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A new class of nonlinear Gronwall–Bellman delay integral inequalities with power and its applications

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Abstract

In this paper, we establish some new delay Gronwall–Bellman integral inequalities with power, which can be used as a convenient tool to study the qualitative properties of solutions to differential and integral equations. We also give some examples to illustrate the application of our results to obtain the estimation for the solution of the integral and differential equations.

Keywords: Delay; Integral; Inequalities; Gronwall–Bellman; Power; Nonlinear

1 Introduction

The field of differential equations has developed a perfect structure. Since the 20th century, inequality theory has been an active research field, a series of basic theories of inequalities have been also established [1–4]. Since for most differential equations it is difficult to find the exact form of expression, people turn to studying the qualitative nature of the solutions of differential equations, for example, the existence, uniqueness, asymptotic property, boundedness and vibration of solutions of differential equations and difference equations; inequalities have become important tools to study the qualitative properties of differential equations. In recent decades, related studies on integral inequalities have produced many results (see [5–26] and the references therein). In 1919, Gronwall [1] established the following important integral inequality for a continuous function u:

$$u(t) \leq c + \int_a^t f(\xi)u(\xi)\,d\xi.$$

In 1943, Bellman [2] obtained the estimation of the unknown function u for some constant $c \geq 0$,

$$u(t) \leq c\exp\left(\int_a^t f(\xi)\,d\xi\right).$$

In 1975, Pachpatte [3] studied the following integral inequality:

$$u(t) \le a(t) + g(t)\int_0^t f(\xi)u(\xi)\,d\xi,$$

where u, f, and g are real-valued nonnegative continuous functions defined on $I = [0,\infty)$, and $a(t)$ is a positive, monotonic, nondecreasing continuous function defined on I.

In 1999, Owaidy et al. [5] discussed the following inequality:

$$u(t) \le u_0 + \int_0^t f(\xi)\left(u^p(\xi) + \int_0^\xi g(\tau)u(\tau)\,d\tau\right)d\xi, \quad t \in [0,\infty),$$

where u, f and g are real-valued nonnegative continuous functions defined on $I = [0,\infty)$.

In recent years, the time-delay dynamic equation has attracted much interest, Lipovan et al. [6] assume $u, f, g \in C([t_0,T),R_+)$, and $\alpha \in C^1([t_0,T),(t_0,T))$ are nondecreasing with $\alpha(t) \le t$ on $[t_0,T)$, and $w \in C(R_+,R_+)$ are nondecreasing with $w(u) > 0$ for $u > 0$, then they studied the following retarded integral inequality in 2000:

$$u(t) \le a + \int_{t_0}^t f(\xi)w\big(u(\xi)\big)\,d\xi + \int_{\alpha(t_0)}^{\alpha(t)} g(\xi)w\big(u(\xi)\big)\,d\xi.$$

In 2005, Agarwal et al. [7] discussed the following n-term delay integral inequality:

$$u(t) \le a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} g_i(t,\xi)w_i(t,\xi)\,d\xi,$$

where u is a continuous and nonnegative function on $[t_0,t_1)$.

In 2011, Abdeldaim et al. [8] studied the following Gronwall–Bellman type inequality with power:

$$u(t) \le u_0 + \int_0^t f(\xi)u(\xi)\left[u(\xi) + \int_0^\xi g(\lambda)u(\lambda)\,d\lambda\right]^p d\xi,$$

where u, f and h are nonnegative real-valued continuous functions defined on $[0,\infty)$, and u_0 and p are positive constants.

In 2019, Li et al. [9] made the following improvement on the basis of the above inequality:

$$u(t) \le a(t) + \int_0^{\sigma(t)} f(\xi)u(\xi)\left[u^2(\xi) + \int_0^\xi g(\lambda)u(\lambda)\,d\lambda\right]^p d\xi,$$

where $u, a, f \in C(R_+,R_+)$, $a(t) \ge 1$, and α is a continuous, differentiable and increasing function on $[t_0,\infty)$ with $\alpha(t) \le t$, $\alpha(t_0) = t_0$.

In this paper, inspired by the above work, we mainly establish the following nonlinear Gronwall–Bellman inequalities:

$$u^q(t) \le a(t) + \int_{t_0}^{\sigma(t)} f(\xi)\left[u^\alpha(\xi) + \int_{t_0}^\xi h(\lambda)u^\beta(\lambda)\,d\lambda\right]^p d\xi, \tag{1}$$

$$u(t) \le u_0 + \int_{t_0}^{\sigma(t)} g(\xi)u^r(\xi)\left[u^m(\xi) + \int_{t_0}^\xi h(\lambda)u^m(\lambda)\,d\lambda\right]^p d\xi, \quad u_0 > 0, \tag{2}$$

$$\varphi(u(t)) \leq a(t) + \int_{t_0}^{t} g_1(t,\xi)h_1(u(\xi))\,d\xi + \int_{t_0}^{t} g_2(t,\xi)h_2(u(\xi))\,d\xi + \int_{t_0}^{t} g_3(t,\xi)h_3(u(\xi))\,d\xi + \int_{t_0}^{t} g(t,\xi)\left(\int_{t_0}^{\xi} f(\xi,\theta)h_4(u(\theta))\,d\theta\right)d\xi, \quad t_0 \geq 0. \tag{3}$$

The structure of this paper is as follows: In Sect. 2, we illustrates some basic lemmas, which will be used in later sections. In Sect. 3, we give some new nonlinear Gronwall–Bellman inequalities. In Sect. 4, we give two examples to illustrate the application of the obtained results in the qualitative research of differential equation solutions. In Sect. 5, we conclude our results.

2 Preliminaries

First, we explain some symbols will to be used: **R** denotes the set of real numbers and $\mathbf{R}_+ = [0,\infty)$, and $C(M,S)$ denotes the class of all continuous functions on the set M with range in the set S.

Here are some very useful lemmas.

Lemma 2.1 ([11]) *Assume* $a \geq 0$, $p \geq q \geq 0$ *and* $p \neq 0$, *we have*

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}, \quad K > 0. \tag{4}$$

We can get the following exceptional cases.

Let $K = 1$, *we have*

$$a^{\frac{q}{p}} \leq \frac{q}{p}a + \frac{p-q}{p}, \quad a \geq 0, p \geq q > 0. \tag{5}$$

Let $K = 1$, $p = 1$, *we have*

$$a^{q} \leq qa + (1-q), \quad a \geq 0, 0 < q \leq 1. \tag{6}$$

Lemma 2.2 *Let* $u,g,\sigma \in C(R_+,R_+)$, $\sigma'(t) \geq 0$ *and* $\sigma(t) \leq t$, $\sigma(t_0) = t_0$, $r \in (0,1]$, $u_0 > 0$. *If* $u(t)$ *satisfies the inequality*

$$u(t) \leq u_0 + \int_{t_0}^{\sigma(t)} g(\xi)u^r(\xi)\,d\xi, \tag{7}$$

then

$$u(t) \leq \exp\left(r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi\right)\left[u_0 + \int_{t_0}^{\sigma(t)} (1-r)g(\xi)\exp\left(-r\int_{t_0}^{\xi} g(\lambda)\,d\lambda\right)d\xi\right]. \tag{8}$$

Proof We assume that

$$v(t) = u_0 + \int_{t_0}^{t} g(\xi)u^r(\xi)\,d\xi,$$

then

$$v(\sigma(t)) = u_0 + \int_{t_0}^{\sigma(t)} g(\xi)u^r(\xi)\,d\xi, \tag{9}$$

and $u(t) \le v(\sigma(t)) \le v(t)$, $v(\sigma(t_0)) = u_0$. Differentiating with respect to t of (9) and using (6), we get

$$\begin{aligned}\sigma'(t)v'(\sigma(t)) &= \sigma'(t)g(\sigma(t))u^r(\sigma(t))\\ &\le \sigma'(t)g(\sigma(t))v^r(\sigma(t))\\ &\le \sigma'(t)g(\sigma(t))\bigl[rv(\sigma(t)) + (1-r)\bigr],\end{aligned}$$

then

$$\sigma'(t)v'(\sigma(t)) - r\sigma'(t)g(\sigma(t))v(\sigma(t)) \le (1-r)\sigma'(t)g(\sigma(t)).$$

Multiplying by $\exp(-r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi)$ on both sides of the above inequality, we can get

$$\left[v(\sigma(t))\exp\left(-r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi\right)\right]' \le (1-r)\sigma'(t)g(\sigma(t))\exp\left(-r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi\right).$$

Next, integrating t from t_0 to t for the above inequality, we get

$$\begin{aligned}&v(\sigma(t))\exp\left(-r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi\right) - v(\sigma(t_0))\\ &\quad\le \int_{t_0}^{t} (1-r)\sigma'(\xi)g(\sigma(\xi))\exp\left(-r\int_{t_0}^{\sigma(\xi)} g(\lambda)\,d\lambda\right)d\xi\\ &\quad\le \int_{t_0}^{\sigma(t)} (1-r)g(\xi)\exp\left(-r\int_{t_0}^{\xi} g(\lambda)\,d\lambda\right)d\xi.\end{aligned}$$

Since $v(\sigma(t_0)) = u_0$, we can get the estimation

$$\begin{aligned}u(t) &\le v(\sigma(t))\\ &\le \exp\left(r\int_{t_0}^{\sigma(t)} g(\xi)\,d\xi\right)\left[u_0 + \int_{t_0}^{\sigma(t)} (1-r)g(\xi)\exp\left(-r\int_{t_0}^{\xi} g(\lambda)\,d\lambda\right)d\xi\right].\end{aligned}$$

This completes the proof. □

3 Main result and proof

In this section, we establish and prove a new class of nonlinear Gronwall–Bellman type delay integral inequalities with power.

Theorem 3.1 *We assume* $u, a, f, h \in C(R_+, R_+)$, *and let* $\sigma(t) \in C[t_0, \infty)$, $\sigma'(t) \ge 0$ *and* $\sigma(t) \le t$, $\sigma(t_0) = t_0$. $q \ge \alpha > 0$, $q \ge \beta > 0$, $p \in (0, 1]$ *and* $q \ge p$. *If u satisfies the inequal-*

ity (1), then we can get

$$u(t) \le \left[a(t) + B(t)\exp\left(\int_{t_0}^{\sigma(t)} \frac{p\alpha}{q} f(\xi)\,d\xi + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q} f(\xi)\left(\int_{t_0}^{\xi} h(\lambda)\,d\lambda\right)d\xi\right)\right]^{\frac{1}{q}}, \quad \forall t \in R_+, \tag{10}$$

where

$$B(t) = \int_{t_0}^{\sigma(t)} f(\xi)\left((1-p) + \frac{p\alpha}{q}a(\xi) + \frac{qp-p\alpha}{q}\right)d\xi + \int_{t_0}^{\sigma(t)} f(\xi)\left[\int_{t_0}^{\xi} h(\lambda)\left(\frac{p\alpha}{q}a(\lambda) + \frac{qp-p\beta}{q}\right)d\lambda\right]d\xi.$$

Proof Using (6), we have

$$\left[u^{\alpha}(s) + \int_{t_0}^{s} h(\lambda)u^{\beta}(\lambda)\,d\lambda\right]^{p} \le p\left[u^{\alpha}(s) + \int_{t_0}^{s} h(\lambda)u^{\beta}(\lambda)\,d\lambda\right] + (1-p). \tag{11}$$

Plugging (11) into (1), we can get

$$u^{q}(t) \le a(t) + \int_{t_0}^{\sigma(t)} f(\xi)\left[p\left(u^{\alpha}(\xi) + \int_{t_0}^{\xi} h(\lambda)u^{\beta}(\lambda)\,d\lambda\right) + (1-p)\right]d\xi. \tag{12}$$

Now, we define $v(t)$ by

$$v(t) = \int_{t_0}^{\sigma(t)} f(\xi)\left[p\left(u^{\alpha}(\xi) + \int_{t_0}^{\xi} h(\lambda)u^{\beta}(\lambda)\,d\lambda\right) + (1-p)\right]d\xi, \tag{13}$$

then $v(t)$ is a nondecreasing function, using (12) and (13), we obtain

$$u(t) \le \left[a(t) + v(t)\right]^{\frac{1}{q}}.$$

Using (5), from the above inequality we get

$$u^{\alpha}(t) \le \left(a(t) + v(t)\right)^{\frac{\alpha}{q}} \le \frac{\alpha}{q}\left(a(t) + v(t)\right) + \left(1 - \frac{\alpha}{q}\right) \tag{14}$$

and

$$u^{\beta}(t) \le \left(a(t) + v(t)\right)^{\frac{\beta}{q}} \le \frac{\beta}{q}\left(a(t) + v(t)\right) + \left(1 - \frac{\beta}{q}\right). \tag{15}$$

Plugging (14) and (15) into (13), we can obtain

$$v(t) \le \int_{t_0}^{\sigma(t)} f(\xi)\left[p\left(\frac{\alpha}{q}\left(a(\xi) + v(\xi)\right) + \left(1 - \frac{\alpha}{q}\right) + \int_{t_0}^{\xi} h(\lambda)\left(\frac{\beta}{q}\left(a(\lambda) + v(\lambda)\right) + \left(1 - \frac{\beta}{q}\right)\right)d\lambda\right) + (1-p)\right]d\xi$$

$$
\begin{aligned}
&\le \int_{t_0}^{\sigma(t)} f(\xi)\left((1-p)+\frac{p\alpha}{q}a(\xi)+\frac{qp-p\alpha}{q}\right)d\xi \\
&\quad + \int_{t_0}^{\sigma(t)} f(\xi)\left[\int_{t_0}^{\xi} h(\lambda)\left(\frac{p\alpha}{q}a(\lambda)+\frac{qp-p\beta}{q}\right)d\lambda\right]d\xi \\
&\quad + \int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(\xi)v(\xi)\,d\xi + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(\xi)\left(\int_{t_0}^{\xi} h(\lambda)v(\lambda)\,d\lambda\right)d\xi \\
&\le B(t) + \int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(\xi)v(\xi)\,d\xi + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(\xi)\left(\int_{t_0}^{\xi} h(\lambda)v(\lambda)\,d\lambda\right)d\xi \\
&\le B(T) + \int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(\xi)v(\xi)\,d\xi + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(\xi)\left(\int_{t_0}^{\xi} h(\lambda)v(\lambda)\,d\lambda\right)d\xi,
\end{aligned}
$$

where $t \in [t_0, T]$, $T \in R_+$, and

$$
\begin{aligned}
B(t) = {}& \int_{t_0}^{\sigma(t)} f(\xi)\left((1-p)+\frac{p\alpha}{q}a(\xi)+\frac{qp-p\alpha}{q}\right)d\xi \\
&+ \int_{t_0}^{\sigma(t)} f(\xi)\left[\int_{t_0}^{\xi} h(\lambda)\left(\frac{p\alpha}{q}a(\lambda)+\frac{qp-p\beta}{q}\right)d\lambda\right]d\xi.
\end{aligned}
$$

Let

$$
y(t) = B(T) + \int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(\xi)v(\xi)\,d\xi + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(\xi)\left(\int_{t_0}^{\xi} h(\lambda)v(\lambda)\,d\lambda\right)d\xi.
$$

Then we can get $y(t)$ is a nondecreasing and positive function, and $v(t) \le y(t)$, $y(t_0) = B(T)$. Differentiating $y(t)$ with respect to t and using $\sigma(t) \le t$, we have

$$
\begin{aligned}
y'(t) &= \frac{p\alpha}{q}\sigma'(t)f\bigl(\sigma(t)\bigr)v\bigl(\sigma(t)\bigr) + \frac{p\beta}{q}\sigma'(t)f\bigl(\sigma(t)\bigr)\int_{t_0}^{\sigma(t)} h(\xi)v(\xi)\,d\xi \\
&\le v(t)\left(\frac{p\alpha}{q}\sigma'(t)f\bigl(\sigma(t)\bigr) + \frac{p\beta}{q}\sigma'(t)f\bigl(\sigma(t)\bigr)\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi\right) \\
&\le y(t)\left(\frac{p\alpha}{q}\sigma'(t)f\bigl(\sigma(t)\bigr) + \frac{p\beta}{q}\sigma'(t)f\bigl(\sigma(t)\bigr)\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi\right).
\end{aligned}
$$

From the above inequality we get

$$
\frac{y'(t)}{y(t)} \le \frac{p\alpha}{q}\sigma'(t)f\bigl(\sigma(t)\bigr) + \frac{p\beta}{q}\sigma'(t)f\bigl(\sigma(t)\bigr)\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi.
$$

Integrating both side of the above inequality from t_0 to t, then we can obtain the estimation for $y(t)$:

$$
y(t) \le B(T)\exp\left(\int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(s)\,ds + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(s)\left(\int_{t_0}^{s} h(\lambda)\,d\lambda\right)ds\right),
$$

by $v(t) \le y(t)$ and $u(t) \le [a(t)+v(t)]^{\frac{1}{q}}$, we can obtain

$$
u(t) \le \left[a(t) + B(T)\exp\left(\int_{t_0}^{\sigma(t)} \frac{p\alpha}{q}f(s)\,ds + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q}f(s)\left(\int_{t_0}^{s} h(\lambda)\,d\lambda\right)ds\right)\right]^{\frac{1}{q}}.
$$

Thus

$$u(T) \le \left[a(T) + B(T)\exp\left(\int_{t_0}^{\sigma(t)} \frac{p\alpha}{q} f(s)\,ds + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q} f(s)\left(\int_{t_0}^{s} h(\lambda)\,d\lambda\right)ds\right)\right]^{\frac{1}{q}}.$$

Because of the arbitrariness of T, we can obtain

$$u(t) \le \left[a(t) + B(t)\exp\left(\int_{t_0}^{\sigma(t)} \frac{p\alpha}{q} f(s)\,ds + \int_{t_0}^{\sigma(t)} \frac{p\beta}{q} f(s)\left(\int_{t_0}^{s} h(\lambda)\,d\lambda\right)ds\right)\right]^{\frac{1}{q}}.$$

The proof is complete. □

Remark 1 If $q = 1$, Theorem 3.1 reduces to Theorem 2.1 in [9]. If $q = p$, $a(t) = x_0$, $\sigma(t) = t$, $p = \beta = 1$, $\alpha = q$, Theorem 3.1 reduces to Theorem 2.3 in [10]. If $q = 1$, $a(t) = x_0$, $\sigma(t) = t$, $p = \beta = 1$, $\alpha = 2 - p$, $\beta = q$, Theorem 3.1 reduces to Theorem 2.5 in [10]. If $q = 1$, $a(t) = x_0$, $\sigma(t) = t$, $\alpha = p$, $\beta = 2p - 1$, Theorem 3.1 reduces to Theorem 2.8 in [10].

Theorem 3.2 *We assume* $u, g, h \in C(R_+, R_+)$, $\sigma(t) \in [t_0, \infty)$, $\sigma'(t) \ge 0$, $\sigma(t) \le t$, $\sigma(t_0) = t_0$, $r \in (0, 1]$, $p > 1$, $m > 1$. *If u satisfies the inequality* (2), *then*

$$u(t) \le \exp\left(r\int_{t_0}^{\sigma(t)} g(\xi)\beta(\xi)\,d\xi\right) \times \left[u_0 + \int_{t_0}^{\sigma(t)} (1-r)g(\xi)\beta(\xi)\exp\left(-r\int_{t_0}^{\xi} g(\lambda)\beta(\lambda)\,d\lambda\right)d\xi\right], \tag{16}$$

where

$$\beta(t) = \left[\frac{\Omega(t)}{1 + (1 - r - mp)\int_{t_0}^{t} g(\xi)\Omega(\xi)\,d\xi}\right]^{\frac{mp}{mp+r-1}},$$

$$\Omega(t) = u_0^{r+mp-1}\exp\left(\frac{r + mp - 1}{m}\int_{t_0}^{t} h(\xi)\,d\xi\right),$$

$$1 + (1 - r - mp)\int_{t_0}^{\sigma(t)} g(\xi)\Omega(\xi)\,d\xi > 0.$$

Proof First, we denote

$$J(t) = u_0 + \int_{t_0}^{t} g(\xi)u^r(\xi)\left[u^m(\xi) + \int_{t_0}^{\xi} h(\lambda)u^m(\lambda)\,d\lambda\right]^p d\xi,$$

and $J(t)$ is a nondecreasing and nonnegative continuous function, then

$$J(\sigma(t)) = u_0 + \int_{t_0}^{\sigma(t)} g(\xi)u^r(\xi)\left[u^m(\xi) + \int_{t_0}^{\xi} h(\lambda)u^m(\lambda)\,d\lambda\right]^p d\xi, \tag{17}$$

and $u(t) \leq J(\sigma(t)) \leq J(t)$, $J(\sigma(t_0)) = J(t_0) = u_0$. Differentiating with respect to t of the above equation, we get

$$\begin{aligned}\sigma'(t)\frac{dJ(\sigma(t))}{d\sigma} &= \sigma'(t)g\bigl(\sigma(t)\bigr)u^r\bigl(\sigma(t)\bigr)\biggl[u^m\bigl(\sigma(t)\bigr) + \int_{t_0}^{\sigma(t)} h(\lambda)u^m(\lambda)\,d\lambda\biggr]^p \\ &\leq \sigma'(t)g\bigl(\sigma(t)\bigr)J^r\bigl(\sigma(t)\bigr)\biggl[J^m\bigl(\sigma(t)\bigr) + \int_{t_0}^{\sigma(t)} h(\lambda)J^m(\lambda)\,d\lambda\biggr]^p \\ &= \sigma'(t)g\bigl(\sigma(t)\bigr)J^r\bigl(\sigma(t)\bigr)Y^p\bigl(\sigma(t)\bigr),\end{aligned}$$

which means

$$\frac{dJ(\sigma(t))}{d\sigma} \leq g\bigl(\sigma(t)\bigr)J^r\bigl(\sigma(t)\bigr)Y^p\bigl(\sigma(t)\bigr), \tag{18}$$

where $Y(t) = J^m(t) + \int_{t_0}^t h(\lambda)J^m(\lambda)\,d\lambda$, then $Y(\sigma(t)) = J^m(\sigma(t)) + \int_{t_0}^{\sigma(t)} h(\lambda)J^m(\lambda)\,d\lambda$, hence $Y(\sigma(t_0)) = J^m(\sigma(t_0)) = J^m(t_0) = {u_0}^m$, we can conclude that

$$J\bigl(\sigma(t)\bigr) \leq Y^{\frac{1}{m}}\bigl(\sigma(t)\bigr).$$

Differentiating with respect to t of $Y(\sigma(t))$, we get

$$\begin{aligned}\sigma'(t)\frac{dY(\sigma(t))}{d\sigma} &= m\sigma'(t)J^{m-1}\bigl(\sigma(t)\bigr)\frac{dJ(\sigma(t))}{d\sigma} + \sigma'(t)h\bigl(\sigma(t)\bigr)J^m\bigl(\sigma(t)\bigr) \\ &\leq m\sigma'(t)J^{m-1}\bigl(\sigma(t)\bigr)\bigl[g\bigl(\sigma(t)\bigr)J^r\bigl(\sigma(t)\bigr)Y^p\bigl(\sigma(t)\bigr)\bigr] + \sigma'(t)h\bigl(\sigma(t)\bigr)J^m\bigl(\sigma(t)\bigr) \\ &\leq m\sigma'(t)Y^{\frac{m-1}{m}}\bigl(\sigma(t)\bigr)\bigl[g\bigl(\sigma(t)\bigr)Y^{\frac{r+mp}{m}}\bigl(\sigma(t)\bigr)\bigr] + \sigma'(t)h\bigl(\sigma(t)\bigr)Y\bigl(\sigma(t)\bigr),\end{aligned}$$

then

$$\frac{dY(\sigma(t))}{d\sigma} \leq mg\bigl(\sigma(t)\bigr)Y^{\frac{r+mp+m-1}{m}}\bigl(\sigma(t)\bigr) + h\bigl(\sigma(t)\bigr)Y\bigl(\sigma(t)\bigr),$$

from the above inequality, we can get

$$Y^{\frac{1-r-mp-m}{m}}\bigl(\sigma(t)\bigr)\frac{dY(\sigma(t))}{d\sigma} - h\bigl(\sigma(t)\bigr)Y^{\frac{1-r-mp}{m}}\bigl(\sigma(t)\bigr) \leq mg\bigl(\sigma(t)\bigr). \tag{19}$$

Denote

$$\Gamma(t) = Y^{\frac{1-r-mp}{m}}(t),$$

then $\Gamma(\sigma(t)) = Y^{\frac{1-r-mp}{m}}(\sigma(t))$, $\frac{d\Gamma(\sigma(t))}{dt} = \frac{1-r-mp}{m}Y^{\frac{1-m-r-mp}{m}}(\sigma(t))\sigma'(t)\frac{dY(\sigma(t))}{d\sigma}$, and $\Gamma(\sigma(t_0)) = Y^{\frac{1-r-mp}{m}}(\sigma(t_0)) = u_0^{1-r-mp}$, from $1 - r - mp < 0$ and (19), we have

$$\frac{d\Gamma(\sigma(t))}{dt} - \frac{1-r-mp}{m}\sigma'(t)\Gamma\bigl(\sigma(t)\bigr)h\bigl(\sigma(t)\bigr) \geq (1-r-mp)\sigma'(t)g\bigl(\sigma(t)\bigr),$$

Multiplying by $\exp(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi)$ on both sides of the above inequality, we get

$$\begin{aligned}&\left[\Gamma\big(\sigma(t)\big)\exp\left(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi\right)\right]'\\&\quad\geq(1-r-mp)\sigma'(t)g\big(\sigma(t)\big)\exp\left(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi\right).\end{aligned}$$

Integrating both sides of the above inequality from t_0 to t, we can get

$$\begin{aligned}&\Gamma\big(\sigma(t)\big)\exp\left(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi\right)-\Gamma\big(\sigma(t_0)\big)\\&\quad\geq(1-r-mp)\int_{t_0}^{t}\sigma'(\xi)g\big(\sigma(\xi)\big)\exp\left(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(\xi)} h(\lambda)\,d\lambda\right)d\xi\\&\quad=(1-r-mp)\int_{t_0}^{\sigma(t)} g(\xi)\exp\left(\frac{r+mp-1}{m}\int_{t_0}^{\xi} h(\lambda)\,d\lambda\right)d\xi.\end{aligned}$$

Since $\Gamma(\sigma(t_0))=Y^{\frac{1-r-mp}{m}}(\sigma(t_0))=u_0^{1-r-mp}$, we get

$$\Gamma\big(\sigma(t)\big)\geq\frac{u_0^{1-r-mp}+(1-r-mp)\int_{t_0}^{\sigma(t)} g(\xi)\exp(\frac{r+mp-1}{m}\int_{t_0}^{\xi} h(\lambda)\,d\lambda)\,d\xi}{\exp(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi)},$$

which means

$$\Gamma\big(\sigma(t)\big)\geq\frac{1+(1-r-mp)u_0^{r+mp-1}\int_{t_0}^{\sigma(t)} g(\xi)\exp(\frac{r+mp-1}{m}\int_{t_0}^{\xi} h(\lambda)\,d\lambda)\,d\xi}{u_0^{r+mp-1}\exp(\frac{r+mp-1}{m}\int_{t_0}^{\sigma(t)} h(\xi)\,d\xi)}.$$

Let $\Omega(t)=u_0^{r+mp-1}\exp(\frac{r+mp-1}{m}\int_{t_0}^{t} h(\xi)\,d\xi)$, using $\Gamma(\sigma(t))=Y^{\frac{1-r-mp}{m}}(\sigma(t))$, we can get

$$Y^p\big(\sigma(t)\big)\leq\left[\frac{\Omega(\sigma(t))}{1+(1-r-mp)\int_{t_0}^{\sigma(t)} g(\xi)\Omega(\xi)\,d\xi}\right]^{\frac{mp}{mp+r-1}},$$

where $1+(1-r-mp)\int_{t_0}^{\sigma(t)} g(\xi)\Omega(\xi)\,d\xi>0$.

By the definition of $\beta(t)$, plugging the above inequality into (18), we can get

$$\frac{dJ(\sigma(t))}{d\sigma}\leq g\big(\sigma(t)\big)\beta\big(\sigma(t)\big)J^r\big(\sigma(t)\big).$$

Integrating both sides of the above inequality from t to t_0, we get

$$J\big(\sigma(t)\big)\leq u_0+\int_{t_0}^{\sigma(t)} g(\xi)\beta(\xi)J^r(\xi)\,d\xi.$$

Therefore, from Lemma 2.2 we can get

$$\begin{aligned} u(t) &\le J\big(\sigma(t)\big) \\ &\le \exp\left(r\int_{t_0}^{\sigma(t)} g(\xi)\beta(\xi)\,d\xi\right) \\ &\quad \times \left[u_0 + \int_{t_0}^{\sigma(t)} (1-r)g(\xi)\beta(\xi)\exp\left(-r\int_{t_0}^{\xi} g(\lambda)\beta(\lambda)\,d\lambda\right)d\xi\right]. \end{aligned}$$

The proof is completed. □

Remark 2 If $u_0 = x_0$, $\sigma(t) = t$, and $r = m = 1$, Theorem 3.2 reduces to Theorem 3.2 in [8].

In the following, we discuss the inequality (3). First we assume that the following conditions are satisfied;

(C_1) $\varphi(u)$ is a positive continuous and strictly increasing function on $[0,\infty)$.

(C_2) $h_j(u)$, $(j = 1,2,3,4)$ are positive, continuous and increasing functions on $[0,\infty)$, and $\frac{h_{j+1}(t)}{h_j(t)}$, $(j = 1,2,3)$ are nondecreasing functions. Moreover, let

$$y_j(t) = \frac{h_j(t)}{h_1(t)}, \quad j = 1,2,3,4, \tag{20}$$

thus $y_j(t)$ are nondecreasing functions, $y_1(t) = 1$ and

$$\frac{y_{j+1}(t)}{y_j(t)} = \frac{h_{j+1}(t)}{h_j(t)}, \quad j = 1,2,3, \tag{21}$$

then $\frac{y_{j+1}(t)}{y_j(t)}$ are nondecreasing, positive and continuous functions.

(C_3) We define the following functions:

$$H_j(u) = \int_1^u \frac{d\xi}{h_j(\varphi^{-1}(\xi))}, \quad j = 1,2,3,4. \tag{22}$$

Then H_j are positive continuous and strictly increasing functions on $[0,\infty)$. We assume that H_j^{-1} define the inverse function of H_j, which are also continuous nondecreasing functions.

(C_4) $a(t)$ is a continuous function on $[t_0,\infty)$, $a(t) \ge 0$, $a(t_0) \ne 0$, and $g_j(t,\xi)$ $(j = 1,2,3)$ and $f(t,\xi)$ are continuous functions on $[t_0,\infty) \times [t_0,\infty)$.

(C_5) We assume that $g(t,\xi)$, $f(t,\xi)$ are nondecreasing and continuous functions on $[t_0,\infty) \times [t_0,\infty)$, and

$$g_4(t,\xi) = \int_\xi^t g(t,\theta)f(\theta,\xi)\,d\theta, \quad t,\xi \in [t_0,\infty).$$

(C_6) $a(t) + \sum_{j=1}^4 g_j(t,\xi)h_j(u(\xi)) > 0$.

Theorem 3.3 *Suppose the conditions* (C_1)–(C_6) *are satisfied, u is a positive and continuous function on $t \ge t_0 \ge 0$, if u satisfies* (3), *we can get the following estimation for u*:

$$u(t) \le \varphi^{-1}\big(H_4^{-1}\big(A_5(t)\big)\big), \quad t \in [t_0,\infty),$$

where

$$\begin{aligned} &A_1(t) = a(t), \\ &A_2(t) = H_1\big(a(t)\big) + \int_{t_0}^{t} g_1(t,\xi)\,d\xi, \\ &A_{j+1}(t) = H_j\big(H_{j-1}^{-1}\big(A_j(t)\big)\big) + \int_{t_0}^{t} g_j(t,\xi)\,d\xi, \quad j = 2,3,4. \end{aligned} \tag{23}$$

Proof Since $g(t,\xi), f(t,\xi), h_4(u(t))$ are nondecreasing and continuous functions, by (C_5), we can get

$$\begin{aligned} &\int_{t_0}^{t} g(t,\xi)\left(\int_{t_0}^{\xi} f(\xi,\theta)h_4\big(u(\theta)\big)\,d\theta\right)d\xi \\ &\quad = \int_{t_0}^{t} h_4\big(u(\theta)\big)\int_{\theta}^{t} g(t,\xi)f(\xi,\theta)\,d\xi\,d\theta \\ &\quad = \int_{t_0}^{t} h_4\big(u(\xi)\big)\int_{\xi}^{t} g(t,\theta)f(\theta,\xi)\,d\theta d\xi \\ &\quad = \int_{t_0}^{t} g_4(t,\xi)h_4\big(u(\xi)\big)\,d\xi, \end{aligned}$$

where the first equality is obtained by swapping the order of double integral, the second equality is obtained by $\theta = \xi$, the third equality is a simplification of the above equation obtained by (C_5). Plugging (24) into (3), we can write (3) as

$$\varphi\big(u(t)\big) \le a(t) + \sum_{j=1}^{4}\int_{t_0}^{t} g_j(t,\xi)h_j\big(u(\xi)\big)\,d\xi.$$

For any fixed $T \in [t_0,\infty)$ and for $t \in [t_0,T]$, from the above inequality, we have

$$\varphi\big(u(t)\big) \le a(t) + \sum_{j=1}^{4}\int_{t_0}^{t} g_j(T,\xi)h_j\big(u(\xi)\big)\,d\xi. \tag{24}$$

We assume that

$$z_1(t) = a(t) + \sum_{j=1}^{4}\int_{t_0}^{t} g_j(T,\xi)h_j\big(u(\xi)\big)\,d\xi, \tag{25}$$

thus $z_1(t)$ is a nondecreasing and nonnegative continuous function, and we have $\varphi(u(t)) \le z_1(t)$, $u(t) \le \varphi^{-1}(z_1(t))$, $z_1(t_0) = a(t_0)$, $z_1(t) \ge a(t)$. We can take the derivative with respect to t in (25), then

$$z_1'(t) = a'(t) + \sum_{j=1}^{4} g_j(T,t)h_j\big(u(t)\big).$$

Multiplying both sides of the above inequality by $\frac{1}{h_1(\varphi^{-1}(z_1(t)))}$, meanwhile using (20), we have

$$\begin{aligned}\frac{z_1'(t)}{h_1(\varphi^{-1}(z_1(t)))} &= \frac{a'(t)+\sum_{j=1}^4 h_j(u(t))g_j(T,t)}{h_1(\varphi^{-1}(z_1(t)))} \\ &\le \frac{a'(t)+\sum_{j=1}^4 h_j(\varphi^{-1}(z_1(t)))g_j(T,t)}{h_1(\varphi^{-1}(z_1(t)))} \\ &= \frac{a'(t)}{h_1(\varphi^{-1}(z_1(t)))}+g_1(T,t)+\frac{\sum_{j=2}^4 h_j(\varphi^{-1}(z_1(t)))g_j(T,t)}{h_1(\varphi^{-1}(z_1(t)))} \\ &\le \frac{a'(t)}{h_1(\varphi^{-1}(a(t)))}+g_1(T,t)+\sum_{j=1}^{3} g_{j+1}(T,t)y_{j+1}\bigl(\varphi^{-1}\bigl(z_1(t)\bigr)\bigr),\end{aligned}$$

integrating both sides of the above inequality from t_0 to t, and using the definition of (22), we obtain

$$\begin{aligned}H_1\bigl(z_1(t)\bigr)-H_1\bigl(z_1(t_0)\bigr) \le{}& H_1\bigl(a(t)\bigr)-H_1\bigl(a(t_0)\bigr)+\int_{t_0}^{t} g_1(T,\xi)\,d\xi \\ &+\sum_{j=1}^{3}\int_{t_0}^{t} g_{j+1}(T,\xi)y_{j+1}\bigl(\varphi^{-1}\bigl(z_1(\xi)\bigr)\bigr)\,d\xi,\end{aligned}$$

which means

$$H_1\bigl(z_1(t)\bigr) \le H_1\bigl(a(t)\bigr)+\int_{t_0}^{t} g_1(T,\xi)\,d\xi+\sum_{j=1}^{3}\int_{t_0}^{t} g_{j+1}(T,\xi)y_{j+1}\bigl(\varphi^{-1}\bigl(z_1(\xi)\bigr)\bigr)\,d\xi. \tag{26}$$

We assume that

$$\eta_1(t)=H_1\bigl(z_1(t)\bigr) \tag{27}$$

and

$$A_2(t)=H_1\bigl(A_1(t)\bigr)+\int_{t_0}^{t} g_1(T,\xi)\,d\xi, \tag{28}$$

from (27) and (28), (26) can be written as

$$\begin{aligned}\eta_1(t) &\le A_2(t)+\sum_{j=1}^{3}\int_{t_0}^{t} g_{j+1}(T,\xi)y_{j+1}\bigl(\varphi^{-1}\bigl(z_1(\xi)\bigr)\bigr)\,d\xi \\ &= A_2(t)+\sum_{j=1}^{3}\int_{t_0}^{t} g_{j+1}(T,\xi)y_{j+1}\bigl(\varphi^{-1}\bigl(H_1^{-1}\bigl(\eta_1(\xi)\bigr)\bigr)\bigr)\,d\xi.\end{aligned}$$

Then we assume that

$$z_2(t)=A_2(t)+\sum_{j=1}^{3}\int_{t_0}^{t} g_{j+1}(T,\xi)y_{j+1}\bigl(\varphi^{-1}\bigl(H_1^{-1}\bigl(\eta_1(\xi)\bigr)\bigr)\bigr)\,d\xi, \tag{29}$$

thus $z_2(t)$ is a nondecreasing and continuous function, and $\eta_1(t) \le z_2(t)$, $z_2(t_0) = A_2(t_0)$, $A_2(t) \le z_2(t)$.

We define a function as

$$Y_{j+1}(u) = \int_0^u \frac{y_j(\varphi^{-1}(H_j^{-1}(\xi)))}{y_{j+1}(\varphi^{-1}(H_j^{-1}(\xi)))}\,d\xi, \quad j = 1,2,3, \tag{30}$$

then, by (21) and (30), we can obtain

$$\begin{aligned} Y_{j+1}(u) &= \int_0^u \frac{y_j(\varphi^{-1}(H_j^{-1}(\xi)))}{y_{j+1}(\varphi^{-1}(H_j^{-1}(\xi)))}\,d\xi \\ &= \int_0^u \frac{h_j(\varphi^{-1}(H_j^{-1}(\xi)))}{h_{j+1}(\varphi^{-1}(H_j^{-1}(\xi)))}\,d\xi \\ &= \int_{H_j^{-1}(0)}^{H_j^{-1}(u)} \frac{h_j(\varphi^{-1}(t))}{h_{j+1}(\varphi^{-1}(t))}\bigl(H_j(t)\bigr)'\,dt \\ &= \int_1^{H_j^{-1}(u)} \frac{1}{h_{j+1}(\varphi^{-1}(t))}\,dt \\ &= H_{j+1}\bigl(H_j^{-1}(u)\bigr), \quad j = 1,2,3, \end{aligned}$$

from (22), we have $H_j(1) = 0$, $H_j^{-1}(0) = 1$, $(H_j(t))' = \frac{1}{h_j(\varphi^{-1}(t))}$. Taking the derivative with respect to t in (29), then multiplying both sides of it by $\frac{1}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))}$, by $y_1(t) = 1$, we have

$$\begin{aligned} \frac{z_2{}'(t)}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))} &= \frac{A_2{}'(t) + \sum_{j=1}^3 g_{j+1}(T,t)y_{j+1}(\varphi^{-1}(H_1^{-1}(\eta_1(t))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))} \\ &\le \frac{A_2{}'(t) + \sum_{j=1}^3 g_{j+1}(T,t)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(t))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))} \\ &= \frac{A_2{}'(t)}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))} + g_2(T,t) \\ &\quad + \sum_{j=2}^3 \frac{g_{j+1}(T,t)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(t))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(t))))}. \end{aligned}$$

Again, integrating both sides of (31) from t_0 to t, and by the definition in (30) and (20), we can obtain

$$\begin{aligned} Y_2\bigl(z_2(t)\bigr) - Y_2\bigl(z_2(t_0)\bigr) &\le \int_{t_0}^t \frac{A_2{}'(\xi)}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi + \int_{t_0}^t g_2(T,\xi)\,d\xi \\ &\quad + \sum_{j=2}^3 \int_{t_0}^t \frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi \\ &\le \int_{t_0}^t \frac{A_2{}'(\xi)}{y_2(\varphi^{-1}(H_1^{-1}(A_2(\xi))))}\,d\xi + \int_{t_0}^t g_2(T,\xi)\,d\xi \\ &\quad + \sum_{j=2}^3 \int_{t_0}^t \frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi \end{aligned}$$

$$\le Y_2\big(A_2(t)\big) - Y_2\big(A_2(t_0)\big) + \int_{t_0}^{t} g_2(T,\xi)\,d\xi + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi,$$

using $z_2(t_0) = A_2(t_0)$, we have

$$Y_2\big(z_2(t)\big) \le Y_2\big(A_2(t)\big) + \int_{t_0}^{t} g_2(T,\xi)\,d\xi + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi.$$

From (31), the above inequality can be written as

$$H_2\big(H_1^{-1}\big(z_2(t)\big)\big) \le H_2\big(H_1^{-1}\big(A_2(t)\big)\big) + \int_{t_0}^{t} g_2(T,\xi)\,d\xi + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi.$$

Let

$$\eta_2(t) = H_2\big(H_1^{-1}\big(z_2(t)\big)\big) \tag{31}$$

and

$$A_3(t) = H_2\big(H_1^{-1}\big(A_2(t)\big)\big) + \int_{t_0}^{t} g_2(T,\xi)\,d\xi, \tag{32}$$

thus $H_1^{-1}(z_2(t)) = H_2^{-1}(\eta_2(t))$, and by (31) and (32), the inequality (31) can be written as

$$\begin{aligned}\eta_2(t) &\le A_3(t) + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}{y_2(\varphi^{-1}(H_1^{-1}(z_2(\xi))))}\,d\xi \\ &= A_3(t) + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_2^{-1}(\eta_2(\xi))))}{y_2(\varphi^{-1}(H_2^{-1}(\eta_2(\xi))))}\,d\xi.\end{aligned}$$

Again, we assume that

$$z_3(t) = A_3(t) + \sum_{j=2}^{3}\int_{t_0}^{t}\frac{g_{j+1}(T,\xi)y_{j+1}(\varphi^{-1}(H_2^{-1}(\eta_2(\xi))))}{y_2(\varphi^{-1}(H_2^{-1}(\eta_2(\xi))))}\,d\xi, \tag{33}$$

we can see that $z_3(t)$ is a nondecreasing function on $[t_0,t]$, and $\eta_2(t) \le z_3(t)$, $z_3(t) \ge A_3(t)$, $z_3(t_0) = A_3(t_0)$. Differentiating $z_3(t)$ with respect to t, we can obtain

$$\begin{aligned}z_3{}'(t) &= A_3{}'(t) + \sum_{j=2}^{3}\frac{g_{j+1}(T,t)y_{j+1}(\varphi^{-1}(H_2^{-1}(\eta_2(t))))}{y_2(\varphi^{-1}(H_2^{-1}(\eta_2(t))))} \\ &\le A_3{}'(t) + \sum_{j=2}^{3}\frac{g_{j+1}(T,t)y_{j+1}(\varphi^{-1}(H_2^{-1}(z_3(t))))}{y_2(\varphi^{-1}(H_2^{-1}(z_3(t))))},\end{aligned}$$

multiplying by $\frac{y_2(\varphi^{-1}(H_2^{-1}(z_3(t))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(t))))}$, then integrating both sides from t_0 to t, and using (C_2), we can obtain

$$\begin{aligned}\int_{t_0}^{t}\frac{y_2(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}z_3{}'(\xi)\,d\xi \le \int_{t_0}^{t}\frac{y_2(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}A_3{}'(\xi)\,d\xi + \int_{t_0}^{t}g_3(T,\xi)\,d\xi \\ + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi,\end{aligned}$$

then

$$\begin{aligned}Y_3\bigl(z_3(t)\bigr) - Y_3\bigl(z_3(t_0)\bigr) &\le \int_{t_0}^{t}\frac{y_2(\varphi^{-1}(H_2^{-1}(A_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(A_3(\xi))))}A_3{}'(\xi)\,d\xi + \int_{t_0}^{t}g_3(T,\xi)\,d\xi \\ &\quad + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi \\ &\le Y_3\bigl(A_3(t)\bigr) - Y_3\bigl(A_3(t_0)\bigr) + \int_{t_0}^{t}g_3(T,\xi)\,d\xi \\ &\quad + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi,\end{aligned}$$

by $z_3(t_0) = A_3(t_0)$, we can obtain

$$Y_3\bigl(z_3(t)\bigr) \le Y_3\bigl(A_3(t)\bigr) + \int_{t_0}^{t}g_3(T,\xi)\,d\xi + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi.$$

Using (31), we can obtain

$$\begin{aligned}H_3\bigl(H_2^{-1}\bigl(z_3(t)\bigr)\bigr) &\le H_3\bigl(H_2^{-1}\bigl(A_3(t)\bigr)\bigr) + \int_{t_0}^{t}g_3(T,\xi)\,d\xi \\ &\quad + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi.\end{aligned} \tag{34}$$

Again, let

$$\eta_3(t) = H_3\bigl(H_2^{-1}\bigl(z_3(t)\bigr)\bigr), \tag{35}$$

$$A_4(t) = H_3\bigl(H_2^{-1}\bigl(A_3(t)\bigr)\bigr) + \int_{t_0}^{t}g_3(T,\xi)\,d\xi, \tag{36}$$

thus $H_2^{-1}(z_3(t)) = H_3^{-1}(\eta_3(t))$, using (35) and (36), the inequality (34) can be written as

$$\begin{aligned}\eta_3(t) &\le A_4(t) + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}{y_3(\varphi^{-1}(H_2^{-1}(z_3(\xi))))}\,d\xi \\ &= A_4(t) + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_3^{-1}(\eta_3(\xi))))}{y_3(\varphi^{-1}(H_3^{-1}(\eta_3(\xi)))))}\,d\xi.\end{aligned}$$

We assume that

$$z_4(t) = A_4(t) + \int_{t_0}^{t}\frac{g_4(T,\xi)y_4(\varphi^{-1}(H_3^{-1}(\eta_3(\xi))))}{y_3(\varphi^{-1}(H_3^{-1}(\eta_3(\xi)))))}\,d\xi, \tag{37}$$

then we see that $z_4(t)$ is a nondecreasing function on $[t_0, t]$, and $\eta_3(t) \leq z_4(t)$, $z_4(t) \geq A_4(t)$, $z_4(t_0) = A_4(t_0)$.

Differentiating (37) with respect to t, we can obtain

$$z_4'(t) = A_4'(t) + \frac{g_4(T,t)y_4(\varphi^{-1}(H_3^{-1}(\eta_3(t))))}{y_3(\varphi^{-1}(H_3^{-1}(\eta_3(t)))))} \leq A_4'(t) + \frac{g_4(T,t)y_4(\varphi^{-1}(H_3^{-1}(z_4(t))))}{y_3(\varphi^{-1}(H_3^{-1}(z_4(t)))))}.$$

Now, multiplying both sides of it by $\frac{y_3(\varphi^{-1}(H_3^{-1}(z_4(t))))}{y_4(\varphi^{-1}(H_3^{-1}(z_4(t))))}$, then integrating both sides of it from t_0 to t, and using (C_2), we can obtain

$$\int_{t_0}^{t} \frac{y_3(\varphi^{-1}(H_3^{-1}(z_4(\xi))))}{y_4(\varphi^{-1}(H_3^{-1}(z_4(\xi))))} z_4'(\xi)\,d\xi \leq \int_{t_0}^{t} \frac{y_3(\varphi^{-1}(H_3^{-1}(z_4(\xi))))}{y_4(\varphi^{-1}(H_3^{-1}(z_4(\xi))))} A_4'(\xi)\,d\xi + \int_{t_0}^{t} g_4(T,\xi)\,d\xi$$

thus

$$\begin{aligned} Y_4\big(z_4(t)\big) - Y_4\big(z_4(t_0)\big) &\leq \int_{t_0}^{t} \frac{y_3(\varphi^{-1}(H_3^{-1}(A_4(\xi))))}{y_4(\varphi^{-1}(H_3^{-1}(A_4(\xi))))} A_4'(\xi)\,d\xi + \int_{t_0}^{t} g_4(T,\xi)\,d\xi \\ &\leq Y_4\big(A_4(t)\big) - Y_4\big(A_4(t_0)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi, \end{aligned}$$

by $z_4(t_0) = A_4(t_0)$, we can obtain

$$Y_4\big(z_4(t)\big) \leq Y_4\big(A_4(t)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi.$$

Using the definition of (31), we can get

$$H_4\big(H_3^{-1}\big(z_4(t)\big)\big) \leq H_4\big(H_3^{-1}\big(A_4(t)\big)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi,$$

thus

$$z_4(t) \leq H_3\left[H_4^{-1}\left(H_4\big(H_3^{-1}\big(A_4(t)\big)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi\right)\right]. \tag{38}$$

Using (27), (31), (33), (35) and (38), we can obtain

$$\begin{aligned} z_1(t) &= H_1^{-1}\big(\eta_1(t)\big) \leq H_1^{-1}\big(z_2(t)\big) = H_2^{-1}\big(\eta_2(t)\big) \leq H_2^{-1}\big(z_3(t)\big) = H_3^{-1}\big(\eta_3(t)\big) \\ &\leq H_3^{-1}\big(z_4(t)\big) \leq H_4^{-1}\left(H_4\big(H_3^{-1}\big(A_4(t)\big)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi\right), \end{aligned}$$

then we have

$$\begin{aligned} u(t) &\leq \varphi^{-1}\big(z_1(t)\big) \\ &\leq \varphi^{-1}\left[H_4^{-1}\left(H_4\big(H_3^{-1}\big(A_4(t)\big)\big) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi\right)\right] \\ &\leq \varphi^{-1}\big(H_4^{-1}\big(A_5(t)\big)\big), \quad t \in [t_0, T], \end{aligned}$$

where $A_5(t) = H_4(H_3^{-1}(A_4(t))) + \int_{t_0}^{t} g_4(T,\xi)\,d\xi$, because of T being arbitrary, we can obtain

$$\begin{aligned} u(t) &\le \varphi^{-1}\bigl(z_1(t)\bigr) \\ &\le \varphi^{-1}\biggl[H_4^{-1}\biggl(H_4\bigl(H_3^{-1}\bigl(A_4(t)\bigr)\bigr) + \int_{t_0}^{t} g_4(t,\xi)\,d\xi\biggr)\biggr] \\ &\le \varphi^{-1}\bigl(H_4^{-1}\bigl(A_5(t)\bigr)\bigr), \quad t \in [t_0, T]. \end{aligned}$$

The proof is complete. □

Remark 3 If $h_3 \equiv 0$, we can see that Theorem 3.3 reduces to Theorem 2.3 in [9]. If $\varphi(u(t)) = x^p(t)$, $a(t) = x_0$, $g_1(t,\xi) = f(s)$, $h_1(u(t)) = x^p(t)$, $g_2(t,\xi) = h(s)$, $h_2(u(t)) = x^q(t)$, and $g(t,\xi) = g_3(t,\xi) = 0$, Theorem 3.3 reduces to Theorem 3.1 in [8]. If $h_1(u(t)) = \eta(u(s))w(u(s))$, $h_2(u(t)) = \eta(u(s))$, and $g(t,\xi) = g_3(t,\xi) = 0$, Theorem 3.3 reduces to Theorem 1 in [12].

4 Applications of the result

In this section, we apply the results of the previous section to study the boundedness of solutions of differential equations and integral equations.

1. First, let us consider the Volterra type retarded integral equation

$$\chi^4(t) = b(t) + \int_{t_0}^{\sigma(t)} g(\xi)\biggl[\chi^2(\xi) + \int_{t_0}^{\xi} w(\lambda)\chi^2(\lambda)\,d\lambda\biggr]^{\frac{1}{3}} d\xi, \tag{39}$$

which often occurs in physical and mechanical applications.

Example 4.1 We assume $\chi(t), b(t), g(t), w(t) \in C(R_+, R_+)$, and let $\sigma(t) \in C[t_0,\infty)$, $\sigma'(t) \ge 0$ and $\sigma(t) \le t$, $\sigma(t_0) = t_0$. We can obtain the estimate for $\chi(t)$ as follows:

$$\bigl|\chi(t)\bigr| \le \biggl[\bigl|b(t)\bigr| + B(t)\exp\biggl(\int_{t_0}^{\sigma(t)} \frac{1}{6}\bigl|g(\xi)\bigr|\,d\xi + \int_{t_0}^{\sigma(t)} \frac{1}{6}\bigl|g(\xi)\bigr|\biggl(\int_{t_0}^{s}\bigl|w(\lambda)\bigr|\,d\lambda\biggr)\,d\xi\biggr)\biggr]^{\frac{1}{4}},$$

where

$$\begin{aligned} B(t) &= \int_{t_0}^{\sigma(t)} \bigl|g(\xi)\bigr|\biggl(\frac{2}{3} + \frac{1}{6}\bigl|b(\xi)\bigr| + \frac{1}{6}\biggr)\,d\xi \\ &\quad + \int_{t_0}^{\sigma(t)} \bigl|g(\xi)\bigr|\biggl[\int_{t_0}^{\xi}\bigl|w(\lambda)\bigr|\biggl(\frac{1}{6}\bigl|b(\lambda)\bigr| + \frac{1}{6}\biggr)\,d\lambda\biggr]\,d\xi. \end{aligned}$$

Proof By (39), we have

$$\bigl|\chi(t)\bigr|^4 \le \bigl|b(t)\bigr| + \int_{t_0}^{\sigma(t)} \bigl|g(\xi)\bigr|\biggl[\bigl|\chi(\xi)\bigr|^2 + \int_{t_0}^{\xi}\bigl|w(\lambda)\bigr|\bigl|\chi(\lambda)\bigr|^2\,d\lambda\biggr]^{\frac{1}{3}} d\xi, \tag{40}$$

taking $|\chi(t)| = u(t)$, (40) can be written as

$$u^4(t) \le \bigl|b(t)\bigr| + \int_{t_0}^{\sigma(t)} \bigl|g(\xi)\bigr|\biggl[u^2(t) + \int_{t_0}^{\xi}\bigl|w(\lambda)\bigr|u^2(\lambda)\,d\lambda\biggr]^{\frac{1}{3}} d\xi. \tag{41}$$

Here, we can conclude that (41) satisfies the conditions of Theorem 3.1 with $q = 4$, $\alpha = \beta = 2$, $p = \frac{1}{3}$, $a(t) = |b(t)|$, $h(t) = |w(t)|$, $f(t) = |g(t)|$, using Theorem 3.1, our conclusion obviously holds. □

2. Next, we consider the following integral equation:

$$\chi(t) = \chi_0 + \int_{t_0}^{\sigma(t)} f(\xi)\chi^{\frac{1}{5}}(\xi)\left[\chi^3(\xi) + \int_{t_0}^{\xi} w(\lambda)\chi^3(\lambda)\,d\lambda\right]^4 d\xi. \tag{42}$$

Example 4.2 We assume $\chi(t), f(t), w(t) \in C(R_+, R_+)$, $\chi'(t) \geq 0$, $\sigma(t) \in [t_0, \infty)$, $\sigma'(t) \geq 0$, $\sigma(t) \leq t$, $\sigma(t_0) = t_0$, then we can get

$$\begin{aligned}\left|\chi(t)\right| \leq{}& \exp\left(\frac{1}{5}\int_{t_0}^{\sigma(t)} \left|f(\xi)\right|\beta(\xi)\,d\xi\right) \\ &\times \left[\chi_0 + \int_{t_0}^{\sigma(t)} \left(\frac{4}{5}\right)\left|f(\xi)\right|\beta(\xi)\exp\left(-\frac{1}{5}\int_{t_0}^{\xi} \left|f(\lambda)\right|\beta(\lambda)\,d\lambda\right)d\xi\right],\end{aligned} \tag{43}$$

where

$$\beta(t) = \left[\frac{\Omega(t)}{1-(\frac{64}{5})\int_{t_0}^{t} |f(\xi)|\Omega(\xi)\,d\xi}\right]^{\frac{15}{14}},$$

$$\Omega(t) = \chi_0^{\frac{56}{5}}\exp\left(\frac{56}{15}\int_{t_0}^{t}\left|w(\xi)\right|d\xi\right).$$

Proof Using (42), we have

$$\left|\chi(t)\right| \leq \chi_0 + \int_{t_0}^{\sigma(t)} \left|f(\xi)\right|\left|\chi(\xi)\right|^{\frac{1}{5}}\left[\left|\chi(\xi)\right|^3 + \int_{t_0}^{\xi}\left|w(\lambda)\right|\left|\chi(\lambda)\right|^3 d\lambda\right]^4 d\xi,$$

let $|\chi(t)| = u(t)$, the above inequality is written as

$$u(t) \leq \chi_0 + \int_{t_0}^{\sigma(t)} \left|f(\xi)\right|u^{\frac{1}{5}}(\xi)\left[u^3(\xi) + \int_{t_0}^{\xi}\left|w(\lambda)\right|u^3(\xi)\,d\lambda\right]^4 d\xi. \tag{44}$$

Here, we can conclude that (44) satisfies the conditions of Theorem 3.2 with $m = m = 3$, $p = 4$, $r = \frac{1}{5}$, $u_0 = \chi_0$, $h(t) = |w(t)|$, $g(t) = |f(t)|$, using Theorem 3.2, our conclusion obviously holds. □

3. We consider the following differential system:

$$\begin{cases} s'(t) = G(t,s), & t \in [0,\infty), \\ s(0) = a_0, \end{cases} \tag{45}$$

where $G(t,s)$ is a continuous function on $[0,\infty) \times (-\infty, -e\sqrt{e}] \cup [e\sqrt{e}, +\infty)$, $a_0 > 0$. We assume that $G(t,s)$ satisfies the following inequality:

$$\left|G(t,s)\right| \leq t^2\sqrt[5]{|s|} + \frac{|s|}{3} - \frac{|s|\ln|s|}{2} + \frac{e^{|s|}}{4}. \tag{46}$$

Example 4.3 Let $G(t,s)$ satisfy the condition of inequality (46), all solutions of differential system (45) satisfy the following estimates:

$$\left|s(t)\right| \le -\ln\left(\exp\left(-\left(\left(a_0^{\frac{4}{5}}+\frac{4t^3}{15}\right)^{\frac{5}{4}}e^{\frac{t}{3}}\right)^{e^{-\frac{t}{2}}}\right)+\frac{t}{4}\right), \quad \forall t\in[0,\infty). \tag{47}$$

Proof Integrating the differential system (45) from 0 to t, we can get

$$s(t)=a_0+\int_0^t G\bigl(\xi,s(\xi)\bigr)\,d\xi, \tag{48}$$

by (46), we can obtain

$$\left|s(t)\right|\le a_0+\int_0^t \xi^2\sqrt[5]{\left|s(\xi)\right|}\,d\xi+\int_0^t\frac{|s(\xi)|}{3}\,d\xi-\int_0^t\frac{|s(\xi)|\ln|s(\xi)|}{2}\,d\xi+\int_0^t\frac{e^{|s|}}{4}\,d\xi,$$

taking $|s(t)|=u(t)$, the above inequality can be written as

$$u(t)\le a_0+\int_0^t \xi^2\sqrt[5]{u(\xi)}\,d\xi+\int_0^t\frac{u(\xi)}{3}\,d\xi-\int_0^t\frac{u(\xi)\ln(u(\xi))}{2}\,d\xi+\int_0^t\frac{e^{|s|}}{4}\,d\xi, \tag{49}$$

we can see that Eq. (49) satisfies (24) : $a(t)=a_0$, $g_1(t,\xi)=t^2$, $g_2(t,\xi)=\frac{1}{3}$, $g_3(t,\xi)=-\frac{1}{2}$, $g_4(t,\xi)=\frac{1}{4}$, $h_1(u)=\sqrt[5]{|u|}$, $h_2(u)=|u|$, $h_3(u)=|u|\ln|u|$, $h_4(u)=e^{|s|}$, $\frac{h_2(t)}{h_1(t)}=\frac{|u|}{\sqrt[5]{|u|}}=|u|^{\frac{4}{5}}$, $\frac{h_3(t)}{h_2(t)}=\ln|u|$, $\frac{h_4(t)}{h_3(t)}=\frac{e^{|s|}}{|u|\ln|u|}$, then we can see that $\frac{h_{j+1}(t)}{h_j(t)}$, $(j=1,2,3)$ is a nondecreasing function for $u>0$. then we can obtain

$$\begin{aligned}
H_1(u)&=\int_{u_0}^u\frac{d\xi}{\sqrt[5]{\xi}}=\frac{5}{4}\bigl(u^{\frac{4}{5}}-u_0^{\frac{4}{5}}\bigr), \qquad H_1^{-1}(u)=\left(\frac{4}{5}u+u_0^{\frac{4}{5}}\right)^{\frac{5}{4}},\\
H_2(u)&=\int_{u_1}^u\frac{d\xi}{\xi}=\ln\frac{u}{u_1}, \qquad H_2^{-1}(u)=u_1e^u,\\
H_3(u)&=\int_e^u\frac{d\xi}{\xi\ln\xi}=\ln\bigl(\ln(u)\bigr), \qquad H_3^{-1}(u)=e^{e^u},\\
H_4(u)&=\int_1^u\frac{d\xi}{e^\xi}=-\bigl(e^{-u}-e^{-1}\bigr), \qquad H_4^{-1}(u)=-\ln\bigl(e^{-1}-u\bigr).
\end{aligned}$$

Using Eq. (23) of Theorem 3.3, we have

$$\begin{aligned}
A_1(t)&=a_0,\\
A_2(t)&=H_1\bigl(A_1(t)\bigr)+\int_0^t\xi^2\,d\xi\\
&=\frac{5}{4}\bigl(a_0^{\frac{4}{5}}-u_0^{\frac{4}{5}}\bigr)+\frac{t^3}{3},\\
A_3(t)&=H_2\bigl(H_1^{-1}\bigl(A_2(t)\bigr)\bigr)+\int_0^t\frac{1}{3}\,d\xi\\
&=H_2\left[\left(a_0^{\frac{4}{5}}+\frac{4t^3}{15}\right)^{\frac{5}{4}}\right]+\frac{t}{3}
\end{aligned}$$

$$
\begin{aligned}
&= \ln \frac{(a_0^{\frac{4}{5}} + \frac{4t^3}{15})^{\frac{5}{4}}}{u_1} + \frac{t}{3}, \\
A_4(t) &= H_3^{-1}\bigl(H_2\bigl(A_3(t)\bigr)\bigr) - \int_0^t \frac{1}{2}\,d\xi \\
&= H_3\biggl[u_1 \exp\biggl(\ln \frac{(a_0^{\frac{4}{5}} + \frac{4t^3}{15})^{\frac{5}{4}}}{u_1} + \frac{t}{3}\biggr)\biggr] - \frac{t}{2} \\
&= H_3\biggl[\biggl(a_0^{\frac{4}{5}} + \frac{4t^3}{15}\biggr)^{\frac{5}{4}} e^{\frac{t}{3}}\biggr] - \frac{t}{2} \\
&= \ln\biggl[\ln\biggl(\biggl(a_0^{\frac{4}{5}} + \frac{4t^3}{15}\biggr)^{\frac{5}{4}} e^{\frac{t}{3}}\biggr)\biggr] - \frac{t}{2}, \\
A_5 &= H_4\bigl[H_3^{-1}\bigl(A_4(t)\bigr)\bigr] + \int_0^t \frac{1}{4}\,d\xi \\
&= H_4\biggl[\biggl(\biggl(a_0^{\frac{4}{5}} + \frac{4t^3}{15}\biggr)^{\frac{5}{4}} e^{\frac{t}{3}}\biggr)^{e^{-\frac{t}{2}}}\biggr] + \frac{t}{4} \\
&= e^{-1} - \exp\biggl(-\biggl(\biggl(a_0^{\frac{4}{5}} + \frac{4t^3}{15}\biggr)^{\frac{5}{4}} e^{\frac{t}{3}}\biggr)^{e^{-\frac{t}{2}}}\biggr) + \frac{t}{4},
\end{aligned}
$$

then

$$
\begin{aligned}
u(t) &\le H_4^{-1}\bigl(A_5(t)\bigr) \\
&= -\ln\biggl(\exp\biggl(-\biggl(\biggl(a_0^{\frac{4}{5}} + \frac{4t^3}{15}\biggr)^{\frac{5}{4}} e^{\frac{t}{3}}\biggr)^{e^{-\frac{t}{2}}}\biggr) + \frac{t}{4}\biggr),
\end{aligned}
$$

which means that $u(t)$ is bounded, for $t \in [0, \infty)$. The proof is completed. □

5 Conclusion

In this paper, we first give a new lemma about the nonlinear Gronwall–Bellman delay integral inequality, then we establish some new delay Gronwall–Bellman integral inequalities with power. And the inequalities obtained in this paper are further generalizations of some results obtained by Li et al. [9]. The results of this paper contribute to the study of the qualitative properties of solutions of differential and integral equations. By the method of Theorem 3.3 in this paper, we can further generalize Eq. (3) to

$$
\begin{aligned}
\varphi\bigl(u(t)\bigr) \le{}& a(t) + \sum_{j=1}^{n} \int_{t_0}^{t} g_j(t,\xi) h_j\bigl(u(\xi)\bigr)\,d\xi \\
&+ \int_{t_0}^{t} g(t,\xi)\biggl(\int_{t_0}^{\xi} f(\xi,\theta) h_{n+1}\bigl(u(\theta)\bigr)\,d\theta\biggr)\,d\xi,
\end{aligned}
$$

then we can get similar results for the estimations on $u(t)$.

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Availability of data and materials
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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BF carried out the main results and completed the corresponding proof. YL participated in the proof of Theorem 3.2, RX participated in the proof and help to complete Sects. 4 and 5. All authors read and approved the final manuscript.

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