mathcal{N}\left(\frac{1}{2} g(\gamma,2t) - g(\gamma,t),\tau\right) \succeq \varphi\big((t,\dots,t,0,0),2n\tau\big)$$

for all $t \in \mathcal{B}, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Then $\delta(\Lambda g,g) \leq \frac{1}{2n}$. Applying Theorem 1, we get that $\{\Lambda^m g\}$ converges to a unique fixed point $\mathfrak{H} \in M$ of Λ such that $\mathfrak{H}(\gamma,2t) = 2\mathfrak{H}(\gamma,t)$, i.e.,

$$\mathfrak{H}(\gamma,t) = \lim_{m\to\infty} \frac{1}{2^m} g\big(\gamma,2^m t\big) \tag{3.13}$$

for all $t \in \mathcal{B}$. Also

$$\delta(g,\mathfrak{H}) \leq \frac{1}{1-\frac{\ell^k}{2^{1-\frac{1}{k}}}}\delta(g,\Lambda g) \leq \frac{1}{2n(1-\frac{\ell^k}{2^{1-\frac{1}{k}}})} = \frac{\sqrt[k]{2^{k-1}}}{2n(\sqrt[k]{2^{k-1}}-\ell^k)},$$

i.e., (3.5) is true for every $t \in \mathcal{B}$. Then (3.11) is true. Using Theorem 2, we can complete the proof. □

Example 8 Let a random operator $g : \Gamma \times \mathcal{B} \to \mathcal{B}$ satisfy

$$\begin{aligned}
&\mathcal{N}\big(\Delta_\eta g(\gamma,t_1,t_2) + \Delta_{\Theta,\Upsilon,\Xi} g(\gamma,t_3,t_4),\tau\big) \\
&\quad\succeq \operatorname{diag}\left[\frac{\tau}{\tau+\sum_{j=1}^4 \|t_j\|^{\frac{1}{6}}}, \exp\left(-\frac{\sum_{j=1}^4 \|t_j\|^{\frac{1}{6}}}{\tau}\right)\right]
\end{aligned} \tag{3.14}$$

for all $t_1,\dots,t_4 \in \mathcal{B}, \gamma \in \Gamma$, all $\eta \in \Omega$ in which $\Omega \in D_{\mathbb{C}}$ is a bounded set, $\Theta,\Upsilon,\Xi \in \mathbb{C}$ and $\tau \in \mathcal{G}^\circ$. Then there is a unique C^*VAF Lie (Θ,Υ,Ξ)-derivation $\mathfrak{H} : \Gamma \times \mathcal{B} \to \mathcal{B}$ which satisfies $\Delta_\upsilon g = 0$ and the inequality

$$\mathcal{N}\big(g(\gamma,w) - \mathfrak{H}(\gamma,w),\tau\big) \succeq \operatorname{diag}\left[\frac{8(\sqrt[6]{32}-1)\tau}{8(\sqrt[6]{32}-1)\tau + 2\sqrt[6]{32}\|w\|^{\frac{1}{6}}}, \exp\left(-\frac{\sqrt[6]{32}\|w\|^{\frac{1}{6}}}{4(\sqrt[6]{32}-1)\tau}\right)\right] \tag{3.15}$$

for all $w \in \mathcal{B}$, $\gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$.

Define

$$\varphi\big((t_1,t_2,t_3,t_4),\tau\big) = \left[\frac{\tau}{\tau + \sum_{j=1}^4 \|t_j\|^{\frac{1}{6}}}, \exp\left(-\frac{\sum_{j=1}^4 \|t_j\|^{\frac{1}{6}}}{\tau}\right)\right]$$

for all $t_1,t_2,t_3,t_4 \in \mathbb{R}$ and $\tau \in \mathcal{G}^\circ$. Put $\ell = \frac{1}{\sqrt[18]{2}}$. Then φ is a $(4,3)$-contractively subhomogeneous function. Now, applying Theorem 3, we get (3.15).

4 C^*-ternary algebra stochastic homomorphism

A $\mathbb{C}$-linear random operator $\eta : \Gamma \times T \to S$ is said to be a *C^*-ternary algebra stochastic homomorphism* (C^*-tash) if

$$\eta\big(\gamma,[t,s,p]\big) = \big[\eta(\gamma,t),\eta(\gamma,s),\eta(\gamma,p)\big]$$

for all $t,s,p \in T$ and $\gamma \in \Gamma$ (see [6, 24]).

Consider a random operator $g : \Gamma \times T \to S$ and define

$$\Xi_\xi g(\gamma,t_1,\ldots,t_p,s_1,\ldots,s_d) := 2g\left(\gamma, \frac{\sum_{j=1}^p \xi t_j}{2} + \sum_{j=1}^d \xi s_j\right) - \sum_{j=1}^p \xi g(\gamma,t_j) - 2\sum_{j=1}^d \xi g(\gamma,s_j)$$

for all $\xi \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $t_1,\ldots,t_p,s_1,\ldots,s_d \in T$ and $\gamma \in \Gamma$.

It is easy to show that a random operator $g : \Gamma \times T \to S$ satisfies

$$\Xi_\xi g(\gamma,t_1,\ldots,t_p,s_1,\ldots,s_d) = 0$$

for all $\xi \in \mathbb{T}^1$, $t_1,\ldots,t_p,s_1,\ldots,s_d \in T$ and $\gamma \in \Gamma$ if and only if

$$g(\gamma,\xi t + \lambda s) = \xi g(\gamma,t) + \lambda g(\gamma,s)$$

for all $\xi,\lambda \in \mathbb{T}^1$, $t,s \in T$ and $\gamma \in \Gamma$.

Theorem 4 *Consider q and σ such that $q < 1$ and $\sigma < 3$. Let $\varphi : T^{p+d} \times (0,\infty) \to \mathcal{A}^+$ $(d \geq 2)$ and $\psi : T^3 \times (0,\infty) \to \mathcal{A}^+$ be a C^*-AVF control function satisfying*

$$\varphi\big(a(t_1,\ldots,t_p,s_1,\ldots,s_d),\tau\big) = \varphi\left((t_1,\ldots,t_p,s_1,\ldots,s_d), \frac{\tau}{a^q}\right), \tag{4.1}$$

$$\psi\big(a(t,s,p),\tau\big) = \psi\left((t,s,p), \frac{\tau}{a^\sigma}\right) \tag{4.2}$$

and

$$\lim_{\mu\to\infty}\varphi\big((t_1,\dots,t_p,s_1,\dots,s_d),\mu\big)=\lim_{\mu\to\infty}\psi\big((t,s,p),\mu\big)=1 \tag{4.3}$$

for all $t_1,\dots,t_p,s_1,\dots,s_d,t,s,p\in T$, $a>0$, *and* $\tau,\nu\in\mathcal{G}^\circ$. *Suppose that* $g:\Gamma\times T\to S$ *is a random operator with* $g(\gamma,0)=0$ *satisfying*

$$\mathcal{N}\big(\Xi_\eta g(\gamma,t_1,\dots,t_p,s_1,\dots,s_d),\tau\big)\succeq\varphi\big((t_1,\dots,t_p,s_1,\dots,s_d),\tau\big) \tag{4.4}$$

and

$$\mathcal{N}\big(g(\gamma,[t,s,p])-\big[g(\gamma,t),g(\gamma,s),g(\gamma,p)\big],\tau\big)\succeq\psi\big((t,s,p),\tau\big) \tag{4.5}$$

for all $\eta\in\mathbb{T}^1$ *and all* $t_1,\dots,t_p,s_1,\dots,s_d,t,s,p\in T$ *and* $\gamma\in\Gamma$ *and* $\tau\in\mathcal{G}^\circ$. *Then there exists a unique* C^**-tash* $\mathfrak{H}:\Gamma\times T\to S$ *such that*

$$\mathcal{N}\big(g(\gamma,t)-\mathfrak{H}(\gamma,t),\tau\big)\succeq\varphi\big((\overbrace{0,\dots,0,t,\dots,t}^{n+d\text{-}times}),2\tau\big(d-d^q\big)\big) \tag{4.6}$$

for all $t\in T,\gamma\in\Gamma$ *and* $\tau\in\mathcal{G}^\circ$.

Proof Let $0<q<1$ and $0<\sigma<3$ (the other cases are similar).

Putting $\eta=1$, $t_1=\cdots=t_p=0$ and $s_1=\cdots=s_d=t$ in (4.4), we get

$$\mathcal{N}\big(2g(\gamma,dt)-2dg(\gamma,t),\tau\big)\succeq\varphi\big((\overbrace{0,\dots,0}^{p},\overbrace{t,\dots,t}^{d}),\tau\big) \tag{4.7}$$

for all $t\in T,\gamma\in\Gamma$ and $\tau\in\mathcal{G}^\circ$. Replacing t by d^nt in (4.7), we get

$$\mathcal{N}\left(\frac{1}{d^{n+1}}g\big(\gamma,d^{n+1}t\big)-\frac{1}{d^n}g\big(\gamma,d^nt\big),\tau\right)\succeq\varphi\big((\overbrace{0,\dots,0}^{p},\overbrace{t,\dots,t}^{d}),2d\tau d^{(1-q)n}\big)$$

for all $t\in T,\gamma\in\Gamma$, all nonnegative integers n and $\tau\in\mathcal{G}^\circ$. Therefore,

$$\begin{aligned}&\mathcal{N}\left(\frac{1}{d^{n+m}}g\big(\gamma,d^{n+m}t\big)-\frac{1}{d^m}g\big(\gamma,d^mt\big),\tau\right)\\&\quad\succeq\varphi\left((\overbrace{0,\dots,0}^{p},\overbrace{t,\dots,t}^{d}),\frac{2d\tau}{\sum_{k=m}^{m+n}d^{(q-1)k}}\right)\end{aligned} \tag{4.8}$$

for all $t\in T$, $n,m\in\mathbb{N}$ and $\tau\in\mathcal{G}^\circ$, and it follows that $\{\frac{1}{d^n}g(\gamma,d^nt)\}$ is a Cauchy sequence for every $t\in A$. The completeness of B implies that $\{\frac{1}{d^n}g(\gamma,d^nt)\}$ converges. Thus we can define the random operator $\mathfrak{H}:\Gamma\times T\to S$ by

$$\mathfrak{H}(\gamma,t):=\lim_{n\to\infty}\frac{1}{d^n}g\big(\gamma,d^nt\big)$$

for all $t \in T, \gamma \in \Gamma$. Putting $m = 0$ and letting $n \to \infty$ in (4.8), we get (4.6). We conclude from (4.1), (4.3), and (4.4) that

$$\begin{aligned}
&\mathcal{N}\left(2\mathfrak{H}\left(\gamma, \frac{\sum_{j=1}^{p} \eta t_j}{2} + \sum_{j=1}^{d} \eta s_j\right) - \sum_{j=1}^{p} \eta \mathfrak{H}(\gamma, t_j) - 2\sum_{j=1}^{d} \eta \mathfrak{H}(\gamma, s_j), \tau\right) \\
&\quad = \lim_{n\to\infty} \mathcal{N}(\frac{1}{d^n}\left(2g\left(\gamma, d^n \frac{\sum_{j=1}^{p} \eta t_j}{2} + d^n \sum_{j=1}^{d} \eta s_j\right)\right. \\
&\qquad \left. - \sum_{j=1}^{p} \eta g(\gamma, d^n t_j) - 2\sum_{j=1}^{d} \eta g(\gamma, d^n s_j), \tau\right) \\
&\quad \succeq \lim_{n\to\infty} \varphi\big(\big(d^n(t_1, \ldots, t_p, s_1, \ldots, s_d)\big), d^n \tau\big) \\
&\quad = \lim_{n\to\infty} \varphi\left((t_1, \ldots, t_p, s_1, \ldots, s_d), \frac{d^n}{d^{nq}} \tau\right) \\
&\quad = 1
\end{aligned}$$

for all $\eta \in \mathbb{T}^1$, $t_1, \ldots, t_p, s_1, \ldots, s_d \in T$, $\gamma \in \Gamma$, and $\tau \in \mathcal{G}^\circ$. Hence

$$2\mathfrak{H}\left(\gamma, \frac{\sum_{j=1}^{p} \eta t_j}{2} + \sum_{j=1}^{d} \eta s_j\right) = \sum_{j=1}^{p} \eta \mathfrak{H}(\gamma, t_j) + 2\sum_{j=1}^{d} \eta \mathfrak{H}(\gamma, s_j)$$

for all $\eta \in \mathbb{T}^1$ and all $t_1, \ldots, t_p, s_1, \ldots, s_d \in T$. Thus $\mathfrak{H}(\lambda t + \eta s) = \lambda \mathfrak{H}(\gamma, t) + \eta \mathfrak{H}(\gamma, s)$ for all $\lambda, \eta \in \mathbb{T}^1$ and all $t, s \in T$.

Therefore, from Lemma 4 the random operator $\mathfrak{H} : \Gamma \times T \to S$ is $\mathbb{C}$-linear.

We conclude from (4.2), (4.3), and (4.5) that

$$\begin{aligned}
&\mathcal{N}\big(\mathcal{H}(\gamma, [t, s, p]) - [\mathcal{H}(\gamma, t), \mathcal{H}(\gamma, s), \mathcal{H}(\gamma, p)], \tau\big) \\
&\quad = \lim_{n\to\infty} \mathcal{N}\left(\frac{1}{d^{3n}}\big(g(\gamma, [d^n t, d^n s, d^n p]) - [g(\gamma, d^n t), g(\gamma, d^n s), g(\gamma, d^n p)]\big), \tau\right) \\
&\quad = \lim_{n\to\infty} \mathcal{N}\big(g(\gamma, [d^n t, d^n s, d^n p]) - [g(\gamma, d^n t), g(\gamma, d^n s), g(\gamma, d^n p)], d^{3n} \tau\big) \\
&\quad \succeq \lim_{n\to\infty} \psi\big((d^n t, d^n s, d^n p), d^{3n} \tau\big) \\
&\quad = \lim_{n\to\infty} \psi\left((t, s, p), \frac{d^{3n}}{d^{n\sigma}} \tau\right) = 1
\end{aligned}$$

for all $t, s, p \in T, \gamma \in \Gamma$, and $\tau \in \mathcal{G}^\circ$. Thus

$$\mathcal{H}(\gamma, [t, s, p]) = [\mathcal{H}(\gamma, t), \mathcal{H}(\gamma, s), \mathcal{H}(\gamma, p)]$$

for all $t, s, p \in T$ and $\gamma \in \Gamma$.

Consider another generalized Cauchy–Jensen additive random operator $\mathcal{K} : \Gamma \times T \to S$ satisfying (4.6). Then we have

$$\mathcal{N}\big(\mathcal{H}(\gamma, t) - \mathcal{K}(\gamma, t), \tau\big) = \lim_{n\to\infty} \mathcal{N}\left(\frac{1}{d^n}\big(g(\gamma, d^n t) - \mathcal{K}(\gamma, d^n t)\big), \tau\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathcal{N}\bigl(g(\gamma,d^n t) - \mathcal{K}(\gamma,d^n t), d^n\tau\bigr)\\
&\succeq \lim_{n\to\infty} \varphi\bigl((\overbrace{0,\ldots,0}^{p},\overbrace{d^n t,\ldots,d^n t}^{d}), 2\tau d^n\bigl(d-d^q\bigr)\bigr)\\
&= \lim_{n\to\infty} \varphi\biggl((\overbrace{0,\ldots,0}^{p},\overbrace{t,\ldots,t}^{d}), \biggl(\frac{2\tau d^n(d-d^q)}{d^{nq}}\biggr)\biggr)\\
&= 1
\end{aligned}
$$

for all $t \in T, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Then $\mathcal{H}(\gamma,t) = \mathcal{K}(\gamma,t)$ for all $t \in T$. Thus the random operator $\mathcal{H} : \Gamma \times T \to S$ is a unique C^*-tash satisfying (4.6), as desired. □

Theorem 5 *Let $q < 1$ and $\sigma < 2$. Let $g : \Gamma \times T \to S$ be a random operator satisfying* (4.1), (4.2), (4.3), (4.4), *and* (4.5). *If there exist a real number $\lambda > 1 (0 < \lambda < 1)$ and an element $t_0 \in T$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} g(\gamma, \lambda^n t_0) = e' (\lim_{n\to\infty} \lambda^n g(\gamma, \frac{t_0}{\lambda^n}) = e')$ (identity element), then the random operator $g : \Gamma \times T \to S$ is a C^*-tash.*

Proof Applying Theorem 4, we get that there exists a unique C^*-tash $\mathcal{H} : \Gamma \times T \to S$ satisfying (4.6). Now,

$$
\mathcal{H}(\gamma,t) = \lim_{n\to\infty} \frac{1}{\lambda^n} g\bigl(\gamma,\lambda^n t\bigr), \quad \biggl(\mathcal{H}(\gamma,t) = \lim_{n\to\infty} \lambda^n g\biggl(\gamma,\frac{t}{\lambda^n}\biggr)\biggr) \tag{4.9}
$$

for all $t \in T$ and all real numbers $\lambda > 1 (0 < \lambda < 1)$. Therefore, from the assumption we get that $\mathcal{H}(\gamma,t_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} g(\gamma,\lambda^n t_0) = e'$. It follows from (4.5) and (4.9) that

$$
\begin{aligned}
&\mathcal{N}\bigl(\bigl[\mathcal{H}(\gamma,t),\mathcal{H}(\gamma,s),\mathcal{H}(\gamma,p)\bigr] - \bigl[\mathcal{H}(\gamma,t),\mathcal{H}(\gamma,s),g(\gamma,p)\bigr],\tau\bigr)\\
&\quad= \mathcal{N}\bigl(\mathcal{H}[\gamma,t,s,p] - \bigl[\mathcal{H}(\gamma,t),\mathcal{H}(\gamma,s),\mathcal{H}(\gamma,p)\bigr],\tau\bigr)\\
&\quad= \lim_{n\to\infty} \mathcal{N}\biggl(\frac{1}{\lambda^{2n}}\bigl(g\bigl(\bigl[\gamma,\lambda^n t,\lambda^n s,p\bigr]\bigr) - \bigl[g\bigl(\gamma,\lambda^n t\bigr),g\bigl(\lambda^n s\bigr),g(\gamma,z)\bigr]\bigr),\tau\biggr)\\
&\quad= \lim_{n\to\infty} \mathcal{N}\bigl(g\bigl(\bigl[\gamma,\lambda^n t,\lambda^n s,p\bigr]\bigr) - \bigl[g\bigl(\gamma,\lambda^n t\bigr),g\bigl(\gamma,\lambda^n s\bigr),g(\gamma,p)\bigr],\lambda^{2n}\tau\bigr)\\
&\quad\succeq \lim_{n\to\infty} \psi\bigl(\bigl(\lambda^t,\lambda^s,\lambda^p\bigr),\lambda^{2n}\tau\bigr)\\
&\quad= \lim_{n\to\infty} \psi\biggl((t,s,p),\frac{\lambda^{2n}}{\lambda^{2n\sigma}}\tau\biggr)\\
&\quad= 1
\end{aligned}
$$

for all $t \in T, \gamma \in \Gamma$ and $\tau \in \mathcal{G}^\circ$. Thus $[\mathcal{H}(\gamma,t),\mathcal{H}(\gamma,s),\mathcal{H}(\gamma,p)] = [\mathcal{H}(\gamma,t),\mathcal{H}(\gamma,s),g(\gamma,p)]$ for all $t,s,p \in T$. Letting $t = s = t_0$ in the last equality, we get $g(\gamma,t) = \mathcal{H}(\gamma,p)$ for all $p \in T$.

Similarly, one can show that $\mathcal{H}(\gamma,t) = g(\gamma,t)$ for all $t \in T$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \lambda^n g(\gamma,\frac{t_0}{\lambda^n}) = e'$. Therefore, the random operator $g : \Gamma \times T \to S$ is a C^*-tash. □

Theorem 6 *Let $q > 1$ and $\sigma > 3$. Let $g : \Gamma \times T \to S$ be a random operator satisfying* (4.4) *and* (4.5). *If there exist a real number $0 < \lambda < 1$ $(\lambda > 1)$ and an element $t_0 \in T$ such that*

$\lim_{n\to\infty}\frac{1}{\lambda^n}g(\gamma,\lambda^n t_0)=e'(\lim_{n\to\infty}\lambda^n g(\gamma,\frac{t_0}{\lambda^n})=e')$, *then the random operator* $g:\Gamma\times T\to S$ *is a* C^**-tash.*

Proof The proof is similar to the proof of Theorem 5, and so we omit it. □

Example 9 Consider q and σ such that $q<1$ and $\sigma<3$. Suppose that $g:\Gamma\times T\to S$ is a random operator with $g(\gamma,0)=0$ satisfying

$$\begin{aligned}&\mathcal{N}\big(\Xi_\eta g(\gamma,t_1,\ldots,t_p,s_1,\ldots,s_d),\tau\big)\\&\quad\succeq\operatorname{diag}\biggl[\frac{\tau}{\tau+(\sum_{j=1}^{p}\|t_j\|^q+\sum_{j=1}^{d}\|s_j\|^q)},\exp\biggl(-\frac{\sum_{j=1}^{p}\|t_j\|^q+\sum_{j=1}^{d}\|s_j\|^q}{\tau}\biggr)\biggr]\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}&\mathcal{N}\big(g(\gamma,[t,s,p])-\big[g(\gamma,t),g(\gamma,s),g(\gamma,p)\big],\tau\big)\\&\quad\succeq\operatorname{diag}\biggl[\frac{\tau}{\tau(\|t\|^q+\|s\|^q)},\exp\biggl(-\frac{\|t\|^q+\|s\|^q}{\tau}\biggr)\biggr]\end{aligned}\tag{4.11}$$

for all $\eta\in\mathbb{T}^1$ and all $t_1,\ldots,t_p,s_1,\ldots,s_d,t,s,p\in T$ and $\gamma\in\Gamma$ and $\tau\in\mathcal{G}^\circ$. Then there exists a unique C^*-tash $\mathfrak{H}:\Gamma\times T\to S$ such that

$$\mathcal{N}\big(g(\gamma,t)-\mathfrak{H}(\gamma,t),\tau\big)\succeq\operatorname{diag}\biggl[\frac{2\tau(d-d^q)}{2\tau(d-d^q)+(d\|t\|^q)},\exp\biggl(-\frac{d\|t\|^q}{2\tau(d-d^q)}\biggr)\biggr]\tag{4.12}$$

for all $t\in T,\gamma\in\Gamma$ and $\tau\in\mathcal{G}^\circ$.

To see this, put

$$\begin{aligned}&\varphi\big((t_1,\ldots,t_p,s_1,\ldots,s_d),\tau\big)\\&\quad=\operatorname{diag}\biggl[\frac{\tau}{\tau+(\sum_{j=1}^{p}\|t_j\|^q+\sum_{j=1}^{d}\|s_j\|^q)},\exp\biggl(-\frac{\sum_{j=1}^{p}\|t_j\|^q+\sum_{j=1}^{d}\|s_j\|^q}{\tau}\biggr)\biggr]\end{aligned}\tag{4.13}$$

and

$$\psi\big((t,s,p),\tau\big)=\operatorname{diag}\biggl[\frac{\tau}{\tau(\|t\|^q+\|s\|^q)},\exp\biggl(-\frac{\|t\|^q+\|s\|^q}{\tau}\biggr)\biggr]\tag{4.14}$$

for all $t_1,\ldots,t_p,s_1,\ldots,s_d,t,s,p\in T$ and $\gamma\in\Gamma$ and $\tau\in\mathcal{G}^\circ$. Now, applying Theorem 4, we get (4.12).

Example 10 Let $q<1$ and $\sigma<2$. Let $g:\Gamma\times T\to S$ be a random operator satisfying (4.10), (4.11). If there exist a real number $\lambda>1(0<\lambda<1)$ and an element $t_0\in T$ such that $\lim_{n\to\infty}\frac{1}{\lambda^n}g(\gamma,\lambda^n t_0)=e'(\lim_{n\to\infty}\lambda^n g(\gamma,\frac{t_0}{\lambda^n})=e')$ (identity element), then the random operator $g:\Gamma\times T\to S$ is a C^*-tash.

Define control functions φ and ψ as in (4.13) and (4.14). Theorem 5 guarantees the result.

5 Conclusion

In this paper we defined a new generalization of uncertain normed spaces by replacing the classical range by C^*-AV fuzzy sets and using triangular norms defined on the positive section of an order commutative C^*-algebra, named C^*-AVF-spaces. Also, by a super C^*-AVF controller, we considered Hyers–Ulam–Rassias stability of stochastic (Θ, Υ, Ξ)-derivations on C^*-AVF Lie C^*-algebras.

Acknowledgements
We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that helped to improve the quality of the manuscript.

Funding
This work was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

Availability of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Publisher's Note

Received: 22 November 2020 Accepted: 10 February 2021 Published online: 04 March 2021

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