

RESEARCH

Open Access



Dynamics of a stochastic population model with predation effects in polluted environments

Guanghai Song^{1*}

*Correspondence:
hhusongguanghai@163.com
¹Huaiyin Normal University, Huaian,
223300, P.R. China

Abstract

The present paper puts forward and probes a stochastic single-species model with predation effect in a polluted environment. We propose a threshold between extermination and weak persistence of the species and provide sufficient conditions for the stochastic persistence of the species. In addition, we evaluate the growth rates of the solution. Theoretical findings are expounded by some numerical simulations.

MSC: 60H10; 60H30; 92D25

Keywords: Predation effects; Random perturbations; Environmental pollution; Persistence

1 Introduction

Environmental pollution because of industry, commerce, agriculture, and the rapid growth of human population has been increasingly prominent and has become an global problem. Pollutants have threatened the health of living organisms. For instance, Amoco Cadiz incident carried off more than 260 thousands tonnes of sea animals in one month [1]; Bhopal disaster killed about 15 thousands people and injured about 560 thousands people [2]; according to the United States Fish and Wildlife Service, pesticides carried off more than 72 million birds every year in US [3].

To probe the influence of pollutants on the evolution of populations, many mathematical models have been put forward. In the 1980s, Hallam and his co-workers [4–6] first constructed a series of deterministic models with pollution and uncovered that environmental pollution has serious influence on the persistence and extermination of the species. These results were improved and extended [7–17]. Particularly, motivated by the fact that the evolution of populations often encounters environmental perturbations [18], several authors (see, e.g., [9, 12–17]) paid attention to stochastic population models with pollution and uncovered that environmental perturbations have vital functions on the persistence and extermination of the species.

On the other hand, most populations have parasites and predators [19]. In general, these predation effects have negative functions on the growth of populations. Therefore we need

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

to test population models with predation effects in polluted environments. However, little research has been conducted to exploit this problem (even for deterministic models).

The objective of this paper is testing the above problem. We construct a population model with predation effects in polluted environments in Sect. 2 and testify that the model has a unique global positive solution in Sect. 3. Then we offer a threshold between extermination and weak persistence of the species and provide conditions under which the species is stochastically persistent in Sect. 4. In Sect. 5, we estimate the growth rates of the solution of the model. Finally, we give the conclusions of this paper and numerically expound the theoretical findings in Sect. 6.

2 The model

Without pollution and predation effects, suppose that the growth of the species follows the logistic role:

$$\frac{d\Phi(t)}{dt} = \Phi(t)[b - \xi \Phi(t)], \quad (1)$$

where $\Phi(t)$ is the population size at time t , and $b > 0$ and $\xi > 0$ stand for the growth rate and the intraspecific competition rate, respectively.

We consider the predation effects. In general, the predation effects saturate at high prey density and vanish quadratically as the prey density tends to zero [19]. Therefore it is reasonable to model the predation effects by the following function:

$$\frac{\lambda \Phi^2(t)}{\rho^2 + \Phi^2(t)},$$

where λ is the upper limit of the predation effects, and ρ^2 measures the saturate effect. Then model (1) is replaced by

$$\frac{d\Phi(t)}{dt} = \Phi(t)[b - \xi \Phi(t)] - \frac{\lambda \Phi^2(t)}{\rho^2 + \Phi^2(t)}. \quad (2)$$

As said before, the evolution of populations often encounters environmental perturbations [18]. In general, we can take advantage of a color noise process to portray the environmental perturbations [20], and it is suitable to utilize a Gaussian white noise process to depict a weakly correlated color noise [21]. Various ways were developed to incorporate the white noise into deterministic population models. A widely accepted way is to suppose that some parameters in the model are influenced by the white noise (see, e.g., [22–32]). Adopting these approaches,

$$b \rightarrow b + \beta_1 \dot{\psi}_1(t), \quad -\xi \rightarrow -\xi + \beta_2 \dot{\psi}_2(t), \quad -\lambda \rightarrow \lambda + \beta_3 \dot{\psi}_3(t),$$

where $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are independent standard Wiener processes defined on a certain complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, and β_i , $i = 1, 2, 3$, stand for the intensities. As a result, model (2) is replaced by

$$\begin{aligned} d\Phi(t) = & \Phi(t) \left(b - \xi \Phi(t) - \frac{\lambda \Phi(t)}{\rho^2 + \Phi^2(t)} \right) dt \\ & + \beta_1 \Phi(t) d\psi_1(t) + \beta_2 \Phi^2(t) d\psi_2(t) + \frac{\beta_3 \Phi^2(t)}{\rho^2 + \Phi^2(t)} d\psi_3(t). \end{aligned} \quad (3)$$

To characterize the influence of pollution, we hypothesize that the populations suck up the pollutants into their bodies [4–6]. Denote by $T(t)$ the concentration of pollutants in the species, which can lead to a descent of the growth rate [4]:

$$b \rightarrow b - H_1(T(t)),$$

where $H_1(T)$ is a positive continuous increasing function of T . Accordingly, model (3) is replaced by

$$\begin{aligned} d\Phi(t) = & \Phi(t) \left(b - H_1(T(t)) - \xi \Phi(t) - \frac{\lambda \Phi(t)}{\rho^2 + \Phi^2(t)} \right) dt \\ & + \beta_1 \Phi(t) d\psi_1(t) + \beta_2 \Phi^2(t) d\psi_2(t) + \frac{\beta_3 \Phi^2(t)}{\rho^2 + \Phi^2(t)} d\psi_3(t). \end{aligned} \quad (4)$$

To depict the pollutants in the species, we pay attention to the following equation:

$$\frac{dT(t)}{dt} = H_2(T_e(t)) - H_3(T(t)), \quad (5)$$

where $T_e(t)$ represents the concentration of pollutants in the environment, $H_2(T_e(t)) > 0$ characterizes the suck up of pollutants from the environment, $H_3(T(t)) > 0$ measures the loss of pollutants because of excretion and detoxication. Both $H_2 \in C^1$ and $H_3 \in C^1$ are increasing functions.

Finally, we portray the changes of $T_e(t)$. Denote by $u(t)$ a continuous and bounded function of t , the input of pollutants from the outside of the environment. Suppose that the changes of $T_e(t)$ are governed by the following equation:

$$\frac{dT_e(t)}{dt} = u(t) - H_4(T_e(t)), \quad (6)$$

where $H_4 \in C^1$, measuring the loss of the pollutants from the environment, is an increasing positive function of T_e .

According to (4)–(6), we derive the following model:

$$\begin{cases} d\Phi(t) = \Phi(t) \left(b - H_1(T(t)) - \xi \Phi(t) - \frac{\lambda \Phi(t)}{\rho^2 + \Phi^2(t)} \right) dt \\ \quad + \beta_1 \Phi(t) d\psi_1(t) + \beta_2 \Phi^2(t) d\psi_2(t) + \frac{\beta_3 \Phi^2(t)}{\rho^2 + \Phi^2(t)} d\psi_3(t), \\ \frac{dT(t)}{dt} = H_2(T_e(t)) - H_3(T(t)), \\ \frac{dT_e(t)}{dt} = u(t) - H_4(T_e(t)). \end{cases} \quad (7)$$

The objectives of this paper is probing some dynamical properties of $\Phi(t)$. Note that the last two equations in model (7) do not depend on $\Phi(t)$, and they have a unique solution $(T(t), T_e(t))$ for certain initial value. As a result, from now on we concentrate on Eq. (4).

3 The existence and uniqueness of the solution

Theorem 1 *For any initial value $\Phi(0) > 0$, Eq. (4) possesses a unique global positive solution $\Phi(t)$ almost surely (a.s.).*

Proof We first concentrate on the equation

$$\begin{aligned} dx(t) = & \left[b - H_1(T(t)) - \frac{\beta_1^2}{2} - \xi e^{x(t)} - \frac{\lambda e^{x(t)}}{\rho^2 + e^{2x(t)}} - \frac{\beta_2^2}{2} e^{2x(t)} - \frac{\beta_3^2 e^{2x(t)}}{2(\rho^2 + e^{2x(t)})^2} \right] dt \\ & + \beta_1 d\psi_1(t) + \beta_2 e^{x(t)} d\psi_2(t) + \frac{\beta_3 e^{x(t)}}{\rho^2 + e^{2x(t)}} d\psi_3(t) \end{aligned} \quad (8)$$

with $x(0) = \ln \Phi(0)$. By the locally Lipschitz continuity of the coefficients in Eq. (8) we deduce that Eq. (8) possesses a unique solution on $[0, \tau_e)$, where $\tau_e \leq +\infty$. It then follows from Itô's formula (see [33], p. 32, Theorem 6.2) that $\Phi(t) = e^{x(t)}$ is the unique positive solution of (4).

Now we testify that $\tau_e = +\infty$. Let m_0 be a positive constant such that $m_0 > \Phi(0)$. For each integer m , define

$$\tau_m = \inf\{t \in [0, \tau_e) : \Phi(t) \geq m\}.$$

Let $\tau_\infty = \lim_{m \rightarrow +\infty} \tau_m$. We can deduce that $\tau_\infty \leq \tau_e$. If $\tau_e < +\infty$, then we can find out constants $\hat{T} > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq \hat{T}\} > \epsilon.$$

It then follows that we can find out an integer $m_1 \geq m_0$ such that for any $m \geq m_1$,

$$P\{\Omega_m\} \geq \epsilon, \quad (9)$$

where $\Omega_m = \{\omega : \tau_m \leq \hat{T}\}$. Define

$$W_1(\Phi) = \Phi^\gamma, \quad \Phi > 0, 0 < \gamma < 1.$$

Then we deduce by Itô's formula that

$$\begin{aligned} dW_1(\Phi) &= \gamma \Phi^\gamma \left[b - H_1(T(t)) - \xi \Phi - \frac{\lambda \Phi}{\rho^2 + \Phi^2} + \frac{\gamma - 1}{2} \beta_1^2 + \frac{\gamma - 1}{2} \beta_2^2 \Phi^2 + \frac{(\gamma - 1) \beta_3^2 \Phi^2}{2(\rho^2 + \Phi^2)^2} \right] dt \\ &\quad + \gamma \beta_1 \Phi^\gamma d\psi_1(t) + \gamma \beta_2 \Phi^{\gamma+1} d\psi_2(t) + \frac{\gamma \beta_3 \Phi^{\gamma+1}}{\rho^2 + \Phi^2} d\psi_3(t) \\ &\leq \gamma b W_1(\Phi) dt + \gamma W_1(\Phi) \left[\beta_1 d\psi_1(t) + \beta_2 \Phi d\psi_2(t) + \frac{\beta_3 \Phi}{\rho^2 + \Phi^2} d\psi_3(t) \right]. \end{aligned}$$

Accordingly,

$$\begin{aligned} \mathbb{E} W_1(\Phi(\tau_m \wedge \hat{T})) &\leq W_1(\Phi(0)) + \gamma b \int_0^{\tau_m \wedge \hat{T}} \mathbb{E} W_1(\Phi(s)) ds \\ &\leq W_1(\Phi(0)) + \gamma b \int_0^{\hat{T}} \mathbb{E} W_1(\Phi(\tau_m \wedge s)) ds. \end{aligned}$$

By Gronwall's inequality (see [33], p. 45, Theorem 8.1) we get

$$\mathbb{E} W_1(\Phi(\tau_m \wedge \hat{T})) \leq W_1(\Phi(0)) e^{\gamma b \hat{T}}. \quad (10)$$

For $\omega \in \Omega_m$, $W_1(\Phi(\tau_m, \omega)) \geq m^\gamma$. Then by (9) and (10) we can see that

$$W_1(\Phi(0)) e^{\gamma b \hat{T}} \geq \mathbb{E}[1_{\Omega_m}(\omega) W_1(\Phi(\tau_m, \omega))] \geq \epsilon m^\gamma.$$

Letting $m \rightarrow +\infty$ gives

$$W_1(\Phi(0)) e^{\gamma b \hat{T}} = \infty. \quad (11)$$

However, by the definitions of $W_1(\Phi)$ and \hat{T} we have

$$W_1(\Phi(0)) e^{\gamma b \hat{T}} < +\infty,$$

which is a contradiction with (11). Thereby, $\tau_e = +\infty$. \square

4 Extinction and persistence

Theorem 2 *If $\Lambda < 0$, then $\lim_{t \rightarrow +\infty} \Phi(t) = 0$ a.s., that is, the species is extinct, where*

$$\Lambda = \Lambda_1 - \liminf_{t \rightarrow +\infty} t^{-1} \int_0^t H_1(T(s)) \, ds, \quad \Lambda_1 = b - \frac{1}{2} \beta_1^2.$$

Proof We can deduce from Itô's formula that

$$\begin{aligned} d \ln \Phi = & \left[\Lambda_1 - H_1(T(t)) - \xi \Phi - \frac{\lambda \Phi}{\rho^2 + \Phi^2} - \frac{\beta_2^2}{2} \Phi^2 - \frac{\beta_3^2 \Phi^2}{2(\rho^2 + \Phi^2)^2} \right] dt \\ & + \beta_1 d\psi_1(t) + \beta_2 \Phi d\psi_2(t) + \frac{\beta_3 \Phi}{\rho^2 + \Phi^2} d\psi_3(t), \end{aligned}$$

that is,

$$\begin{aligned} \ln \Phi(t) - \ln \Phi(0) &= \Lambda_1 t - \int_0^t H_1(T(s)) \, ds \\ &\quad - \int_0^t \left[\xi \Phi(s) + \frac{\lambda \Phi(s)}{\rho^2 + \Phi^2(s)} + \frac{\beta_2^2}{2} \Phi^2(s) + \frac{\beta_3^2 \Phi^2(s)}{2(\rho^2 + \Phi^2(s))^2} \right] ds \\ &\quad + \Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t), \end{aligned} \quad (12)$$

where

$$\Gamma_1(t) = \beta_1 \psi_1(t), \quad \Gamma_2(t) = \int_0^t \beta_2 \Phi(s) d\psi_2(s), \quad \Gamma_3(t) = \int_0^t \frac{\beta_3 \Phi(s)}{\rho^2 + \Phi^2(s)} d\psi_3(s).$$

The quadratic variations of $\Gamma_2(t)$ and $\Gamma_3(t)$ are

$$\langle \Gamma_2(t), \Gamma_2(t) \rangle = \int_0^t \beta_2^2 \Phi^2(s) \, ds$$

and

$$\langle \Gamma_3(t), \Gamma_3(t) \rangle = \int_0^t \frac{\beta_3^2 \Phi^2(s)}{(\rho^2 + \Phi^2(s))^2} ds \leq \frac{\beta_3^2}{2\rho^2} t,$$

respectively. We can easily obtain that

$$\lim_{t \rightarrow +\infty} t^{-1} \Gamma_i(t) = 0 \quad \text{a.s., } i = 1, 3. \quad (13)$$

By the exponential martingale inequality (see [33], p. 44, Theorem 7.4) we deduce that

$$P \left\{ \sup_{0 \leq t \leq M} \left[\Gamma_2(t) - \frac{1}{2} \langle \Gamma_2(t), \Gamma_2(t) \rangle \right] > 2 \ln M \right\} \leq 1/M^2.$$

By the Borel–Cantelli lemma (see [33], p. 7, Lemma 2.4), for almost all $\omega \in \Omega$, there exists an integer $M_0 = M_0(\omega)$ such that for $M \geq M_0$,

$$\sup_{0 \leq t \leq M} \left[\Gamma_2(t) - \frac{1}{2} \langle \Gamma_2(t), \Gamma_2(t) \rangle \right] \leq 2 \ln M.$$

Therefore, for any $0 \leq t \leq M, M \geq M_0$,

$$\Gamma_2(t) \leq 2 \ln M + \frac{1}{2} \langle \Gamma_2(t), \Gamma_2(t) \rangle = 2 \ln M + 0.5 \int_0^t \beta_2^2 \Phi^2(s) ds. \quad (14)$$

Then we deduce from (12) and (14) that for any $0 \leq t \leq M, M \geq M_0$,

$$\ln \Phi(t) - \ln \Phi(0) \leq \Lambda_1 t - \int_0^t H_1(T(s)) ds - \xi \int_0^t \Phi(s) ds + 2 \ln M + \Gamma_1(t) + \Gamma_3(t). \quad (15)$$

For $0 < M - 1 \leq t \leq M, M \geq M_0$, we have

$$t^{-1} \{ \ln \Phi(t) - \ln \Phi(0) \} \leq \Lambda_1 - t^{-1} \int_0^t H_1(T(s)) ds + 2(M-1)^{-1} \ln M + t^{-1} \Gamma_1(t) + t^{-1} \Gamma_3(t).$$

Then (13) shows that

$$\limsup_{t \rightarrow +\infty} t^{-1} \ln \Phi(t) \leq \Lambda_1 - \liminf_{t \rightarrow +\infty} t^{-1} \int_0^t H_1(T(s)) ds = \Lambda < 0.$$

Therefore, for any $\epsilon \in (0, -\Lambda)$, there is T_1 such that for all $t > T_1$,

$$t^{-1} \ln \Phi(t) \leq \epsilon + \Lambda < 0,$$

That is, for all $t > T_1$

$$\Phi(t) \leq e^{(\epsilon + \Lambda)t}.$$

Hence $\lim_{t \rightarrow +\infty} \Phi(t) = 0$ a.s. □

Theorem 3 If $\Lambda \geq 0$, then

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t \Phi(s) \, ds \leq \frac{\Lambda}{\xi} \quad a.s. \quad (16)$$

In particular, $\Lambda = 0 \Rightarrow \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \Phi(s) \, ds = 0$, that is, the species is nonpersistent in the mean.

Proof For arbitrary $\epsilon > 0$, we can find $\tilde{T} > 0$ such that for $0 < \tilde{T} < M - 1 \leq t \leq M, M \geq M_0$,

$$t^{-1} \left[\ln \Phi(0) + \Lambda_1 - \int_0^t H_1(T(s)) \, ds + 2 \ln M + \Gamma_1(t) + \Gamma_3(t) \right] \leq \Lambda + \epsilon.$$

Set $\Xi = \Lambda + \epsilon$. In view of (15), for arbitrary $0 < \tilde{T} < M - 1 \leq t \leq M, M \geq M_0$, we obtain

$$\begin{aligned} \ln \Phi(t) &\leq \ln \Phi(0) + \Lambda_1 t - \int_0^t H_1(T(s)) \, ds - \xi \int_0^t \Phi(s) \, ds + 2 \ln M + \Gamma_1(t) + \Gamma_3(t) \\ &\leq \Xi t - \xi \int_0^t \Phi(s) \, ds. \end{aligned}$$

Set $\alpha(t) = \int_0^t \Phi(s) \, ds$. Then we can see that

$$e^{\xi \alpha(t)} (d\alpha/dt) \leq e^{\Xi t}, \quad t \geq \tilde{T}.$$

Integrating both sides from \tilde{T} to t , we get

$$\int_{\tilde{T}}^t e^{\xi \alpha(s)} \, d\alpha(s) \leq \int_{\tilde{T}}^t e^{\Xi s} \, ds.$$

Therefore

$$e^{\xi \alpha(t)} \leq e^{\xi \alpha(\tilde{T})} + \xi \Xi^{-1} e^{\Xi t} - \xi \Xi^{-1} e^{\Xi \tilde{T}}.$$

Taking logarithms leads to

$$\alpha(t) \leq (\xi)^{-1} \ln \{ \xi \Xi^{-1} e^{\Xi t} + e^{\xi \alpha(\tilde{T})} - \xi \Xi^{-1} e^{\Xi \tilde{T}} \}.$$

As a result,

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t \Phi(s) \, ds \leq \xi^{-1} \limsup_{t \rightarrow +\infty} \{ t^{-1} \ln \{ \xi \Xi^{-1} e^{\Xi t} + e^{\xi \alpha(\tilde{T})} - \xi \Xi^{-1} e^{\Xi \tilde{T}} \} \}.$$

It then follows from L'Hospital's rule that

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t \Phi(s) \, ds \leq \frac{\Xi}{\xi} = \frac{\Lambda + \epsilon}{\xi}.$$

Since ϵ is arbitrary, we get (16). \square

Theorem 4 *If $\Lambda > 0$, then $\limsup_{t \rightarrow +\infty} \Phi(t) > 0$, that is, the species is weakly persistent.*

Proof Denote

$$L = \left\{ \omega : \lim_{t \rightarrow +\infty} \Phi(t, \omega) = 0 \right\}.$$

We hypothesize that $P\{L\} > 0$. For arbitrary $\omega \in L$, $\lim_{t \rightarrow +\infty} \Phi(t, \omega) = 0$. Therefore, for any $\epsilon \in (0, 1)$, there is T such that $\Phi(t, \omega) \leq \epsilon$ for all $t \geq T$. As a result, for sufficiently large t ,

$$\begin{aligned} t^{-1} [\ln \Phi(t, \omega) - \ln \Phi(0)] &\leq \ln \epsilon / t \leq 0, \\ \int_0^t \beta_2^2 \Phi^2(s, \omega) \, ds &= \int_0^T \beta_2^2 \Phi^2(s, \omega) \, ds + \int_T^t \beta_2^2 \Phi^2(s, \omega) \, ds \\ &\leq \beta_2^2 M_1 + \beta_2^2 \epsilon (t - T) \leq 2\beta_2^2 \epsilon t, \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \left[\xi \Phi(s, \omega) + \frac{\lambda \Phi(s, \omega)}{\rho^2 + \Phi^2(s, \omega)} + \frac{\beta_2^2}{2} \Phi^2(s, \omega) + \frac{\beta_3^2 \Phi^2(s, \omega)}{2(\rho^2 + \Phi^2(s, \omega))^2} \right] \, ds \\ &= \int_0^T \left[\xi \Phi(s, \omega) + \frac{\lambda \Phi(s, \omega)}{\rho^2 + \Phi^2(s, \omega)} + \frac{\beta_2^2}{2} \Phi^2(s, \omega) + \frac{\beta_3^2 \Phi^2(s, \omega)}{2(\rho^2 + \Phi^2(s, \omega))^2} \right] \, ds \\ &\quad + \int_T^t \left[\xi \Phi(s, \omega) + \frac{\lambda \Phi(s, \omega)}{\rho^2 + \Phi^2(s, \omega)} + \frac{\beta_2^2}{2} \Phi^2(s, \omega) + \frac{\beta_3^2 \Phi^2(s, \omega)}{2(\rho^2 + \Phi^2(s, \omega))^2} \right] \, ds \\ &\leq M_2 + \left(\xi + \lambda / \rho^2 + \frac{\beta_2^2}{2} \epsilon + \frac{\beta_3^2}{2\rho^4} \epsilon \right) \epsilon (t - T) \\ &\leq 2 \left(\xi + \lambda / \rho^2 + \frac{\beta_2^2}{2} + \frac{\beta_3^2}{2\rho^4} \right) \epsilon t, \end{aligned}$$

where M_1 and M_2 are positive constants. Consequently,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} t^{-1} [\ln \Phi(t, \omega) - \ln \Phi(0)] &\leq 0, \quad \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \beta_2 \Phi(s, \omega) \, d\psi_2(s) = 0, \\ \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \left[\xi \Phi(s, \omega) + \frac{\lambda \Phi(s, \omega)}{\rho^2 + \Phi^2(s, \omega)} + \frac{\beta_2^2}{2} \Phi^2(s, \omega) + \frac{\beta_3^2 \Phi^2(s, \omega)}{2(\rho^2 + \Phi^2(s, \omega))^2} \right] \, ds &= 0. \end{aligned}$$

We then deduce from (12) and (13) that

$$0 > \limsup_{t \rightarrow +\infty} t^{-1} \ln \Phi(t, \omega) = \Lambda > 0,$$

a contradiction. As a result, $P\{L\} = 0$.

Denote

$$\tilde{L} = \left\{ \omega : \limsup_{t \rightarrow +\infty} \Phi(t, \omega) > 0 \right\},$$

and hence $\tilde{L} \cap L = \emptyset$. Note that $\Phi(t) > 0$, and therefore $\tilde{L} \cup L = \Omega$. Then we deduce from $P\{L\} = 0$ that $P\{\tilde{L}\} = 1$, which is the required statement. \square

Theorems 2–4 uncover that Λ is the threshold between extermination and weak persistence of the species. Now let us test the permanence of the species.

Theorem 5 *If $\bar{\Lambda} := \Lambda_1 - \limsup_{t \rightarrow +\infty} H_1(T(t)) > 0$, then the species is stochastically permanent, that is, for any $\epsilon \in (0, 1)$, we can find two constants $\sigma_1 = \sigma_1(\epsilon) > 0$ and $\sigma_2 = \sigma_2(\epsilon) > 0$ such that*

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{\Phi(t) \geq \sigma_1\} \geq 1 - \epsilon, \quad \liminf_{t \rightarrow +\infty} \mathbb{P}\{\Phi(t) \leq \sigma_2\} \geq 1 - \epsilon.$$

Proof Set

$$W_2(\Phi) = 1/\Phi^2, \quad \Phi > 0.$$

Then we deduce by Itô's formula that

$$\begin{aligned} dW_2(\Phi) &= 2W_2(\Phi) \left[\xi \Phi + \frac{\lambda \Phi}{\rho^2 + \Phi^2} - b + H_1(T(t)) \right] dt \\ &\quad + 3\beta_1^2 W_2(\Phi) dt + 3\beta_2^2 dt + \frac{3\beta_3^2}{(\rho^2 + \Phi^2)^2} dt \\ &\quad - 2\beta_1 W_2(\Phi) d\psi_1(t) - 2\beta_2 \Phi^{-1} d\psi_2(t) - \frac{2\beta_3 \sqrt{W_2(\Phi)}}{\rho^2 + \Phi^2} d\psi_3(t) \\ &= 2W_2(\Phi) \left\{ 1.5\beta_2^2 \Phi^2 + \xi \Phi - b + H_1(T(t)) + 1.5\beta_1^2 + \frac{\lambda \Phi}{\rho^2 + \Phi^2} + \frac{1.5\beta_3^2 \Phi^2}{(\rho^2 + \Phi^2)^2} \right\} dt \\ &\quad - 2\beta_1 W_2(\Phi) d\psi_1(t) - 2\beta_2 \Phi^{-1} d\psi_2(t) - \frac{2\beta_3 \sqrt{W_2(\Phi)}}{\rho^2 + \Phi^2} d\psi_3(t). \end{aligned}$$

Choose a constant $\varpi \in (0, 1)$ that satisfies

$$\varpi < \bar{\Lambda}/\beta_1^2. \quad (17)$$

Set

$$W_3(\Phi) = (1 + W_2(\Phi))^\varpi.$$

Then we deduce by Itô's formula that for sufficiently large t ,

$$\mathbb{E} W_3(\Phi(t)) = W_3(\Phi(0)) + \mathbb{E} \int_0^t \mathcal{L} W_3(\Phi(s)) ds,$$

where

$$\begin{aligned} \mathcal{L} W_3(\Phi) &= 2\varpi (1 + W_2(\Phi))^{\varpi-2} \left\{ (W_2(\Phi) + W_2^2(\Phi)) \left[1.5\beta_2^2 \Phi^2 + \xi \Phi + \frac{\lambda \Phi}{\rho^2 + \Phi^2} \right] \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1.5\beta_3^2\Phi^2}{(\rho^2 + \Phi^2)^2} - b + H_1(T(t)) + 1.5\beta_1^2 \Big] \\
& + (\varpi - 1)W_2^2(\Phi)\beta_1^2 + (\varpi - 1)W_2(\Phi)\beta_2^2 + \frac{(\varpi - 1)W_2(\Phi)\beta_3^2}{(\rho^2 + \Phi^2)^2} \Big\} \\
& = 2\varpi(1 + W_2(\Phi))^{\varpi-2} \Big\{ \Big[-b + H_1(T(t)) + \frac{\beta_1^2}{2} + \varpi\beta_1^2 \Big] W_2^2(\Phi) \\
& + \xi W_2^{1.5}(\Phi) + \frac{\lambda}{\rho^2 + \Phi^2} W_2^{1.5}(\Phi) \\
& + \Big[-b + H_1(T(t)) + 1.5\beta_1^2 + \frac{\lambda\Phi}{\rho^2 + \Phi^2} + \frac{(\varpi + 0.5)\beta_3^2}{(\rho^2 + \Phi^2)^2} + (\varpi + 0.5)\beta_2^2 \Big] W_2(\Phi) \\
& + \xi W_2^{0.5}(\Phi) + 1.5\beta_2^2 + \frac{1.5\beta_3^2}{(\rho^2 + \Phi^2)^2} \Big\} \\
& \leq 2\varpi(1 + W_2(\Phi))^{\varpi-2} \Big\{ \Big[-\Lambda_1 + \limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + \varpi\beta_1^2 \Big] W_2^2(\Phi) \\
& + \Big(\xi + \frac{\lambda}{\rho^2} \Big) W_2^{1.5}(\Phi) \\
& + \Big[\limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + 1.5(\beta_1^2 + \beta_2^2 + \beta_3^2/\rho^4) + \frac{\lambda}{2\rho} \Big] W_2(\Phi) \\
& + \xi W_2^{0.5}(\Phi) + 1.5\beta_2^2 + 1.5\beta_3^2/\rho^4 \Big\} \\
& = 2\varpi(1 + W_2(\Phi))^{\varpi-2} \Big\{ \Big[\bar{\Lambda} - \varpi\beta_1^2 - \epsilon \Big] W_2^2(a) + \Big(\xi + \frac{\lambda}{\rho^2} \Big) W_2^{1.5}(\Phi) \\
& + \Big[\limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + 1.5(\beta_1^2 + \beta_2^2 + \beta_3^2/\rho^4) + \frac{\lambda}{2\rho} \Big] W_2(\Phi) \\
& + \xi W_2^{0.5}(\Phi) + 1.5\beta_2^2 + 1.5\beta_3^2/\rho^4 \Big\}.
\end{aligned}$$

Here $\epsilon < \bar{\Lambda} - \varpi\beta_1^2$. Choose a constant $\nu > 0$ that satisfies

$$\bar{\Lambda} - \varpi\beta_1^2 - \epsilon - \frac{\nu}{2\varpi} > 0. \quad (18)$$

Set

$$W_4(\Phi) = e^{\nu t} W_3(\Phi).$$

Then we deduce by Itô's formula that for sufficiently large t ,

$$\mathbb{E} W_4(\Phi(t)) = W_3(\Phi(0)) + \mathbb{E} \int_0^t \mathcal{L} W_4(\Phi(s)) \, ds,$$

where

$$\begin{aligned}
\mathcal{L} W_4(\Phi) & = \nu e^{\nu t} (1 + W_2(\Phi))^{\varpi} + e^{\nu t} \mathcal{L} W_3(\Phi) \\
& \leq 2e^{\nu t} \varpi (1 + W_2(\Phi))^{\varpi-2} \Big\{ \Big[\bar{\Lambda} - \varpi\beta_1^2 - \frac{\nu}{2\varpi} - \epsilon \Big] W_2^2(\Phi)
\end{aligned}$$

$$\begin{aligned}
& + \left(\xi + \frac{\lambda}{\rho^2} \right) W_2^{1.5}(\Phi) \\
& + \left[\limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + 1.5(\beta_1^2 + \beta_2^2 + \beta_3^2/\rho^4) + \frac{\lambda}{2\rho} + \frac{\nu}{\varpi} \right] W_2(\Phi) \\
& + \xi W_2^{0.5}(\Phi) + 1.5\beta_2^2 + 1.5\beta_3^2/\rho^4 + \frac{\nu}{2\varpi} \Big\} \\
& =: e^{\nu t} \sigma(\Phi).
\end{aligned}$$

According to (18),

$$\tilde{\sigma} := \sup_{\Phi > 0} \sigma(\Phi) < +\infty.$$

Thereby

$$\mathbb{E}[e^{\nu t} (1 + W_2(\Phi(t)))^{\varpi}] \leq (1 + \Phi^{-2}(0))^{\varpi} + \nu^{-1} \tilde{\sigma} (e^{\nu t} - 1).$$

Accordingly,

$$\begin{aligned}
\limsup_{t \rightarrow +\infty} \mathbb{E}[\Phi^{-2\varpi}(t)] &= \limsup_{t \rightarrow +\infty} \mathbb{E}[W_2^{\varpi}(\Phi(t))] \\
&\leq \limsup_{t \rightarrow +\infty} \mathbb{E}[(1 + W_2(\Phi(t)))^{\varpi}] \leq \tilde{\sigma}.
\end{aligned} \tag{19}$$

For any $\epsilon > 0$, set $\sigma_1 = (\epsilon/\tilde{\sigma})^{\frac{1}{2\varpi}}$. By Chebyshev's inequality (see [33], p. 5),

$$\mathbb{P}\{\Phi(t) < \sigma_1\} = \mathbb{P}\{\Phi^{-2\varpi}(t) > \sigma_1^{-2\varpi}\} \leq \frac{\mathbb{E}[\Phi^{-2\varpi}(t)]}{\sigma_1^{-2\varpi}} = \sigma_1^{2\varpi} \mathbb{E}[\Phi^{-2\varpi}(t)].$$

Thereby

$$\limsup_{t \rightarrow +\infty} \mathbb{P}\{\Phi(t) < \sigma_1\} \leq \epsilon.$$

Accordingly,

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{\Phi(t) \geq \sigma_1\} \geq 1 - \epsilon.$$

Now we testify

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{\Phi(t) \leq \sigma_2\} \geq 1 - \epsilon. \tag{20}$$

Set

$$W_5(\Phi) = \Phi^c, \quad \Phi > 0, 0 < c < 1.$$

Then we deduce by Itô's formula that

$$d(e^t W_5(\Phi))$$

$$\begin{aligned}
&= e^t \left\{ \Phi^c + c\Phi^c \left[b - \xi\Phi - \frac{\lambda\Phi}{\rho^2 + \Phi^2} + \frac{c-1}{2}\beta_1^2 + \frac{c-1}{2}\beta_2^2\Phi^2 + \frac{(c-1)\beta_3^2\Phi^2}{2(\rho^2 + \Phi^2)^2} \right] \right\} dt \\
&\quad + c\beta_1 e^t \Phi^c d\psi_1(t) + c\beta_2 e^t \Phi^{c+1} d\psi_2(t) + c\beta_3 e^t \frac{\Phi^{c+1}}{\rho^2 + \Phi^2} d\psi_3(t) \\
&\leq e^t \sigma_3 dt + ce^t \Phi^c \left[\beta_1 d\psi_1(t) + \beta_2 \Phi d\psi_2(t) + \frac{\beta_3 \Phi}{\rho^2 + \Phi^2} d\psi_3(t) \right],
\end{aligned}$$

where $\sigma_3 > 0$ is a constant. Accordingly,

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[\Phi^c(t)] \leq \sigma_3.$$

By Chebyshev's inequality we derive (20). □

5 Upper- and lower-growth rates

Theorem 6 For model (4), we have

$$\limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{\ln t} \leq 1, \quad a.s. \quad (21)$$

Proof We deduce from Itô's formula that

$$\begin{aligned}
&d[e^t \ln \Phi] \\
&= e^t \left[\ln \Phi + b - H_1(T(t)) - \frac{\beta_1^2}{2} - \xi\Phi - \frac{\beta_2^2}{2}\Phi^2 - \frac{\lambda\Phi}{\rho^2 + \Phi^2} - \frac{\beta_3^2\Phi^2}{2(\rho^2 + \Phi^2)^2} \right] dt \\
&\quad + \beta_1 e^t d\psi_1(t) + \beta_2 e^t \Phi d\psi_2(t) + \frac{\beta_3 e^t \Phi}{\rho^2 + \Phi^2} d\psi_3(t).
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&e^t \ln \Phi(t) - \ln \Phi(0) \\
&= \int_0^t e^s \left[\ln \Phi(s) + b - H_1(T(s)) - \frac{\beta_1^2}{2} - \xi\Phi(s) - \frac{\beta_2^2}{2}\Phi^2(s) - \frac{\lambda\Phi(s)}{\rho^2 + \Phi^2(s)} \right. \\
&\quad \left. - \frac{\beta_3^2\Phi^2(s)}{2(\rho^2 + \Phi^2(s))^2} \right] ds + \Gamma_4(t) + \Gamma_5(t) + \Gamma_6(t),
\end{aligned} \quad (22)$$

where

$$\begin{aligned}
\Gamma_4(t) &= \int_0^t \beta_1 e^s d\psi_1(s), & \Gamma_5(t) &= \int_0^t \beta_2 e^s \Phi(s) d\psi_2(s), \\
\Gamma_6(t) &= \int_0^t \frac{\beta_3 e^s \Phi(s)}{\rho^2 + \Phi^2(s)} d\psi_3(s).
\end{aligned}$$

Set

$$\Gamma(t) = \Gamma_4(t) + \Gamma_5(t) + \Gamma_6(t).$$

Thereby

$$\langle \Gamma(t), \Gamma(t) \rangle = \int_0^t e^{2s} \left[\beta_1^2 + \beta_2^2 \Phi^2(s) + \frac{\beta_3^2 \Phi^2(s)}{(\rho^2 + \Phi^2(s))^2} \right] ds.$$

By the exponential martingale inequality (see [33], p. 44, Theorem 7.4), for any $\kappa > 1$ and $\varrho > 0$,

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq \varrho m} \left[\Gamma(t) - \frac{e^{-\varrho m}}{2} \langle \Gamma(t), \Gamma(t) \rangle \right] > \kappa e^{\varrho m} \ln m \right\} \leq m^{-\kappa}.$$

By Borel–Cantelli’s lemma (see [33], p. 7, Lemma 2.4) we deduce that for almost all $\omega \in \Omega$, there is m_2 such that for any $m \geq m_2$,

$$\Gamma(t) \leq \frac{e^{-\varrho m}}{2} \langle \Gamma(t), \Gamma(t) \rangle + e^{\varrho m} \ln m, \quad 0 \leq t \leq \varrho m.$$

Accordingly, for $m \geq m_2$ and $0 \leq t \leq \varrho m$,

$$\Gamma(t) \leq \frac{e^{-\varrho m}}{2} \int_0^t e^{2s} \left[\beta_1^2 + \beta_2^2 \Phi^2(s) + \frac{\beta_3^2 \Phi^2(s)}{(\rho^2 + \Phi^2(s))^2} \right] ds + \kappa e^{\varrho m} \ln m.$$

Then from (22) it follows that for $m \geq m_2$ and $0 \leq t \leq \varrho m$,

$$\begin{aligned} & e^t \ln \Phi(t) - \ln \Phi(0) \\ & \leq \int_0^t e^s \left[\ln \Phi(s) + b - H_1(T(s)) - \frac{\beta_1^2}{2} - \xi \Phi(s) - \frac{\beta_2^2}{2} \Phi^2(s) \right. \\ & \quad \left. - \frac{\lambda \Phi(s)}{\rho^2 + \Phi^2(s)} - \frac{\beta_3^2 \Phi^2(s)}{2(\rho^2 + \Phi^2(s))^2} \right] ds \\ & \quad + \frac{e^{-\varrho m}}{2} \int_0^t e^{2s} \left[\beta_1^2 + \beta_2^2 \Phi^2(s) + \frac{\beta_3^2 \Phi^2(s)}{(\rho^2 + \Phi^2(s))^2} \right] ds + \kappa e^{\varrho m} \ln m \\ & = \int_0^t e^s \left[\ln \Phi(s) + b - H_1(T(s)) - \xi \Phi(s) \right. \\ & \quad \left. - \frac{1 - e^{s-\varrho m}}{2} \left(\beta_1^2 + \beta_2^2 \Phi^2(s) + \frac{\beta_3^2 \Phi^2(s)}{\rho^2 + \Phi^2(s)} \right) \right] ds + \kappa e^{\varrho m} \ln m \\ & \leq \int_0^t e^s [\ln \Phi(s) + b - \xi \Phi(s)] ds + \kappa e^{\varrho m} \ln m \\ & \leq \sigma_4 (e^t - 1) + \kappa e^{\varrho m} \ln m, \end{aligned}$$

where $\sigma_4 = \max\{1, -\ln \xi + b - 1\}$. As a result, for $0 < \varrho(m-1) \leq t \leq \varrho m$ and $m \geq m_2$, we derive

$$\frac{\ln \Phi(t)}{\ln t} \leq \frac{e^{-t} \ln \Phi(0)}{\ln t} + \frac{\sigma_4(1 - e^{-t})}{\ln t} + \frac{\kappa e^{-\varrho(m-1)} e^{\varrho m} \ln m}{\ln(\varrho(m-1))}.$$

Note that

$$\lim_{m \rightarrow +\infty} \frac{\ln m}{\ln(\varrho(m-1))} = \lim_{m \rightarrow +\infty} \frac{\ln m}{\ln \varrho + \ln(m-1)} = 1.$$

Accordingly,

$$\limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{\ln t} \leq \kappa e^{\varrho}.$$

Letting $\kappa \rightarrow 1$ and $\varrho \rightarrow 0$, we obtain (21). \square

Theorem 6 probes the upper-growth rate of $\Phi(t)$. Now let us consider the lower-growth rate of $\Phi(t)$.

Theorem 7 *If $\bar{\Lambda} > 0$, then*

$$\liminf_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{\ln t} \geq -\frac{1}{2\varpi}. \quad (23)$$

Proof We can deduce from (19) that there is a constant $\sigma_5 > 0$ such that

$$\mathbb{E}[(1 + W_2(\Phi(t)))^{\varpi}] \leq \sigma_5, \quad t \geq 0. \quad (24)$$

By Itô's formula,

$$\begin{aligned} & d[(1 + W_2(\Phi))^{\varpi}] \\ &= 2\varpi(1 + W_2(\Phi))^{\varpi-2} \left\{ \left[-b + H_1(T(t)) + \frac{\beta_1^2}{2} + \frac{\lambda\Phi}{\rho^2 + \Phi^2} + \varpi\beta_1^2 \right] W_2^2(\Phi) \right. \\ & \quad + \xi W_2^{1.5}(\Phi) + \left[-b + H_1(T(t)) + 1.5\beta_1^2 + \frac{\lambda\Phi}{\rho^2 + \Phi^2} \right. \\ & \quad + \left. \frac{(\varpi + 1/2)\beta_3^2}{(\rho^2 + \Phi^2)^2} + (\varpi + 1/2)\beta_2^2 \right] W_2(\Phi) \\ & \quad + \left. \xi W_2^{0.5}(\Phi) + 1.5\beta_2^2 + \frac{1.5\beta_3^2}{(\rho^2 + \Phi^2)^2} \right\} \\ & \quad - 2\varpi(1 + W_2(\Phi))^{\varpi-1} \left[\beta_1 W_2(\Phi) d\psi_1(t) + \beta_2 \Phi^{-1} d\psi_2(t) + \frac{\beta_3 \Phi^{-1}}{\rho^2 + \Phi^2} d\psi_3(t) \right] \\ & \leq 2\varpi(1 + W_2(\Phi))^{\varpi-2} \{ \mu_1 W_2^2(\Phi) + \mu_2 W_2^{1.5}(\Phi) + \mu_3 W_2(\Phi) + \mu_2 W_2^{0.5}(\Phi) + \mu_4 \} \\ & \quad - 2\varpi(1 + W_2(\Phi))^{\varpi-1} \left[\beta_1 W_2(\Phi) d\psi_1(t) + \beta_2 \Phi^{-1} d\psi_2(t) + \frac{\beta_3 \Phi^{-1}}{\rho^2 + \Phi^2} d\psi_3(t) \right], \end{aligned}$$

where

$$\begin{aligned} \mu_1 &= -b + \limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + \left(\varpi + \frac{1}{2} \right) \beta_1^2 + \frac{\lambda}{2\rho}, \quad \mu_2 = \xi, \\ \mu_3 &= -b + \limsup_{t \rightarrow +\infty} H_1(T(t)) + \epsilon + 1.5 \left(\beta_1^2 + \frac{\beta_3^2}{2\rho^2} \right) + \frac{\lambda}{2\rho} + \left(\varpi + \frac{1}{2} \right) \beta_2^2, \\ \mu_4 &= 1.5 \left\{ \beta_2^2 + \frac{\beta_3^2}{2\rho^2} \right\}. \end{aligned}$$

Choose a positive constant σ_6 such that

$$2\varpi(\mu_1 W_2^2(\Phi) + \mu_2 W_2^{1.5}(\Phi) + \mu_3 W_2(\Phi) + \mu_2 W_2^{0.5}(\Phi) + \mu_4) \leq \sigma_6(1 + W_2(\Phi))^2.$$

Accordingly,

$$\begin{aligned} d((1 + W_2(\Phi))^{\varpi}) &\leq \sigma_6 (1 + W_2(\Phi))^{\varpi} dt - 2\varpi (1 + W_2(\Phi))^{\varpi-1} \\ &\quad \times \left[\beta_1 W_2(\Phi) d\psi_1(t) + \beta_2 \Phi^{-1} d\psi_2(t) + \frac{\beta_3 \Phi^{-1}}{\rho^2 + \Phi^2} d\psi_3(t) \right]. \end{aligned} \quad (25)$$

Choose a positive constant θ such that

$$\sigma_6 \theta + 12\varpi \theta^{0.5} (\beta_1 + \beta_2 + \beta_3 / \rho^2) < \frac{1}{2}. \quad (26)$$

Let $N = 1, 2, \dots$. Then from (25) we deduce that

$$\begin{aligned} &\mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^{\varpi} \right) \\ &\leq \mathbb{E} (1 + W_2(\Phi((N-1)\theta)))^{\varpi} \\ &\quad + \sigma_6 \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi} ds \right| \right) \\ &\quad + 2\varpi \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi-1} \left[\beta_1 W_2(\Phi(s)) d\psi_1(s) \right. \right. \right. \\ &\quad \left. \left. + \beta_2 \Phi^{-1}(s) d\psi_2(s) + \frac{\beta_3 \Phi^{-1}(s)}{\rho^2 + \Phi^2(s)} d\psi_3(s) \right] \right| \right). \end{aligned} \quad (27)$$

By the Burkholder–Davis–Gundy inequality (see [33], p. 40, Theorem 7.3) we have

$$\begin{aligned} &\mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi-1} \left[\beta_1 W_2(\Phi(s)) d\psi_1(s) \right. \right. \right. \\ &\quad \left. \left. + \beta_2 \Phi^{-1}(s) d\psi_2(s) + \frac{\beta_3 \Phi^{-1}(s)}{\rho^2 + \Phi^2(s)} d\psi_3(s) \right] \right| \right) \\ &\leq \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi-1} [\beta_1 W_2(\Phi(s))] d\psi_1(s) \right| \right) \\ &\quad + \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi-1} [\beta_2 \Phi^{-1}(s)] d\psi_2(s) \right| \right) \\ &\quad + \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^{\varpi-1} \left[\frac{\beta_3 \Phi^{-1}(s)}{\rho^2 + \Phi^2(s)} \right] d\psi_3(s) \right| \right) \\ &\leq 6\mathbb{E} \left(\int_{(N-1)\theta}^{N\theta} (1 + W_2(\Phi(s)))^{2\varpi-2} [\beta_1 W_2(\Phi(s))]^2 ds \right)^{0.5} \\ &\quad + 6\mathbb{E} \left(\int_{(N-1)\theta}^{N\theta} (1 + W_2(\Phi(s)))^{2\varpi-2} [\beta_2 \Phi^{-1}(s)]^2 ds \right)^{0.5} \\ &\quad + 6\mathbb{E} \left(\int_{(N-1)\theta}^{N\theta} (1 + W_2(\Phi(s)))^{2\varpi-2} \left[\frac{\beta_3}{\rho^2} \Phi^{-1}(s) \right]^2 ds \right)^{0.5} \\ &\leq 6\theta^{0.5} [\beta_1 + \beta_2 + \beta_3 / \rho^2] \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^{\varpi} \right). \end{aligned} \quad (28)$$

Moreover,

$$\begin{aligned} & \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} \left| \int_{(N-1)\theta}^t (1 + W_2(\Phi(s)))^\varpi \, ds \right| \right) \\ & \leq \mathbb{E} \left(\int_{(N-1)\theta}^{N\theta} |(1 + W_2(\Phi(s)))^\varpi| \, ds \right) \\ & \leq \theta \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^\varpi \right). \end{aligned} \quad (29)$$

Then from (27), (28), and (29) it follows that

$$\begin{aligned} & \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^\varpi \right) \\ & \leq \mathbb{E} (1 + W_2(\Phi((N-1)\theta)))^\varpi \\ & \quad + [\sigma_6\theta + 12\varpi\theta^{0.5}(\beta_1 + \beta_2 + \beta_3/\rho^2)] \mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^\varpi \right). \end{aligned}$$

By (24) and (26) we get that

$$\mathbb{E} \left(\sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^\varpi \right) \leq 2\sigma_5.$$

For any $\epsilon > 0$, Chebyshev's inequality suggests that

$$\mathbb{P} \left\{ \sup_{(N-1)\theta \leq t \leq N\theta} (1 + W_2(\Phi(t)))^\varpi > (N\theta)^{1+\epsilon} \right\} \leq \frac{2\sigma_5}{(N\theta)^{1+\epsilon}}, \quad N = 1, 2, \dots$$

By the Borel–Cantelli lemma, for almost all $\omega \in \Omega$, we can find an integer N_0 such that for any $N \geq N_0$ and $(N-1)\theta \leq t \leq N\theta$,

$$\frac{\ln(1 + W_2(\Phi(t)))^\varpi}{\ln t} \leq \frac{(1 + \epsilon) \ln(N\theta)}{\ln((N-1)\theta)}.$$

Accordingly,

$$\limsup_{t \rightarrow +\infty} \frac{\ln(1 + W_2(\Phi(t)))^\varpi}{\ln t} \leq 1 + \epsilon.$$

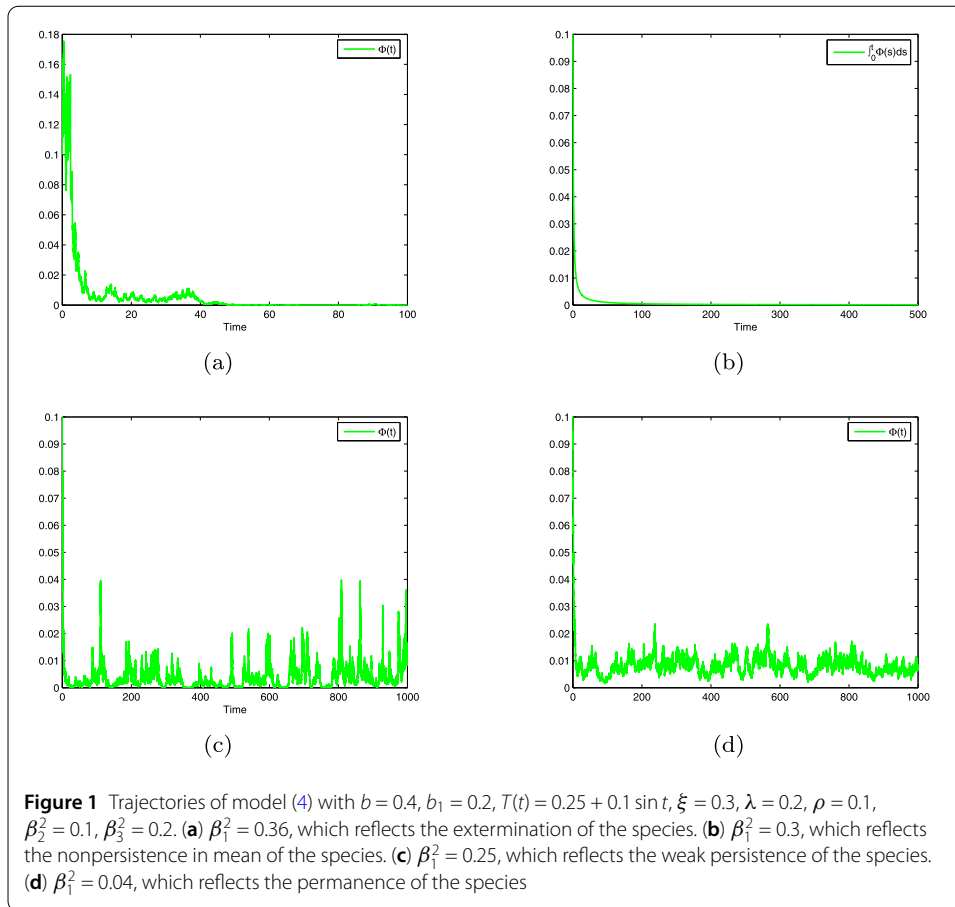
Letting $\epsilon \rightarrow 0$ results in

$$\limsup_{t \rightarrow +\infty} \frac{\ln(1 + W_2(\Phi(t)))^\varpi}{\ln t} \leq 1.$$

Thereby

$$\limsup_{t \rightarrow +\infty} \frac{\ln(\Phi^{-2\varpi}(t))}{\ln t} \leq 1,$$

which is (23). □



6 Conclusions and simulations

In this paper, we have constructed a stochastic single-species model with predator effects in polluted environments. We have probed some dynamical properties of the model, including the existence and uniqueness of the solution (Theorem 1), the threshold between extermination and persistence (Theorems 2–4), stochastic permanence (Theorem 5), and upper- and lower-growth rates (Theorems 6 and 7). To our best knowledge, this paper is the first one to probe population models with predation effect in a polluted environment.

Now let us numerically expound the theoretical findings by the Milstein method [34]. We choose $H(T(t)) = b_1 T(t)$ and pay attention to the following discretization equation of model (4):

$$\begin{aligned} \Phi_{k+1} = & \Phi_k + \Phi_k \left(b - b_1 T(k\Delta t) - \xi \Phi_k - \frac{\lambda \Phi_k}{\rho^2 + \Phi_k^2} \right) \Delta t \\ & + \beta_1 \Phi_k \sqrt{\Delta t} \xi_{1k} + 0.5 \beta_1^2 \Phi_k^2 (\xi_{1k}^2 \Delta t - \Delta t) \\ & + \beta_2 \Phi_k^2 \sqrt{\Delta t} \xi_{2k} + 0.5 \beta_2^2 \Phi_k^4 (\xi_{2k}^2 \Delta t - \Delta t) \\ & + \frac{\beta_3 \Phi_k^2}{\rho^2 + \Phi_k^2} \sqrt{\Delta t} \xi_{3k} + \frac{\beta_3^2 \Phi_k^4}{2(\rho^2 + \Phi_k^2)^2} \sqrt{\Delta t} (\xi_{3k}^2 \Delta t - \Delta t), \end{aligned}$$

where ξ_{1k} , ξ_{2k} , and ξ_{3k} are standard Gaussian random variables. We set $b = 0.4$, $b_1 = 1$, $T(t) = 0.25 + 0.1 \sin t$, $\xi = 0.3$, $\lambda = 0.2$, $\rho = 0.1$, $\beta_2^2 = 0.1$, $\beta_3^2 = 0.2$. For different β_1 , we plot the trajectories of $\Phi(t)$ in Fig. 1(a)–(d).

1. In Fig. 1(a), $\beta_1^2 = 0.36$, and hence

$$\Lambda = b - \frac{1}{2}\beta_1^2 - \liminf_{t \rightarrow +\infty} t^{-1} \int_0^t H_1(T(s)) \, ds = -0.03.$$

Then from Theorem 2 we deduce that $\lim_{t \rightarrow +\infty} \Phi(t) = 0$. See Fig. 1(a).

2. In Fig. 1(b), $\beta_1^2 = 0.3$, and hence $\Lambda = 0$. Then from Theorem 3 we deduce that $\lim_{t \rightarrow +\infty} t^{-1} \int_0^t \Phi(s) \, ds = 0$. See Fig. 1(b).
3. In Fig. 1(c), $\beta_1^2 = 0.25$, and hence $\Lambda = 0.025$. Then from Theorem 4 we deduce that $\limsup_{t \rightarrow +\infty} \Phi(t) > 0$. See Fig. 1(c).
4. In Fig. 1(d), $\beta_1^2 = 0.04$, and hence

$$\overline{\Lambda} = b - \frac{1}{2}\beta_1^2 - \limsup_{t \rightarrow +\infty} H_1(T(t)) = 0.03.$$

Then from Theorem 5 we deduce that $\Phi(t)$ is permanent. See Fig. 1(d).

To finish this paper, we want to point out that our model (7) may be used to describe the effect of pollution in the usual situation, but it may not well describe some extreme pollution cases (for instance, some extreme examples given in Sect. 1). For these extreme cases, more complicate models should be constructed. We leave these problems for further research.

Acknowledgements

The author thanks the editor and the reviewers for their valuable comments.

Funding

Not applicable.

Availability of data and materials

All data are included in this paper.

Competing interests

The author declares that they have no competing interests.

Authors' contributions

The single author is responsible for the complete manuscript. The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 September 2020 Accepted: 11 February 2021 Published online: 25 March 2021

References

1. Maurin, C.: Accidental Oil Spills: Biological and Ecological Consequences of Accidents in French Waters on Commercially Exploitable Living Marine Resources. Wiley, Chichester (1984)
2. Ichimura, Y., Baker, D.: Acute inhalational injury. In: Reference Module in Biomedical Sciences (2019). <https://doi.org/10.1016/B978-0-12-801238-3.11495-3>
3. <http://www.fws.gov/birds/bird-enthusiasts/threats-to-birds.php>
4. Hallam, T., Clark, C., Lassider, R.: Effects of toxicant on population: a qualitative approach I. Equilibrium environmental exposure. *Ecol. Model.* **8**, 291–304 (1983)
5. Hallam, T., Clar, C., Jordan, G.: Effects of toxicant on population: a qualitative approach II. First order kinetics. *J. Math. Biol.* **109**, 411–429 (1983)
6. Hallam, T., Deluna, J.: Effects of toxicant on populations: a qualitative approach III. Environmental and food chain pathways. *J. Theor. Biol.* **109**, 411–429 (1984)

7. Freedman, H., Shukla, J.: Models for the effect of toxicant in single-species and predator–prey systems. *J. Math. Biol.* **30**, 15–30 (1991)
8. Liu, H., Ma, Z.: The threshold of survival for system of two species in a polluted environment. *J. Math. Biol.* **30**, 49–51 (1991)
9. Gard, T.: Stochastic models for toxicant-stressed populations. *Bull. Math. Biol.* **54**, 827–837 (1992)
10. Liu, B., Chen, L., Zhang, Y.: The effects of impulsive toxicant input on a population in a polluted environment. *J. Biol. Syst.* **11**, 265–274 (2003)
11. He, J., Wang, K.: The survival analysis for a population in a polluted environment. *Nonlinear Anal., Real World Appl.* **10**, 1555–1571 (2009)
12. Liu, M., Wang, K.: Survival analysis of stochastic single-species population models in polluted environments. *Ecol. Model.* **220**, 1347–1357 (2009)
13. Liu, M., Wang, K.: Persistence and extinction of a stochastic single-specie model under regime switching in a polluted environment. *J. Theor. Biol.* **264**, 934–944 (2010)
14. Liu, M., Wang, K., Wu, Q.: Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle. *Bull. Math. Biol.* **73**, 1969–2012 (2011)
15. Zhang, S., Tan, D.: Dynamics of a stochastic predator–prey system in a polluted environment with pulse toxicant input and impulsive perturbations. *Appl. Math. Model.* **39**, 6319–6331 (2015)
16. Liu, M.: Survival analysis of a cooperation system with random perturbations in a polluted environment. *Nonlinear Anal. Hybrid Syst.* **18**, 100–116 (2015)
17. Liu, M., Du, C., Deng, M.: Persistence and extinction of a modified Leslie–Gower Holling-type II stochastic predator–prey model with impulsive toxicant input in polluted environments. *Nonlinear Anal. Hybrid Syst.* **27**, 177–190 (2018)
18. May, R.: *Stability and Complexity in Model Ecosystems*. Princeton University Press, NJ (2001)
19. Ludwig, D., Jones, D., Holling, C.: Qualitative analysis of insect outbreak systems: the spruce budworm and forest. *J. Anim. Ecol.* **47**, 315–332 (1978)
20. Braumann, C.A.: Variable effort harvesting models in random environments: generalization to density-dependent noise intensities. *Math. Biosci.* **177–178**, 229–245 (2002)
21. Øksendal, B.: *Stochastic Differential Equations: An Introduction with Applications*, 5th edn. Springer, Berlin (1998)
22. Beddington, J., May, R.: Harvesting natural populations in a randomly fluctuating environment. *Science* **197**, 463–465 (1977)
23. Zhu, C., Yin, G.: On hybrid competitive Lotka–Volterra ecosystems. *Nonlinear Anal.* **71**, e1370–e1379 (2009)
24. Tan, R., Wang, H., Xiang, H., Liu, Z.: Dynamic analysis of a nonautonomous impulsive single-species system in random environments. *Adv. Differ. Equ.* **218**, 1–17 (2015)
25. Liu, Z., Guo, S., Tan, R., Liu, M.: Modeling and analysis of a non-autonomous single-species model with impulsive and random perturbations. *Appl. Math. Model.* **4**, 5510–5531 (2016)
26. Liu, M., Zhu, Y.: Stability of a budworm growth model with random perturbations. *Appl. Math. Lett.* **79**, 13–19 (2018)
27. Lv, H., Liu, Z., Chen, Y., Chen, J., Xu, D.: Stochastic permanence of two impulsive stochastic delay single species systems incorporating predation term. *J. Appl. Math. Comput.* **56**, 691–713 (2018)
28. Liu, M., Bai, C.: Optimal harvesting of a stochastic mutualism model with regime-switching. *Appl. Math. Comput.* **375**, 125040 (2020)
29. Li, D., Liu, M.: Invariant measure of a stochastic food-limited population model with regime switching. *Math. Comput. Simul.* **178**, 16–26 (2020)
30. Ji, W., Hu, G.: Stability and explicit stationary density of a stochastic single-species model. *Appl. Math. Comput.* **390**, 125593 (2021)
31. Ji, W., Wang, Z., Hu, G.: Stationary distribution of a stochastic hybrid phytoplankton model with allelopathy. *Adv. Differ. Equ.* **2020**, 632 (2020)
32. Ji, W., Zhang, Y., Liu, M.: Dynamical bifurcation and explicit stationary density of a stochastic population model with Allee effects. *Appl. Math. Lett.* **111**, 106662 (2021)
33. Mao, X.: *Stochastic Differential Equations and Applications*. Horwood Publishing, Chichester (1997)
34. Higham, D.: An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.* **43**, 525–546 (2001)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)